Buridan's ass and a menu of options

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Abstract. The goal of the paper is to study how a menu of options affects decisions of a rational agent facing uncertainty over future payoff streams. Using the real options approach, we demonstrate that multiple options not only increase the barrier which the underlying stochastic variable has to reach in order investment became optimal, but cause the investor to be inactive even when the cost of investment is vanishing. As a technical contribution, the paper suggests a robust method of solution of a two-point optimal stopping problem.

Key words: Real options, menu of options.

JEL Classification: D81, C61, G31

1 Introduction

> Buridan's ass: A paradox of medieval logic concerning the behavior of an ass who is placed equidistantly from two piles of hay of equal size and quality. Assuming that the behavior of the ass is entirely rational, he has no reason to prefer one pile to the other and therefore cannot reach a decision which pile to eat first, and so remains in his original position and starves to death.

In real life, a decision maker typically faces a menu of options. For example, individuals have to choose among a variety of retirement plans, or insurance companies. Oil companies need to decide whether to invest into conventional (crude oil) or unconventional (tar sands) extraction. Pharmaceutical companies may target development of new drugs or improvement of the quality of existing drugs. British Petroleum (BP) Amoco and ARCO had to choose among several available options before merging together.

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In all the above examples, it is costly to reverse an action, exercising an option generates a stream of benefits which evolve stochastically, and, in many instances, options are mutually exclusive. The goal of this paper is to study how multiple options affect decisions of a rational agent facing uncertainty over future payoff streams. In particular, we want to determine which option (if any) will be exercised if a menu of options is available.

Typically, in the real options literature, the optimal exercise strategy is considered under an assumption that only one option is available at a time, and the answer is of the form: exercise the option when the underlying stochastic variable crosses a certain threshold (see, for example, Dixit and Pindyck (1996) and the bibliography therein). Among exceptions are "polar" options such as expansion-contraction (see, e.g., Abel et al. (1996) for a two-period model, Dixit and Pindyck (2000) for a Gaussian continuous time model, and Boyarchenko and Levendorskiǐ(2000) for a non-Gaussian model) or expansion-abandonment (see, e.g., Mauer and Ott (2000)).

Several studies have considered models when decision makers have some degree of flexibility in terms of switching between different modes of operation. For example, Brennan and Schwartz (1985) construct a model of decision to open, mothball, and close a mine producing a natural resource. Trigeorgis (1993, 1996) develops models that allow for a variety of options interacting within a single project. Kulatilaka (1995) presents a general finite time horizon real options model with multiple operating modes which is solved numerically by time discretization and backwards induction. For further references on switching options and flexibility, see Kulatilaka (1995), Chapter 7 in Dixit and Pindyck (1996), and Brekke and Schieldrop (2000).

Brekke and Schieldrop (2000) present a model which is the closest to the model in this paper. They consider a firm with an option to build a plant producing a single good using one of two input factors with uncertain prices. Input prices follow correlated geometric Brownian motions. The firm chooses not only when to invest, but also a technology: a pure technology that can use only one input factor or a flexible technology that allows switching between two factors. Brekke and Schieldrop (2000) notice that if the flexible technology is unavailable, then the choice between pure technologies makes the firm more reluctant to invest as compared to the case when only one technology is available, because the inaction region (i.e., the set of states where it is optimal to continue waiting for a better investment opportunity) becomes larger. However there is no precise characterization of the inaction region in that paper.

We provide complete characterization of inaction regions for various combinations of the parameters of the model. We show that Brekke and Schieldrop's (2000) conjecture does not hold sometimes, because for some parameter values, existence of an alternative option does not affect investment in a pure technology. For other parameter values, the picture may be richer than the one conjectured in the aforementioned paper.

Problems of timing investment, capital expansion-contraction program, timing new technology adoption and other problems in the real options theory are simplified if a competitive firm is considered, and the price of output is the primitive of the model. Optimal investment/disinvestment rules change (and may change significantly) if the strategic interactions are introduced (see Dixit and Pindyck (1996), Grenadier (2000, 2002), Smit and Trigeorgis (2004), Murto (2004) and the bibliography therein). To separate the dependence of optimal option(s) exercise strategies on a menu of options from the influence of such factors as strategic interactions, we consider decisions of an isolated agent who takes the evolution of payoff streams as given. We leave for the future the analysis of decision making with multiple options and strategic interactions of agents.

The leading example in our paper is a risk-neutral entrepreneur who contemplates investment of a fixed size capital into a technology. We consider the case when two projects are available, so the entrepreneur must decide not only when to invest, but also which technology to choose. In order to keep things as simple as possible, we assume that the projects are mutually exclusive, so that the entrepreneur can invest in at most one project, and it is prohibitively costly to reverse investment. We leave for the future study the analysis of the situation when the investor has an opportunity to diversify and invest a part of the available capital in one project, and the remaining capital in the other one.

