A game theoretic application of inverse limit[*]

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October 21, 2005

Abstract

It is a usual problem of modeling games with incomplete information to handle the hierarchies of beliefs. It is our aim to construct a type space introduced by Harsányi, and to show the connection between the concept of hierarchies of beliefs and the idea of inverse system. Therefore, we give an existence theorem for measure inverse limit, which is more general than the previously known theorems, and by this theorem we build a complete universal type space based on a purely measurable parameter space (i.e. non topological).

1 Introduction

During the process of modeling a given situation in game theory, one usually faces the question of how informed the players in the situation, i.e. what the players believe about the given situation, and what the players believe about that what the players believe about the situation, and so on. This phenomena, i.e. the transparent use of hierarchies of beliefs, could make the model extremely difficult. The above mentioned problem is avoidable if one uses the concept of common knowledge, i.e. one defines a game in which every element is common knowledge, so in which every player knows that every players knows that, ... the parameters of the game.

In many cases the given game is common knowledge, however there are many situations when some part of the game, i.e. some parameters of the game is not common knowledge. In the latter cases, it is also the aim to define a commonly known game, i.e. a model, which does not contain hierarchies of beliefs explicitly. Harsányi [7] avoided the problem of hierarchies of beliefs by introducing the concept of type, which means the "types" of players. According to Harsányi "we can regard the vector c_i as representing certain physical, social, and psychological *attributes* of player i himself in that it summarizes some crucial parameters of player i's own payoff function U_i as well as the main parameters of his beliefs about his social and physical environment ... the rules of the game as such allow

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[*]This work was supported by grant OTKA T046194.

any given player i to belong to any one of a number of possible *types*, corresponding to the alternative values of his vector c_i could take ... Each player is assumed to know his own type, but to be in general ignorant about the other players' actual types."

Heifetz and Samet [9] formalized the concept of type space as follows (we don't give the measurable structure here):

Definition 1. *The type space* $< (T_i, \mathcal{M}_i)_{i \in M \cup \{0\}}, m_{i \in M} >$ *(briefly* $< (T, \mathcal{M}), m) >$*) based on parameter space* S *is as follows (where* M *is the set of players, and* 0 *denotes an extra player):*

1. $T_0 = S$, $(T_i, \mathcal{M}_i)$ *is a measurable space* $\forall i \in M \cup \{0\}$,
2. $f_i : T_i \to (\Delta(T, \mathcal{M}), \mathcal{A}_{HS})$ *is a measurable function* $\forall i \in M$, *where* Δ *stands for the probability measures, and* $T = \times_{j \in M \cup \{0\}} T_j$, $\mathcal{M} = \otimes_{j \in M \cup \{0\}} \mathcal{M}_j$,
3. $marg_{\Delta(T_i, \mathcal{M}_i)} f_i(t_i) = \delta_{t_i}$, *where* δ_{t_i} *is the Dirac measure concentrated on* t_i, $\forall t_i \in T_i$.

It is easy to see that, the two last points in the definition above can be drawn together:

Definition 2. *The type space* $< (T_i, \mathcal{M}_i)_{i \in M \cup \{0\}}, m_{i \in M} >$ *(briefly* $< (T, \mathcal{M}), m) >$*) based on parameter space* S *is as follows:*

1. $T_0 = S$, $(T_i, \mathcal{M}_i)$ *is a measurable space* $\forall i \in M \cup \{0\}$,
2. $f_i : T_i \to (\Delta(T_{-i}, \mathcal{M}_{-i}), \mathcal{A}_{HS})$ *is a measurable function* $\forall i \in M$, *where* $T_{-i} = \times_{j \in (M \cup \{0\}) \setminus \{i\}} T_j$, *and* $\mathcal{M}_{-i} = \otimes_{j \in (M \cup \{0\}) \setminus \{i\}} \mathcal{M}_j$.

We discuss two properties of the type space here. The first is the universality. A type space is universal w.r.t. a model, i.e. the parameter space and the set of all feasible beliefs are fixed, if it is richer (broader) than any other type space in the model, i.e. if it contains all beliefs. The second property is the completeness. A type space is complete if every coherent hierarchy of beliefs is a type in it, i.e. if every hierarchy of beliefs in the model can be regarded as a type in Definition 2.

Neither Harsányi nor Heifetz and Samet constructed type space, they assumed it as given. The construction of type space from coherent hierarchies of beliefs is the topic of the works of Böge and Eisele [2], Mertens and Zamir [13], Brandenburger and Dekel [3], Heifetz [8], Mertens et al. [14], and [16]. It seems that, to get a complete universal type space some kind of compactness is needed (see Heifetz and Samet [10]), so although in different ways, every above mentioned paper uses the concept of compactness. This work based on [16], introduces a model in which the parameter space is purely measurable (in the other papers the parameter space is topological), and the beliefs are probability measures such that, their restrictions on the different order belief spaces are compact regular measures. In this model the universal type space is complete.

The next section gives the details of the mathematical apparatus, and the last section covers the game theoretic application of the mathematical results.

2 The existence of measure inverse limit

In this paper the measures are probability measures. The preordered set is a set with a binary relation such that, it is transitive, and if an element of the set relates to an other element, then it also relates to itself. The right directed set, henceforth directed set is a preoredered set such that, every finite subset of it is bounded above. Let $\mathcal{A} \subseteq \mathcal{P}(\mathcal{X})$ be a set ring, $\mathcal{C} \subseteq \mathcal{A}$, μ additive set function on $\mathcal{A}$; μ is inner $\mathcal{C}$-regular if for arbitrary $\epsilon > 0$ and for arbitrary $A \in \mathcal{A}$, $\exists C \in \mathcal{C}$ such that $C \subseteq A$ and $\mu(A \setminus C) < \epsilon$. For arbitrary $Z \in \mathcal{P}(\mathcal{X})$ $\mu^*(Z) \overset{\circ}{=} \max\{\inf_{(A_n)_{n\in\mathbb{N}}\in\mathcal{A},\ Z\subseteq\cup_n A_n} \sum_n \mu(A_n), \sup_{Z\supseteq A\in\mathcal{A}} \mu(A)\}$[1].

Definition 3. *Let $(I, \leq)$ be a preordered set, and let $(X_i)_{i\in I}$ be a family of nonvoid sets. Moreover, let $f_{ij} : X_j \longrightarrow X_i$, if $i \leq j$.*

1. *$(i \leq j$ and $j \leq k) \Longrightarrow f_{ik} = f_{ij} \circ f_{jk}$,*
2. *$\forall i \in I$ $f_{ii} = id_{X_i}$.*

The system $(X_i, (I, \leq), f_{ij}|_{i\leq j})$, which satisfies points 1., 2. is called inverse system.

The inverse system (projective system) is a system of sets connected to each other in a certain way.

Definition 4. *Let $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij}|_{i\leq j})$ be an inverse system, where $(X_i, \mathcal{A}_i, \mu_i)$ is measure space $\forall i \in I$.*

1. *f_{ij} is measurable function $\forall(i \leq j)$,*
2. *$\mu_i = \mu_j \circ f_{ij}^{-1}, \forall(i \leq j)$.*

The system $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij}|_{i\leq j})$, which satisfies points 1., 2. is called measure inverse system.

Definition 5. *Let $(X_i, (I, \leq), f_{ij}|_{i\leq j})$ be an inverse system. Let $X = \prod_{i\in I} X_i$, and $P = \{x \in X \mid pr_i(x) = f_{ij} \circ pr_j(x),\ \forall(i \leq j)\}$, where pr_i is the coordinate projection from X to X_i $\forall i \in I$. P is called the inverse limit of the inverse system $(X_i, (I, \leq), f_{ij}|_{i\leq j})$, and it is denoted by $P = \varprojlim(X_i, (I, \leq), f_{ij}|_{i\leq j})$. Moreover let $p_i \overset{\circ}{=} pr_i|_P$, so $p_i = f_{ij} \circ p_j$ $\forall(i \leq j)$.*

The inverse limit is the generalization of the Cartesian product. If $\leq$ in $(I, \leq)$ is the empty relation, then the inverse limit is the Cartesian product.