Once implemented, both technologies will produce the same commodity[2]. The output is sold on the spot at the market price that follows a Gaussian process. Each of the investment opportunities generates a stream of payoffs which is an increasing function of the underlying stochastic variable. We rule out the case, when one investment project is superior to the other in all respects relevant to the investment decision, because in the latter case, the answer remains the same as when only the former investment opportunity were present. By assumption, project 1 generates higher profit flows than project 2 when the market price is relatively low. Project 2 becomes more attractive at high levels of the spot price of output.

If only one project is available, then the investment is optimal when the price of output reaches a certain barrier called the investment threshold. If the spot price is below the investment threshold, then it is optimal to wait. In other words, the inaction region is an interval adjacent to zero. If two projects are available, the inaction region may be a union of an interval adjacent to zero, and one or two bounded intervals which we call *Buridan zones*. When the spot price is in a Buridan zone, the investor, as the famous animal, waits unable to choose one of the two competing technologies although in the absence of one of the investment opportunities, the entrepreneur would have invested into an available project. If the price process is modelled as an exponential of the Brownian motion (with drift), then eventually, the spot price will leave Buridan zone, and the entrepreneur will invest. So, unlike the prototype animal, the investor will move, eventually, and will not starve to death.

To be more specific, let us describe in more detail the evolution of the spot price and investment strategy of the rational investor when there is one Buridan zone.

[2]We understand that this assumption may seem unrealistic, but if the goods produced by each technology are close substitutes, then their prices are highly (positively) correlated. Therefore our model can be regarded as the first approximation to a more realistic model.

Let P_0 be the spot price at time 0, when the entrepreneur starts contemplating the investment. For low values of P_0, it is not optimal to invest in either project, so the rational investor waits till the price crosses the investment threshold for project 1 and then invests. If P_0 is above the investment threshold but below the lower boundary of the Buridan zone, the entrepreneur invests immediately in project 1, because, as it will be clear further in the paper, the NPV of project 1 is higher than the NPV of project 2 for such spot price values. Now, suppose that P_0 is in the Buridan zone. Then the investor contemplates both projects and waits. If the spot price drops down to the lower boundary of the Buridan zone, then investment in project 1 becomes optimal. If the price goes up to the upper boundary of this interval, then investment in project 2 becomes optimal. For all values of P_0 above the upper bound of the Buridan zone, immediate investment in project 2 is optimal, because now the NPV of project 2 is bigger than the NPV of project one.

It is shown in the paper that the upper bound of the Buridan zone is bigger than the investment threshold for project 2, when only this project is available, so the option to invest in project 1 delays investment in project 2. If the spot price is in the Buridan zone, then obviously we may say that option to invest in project 2 causes delay of investment in project 1. Should project 2 had not been available, the investor would have immediately invested into the available project. This demonstrates that multiple options make delays more likely.

The rest of the paper is organized as follows. In Section 2, investment delays with a menu of options are demonstrated for the simplest model of uncertainty which is resolved in one period. In Section 3, we consider the timing of investment when the entrepreneur has to choose between two projects, and analyze the conditions which ensure the existence of one or two Buridan zones. In Section 4, we consider in more detail the case of one Buridan zone. The standard approach to a two-point optimal stopping problem in the real options literature is to write down a non-linear system of four equations in four unknowns and to solve it numerically. This is not very convenient because a solution to a non-linear system is not unique, and therefore there is no guarantee that a numerical procedure gives the correct solution. The paper demonstrates how an optimal stopping problem with two barriers can be reduced to finding the smallest of two zeros of a concave function. After that, the investment thresholds and the value of the investment opportunity are given by explicit analytical expressions in terms of the smallest zero and parameters of the model. Numerical results are presented, and the dependence of the Buridan zone on the costs of investment is analyzed. As the fixed cost of investment vanishes, the inaction region adjacent to zero vanishes as well, but the Buridan zone does not. The first observation is a well-known result in the real options theory, whereas the second one seems to be new. In Section 5, we discuss the extension of our results to the case of a diffusion process with embedded jumps. Section 6 concludes.

2 Two period model

2.1 Investment problem

Consider the simplest possible model. Assume that project j ($j = 1, 2$) generates the perpetual profit flow P_t^j, where at any $t \geq 0$, P_t^j is a random variable whose current realization $P_0^j = P$ is the same for both projects and known. The fixed cost of investment in either project is $I > 0$. The entrepreneur anticipates that at $t = 1$, both P^1 and P^2 may go up or down with equal probabilities and will remain at its new level forever (i.e., all the uncertainty is resolved at $t = 1$). To be more specific, for project j, with probability 1/2, for all $t \geq 1$, $P_t^j = (1-d_j)P$, and, also with probability 1/2, for all $t \geq 1$, $P_t^j = (1 + u_j)P$. Let $\beta \in (0, 1)$ be an endogenous discount factor. Assume that the parameters of the model satisfy the following conditions

(i) $1 > u_2 > u_1 > 0$,

(ii) $1 > d_2 > d_1 > 0$,

(iii) $u_1 - d_1 > u_2 - d_2 > 0$,

(iv) $2 + \beta(u_1 - d_1) > \beta(1 + u_2)$.

Clearly, waiting with the decision to invest or not in either project beyond $t = 1$ is not optimal, because it reduces the present value of the potential investment gain without adding new information.

2.2 Optimal investment strategy

Suppose that no investment is made at $t = 0$. Then, at time $t = 1$, the entrepreneur makes a "now-or-never" decision to invest in project 1 or 2. Assume first that only project j is available. It is easy to see that if $I > P_1^j/(1 - \beta)$ then investment is not optimal, and the value of the investment opportunity is zero. Otherwise, the investment is made and the gain is $P_1^j/(1 - \beta) - I$. We conclude that, given the current profit $P_0^j = P$, the expected present value at $t = 0$ of investment in project j at $t = 1$ is

$$\begin{aligned} V_1^j(P) &= \beta E\left[\left(\frac{P_1^j}{1-\beta} - I\right)_+ \mid P_0^j = P\right] \\ &= \frac{1}{2}\beta\left(\frac{(1+u_j)P}{1-\beta} - I\right)_+ + \frac{1}{2}\beta\left(\frac{(1-d_j)P}{1-\beta} - I\right)_+ . \end{aligned}$$

Here we use the standard notation $a_+ = \max\{a, 0\}$. We see that if $P < (1 - q)I/(1+u_j)$, then investment is not optimal in either state. If $(1-q)I/(1+u_j) \leq P < (1 - q)I/(1 - d_j)$, then investment is optimal at $t = 1$ only if P^j goes up. For $P \geq (1 - q)I/(1 - d_j)$ investment is optimal in both states at $t = 1$. Hence

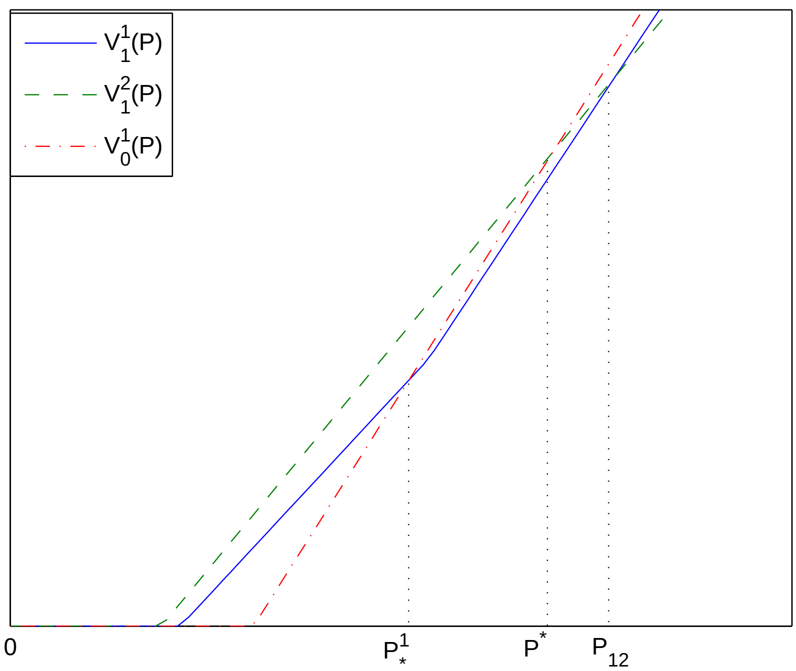

Figure 1: Two-period model. Parameters: $\beta = 0.95$, $I = 5$, $u_1 = 0.6$, $d_1 = 0.35$, $u_2 = d_2 = 0.8$. $V_1^{1,2}(P)$ – the option value of investment in project 1 or 2 at $t = 1$. $V_0^1(P)$ – the value of investment in project 1 at $t = 0$. P^* – the investment threshold when two projects are available. P_*^1 – the investment threshold for project 1, when only this project is available. At P_{12} the investor is indifferent between projects 1 and 2 at $t = 1$.