Definition 6. *Let $((X_i, \mathcal{A}_i, \mu_i), (I, \leq), f_{ij}|_{i\leq j})$ be a measure inverse system, and let $P = \varprojlim(X_i, (I, \leq), f_{ij}|_{i\leq j})$.*

1. *$(P, \mathcal{A})$, where $\mathcal{A}$ is the coarsest σ-algebra w.r.t. p_i is measurable $\forall i \in I$,*
2. *μ is such a measure on $(P, \mathcal{A})$ that $\mu \circ p_i^{-1} = \mu_i$ $\forall i \in I$.*

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[1] This concept stems from the the idea of outer measure used in measure extension. We'd like to get that $\mu^* = \mu$ on $\mathcal{A}$, so since the lack of σ-additivity, we had to change the original concept a little bit.

The $(P,\mathcal{A},\mu)$ measure space, which satisfies points 1., 2. is called measure inverse limit, and it is denoted by $(P,\mathcal{A},\mu)=\varprojlim((X_i,\mathcal{A}_i,\mu_i),(I,\leq),f_{ij}|_{i\leq j})$.

One of the main problems of the existence of measure inverse limit is the σ-additivity of μ. We introduce the following concept to emphasize this problem:

Definition 7. *Let $((X_i,\mathcal{M}_i,\mu_i),(I,\leq),f_{ij}|_{i\leq j})$ be a measure inverse system, and let $P=\varprojlim(X_i,(I,\leq),f_{ij}|i\leq j)$.*

1. *$\mathcal{A}=\cup_i p_i^{-1}(\mathcal{M}_i)$ is an algebra,*
2. *μ is such an additive set function on $\mathcal{A}$, that $\mu\circ p_i^{-1}=\mu_i\ \forall i\in I$.*

$(P,\mathcal{A},\mu)$, which satisfies points 1., 2. is called weak measure inverse limit, and it is denoted by $(P,\mathcal{A},\mu)=w-\varprojlim((X_i,\mathcal{M}_i,\mu_i),(I,\leq),f_{ij}|_{i\leq j})$.

It is necessary to get a "valuable" measure inverse limit, i.e. the inverse limit P is not empty. For this propose Bochner [1] introduced the concept of sequential maximality. Later Millington and Sion [12] weakened this for almost sequential maximality.

Definition 8. *The $((X_i,\mathcal{M}_i,\mu_i),(I,\leq),f_{ij}|_{i\leq j})$ measure inverse system is almost sequentially maximal (a.s.m.), if for arbitrary $i_1\leq i_2\leq\ldots\in I\ \exists A_{i_n}\subseteq X_{i_n}$ such that*

- $f^{-1}_{i_n i_m}(A_{i_n})\subseteq A_{i_m}\ \forall(n\leq m)$,
- $\mu^*_{i_n}(A_{i_n})=0\ \forall n$,
- *if $x_{i_n}\in(X_{i_n}\setminus A_{i_n})$, $x_{i_n}=f_{i_n i_{n+1}}(x_{i_{n+1}})\ \forall n$, then $\exists x\in P=\varprojlim(X_i,(I,\leq),f_{ij}|_{i\leq j})$ such that $x_{i_n}=p_{i_n}(x)\ \forall n$.*

The almost sequential maximality ensures that the inverse limit is not empty, so it substitutes for the Axiom of Choice in this case.

Corollary 9. *Let $(X_n,\mathcal{M}_n)$ be measurable spaces $n\in\mathbb{N}$, and let $(Y_n,\mathcal{N}_n)\doteq(\times_{i=1}^n X_i,\otimes_{i=1}^n\mathcal{M}_i)$, $((Y_n,\mathcal{N}_n,\mu_n),(\mathbb{N},\leq),f_{mn}|_{m\leq n})$, where f_{mn}s are coordinate projections and μ_ns are arbitrary measures such that they satisfy point 2. in Definition 4. In this case the measure inverse system $((Y_n,\mathcal{N}_n,\mu_n),(\mathbb{N},\leq),f_{mn}|_{m\leq n})$ is almost sequentially maximal.*

Proof. It is immediate that $\varprojlim(Y_n,(\mathbb{N},\leq),f_{mn}|_{m\leq n})=\times_{i=1}^\infty Y_i=\times_{i=1}^\infty X_i$. Because of the surjectivity (onto) of the coordinate projections, and the structure of the Cartesian product, we can choose sets A_n as $\emptyset$ to get the almost sequential maximality in Definition 8. Q.E.D.

The following result, which is our main mathematical result, is a generalization of Metivier's [15] (p. 269.) and Mallory and Sion's [11], hence so is it Bochner's [1] and Choksi's [4].

Theorem 10. *Let $((X_i,\mathcal{M}_i,\mathcal{C}_i,\mu_i),f_{ij},(I,\leq))|_{i\leq j}$ be a measure inverse system, where $\mathcal{C}_i\subseteq\mathcal{M}_i$ is σ-compact set system $\forall i\in I$. If*

1. *$(I,\leq)$ is a directed set,*

2. $f_{ij}(\mathcal{C}_j) \subseteq \mathcal{C}_i \ \forall (i \leq j)$*,*

3. $f_{ij}^{-1}(\{x_i\}) \cap \mathcal{C}_j$ *is* σ*-compact set system* $\forall (i \leq j) \ \forall x_i \in X_i$*,*

4. for arbitrary $C_1, C_2 \in \mathcal{C}_i$ $C_1 \cap C_2 \in \mathcal{C}_i \ \forall i \in I$*,*

5. $((X_i, \mathcal{M}_i, \mu_i), (I, \leq), f_{ij}|_{i \leq j})$ *is almost sequentially maximal,*

6. μ_i *is inner* $\mathcal{C}_i$*-regular* $\forall i \in I$*,*

then $(X, \mathcal{M}, \mu) = \varprojlim((X_i, \mathcal{M}_i, \mu_i), (I, \leq), f_{ij}|_{i \leq j})$ *exists, and it is unique.*

For the sake of clarity, we split the proof into parts. First we consider the σ-additivity.

Definition 11. *Let* $\mathcal{A} \subseteq \mathcal{P}(X)$ *be a set ring,* $\mathcal{C} \subseteq \mathcal{A}$ *be a set system, and let* μ *be an additive set function on* $\mathcal{A}$*. The* $\mathcal{C}$ *set system is* μ*-almost* σ*-compact, if for arbitrary* $(C_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}$ *and for arbitrary* $\epsilon > 0$, $\exists A \subseteq X$ $\mu^*(A) < \epsilon$ *such that, if* $\complement A \cap (\cap_n C_n) = \emptyset$*, then* $\exists m \in \mathbb{N}$*, such that* $\complement A \cap (\cap_{n=1}^m C_n) = \emptyset$*.*

Corollary 12. *Any* σ*-compact set system in* $(X, \mathcal{M}, \mu)$ *measure space is* μ*-almost* σ*-compact set system.*

The intuition behind the definition of μ-almost σ-compactness comes from the concept of almost sequential maximality (Definition 8.) and Proposition 16.

Lemma 13. *Let* $\mathcal{A} \subseteq \mathcal{P}(X)$ *be a set ring,* $\mathcal{C} \subseteq \mathcal{A}$ *be a* μ*-almost* σ*-compact set system, where* μ *is an inner* $\mathcal{C}$*-regular additive set function on* $\mathcal{A}$*. Then* μ *is* σ*-additive on* $\mathcal{A}$*.*

Proof. The proof based on the fact, that on set ring the upper σ-continuity at $\emptyset$ is equivalent to the σ-additivity.