(see Figure 1)

$$V_1^j(P) = \begin{cases} 0 & \text{if } P < (1-q)I/(1+u_j), \\ \frac{\beta}{2}\left(\frac{(1+u_j)P}{1-\beta} - I\right) & \text{if } (1-q)I/(1+u_j) \le P < (1-q)I/(1-d_j), \\ \frac{\beta}{2}\left(\frac{(2+u_j-d_j)P}{1-\beta} - 2I\right) & \text{if } P \ge (1-q)I/(1-d_j). \end{cases}$$

If two projects are available, the rational investor chooses the one of the highest value, therefore, at time 0, the option value of investment at $t = 1$ is $V_1(P) = \max\{V_1^1(P), V_1^2(P)\}$. Using assumptions (i)–(iii) on the values of the parameters made earlier and comparing values $V_1^1(P)$ and $V_1^2(P)$, we obtain the following option value of the future investment

$$V_1(P) = \begin{cases} 0 & \text{if } P < (1-q)I/(1+u_2), \\ \frac{\beta}{2}\left(\frac{(1+u_2)P}{1-\beta} - I\right) & \text{if } (1-q)I/(1+u_2) \le P < P_{12}, \\ \frac{\beta}{2}\left(\frac{(2+u_1-d_1)P}{1-\beta} - 2I\right) & \text{if } P \ge P_{12}, \end{cases}$$

where $P_{12} \in ((1-q)I/(1-d_1), (1-q)I/(1-d_2))$ is a solution to

$$\frac{(1+u_2)P}{1-\beta} - I = \frac{(2+u_1-d_1)P}{1-\beta} - 2I.$$

Evidently, P_{12} is the level of profit which makes the investor indifferent between projects 1 and 2 at $t = 1$.

Consider now the value of investment in project j at time $t = 0$.

$$V_0^j(P) = \left(\frac{(1+\beta(u_j-d_j)/2)P}{1-\beta} - I\right)_+.$$

The investor has to choose between investment at $t = 0$ and investment at $t = 1$, i.e., her value function is $V(P) = \max\{V_0^1(P), V_0^2(P)\, V_1(P)\}$. Since $u_1 - d_1 > u_2 - d_2 > 0$, we have $V_0^1(P) > V_0^2(P)$ for all $P > 0$. Therefore if the entrepreneur invests at $t = 0$, she will invest in project 1.

On the strength of assumptions (i) and (iv), there exists $P^* \in ((1-\beta)I/(1+\beta(u_1-d_1)/2), P_{12})$ such that $V_0^1(P^*) = V_1(P^*)$, i.e., at the spot value $P_0^1 = P_0^2 = P^*$, the entrepreneur is indifferent between investment at $t = 0$ and waiting till $t = 1$ (see Figure 1). For all $P > P^*$, $V_0(P) > V_1(P)$, hence the entrepreneur will immediately invest in project 1. If $(1-q)I/(1+u_2) \le P < P^*$, then the entrepreneur will wait till $t = 1$ and then invest in project 2 only if P^2 goes up, otherwise she will not invest at all. If $P < (1-q)I/(1+u_2)$, then no investment will be ever made.

2.3 Investment delays with multiple options

Notice that had only project 1 been available, investment in this project at $t = 0$ would have been optimal for all $P > P_*^1$, where P_*^1 is a solution to $V_0^1(P) = V_1^1(P)$. It is straightforward to check that $P_*^1 < P^*$, hence if the current realization of the state variable is $P \in (P_*^1, P^*)$, no investment activity will be observed when two projects are available as opposed to the case when only project 1 is available. We conclude that the existence of the second project delays investment in project 1 in the sense that it increases the barrier that the state variable (current profit) has to cross in order that the latter investment become optimal. Moreover, if at $t = 1$ the profit goes down, no investment will be ever made, though without project 2 at hand, the entrepreneur would have invested in project 1 at $t = 0$.

Had only project 2 been available, investment in this project at $t = 0$ would have been optimal for all $P > P_*^2$, where P_*^2 is a solution to $V_0^2(P) = V_1^2(P)$. Evidently, $P_*^2 > P^*$, hence the option to invest in project 1 delays investment in project 2, because the latter investment can happen at $t = 1$ only.

3 Existence of a Buridan zone: general analysis

3.1 Model specification

Assume that after the optimal adjustment for flexible factors of production, project j generates the operating profit $G_j P_t^{\alpha_j}$ $(j = 1, 2)$, where P_t is the price of output. Further assume that $G_j > 0$, $\alpha_2 > \alpha_1 > 0$. If the entrepreneur decides to invest in project j at time $\tau \geq 0$, she incurs fixed cost $I_j > 0$ (which also can be interpreted as the present value of the expenditure stream to which the investor commits at time τ) and starts to accrue the stream of operating profits from $t = \tau$ onward. The assumption $\alpha_2 > \alpha_1$ implies that the second project performs better than the first one at high levels of the state variable (spot price of output).