Let $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ be a sequence of sets such that $A_n \supseteq A_{n+1}$, $\cap_n A_n = \emptyset$. Assume that $\lim\limits_{n \to \infty} \mu(A_n) \nrightarrow 0$, i.e. $\exists \delta > 0$ such that $\lim\limits_{n \to \infty} \mu(A_n) > \delta$. Because of μ is an inner $\mathcal{C}$-regular additive set function, for any $\kappa > 0$ $\forall A_n$ $\exists C_n \in \mathcal{C}$ such that $C_n \subseteq A_n$, and $\mu(A_n \setminus C_n) < \frac{\kappa}{2^{n+1}}$. It is immediate that $\forall m \in \mathbb{N}$ $\cap_{n=1}^m C_n \subseteq A_n$, and

$$\mu(A_m \setminus (\cap_{n=1}^m C_n)) \leq \mu(\cup_{n=1}^m (A_n \setminus C_n)) \leq \sum_{n=1}^m \mu(A_n \setminus C_n) < \kappa.$$

Since $\cap_n A_n = \emptyset$, $C_n \subseteq A_n$, so $\cap_n C_n = \emptyset$. $\mathcal{C}$ is μ-almost σ-compact set system, so for arbitrary $\epsilon > 0$ $\exists A \subseteq X$ such that $\mu^*(A) < \epsilon$, and $\exists m^* \in \mathbb{N}$ such that $\complement A \cap (\cap_{n=1}^{m^*} C_n) = \emptyset$. Let $\epsilon = \kappa = \frac{\delta}{2}$, then

$$\mu^*(A_m \setminus (\complement A \cap (\cap_{n=1}^m C_n))) \leq \mu(A_m \setminus (\cap_{n=1}^m C_n)) + \epsilon < \delta \quad \forall m \in \mathbb{N}.$$

Hence

$$\lim_{m \to \infty} \mu^*(A_m \setminus (\complement A \cap (\cap_{n=1}^m C_n))) \leq \lim_{m \to \infty} \mu(A_m \setminus (\cap_{n=1}^m C_n)) + \epsilon \leq \delta,$$

so

$$\lim_{m\to\infty}\mu(A_m) = \lim_{m\to\infty}\mu^*(A_m) = \lim_{m\to\infty}\mu^*(A_m \setminus (\complement A \cap (\cap_{n=1}^{m} C_n))) \le \delta,$$

but $\lim\limits_{m\to\infty}\mu(A_m) > \delta$, which is a contradiction, therefore $\lim\limits_{n\to\infty}\mu(A_n) \to 0$.

Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be arbitrary disjoint sets such that $A = \cup_n A_n$, and $A \in \mathcal{A}$. Let $B_n = A\setminus(\cup_{m=1}^{n} A_m)$, then $B_n \supseteq B_{n+1}$, $B_n \in \mathcal{A}\ \forall n$, and $\cap_n B_n = \emptyset$. Because of the additivity of μ

$$\mu(A) = \mu(B_n) + \mu(\cup_{m=1}^{n} A_m) \quad \forall n \in \mathbb{N},$$

and

$$\mu(A) = \mu(B_n) + \sum_{m=1}^{n}\mu(A_m) \quad \forall n \in \mathbb{N},$$

$$\mu(A) = \lim_{n\to\infty}\mu(B_n) + \lim_{n\to\infty}\sum_{m=1}^{n}\mu(A_m).$$

Since $\lim\limits_{n\to\infty}\mu(B_n) \to 0$:

$$\mu(A) = \sum_n \mu(A_n).$$

Q.E.D.

The next step is the proof of the existence of the weak measure inverse limit. The following result is a generalization of Rao's [17].

Proposition 14. *Let* $((X_i, \mathcal{M}_i, \mu_i), (I, \le), f_{ij}|_{i\le j})$ *be a measure inverse system such that*

1. $(I, \le)$ *is a directed set,*
2. *the measure inverse system* $((X_i, \mathcal{M}_i, \mu_i), (I, \le), f_{ij}|_{i\le j})$ *is almost sequentially maximal.*

Then the weak measure inverse limit $(P, \mathcal{A}, \mu) = w - \varprojlim((X_i, \mathcal{M}_i, \mu_i), (I, \le), f_{ij}|_{i\le j})$ *exists, and it is unique.*

Proof. We know that $\mathcal{A} = \cup_{i\in I} p_i^{-1}(\mathcal{M}_i)$. $\mathcal{A}$ is a field: let $A_1, A_2, \ldots, A_n$ be arbitrary sets from $\mathcal{M}$. $(I, \le)$ is a directed set, so $\exists i \in I$, $\exists(B_1, B_2, \ldots, B_n \in \mathcal{M}_i)$ such that $p_i^{-1}(B_1) = A_1$, $p_i^{-1}(B_2) = A_2, \ldots, p_i^{-1}(B_n) = A_n$. Since $\mathcal{M}_i$ is a σ-algebra, p_i is a measurable function and $A_1, A_2, \ldots, A_n$ are arbitrary sets so $\mathcal{A}$ is a field.

Define μ as $\mu(p_i^{-1}(B)) = \mu_i(B)\ \forall i \in I$, $B \in \mathcal{M}_i$. Let $A \in \mathcal{M}_i$ and $B \in \mathcal{M}_j$ such that $p_i^{-1}(A) = p_j^{-1}(B)$, then $\exists k \in I$ such that $i \le k$ and $j \le k$. Since the concept of measure inverse system $\mu_k(f_{ik}^{-1}(A)) = \mu_i(A)$ and $\mu_k(f_{jk}^{-1}(B)) = \mu_j(B)$. Because of the almost sequential maximality $\exists C \in \mathcal{M}_k$ such that $\mu_k(C) = 1$, and $C \subseteq p_k(P)$. Then $\mu_j(B) = \mu_k(f_{ik}^{-1}(A) \cap f_{jk}^{-1}(B) \cap C) = \mu_i(A)$, so $\mu(p_j^{-1}(B)) = \mu(p_i^{-1}(A))$, therefore μ is well defined.

The proof of the additivity of μ: let $A_1, A_2, \ldots, A_n$ be arbitrary pairwise disjoint sets from $\mathcal{A}$, then since $(I, \le)$ is a directed set $\exists i \in I$, $\exists B_1, B_2, \ldots, B_n \in \mathcal{M}_i$ such that $p_i^{-1}(B_1) = A_1, p_i^{-1}(B_2) = A_2, \ldots, p_i^{-1}(B_n) = A_n$, and $\forall(k, l) \le n$ $\mu_i(B_k \cap B_l) = 0$. Since $\mathcal{M}_i$ is a σ-algebra, μ_i is a σ-additive set function and $A_1, A_2, \ldots, A_n$ be arbitrary pairwise disjoint sets so μ is additive on $\mathcal{A}$. Q.E.D.

In the follows, we look into the structure of the product space.

Lemma 15. *Let $I = \{k, l\}$, X_i be sets, and let $\mathcal{C}_i \subseteq \mathcal{P}(X_i)$ be a σ-compact set system $\forall i \in I$. If*

1. *$f_{ij}(\mathcal{C}_j) \subseteq \mathcal{C}_i$ $\forall (i \leq j)$,*
2. *$f_{ij}^{-1}(\{x_i\}) \cap \mathcal{C}_j$ is σ-compact set system $\forall x_i \in X_i$, $\forall (i \leq j)$,*
3. *for arbitrary $C_1, C_2 \in \mathcal{C}_i$ $C_1 \cap C_2 \in \mathcal{C}_i$ $\forall i \in I$,*

then $\mathcal{C} \circeq p_i^{-1}(\mathcal{C}_i) \cup p_j^{-1}(\mathcal{C}_j)$ is a σ-compact set system in $\mathcal{P}(P = \varprojlim((X_i, (I, \leq), f_{ij}|_{i\leq j}))$.

Proof. We prove that if $(C_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ and $\cap_{n=1}^{m} C_n \neq \emptyset$ $\forall m \in \mathbb{N}$, then $\cap_n C_n \neq \emptyset$.

We distinguish two different cases:

1. k and l are not related to each other.

   Then $P = X_k \times X_l$, so it is the Cartesian product. It is immediate that $\forall C_n \in \mathcal{C}$ either $\exists C_n^k \in \mathcal{C}_k$ such that $C_n = p_k^{-1}(C_n^k)$, or $\exists C_n^l \in \mathcal{C}_l$ such that $C_n = p_l^{-1}(C_n^l)$, hence either $C_n = C_n^k \times X_l$ or $C_n = X_k \times C_n^l$ $\forall n$. Let $N_k = \{n \in \mathbb{N} \mid C_n = p_k^{-1}(C_n^k), C_n^k \in \mathcal{C}_k\}$, and let N_l be defined in the same way. Then $\cap_n C_n = (\cap_{n\in N_k} C_n^k) \times (\cap_{n\in N_l} C_n^l)$, if N_k and N_l sets are non-empty. If N_k is empty, then $\cap_n C_n = X_k \times (\cap_{n\in N_l} C_n^l)$, and if N_l is empty, then $\cap_n C_n = (\cap_{n\in N_k} C_n^k) \times X_j$. Since $\mathcal{C}_k$, $\mathcal{C}_l$ are σ-compact set systems, so $\cap_{n\in N_k} C_n^k \neq \emptyset$ and $\cap_{n\in N_l} C_n^l \neq \emptyset$, hence $\cap_n C_n \neq \emptyset$.

2. Let $k \leq l$ (the discussion of case $l \leq k$ is the same).

   In this case $P = X_l$. Let $N_k = \{n \in \mathbb{N} \mid C_n = p_k^{-1}(C_n^k),\ C_n^k \in \mathcal{C}_k\}$, and let N_l be defined in a similar way. Let $C^m = (\cap_{\{n\in N_k | n\leq m\}} f_{kl}(C_n))$ $\cap f_{kl}(\cap_{\{n\in N_l | n\leq m\}} C_n)$. Then $C^m \neq \emptyset$, and since points *1.* and *3.* $C^m \in \mathcal{C}_k$ $\forall m \in \mathbb{N}$. It is clear that $C^m \supseteq C^{m+1}$ $\forall m \in \mathbb{N}$, so since $\mathcal{C}_k$ is σ-compact set system $\cap_m C^m \neq \emptyset$. Let $x_k \in \cap_m C^m$ be arbitrary fixed, then since point *2.* $f_{kl}^{-1}(\{x_k\}) \cap C_n$ $\forall n \in N_l$ is σ-compact set system. We chose x_k as $\cap_{n=1}^{m}(f_{kl}^{-1}(\{x_k\}) \cap C_n) \neq \emptyset$ $\forall m \in \mathbb{N}$, so $\cap_n(f_{kl}^{-1}(\{x_k\}) \cap C_n) \neq \emptyset$. Let $x_l \in \cap_n(f_{kl}^{-1}(\{x_k\}) \cap C_n)$ be arbitrary fixed, then:

   a: $x_k = f_{kl}(x_l)$,

   b: $x_l \in \cap_n C_n$,

so, $\cap_n C_n \neq \emptyset$. Q.E.D.

Proposition 16. *Let $((X_i, \mathcal{M}_i, \mu_i), (I, \leq), f_{ij}|_{i\leq j})$ be a measure inverse system, and let $\mathcal{C}_i \subseteq \mathcal{M}_i$ be σ-compact set system $\forall i \in I$, and let $J \subseteq I$. If*

1. *$(I, \leq)$ is a directed set,*
2. *every countable subset of J has the least element,*
3. *$f_{ij}(\mathcal{C}_j) \subseteq \mathcal{C}_i$ $\forall (i \leq j) \in I$,*
4. *$f_{ij}^{-1}(\{x_i\}) \cap \mathcal{C}_j$ $\forall (i \leq j)$ is σ-compact set system $\forall x_i \in X_i$, $\forall i \in I$,*
5. *for arbitrary $C_1, C_2 \in \mathcal{C}_i$ $C_1 \cap C_2 \in \mathcal{C}_i$ $\forall i \in J$,*

6. *μ_i is inner $\mathcal{C}_i$-regular $\forall i \in I$,*

7. *$((X_i, \mathcal{M}_i, \mu_i), (I, \le), f_{ij}|_{i\le j})$ measure inverse system is almost sequentially maximal,*

then $\mathcal{C} \stackrel{\circ}{=} \cup_{i\in A} p_i^{-1}(\mathcal{C}_i)$ is μ-almost σ-compact set system in $\mathcal{M}$, where $(P, \mathcal{M}, \mu) = w - \varprojlim((X_i, \mathcal{M}_i, \mu_i), (I, \le), f_{ij}|_{i\le j})$.

Proof. Since points *1.*, *7.* and Proposition 14. $(P, \mathcal{M}, \mu) = w - \varprojlim((X_i, \mathcal{M}_i, \mu_i), (I, \le), f_{ij}|_{i\le j})$ exists, and it is unique. It is enough to see that, for arbitrary $(C_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ and for arbitrary $\epsilon > 0$ $\exists A \subseteq P$ $\mu^*(A) < \epsilon$ such that, if $\forall m \in \mathbb{N}$ $\complement A \cap (\cap_{n=1}^m C_n) \ne \emptyset$, then $\complement A \cap (\cap_n C_n) \ne \emptyset$.

Let $(C_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ and $\epsilon > 0$ be arbitrary fixed. Let $N_i = \{n \in \mathbb{N} \mid C_n = p_i^{-1}(C'_n),\ C'_n \in \mathcal{C}_i,\ i \in J\}$ $\forall i \in J$. Moreover, let $i(n)$ be an arbitrary fixed element of $\{i \in J \mid n \in N_i\}$.

Let i_1 be the least element of the set $\{i(1), i(2), \ldots\}$ (since point *2.* it works). In the same way, let i_m be the least element of the set $J \setminus \{i_n\}_{n\le m}$ $\forall m \in \mathbb{N}$.

Let A_{i_n} be as in Definition 8. (point *7.*), and let $\overline{A}_{i_n} \in \mathcal{M}_{i_n}$ be such that $A_{i_n} \subseteq \overline{A}_{i_n}$ and $\mu_{i_n}(\overline{A}_{i_n}) = 0$ $\forall n$ ($\mathcal{M}_{i_n}$ is σ-algebra, so $\overline{A}_{i_n}$ exists $\forall n$). Then, since point *6.* $\exists K_{i_n} \in \mathcal{C}_{i_n}$ such that $K_{i_n} \subseteq \complement\overline{A}_{i_n}$ and $\mu_{i_n}(K_{i_n}) > 1 - \frac{\epsilon}{2^{n+1}}$ $\forall n$. Let $A = \cup_n p_{i_n}^{-1}(\complement K_{i_n})$.

In the follows we show that $\mu^*(A) < \epsilon$.

Assume indirectly that $\mu^*(A) \ge \epsilon$. Then, since $A = \cup_n p_{i_n}^{-1}(\complement K_{i_n})$ and $\mu_{i_n}(\complement K_{i_n}) \le \frac{\epsilon}{2^{n+1}}$ $\forall n$, $\exists i^* \in I$, $\exists B^{i^*} \in \mathcal{M}_{i^*}$ such that $\mu_{i^*}(B^{i^*}) \ge \frac{4\epsilon}{5}$, and with the notation $B = p_{i^*}^{-1}(B^{i^*})$ $B \subseteq A$.

Let $j_1 \in I$ be such that $j_1 \ge i^*$ and $j_1 \ge i_1$, in general, let $j_n \in I$ be such that $j_n \ge j_{n-1}$ and $j_n \ge i_n$ $\forall n \ge 2$ (point *1.*). Then $\exists C^{j_1} \in \mathcal{C}_{j_1}$ such that $C^{j_1} \subseteq (f_{i^*j_1}^{-1}(B^{i^*}) \setminus (f_{i_1j_1}^{-1}(A_{i_1}) \cup A_{j_1}))$ and $\mu_{j_1}(C^{j_1}) > \frac{2\epsilon}{3}$ (point *6.*), where A_{j_1} is from Definition 8. for chain $j_1 \le j_2 \ldots$. In the same way, $\exists C^{j_n} \in \mathcal{C}_{j_n}$ such that $C^{j_n} \subseteq (f_{i^*j_n}^{-1}(B^{i^*}) \setminus (f_{i_nj_n}^{-1}(A_{i_n}) \cup A_{j_n}))$ and $\mu_{j_n}(C^{j_n}) > \frac{2\epsilon}{3}$ $\forall n$.