Assume that the riskless rate $q > 0$ is fixed, and under the risk-neutral measure, the price of output follows the geometric Brownian motion (with drift): $P_t = e^{X_t}$, that is, the dynamics of X_t is governed by a stochastic differential equation

$$dX_t = bdt + \sigma dW_t,$$

where dW_t is the increment of the standard Brownian motion, $b \in \mathbf{R}$ is the drift, and $\sigma > 0$ is the variance. Introduce

$$\Psi(z) = \frac{\sigma^2}{2} z^2 + bz.$$

Let $P = e^x$ be the current (spot) price and E^x denote the expectation operator under the risk-neutral measure, conditioned on $X_0 = x$. The moment generating function of the Brownian motion admits a representation

$$E\left[e^{\alpha X_t}\right] = E\left[e^{\alpha X_t} \mid X_0 = 0\right] = e^{t\Psi(\alpha)}, \tag{3.1}$$

therefore

$$E^x\left[e^{\alpha X_t}\right] = e^{t\Psi(\alpha)+\alpha x}. \tag{3.2}$$

Equation (3.2) allows one to calculate the NPV of project j started at the spot price P

$$V^j_{\text{ex}}(P) = E^x\left[\int_0^\infty e^{-qt} G_j e^{\alpha_j X_t} dt\right] - I_j = \frac{G_j P^{\alpha_j}}{q - \Psi(\alpha_j)} - I_j. \tag{3.3}$$

The value in equation (3.3) is well-defined iff

$$q - \Psi(\alpha_j) > 0. \tag{3.4}$$

Denote by $\beta^\pm = \beta^\pm(q)$ the positive and negative roots of the fundamental quadratic

$$q - \Psi(z) = 0;$$

under condition (3.4), $\beta^- < 0 < \alpha_1 < \alpha_2 < \beta^+$. Set

$$\kappa^+_q(z) = \frac{\beta^+}{\beta^+ - z}, \quad \kappa^-_q(z) = \frac{\beta^-}{\beta^- - z}. \tag{3.5}$$

Since $q - \Psi(z)$ is a quadratic polynomial, and $\beta^{\pm}$ are its roots, the following identity holds

$$\frac{q}{q - \Psi(z)} = \kappa_q^+(z)\kappa_q^-(z). \tag{3.6}$$

3.2 The case of only one investment opportunity

The classical investment rule is: invest when $P_t = H_{j0}$, where the investment threshold H_{j0} is the solution to the equation

$$\frac{G_j {H_{j0}}^{\alpha_j}}{q - \Psi(\alpha_j)} = \frac{\beta^+}{\beta^+ - \alpha_j} \cdot I_j \tag{3.7}$$

(see, e.g., Dixit and Pindyck (1996), where the result is obtained for the case $\alpha_j = 1$). We set $K_j = qI_j/G_j$ and, using (3.6), rewrite (3.7) as

$$K_j^{-1}\kappa_q^-(\alpha_j){H_{j0}}^{\alpha_j} = 1, \tag{3.8}$$

so that the investment threshold is

$$H_{j0} = \left(K_j/\kappa_q^-(\alpha_j)\right)^{1/\alpha_j}. \tag{3.9}$$

If at time 0, when the entrepreneur starts contemplating investment, $P_0 = P \geq H_{j0}$, then she invests immediately, and the NPV of the project is given by (3.3). Using (3.6) and (3.8), we may simplify (3.3):

$$V^j_{\text{ex}}(P) = I_j\kappa_q^+(\alpha_j)((P/H_{j0})^{\alpha_j} - 1), \quad P \geq H_{j0}. \tag{3.10}$$

If $P_0 = P < H_{j0}$, the investor waits till P_t reaches H_{j0}, and the option value of investing into the project is $V^j_{\text{opt}}(P) = D_j(P/H_{j0})^{\beta^+}$, where the constant D_j can be found from the value-matching condition

$$V^j_{\text{opt}}(H_{j0}) = \frac{G_j {H_{j0}}^{\alpha_j}}{q - \Psi(\alpha_j)} - I_j.$$

Using (3.8), we derive

$$D_j = \frac{qI_j\kappa_q^-(\alpha_j)^{-1}}{q - \Psi(\alpha_j)} - I_j = I_j(\kappa_q^+(\alpha_j) - 1).$$