We chose C^{j_n} in such a way that $\cap_{n=1}^m f_{j_1j_n}(C^{j_n}) \ne \emptyset$ $\forall m$, so since point *3.* $\cap_n f_{j_1j_n}(C^{j_n}) \ne \emptyset$. Let $x_{j_1} \in \cap_n f_{j_1j_n}(C^{j_n})$ be arbitrary fixed. x_{j_1} was chosen in a way that $f_{j_1j_2}^{-1}(\{x_{j_1}\}) \cap (\cap_{n=2}^m f_{j_2j_n}(C^{j_n})) \ne \emptyset$ $\forall m$. Then, since point *4.* $f_{j_1j_2}^{-1}(\{x_{j_1}\}) \cap (\cap_{n\le 2} f_{j_2j_n}(C^{j_n})) \ne \emptyset$. Let $x_{j_2} \in f_{j_1j_2}^{-1}(\{x_{j_1}\}) \cap (\cap_{n\ge 2} f_{j_2j_n}(C^{j_n}))$ be arbitrary fixed. Then

i. $f_{i_1j_1}(x_{j_1}) \notin A_{i_1}$ and $f_{i_2j_2}(x_{j_2}) \notin A_{i_2}$,

ii. $x_{j_1} \in f_{i^*j_1}^{-1}(B^{i^*})$ $x_{j_2} \in f_{i^*j_2}^{-1}(B^{i^*})$.

Define the sequence $(x_{j_n})_{n\in\mathbb{N}}$ in the way above. Then $\forall n$

iii. $f_{i_nj_n}(x_{j_n}) \notin A_{i_n}$,

iv. $x_{j_n} \in f_{i^*j_n}^{-1}(B^{i^*})$.

Since point *7.* $\exists x \in P$ such that $x_{j_n} = p_{j_n}(x)$ $\forall n$. However, since **iii.** $x \notin A$ and since **iv.** $x \in B$, therefore $B \nsubseteq A$, and it is a contradiction.

In the follows we show that, if $\forall m \in \mathbb{N}$ $\complement A \cap (\cap_{n=1}^m C_n) \ne 0$, then $\complement A \cap (\cap_n C_n) \ne \emptyset$.

Let $C^m = K_{i_1} \cap (\cap_{j\in I} f_{i_1 j}(\cap_{\{n\in N_j | n\leq m\}} p_j(C_n))) \ \forall m \in \mathbb{N}$. Then $C^m \neq \emptyset$, and since points *3.*, *5.* $C^m \in \mathcal{C}_i \ \forall m \in \mathbb{N}$. It is clear that $C^m \supseteq C^{m+1} \ \forall m \in \mathbb{N}$, so since $\mathcal{C}_i$ is σ-compact set system $\cap_m C^m \neq \emptyset$.

Let $x_{i_1} \in \cap_m C^m$ be arbitrary fixed. Since point *4.* $f^{-1}_{i_1 i_2}(\{x_{i_1}\}) \cap \mathcal{C}_{i_2}$ is σ-compact set system. x_{i_1} was chosen in a way that $K_{i_2} \cap (\cap_{n=1}^m (f^{-1}_{i_1 i_2}(\{x_{i_1}\}) \cap(\cap_{j\in I\setminus\{i_1\}} f_{i_2 j}(\cap_{\{n\in N_j | n\leq m\}} p_j(C_n))) \neq \emptyset \ \forall m \in \mathbb{N}$, so $K_{i_2} \cap (\cap_n (f^{-1}_{i_1 i_2}(\{x_{i_1}\}) \cap(\cap_{j\in I\setminus\{i_1\}} f_{i_2 j} \ (\cap_{\{n\in N_j | n\leq m\}} p_j(C_n))) \neq \emptyset$. Let $x_{i_2} \in K_{i_2} \cap (\cap_m (f^{-1}_{i_1 i_2}(\{x_{i_1}\}) \cap(\cap_{j\in I\setminus\{i_1\}} f_{i_2 j}(\cap_{\{n\in N_j | \ n\leq m\}} p_j(C_n))) be arbitrary fixed, then

a: $x_{i_1} = f_{i_1 i_2}(x_{i_2})$,

b: $x_{i_2} \in K_{i_2} \cap (\cap_{n\notin N_{i_1}\cup N_{i_2}} p_{i_2}(C_n)) \cap (p_{i_2}(\cap_{n\in N_{i_1}\cup N_{i_2}} C_n))$.

We can apply the method used above to the chain $i_1 \leq i_2 \leq i_3 \leq \ldots$ to get a set of points, which $\forall k \in \mathbb{N}$:

c: $x_{i_k} = f_{i_k i_{k+1}}(x_{i_{k+1}})$,

d: $x_{i_k} \in K_{i_k} \cap (\cap_{n\notin \cup_{j=1}^k N_{i_j}} p_{i_k}(C_n)) \cap (p_{i_k}(\cap_{n\in \cup_{j=1}^k N_{i_j}} C_n))$.

Since **c:**, **d:** $x_{i_k} \in (X_{i_k} \setminus \complement K_{i_k})$, $x_{i_k} = f_{i_k i_{k+1}}(x_{i_{k+1}}) \ \forall k \in \mathbb{N}$, so since $A_{i_k} \subseteq \complement K_{i_k} \ \forall k$ and point *7.* $\exists x \in \varprojlim(X_i, (I, \leq), f_{ij}|_{i\leq j})$ such that $x_{i_k} = p_{i_k}(x)$ $\forall k$. However, $x \in \complement A \cap (\cap_n C_n)$, so $\complement A \cap (\cap_n C_n) \neq \emptyset$. Q.E.D.

Proposition 17. *Let* $((X_i, \mathcal{M}_i, \mathcal{C}_i, \mu_i), (I, \leq), f_{ij}|_{i\leq j})$ *be a measure inverse system, where* $\mathcal{C}_i \subseteq \mathcal{M}_i$ *is* σ*-compact set system* $\forall i \in I$. *If*

1. $(I, \leq)$ *is a directed set,*
2. $(X, \mathcal{A}, \mu) = w - \varprojlim((X_i, \mathcal{M}_i), (I, \leq), f_{ij}|_{i\leq j})$ *exists,*
3. $\forall(i_1 \leq i_2 \leq \ldots)$ *sequence,* $\cup_n p^{-1}_{i_n}(\mathcal{C}_{i_n})$ *is* μ*-almost* σ*-compact set system,*
4. μ_i *is inner* $\mathcal{C}_i$*-regular* $\forall i \in I$,

then $(X, \mathcal{M}, \mu) = \varprojlim((X_i, \mathcal{M}_i, \mu_i), (I, \leq), f_{ij}|_{i\leq j})$ *exists, and it is unique.*

Proof. We prove that μ is σ-additive on $\cup_{i\in I} p_i^{-1}(\mathcal{M}_i)$.

Since condition *4.* μ is inner $\cup_{i\in I} p_i^{-1}(\mathcal{C}_i)$-regular.

Let $A_1, A_2, \ldots, A_n, \ldots \in \cup_{i\in I} p_i^{-1}(\mathcal{M}_i)$ be arbitrary pairwise disjoint sets. Since definition of A_ns $\exists i(n) \in I$, and $\exists A_n^{i(n)} \in \mathcal{M}_{i(n)}$ such that $A_n = p^{-1}_{i(n)}$ $(A_n^{i(n)}) \ \forall n$. Let $i_1 = i(1)$. Since condition *1.* $\exists i^* \in I$ such that $i_1 \leq i^*$, and $i(2) \leq i^*$. Let $i_2 = i^*$. Define the chain $i_1 \leq i_2 \leq \ldots$ by the method above. Since condition *3.*, $A_1, A_2, \ldots, A_n, \ldots \in$ are arbitrary pairwise disjoint sets and Lemma 13. μ is σ-additive on $\cup_{i\in I} p_i^{-1}(\mathcal{M}_i)$. Then this proposition is a direct corollary of the measure extension theorem (see e.g. Halmos [6]), so $(X, \mathcal{M}, \mu) = \varprojlim((X_i, \mathcal{M}_i, \mu_i), (I, \leq), f_{ij}|_{i\leq j})$ exists, and it is unique. Q.E.D.

The proof of Theorem 10. Since Proposition 16. for arbitrary chain $i_1 \leq i_1 \leq \cdots$, $\cup_n p^{-1}_{i_n}(\mathcal{C}_{i_n})$ is μ-almost σ-compact set system. Since Propositions 14., 17. $(X, \mathcal{M}, \mu) = \varprojlim((X_n, \mathcal{M}_n, \mu_n), (I, \leq), f_{ij}|_{i\leq j})$ exists, and it is unique. Q.E.D.

3 Complete, universal type space based on a purely measurable parameter space

At first let S denote the parameter space, i.e. it contains all facts, which are independent from the players, and which have influence on the game, e.g. the description of the game. To model the players' beliefs we have to get the beliefs space generated by S, i.e. we have to care about what the players believe about S, and what the players believe about that what the players believe about S, and so on.

Definition 18. *The parameter space $(S, \mathcal{A}_S)$ is a measurable space, where $\mathcal{A}_S$ is a σ-algebra on S.*

We assume that the parameter space is purely measurable, i.e. it is not topological. In our model the players use only ideas such as event, outcome, probability, so it looks like a purely measure theoretic model. However, it is well known that in a purely measure theoretic model the type space is not necessarily complete and universal (see Heifetz and Samet [10]).

Definition 19. *Denote the set of probability measures on $(S, \mathcal{A}_S)$ as $\Delta(S, \mathcal{A}_S)$, then $\Delta(S, \mathcal{A}_S) \subseteq [0,1]^{\mathcal{A}_S}$. Define the topology of $\Delta(S, \mathcal{A}_S)$ as a subspace of $[0,1]^{\mathcal{A}_S}$, so $(\Delta(S, \mathcal{A}_S), \tau)$ is the point-wise convergence topology. Therefore $(\Delta(S, \mathcal{A}_S), \tau)$ (briefly (Δ, τ)) is a topological space, and let the Baire sets of (Δ, τ) be $B(\Delta, \tau)$.*

For the sake of brevity henceforward we denote $\Delta(S, \mathcal{A}_S)$ as $\Delta(S)$, where it is not ambiguous, so do we in the case of $B(\Delta(S), \tau)$ and $B(\Delta(S))$.

Definition 20. *Define a sequence of sets recursively, where M is the set of players:*

$$\begin{aligned}
V_0 &= (S, \mathcal{A}_S) \\
V_1 &= V_0 \otimes (\Delta(V_0)^M, B(\Delta(V_0)^M)) \\
V_2 &= V_1 \otimes (\Delta(V_1)^M, B(\Delta(V_1)^M)) \\
&= V_0 \otimes (\Delta(V_0)^M, B(\Delta(V_0)^M)) \otimes (\Delta(V_1)^M, B(\Delta(V_1)^M)) \\
&\vdots \\
V_n &= V_{n-1} \otimes (\Delta(V_{n-1})^M, B(\Delta(V_{n-1})^M)) \\
&= V_0 \otimes \otimes_{j=0}^{n-1} (\Delta(V_j)^M, B(\Delta(V_j)^M)) \\
&\vdots
\end{aligned}$$

where $\otimes$ denotes the measurable product.

Let $V_\infty = S \times \times_{j=0}^{\infty} \Delta(V_j)^M$. V_∞ is called belief space, and a point of it is a state of the world.

A point of V_0 is called a value of the parameters, i.e. it is a feasible value of the parameters. A point of V_1 consists of a possible value of the parameters, and the first oder beliefs of the players (i.e. the players' beliefs about S), and so on.

If $v \in V_\infty$, then $v = (s, \mu_1^1, \mu_1^2, \ldots, \mu_2^1, \mu_2^2, \ldots)$, where μ_j^i is jth order belief of player i. Therefore, every point in V_∞ can be regarded as an *hierarchy of beliefs*, $(\mu_1^i, \mu_2^i, \ldots)$ for all players ($\forall i \in M$), and a feasible value of the parameters.

Definition 21. *Let $i \in M$ be arbitrary, fixed. The hierarchy of beliefs $(\mu_1^i, \mu_2^i, \ldots)$ is coherent iff $\forall n \geq 2$*

1. $marg_{V_{n-2}} \mu_n^i = \mu_{n-1}^i$,
2. $marg_{[\Delta(V_{n-2})]^i} \mu_n^i = \delta^i_{\mu^i_{n-1}}$,

where $\mu_n^i \in [\Delta(V_{n-1})]^i$ ($[\Delta(V_{n-1})]^i$ is index i "copy" of $\Delta(V_{n-1})$).

The point *1.* declares that the opinions on the facts of the game do not change in the hierarchy. According to the last point, every players knows her own beliefs (see Harsányi [7]). These two properties is the "logic" of the players, we assume that this "logic" is *common knowledge.*

Definition 22. *Take such a points $(s, \mu_1^1, \mu_1^2, \ldots, \mu_2^1, \mu_2^2, \ldots)$ of V_∞ that, the hierarchies of beliefs $(\mu_1^i, \mu_i^2, \ldots)$ are coherent $\forall i \in M$. Let V_∞^c be the set of all this kind of points, and call V_∞^c* coherent subspace.

We use ^{c} for other spaces in the same meaning, i.e. according to Definition 22.

In our model the hierarchies of beliefs are such a sequence of probability measures that, they are coherent, and their restrictions on the ,,truncated" belief spaces are compact regular measures. It is clear that, we need to redefine the belief space (Definition 20.):

Definition 23. *Let*

$$\begin{aligned}
V_0' &= V_0 \\
V_1' &= V_0' \otimes (\Delta_{MC}(V_0')^M, B(\Delta_{MC}(V_0')^M)) \\
V_2' &= (V_1' \otimes (\Delta_{MC}(V_1')^M, B(\Delta_{MC}(V_1')^M)))^c \\
&\vdots \\
V_n' &= (V_{n-1}' \otimes (\Delta_{MC}(V_{n-1}')^M, B(\Delta_{MC}(V_{n-1}')^M)))^c \\
&= (V_0' \otimes \otimes_{j=0}^{n-1} (\Delta_{MC}(V_j')^M, B(\Delta_{MC}(V_j')^M)))^c \\
&\vdots
\end{aligned}$$

where Δ_{MC} stands for such a set of probability measures that

$$\begin{aligned}
\Delta_{MC}(V_0') &= \Delta(V_0') \\
\Delta_{MC}(V_1') &= \Delta_C(\Delta_{MC}(V_0')^M, B(\Delta_{MC}(V_0')^M)) \\
\Delta_{MC}(V_2') &= \Delta_C((\otimes_{j=0}^{1} (\Delta_{MC}(V_j')^M, B(\Delta_{MC}(V_j')^M)))^c) \\
&\vdots \\
\Delta_{MC}(V_n') &= \Delta_C((\otimes_{j=0}^{n-1} (\Delta_{MC}(V_j')^M, B(\Delta_{MC}(V_j')^M)))^c) \\
&\vdots
\end{aligned}$$

where Δ_C is the set of compact regular (according to the terminology of the previous section: inner regular on the compact sets) probability measures.