Thus,

$$V^j_{\text{opt}}(P) = I_j(\kappa_q^+(\alpha_j) - 1)(P/H_{j0})^{\beta^+}, \quad P \leq H_{j0}. \tag{3.11}$$

The value function of project j is therefore

$$V^j(P) = \begin{cases} V^j_{\text{opt}}(P) & \text{if } P \leq H_{j0}, \\ V^j_{\text{ex}}(P) & \text{if } P \geq H_{j0}. \end{cases}$$

For the extension of (3.9), (3.10) and (3.11) for general Lévy processes (that is, processes with stationary independent increments), and interpretation of the investment rule as Marshallian rule with the NPV calculated under the infimum process instead of the initial one (the record setting bad news principle), see Boyarchenko (2004). Notice that (3.6) is a special case of the Wiener-Hopf factorization, which is valid for any Lévy process. In the general case, formulas for the $\Psi(z)$, definable from (3.1), and factors $\kappa_q^{\pm}(z)$ are more complicated than in the Brownian motion case; however, in the case of a diffusion process with embedded exponentially distributed jumps, $\kappa_q^{\pm}(z)$ can be easily expressed in terms of the roots of the rational function $q - \Psi(z)$.

3.3 Classification of inaction regions when two projects are available

The entrepreneur has the perpetual American option with the payoff $V_{\text{ex}}(P) = \max\{V_{\text{ex}}^1(P), V_{\text{ex}}^2(P)\}$, where P is the current price. The exposition below simplifies if one makes the change of variables $P \mapsto P^{\alpha_2}$ (this is equivalent to setting $\alpha_2 = 1$), which makes V_{ex}^2 a linear function and V_{ex}^1 a concave one. All the stylized figures and numerical examples in the paper are produced under this normalization. The following statements can be easily verified graphically.

I. If $I_2 \geq I_1$, then there exists a unique $H_{12} > 0$ such that

$$\begin{array}{rcll} V_{\text{ex}}^1(P) & > & V_{\text{ex}}^2(P) & \forall\ P < H_{12}, \\ V_{\text{ex}}^1(P) & < & V_{\text{ex}}^2(P) & \forall\ P > H_{12}. \end{array}$$

At $P = H_{12}$, $V_{\text{ex}}(P)$ has a kink.

II. If $I_2 < I_1$, then the following three cases are possible:

1. for all $P > 0$, $V_{\text{ex}}^1(P) < V_{\text{ex}}^2(P)$;
2. there exists H_{12} such that $V_{\text{ex}}^1(H_{12}) = V_{\text{ex}}^2(H_{12})$ but for all other P, $V_{\text{ex}}^1(P) < V_{\text{ex}}^2(P)$;
3. there exist $0 < H_* < H^*$ such that

$$\begin{array}{rcll} V_{\text{ex}}^1(P) & < & V_{\text{ex}}^2(P), & \forall\ P < H_* \text{ and } P > H^*, \\ V_{\text{ex}}^1(P) & > & V_{\text{ex}}^2(P), & \forall\ P \in (H_*, H^*). \end{array}$$

Notice that in the first two cases, $V_{\text{ex}}(P) = V_{\text{ex}}^2(P)$ for all P, and in the third one, $V_{\text{ex}}(P)$ has two kinks at $P = H_*, H^*$.

For small levels of P, it is not optimal to invest in either of the two projects. Hence, there exists an inaction zone of the form $(0, H_0)$. To find H_0, we use the standard argument. The option value in this inaction (sub)region is of the form $V_{\text{opt}}(P) = AP^{\beta^+}$, and the constant A and the boundary H_0 can be found from the value matching and smooth pasting condition. Notice, first, that V_{opt} is convex (recall that $\beta^+ > \alpha_2 > \alpha_1$ and $\alpha_2 = 1$ after the change of variables) and $V_{\text{ex}}(0) = \max\{-I_1, -I_2\} < 0$, therefore for a sufficiently large A, the curve

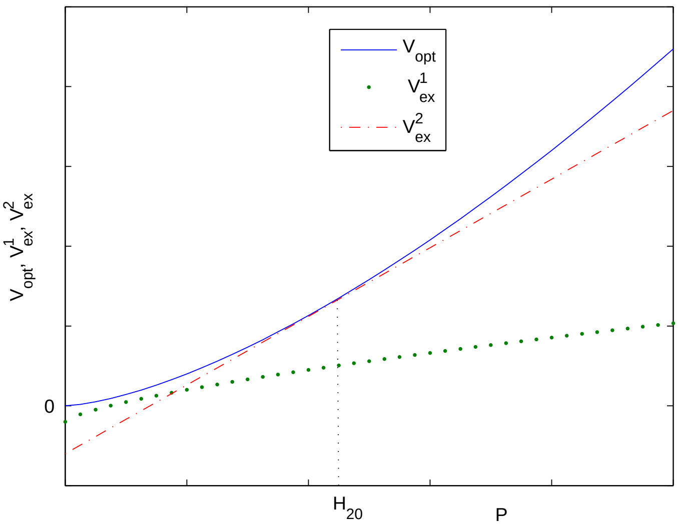

Figure 2: Case I (a): project 1 is redundant.