V_n' contains such a hierarchies of beliefs that: the beliefs are maximum nth order beliefs, coherent and their restrictions on the set of the beliefs are compact regular probability measures. Therefore, V_n' consists of such a beliefs that we'd like to use in our model.

The following definition is the most important step towards our main result.

Definition 24. *Let $i \in M$ be arbitrary fixed. Define the sequence of truncated belief spaces (see Definition 20.):*

$$\begin{array}{rcl}
C_0 & = & (\Delta_{MC}(V_0')^{M\setminus\{i\}}, B(\Delta_{MC}(V_0')^{M\setminus\{i\}})) \\
C_1 & = & (\otimes_{j=0}^{1}(\Delta_{MC}(V_j')^{M\setminus\{i\}}, B(\Delta_{MC}(V_j')^{M\setminus\{i\}})))^c \\
 & \vdots & \\
C_n & = & (\otimes_{j=0}^{n}(\Delta_{MC}(V_j')^{M\setminus\{i\}}, B(\Delta_{MC}(V_j')^{M\setminus\{i\}})))^c \\
 & \vdots &
\end{array}$$

The truncated belief spaces are the topological sub-products of the appropriate belief spaces.

In the next definition we give the measurable spaces of the measure inverse system we use later.

Definition 25. *Let $i \in M$ be arbitrary fixed. Define a sequence of spaces in a recursive way:*

$$\begin{array}{rcl}
T_0 & = & V_0 \\
T_1 & = & V_0' \otimes (\Delta_{MC}(V_0')^{M\setminus\{i\}}, B(\Delta_{MC}(V_0')^{M\setminus\{i\}})) \\
T_2 & = & (V_1' \otimes (\Delta_{MC}(V_1')^{M\setminus\{i\}}, B(\Delta_{MC}(V_1')^{M\setminus\{i\}})))^c \\
 & \vdots & \\
T_n & = & (V_{n-1}' \otimes (\Delta_{MC}(V_{n-1}')^{M\setminus\{i\}}, B(\Delta_{MC}(V_{n-1}')^{M\setminus\{i\}})))^c \\
 & = & (V_0' \otimes \otimes_{j=0}^{n-1}(\Delta_{MC}(V_j')^{M\setminus\{i\}}, B(\Delta_{MC}(V_j')^{M\setminus\{i\}})))^c \\
 & \vdots &
\end{array}$$

We define a given player's set of all feasible hierarchies of beliefs.

Definition 26. *Let $i \in M$ be arbitrary fixed, and let*

$$T^i = (\times_{j=0}^{\infty} \Delta_{MC}(V_j')^i)^c$$

T^i is the type set of player i, and a point of T^i is a feasible type of player i.

T^i is the collection of all coherent hierarchies of beliefs of player i on the coherent hierarchies of beliefs of the other players.

The type set of player i consists of the all coherent hierarchies of beliefs, i.e. if $t \in T^i$, then $t = (\nu_1^i, \nu_2^i, \nu_3^i, \ldots)$, and it is coherent. Because of this property if T^i is in a type space, then it is a *complete type space.* It is clear that in our model the beliefs are not, but the their marginals on the truncated nth order belief spaces are compact regular probability measures.

Corollary 27. *T^i is a subspace of a product space, so its topology is the pointwise convergence topology: (T^i, τ).*

Corollary 28. *Let $i \in M$ be arbitrary, fixed, then*

$$((T_n, \nu_{n+1}^i), (\mathbb{N} \cup \{0\}, \leq), pr_{mn}|_{m \leq n}) \tag{1}$$

is a measure inverse system, where pr_{mn} is coordinate projection from T_n to T_m $\forall (m \leq n)$, and $(\mu_1^i, \ldots, \mu_{n+1}^i, \ldots) \in T^i$.

Proof. The concept of measure inverse system can be found in Definition 4.

- $pr_{mn} = pr_{mk} \circ pr_{kn}$ $\forall (m \leq k \leq n)$, since these are coordinate projections,
- $pr_{nn} = id_{T^c_n}$ $\forall n$ since these are also coordinate projections,
- pr_{mn} is measurable function $\forall (m \leq n)$, since the concept of measurable product structure,
- $\nu^i_{n+1}(pr^{-1}_{mn}(A)) = \nu^i_{m+1}(A)$ $\forall (m \leq n)$ and $\forall A \in T_m$ measurable sets, since the hierarchies of beliefs are coherent.

Q.E.D.

The Corollary 28. above makes connection between the concepts of measure inverse system and belief space. Therefore, the question is the existence of a proper measure inverse limit.

Remark 29. In Corollary 28. one can change (1) to the measure inverse system

$$((C_n, marg_{C_n} \nu^i_{n+2}), (\mathbb{N} \cup \{0\}, \leq), pr_{mn}|_{m \leq n}). \tag{2}$$

The following proposition shows that, the main problem of the existence of measure inverse limit is the σ-additivity of ν.

Proposition 30. *Let $i \in M$ be arbitrary, fixed. The measure inverse system defined in* (1) *has weak measure inverse limit $(T, \mathcal{A}_T, \nu^i)$ (see Definition 7.).*

Proof. See Proposition 14., and Definition 22. Q.E.D.

Proposition 30. concentrates on the additivity of ν^i. There are two usual hurdles of the existence of measure inverse limit. First the richness of the inverse limit, i.e. whether inverse limit contains enough points, Heifetz and Samet's [10] counterexample is based on this problem. The second problem is the σ-additivity of ν^i. Naturally, these two problems are not independent from each other.

To avoid these problems, we use coordinate projections (Definition 8. and Example 9.), and some kind of compact regularity.

Definition 31. *Let $i \in M$ be arbitrary fixed. $\Delta_{MC}(T, \mathcal{A}_T)$ is such a set of probability measures that, if $\nu \in \Delta_{MC}(T, \mathcal{A}_T)$, then $marg_{C_{n-1}} \nu \in \Delta_C(C_{n-1})$ $\forall n$.*

The following theorem is our main contribution.

Theorem 32. *T^i is a universal type space, so there is an homeomorphism $f : T^i \to (\Delta_{MC}(T, \mathcal{A}), \tau)$.*

For the sake of clarity we split the proof into parts. The following lemma is about fitting up measure spaces.

Lemma 33. *Let $(M, \mathcal{A}_M, \mu_M), (N, \mathcal{A}_N, \mu_N)$ be probability measures spaces, and let μ be additive set function on the filed generated by the cylindrical sets $\mathcal{A} \subseteq \mathcal{P}(M \times N)$, moreover let p_M and p_N be coordinate projections. If $\mu \circ p^{-1}_M = \mu_M$ and $\mu \circ p^{-1}_N = \mu_N$, then μ is σ-additive.*

Proof. Is it easy to see that the sets of $\mathcal{A}$ have the forms as follows: $\cup_{j=1}^{m}(M_j \times N_j)$, where $m \in \mathbb{N}$, $M_j \in \mathcal{A}_M$, $N_j \in \mathcal{A}_N$. It is well known that μ is σ-additive on $\mathcal{A}$ iff for arbitrary sequence of sets such that $A_n \supseteq A_{n+1}$ $\cap_n A_n = \emptyset \Longrightarrow \lim\limits_{n\to\infty} \mu(A_n) = 0$.

Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be arbitrary, fixed sequence of sets such that $A_n \supseteq A_{n+1}$, and $\cap_n A_n = \emptyset$. Then $\forall n \in \mathbb{N}$ let $k_n \in \mathbb{N}$ such that $A_n = \cup_{j=1}^{k_n}(M_j^n \times N_j^n)$. Let $F \circeq \{f \in \mathbb{N}^{\mathbb{N}} \mid f(n) \leq k_n \quad \forall n\ \}$, then $\cap_n A_n = \cup_{f\in F} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)$. It is clear that

$$\cap_n A_n = \emptyset \Longrightarrow (\cap_n (M_{f(n)}^n \times N_{f(n)}^n) = \emptyset \quad \forall f \in F). \tag{3}$$

Split $\cap_n (M_{f(n)}^n \times N_{f(n)}^n)$ sets into two groups. Let the first one F_1 contain fs such that $\cap_n M_{f(n)}^n = \emptyset$, and let the others be in F_2.