AP^{β^+} will not touch the curve $V_{\rm ex}(P)$. Now, as A decreases, the touchdown happens eventually. Since the touchdown cannot happen at any of the kinks, the following cases are possible (we use the same enumeration, as above, with additional subcases, whenever necessary).

Case I (a). $I_2 \geq I_1$, and the curves touch at $P = H_{20} > H_{12}$ only. In this case, project 1 is redundant, the action region is $[H_{20}, +\infty)$, and the investment is made in project 2. We plot this case in Figure 2.

Case I (b). $I_2 \geq I_1$, and the curves touch at $P = H_{10} < H_{12}$ only. In this case, if $P_0 = P < H_{10}$, the entrepreneur waits till the price, P_t, reaches H_{10}, and then invests in project 1. If P is sufficiently large, then investment in project 2 is optimal but it cannot be optimal for all $P > H_{12}$. Indeed, if it had been the case, the value of the investment opportunity (with two projects available) would have coincided with $V_{\rm ex}$ on $[H_{10}, +\infty)$. But $V_{\rm ex}(P)$ has a kink at H_{12}, and the value function must be smooth. This implies that in a neighborhood of H_{12}, it must be optimal to wait and not to invest in either of the two projects. The intuitive idea is that by waiting a little bit longer, we can observe the next realizations of P and choose positions on either side of the kink. Indeed, suppose that instead of waiting, the entrepreneur invests in project 1 when $P = H_{12} - \epsilon$ and $V^1_{\rm ex}(H_{12} - \epsilon) > V^2_{\rm ex}(H_{12} - \epsilon)$. It may be the case that the next moment $P = H_{12} + \epsilon$, then project 2 becomes more attractive than project 1. But investment is irreversible, so it was better to delay the investment in the

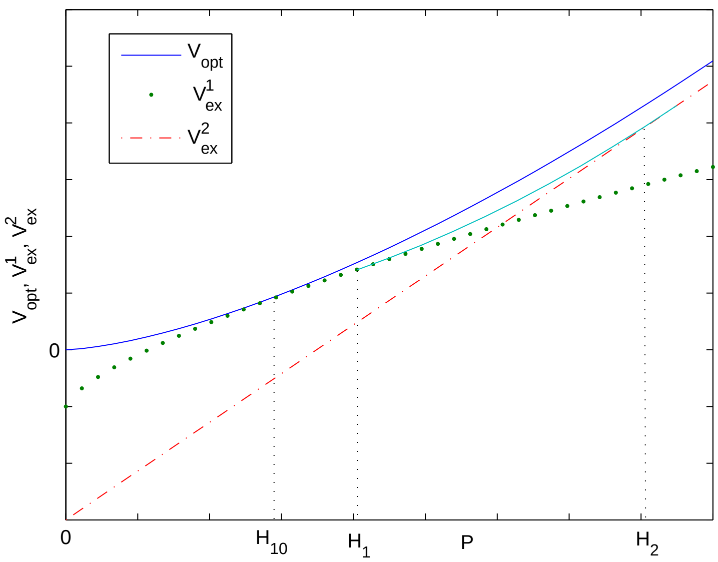

Figure 3: Case I (b): one Buridan zone (H_1, H_2).

first place[3]. To be more specific, in a certain neighborhood of H_{12}, it must be optimal to wait for the state variable to reach a certain level H_2, and then invest in project 2. Recall that the latter is better than the former for high realizations of the state variable. However, if the state variable falls sufficiently low, to a certain level H_1, then investment in the project 1 becomes optimal.

Clearly, $H_{10} \leq H_1 < H_{12} < H_2$. In fact, we may write $H_{10} < H_1$, because if $H_{10} = H_1$ the curves AP^{β^+} and V_{ex} touch at two points - we study this case below. Since $H_{10} < H_1$, project 1 is not redundant, hence there is an additional non-trivial investment opportunity. We conclude that $H_2 > H_{20}$ because an additional investment opportunity must increase the option value of waiting.

To summarize, case I (b) is the case of one Buridan zone described in the Introduction. See Figure 3.