Let $M_n \circeq \cup_{f\in F_1} \cap_{j=1}^{n} M_{f(j)}^{j}$, where n is arbitrary, fixed. For any n M_n consists of finite number of sets from $\mathcal{A}_M$, so $M_n \in \mathcal{A}_M$. It is immediate that $M_n \supseteq M_{n+1}$ $\forall n$, hence $(M_n)_{n\in\mathbb{N}}$ is a monotone sequence of sets. We have to see that $\cap_n M_n = \emptyset$.

$\cap_n M_n = \cup_{f\in F_1} \cap_n M_{f(n)}^n$. Since (3) $\cap_n M_{f(n)}^n = \emptyset$ $\forall f \in F_1$, so $\cap_n M_n = \emptyset$.

It follows that $\cap_n (M_n \times N) \supseteq \cup_{f\in F_1} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)$. Because of μ_M is σ-additive, so $\mu_M(M_n) \to 0$, hence

$$\lim_{n\to\infty} \mu_M(M_n) = \lim_{n\to\infty} \mu(M_n \times N) \geq \lim_{n\to\infty} \mu(\cup_{f\in F_1} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)),$$

therefore $\mu(\cup_{f\in F_1} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)) \to 0$.

The case of F_2 is the same, so $\mu(\cup_{f\in F_2} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)) \to 0$.

μ is additive, hence

$$\begin{aligned} &\mu(\cup_{f\in F_1} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)) + \mu(\cup_{f\in F_2} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)) \\ &\geq \mu((\cup_{f\in F_1} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)) \cup (\cup_{f\in F_2} \cap_n (M_{f(n)}^n \times N_{f(n)}^n))). \end{aligned} \tag{4}$$

Let $\epsilon > 0$ be arbitrary, fixed. Then $\exists n_1 \in \mathbb{N}$ such that $\mu(\cup_{f\in F_1} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)) < \frac{\epsilon}{2}$, $\forall n \geq n_1$, and $\exists n_2 \in \mathbb{N}$, such that $\mu(\cup_{f\in F_2} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)) < \frac{\epsilon}{2}$, $\forall n \geq n_2$. Then $\mu(\cup_{f\in F_1} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)) + \mu(\cup_{f\in F_2} \cap_n (M_{f(n)}^n \times N_{f(n)}^n)) < \epsilon$, $\forall n \geq \max\{n_1, n_2\}$. Since (4), and ϵ is arbitrary $\mu(A_n) \to 0$. Q.E.D.

Definition 34. *Let $g : \Delta_{MC}(T, \mathcal{A}) \to T^i$, i.e. it corresponds ν to such a point $t = (\nu_1^i, \nu_2^i, \ldots, \nu_n^i, \ldots) \in T^i$ that*

$$\nu_n^i = marg_{T_{n-1}} \nu \quad \forall n \in \mathbb{N}.$$

Lemma 35. *g is bijective (one to one and onto).*

Proof. First we show that g is injective (one to one). Any $\nu \in \Delta_{MC}(T, \mathcal{A}_T)$ determines its marginals, so it determines a unique point in T^i.

g is surjective (onto). Because of Example 9., the compact sets, and the compact regularity of marginal measures, the conditions of Theorem 10. are satisfied in the case of the measure inverse system (2). Since Proposition 30., we can apply Lemma 33. for $(S, \mathcal{A}, \nu_1^i)$ and the measure inverse limit of (2).

The measure extension theorem ensures the uniqueness, so the measure inverse inverse system (1) has a unique meaure measure inverse limit. Therefore, for any $t \in T^i$ $\exists \nu \in \Delta_{MC}(T, \mathcal{A}_T)$ (in the measure inverse limit) such that ν's marginals are in t. Q.E.D.

Definition 36. *Let* $f = g^{-1}$.

Lemma 37. f *is a homeomorphism.*

Proof. It is immediate, it is left for the readers to prove. Q.E.D.

The proof of Theorem 32. Let f be defined in Definition 36.

Since Lemma 35. f is a bijection.

Since Lemma 37. f is a homeomorphism. Q.E.D.

Remark 38. The type space in Theorem 32. is universal.

Remark 39. We proved the existence of a homeomorphism only for $(\Delta_{MC}(T, \mathcal{A}_T), \tau)$ but not for $(\Delta_{MC}\ (\sigma(\mathcal{A}_T)), \tau)$. The Example 40. illustrates the reason.

The following counterexample is mentioned in Remark 39.

Example 40. Let $\Omega \overset{\circ}{=} [0,1]^{\{1,1/2,1/3,\ldots,1/n,\ldots\}}$, i.e. be the set of functions defined on points $1, 1/2, 1/3, \ldots, 1/n, \ldots$ with range $[0,1]$.

Let $f_n(x) = \begin{cases} 1, & \text{if } x = 1/n \\ 0 & \text{otherwise} \end{cases}$, $\delta_{f_n}(A) = \begin{cases} 1, & \text{if } f_n \in A \\ 0 & \text{otherwise} \end{cases}$ be Dirac measures. Then Ω is a compact metric space, and its measurable structure generated by the coordinate projections, and the Baire and the Borel structures coincide.

Let $f_0 = 0$ be constant zero function, and let δ_{f_0} be the Dirac measure as defined above. It is easy to see that $\delta_{f_n} \to \delta_{f_0}$ point-wise on the sets of the field (generated by the cylindrical sets), but on $B = \{f_0\}$ (constant zero function), which is not in the field, but in the σ-algebra, $\delta_{f_n}(B) \nrightarrow \delta_{f_0}(B)$.

Example 41. indicates that why our model is more general than the previous works.

Example 41. Let be two players, both players have two strategies. This game in normal form is a point in $\mathbb{R}^8$. There are two random variables, which determine the payoffs of the players. Therefore, the parameter space: $S = {\mathbb{R}^8}^{\mathbb{R}^2}$ (the parameters are functions from $\mathbb{R}^2$ to $\mathbb{R}^8$). S is neither compact, nor Polish, so Mertens and Zamir's and Brandenburger and Dekel's construction do not work in this case. Let the measurable structure of S be the Borel sets of S. In our model, the opinions are the probability measures on S, but these are not necessarily compact regular measures, hence Heifetz's, and Mertens' et al. models are less general, than ours.

The main strength of our model (as we think) that in it the structure of a given order beliefs does not depend directly on the topological structure of the lower order beliefs, so the parameter space can be purely measurable.

Remark 42. We can rewrite our model as follows:

$$(\{S, \Delta(T_n^c)^i\}, ((\{S\} \cup (\mathbb{N} \cup \{0\}) \times M), R), f_{nm}|_{nRm})$$

where R is such a binary relation that $\{S\}$ is in relation only with itself, and $(n \times i)R(m \times j) \Leftrightarrow n \leq m$, moreover $f_{(n\times i)(m\times j)} : \Delta(T_m^c)^j \to \Delta(T_n^c)^i$ is such a function that $f_{(n\times i)(m\times j)}(\mu_m^j) \overset{\circ}{=} marg_{\Delta(T_n^c)^i}\mu_m^j\ \forall (n \times i)R(m \times j)$.

Remark 42. shows that the concept of belief space ”calls for” the cutting the parameter space off the model and the refitting it later.

Two remarks left:

Remark 43. The role of Baire measurable structure in our model is only that it makes the model more similar to a purely measurable model (see [9]). If one changes the Baire sets to Borel sets, then all our results remain valid.

Remark 44. It is commonly accepted that, in the Bayesian framework model itself is commonly known by the players. It is also the case in our model as well, but only up to Remark 39. In a ”nice” model, there is σ-field instead of field, however, in our model, if we use σ-field, then we get a *Bayesian model* in which, *the model itself is not commonly known by the players.*

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