Case I (c). $I_2 \geq I_1$, and the curves touch at $P = H_1 < H_{12}$ and $P = H_{20} > H_{12}$ only. In this case, the following two investment strategies are optimal:

(i) the same as in Case I (a);

(ii) if $P = H_1 = H_{10}$, invest in project 1, if $P \geq H_2$, invest in project 2, if $P \in (0, H_2)$ and $P \neq H_1$, wait.

We may call Case I (c) the case of a *removable* Buridan zone: if we use the first investment strategy, we may forget about the Buridan zone. This case is plotted

[3]For more discussion of the smooth pasting principle in real options, see, for example, Dixit and Pindyck (1996) and references therein.

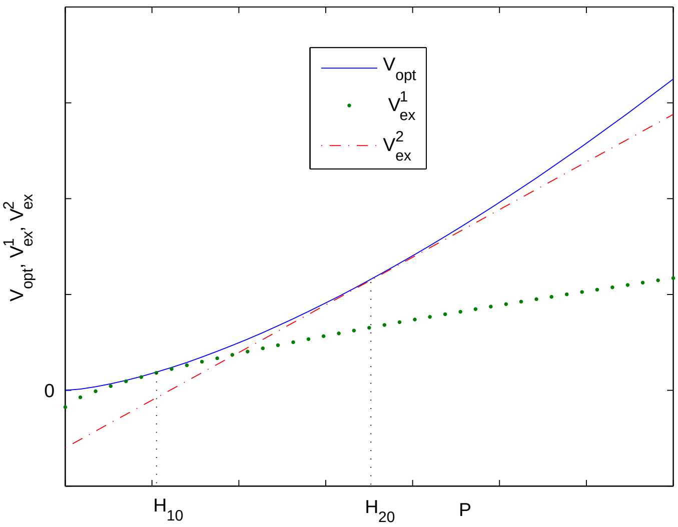

Figure 4: Case I (c): removable Buridan zone.

in Figure 4.

Case II.1. Project 1 is evidently redundant.

Case II.2. The following cases are possible:

(a) The curves touch at $P = H_{20}$ only, and $H_{20} \neq H_{12}$. Project 1 is redundant.

(b) The curves touch at $P = H_{20} = H_{12} = H_{10}$. The optimal strategy is: if $P > H_{12}$, invest in project 2; if $P < H_{12}$, wait, and if $P = H_{12}$, invest in either of the two projects.

Case II.3. The following cases are possible:

(a) The curves touch at $P = H_{20} > H^*$ only. Project 1 is redundant, and the optimal investment strategy is the same as in Case I (a).

(b) The curves touch at $P = H_{10} \in (H_*, H^*)$ only. The optimal investment strategy is as in Case I (b) (one Buridan zone).

(c) The curves touch at $P = H_{10} \in (H_*, H^*)$ and at $P = H_{20} > H^*$ only. The optimal strategy is as in Case I (c) (removable Buridan zone).

(d) The curves touch at $P = H_{20} < H_*$ only. The same arguments which we applied in Case I (b) to a neighborhood of H_{12}, are now applicable to neighborhoods of H_* and H^*, where V_{ex} has kinks. Hence, we have two Buridan zones $(H_{*,2}, H_{*,1})$ and (H_1^*, H_2^*), where

$$H_{20} < H_{*,2} < H_{*,1} < H_1^* < H_2^*.$$

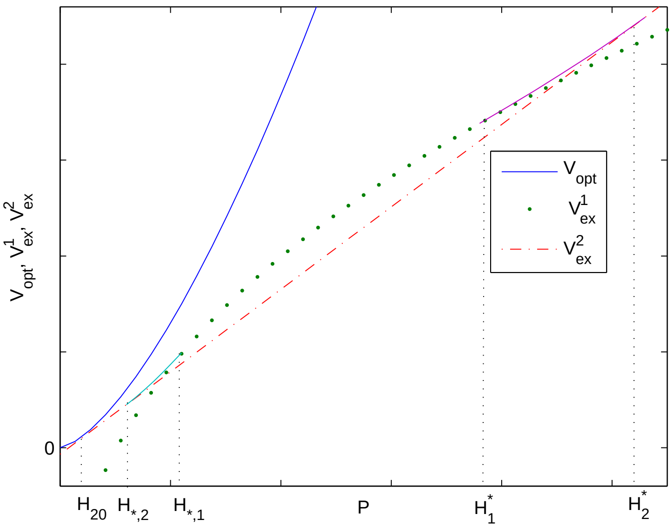

Figure 5: Case II.3 (d): two Buridan zones $(H_{*,2}, H_{*,1})$ and (H_1^*, H_2^*).

> See Figure 5. The optimal exercise rule is: as P_t arrives in $[H_{20}, H_{*,2}]$ or $[H_2^*, +\infty)$, invest in project 2, and as P_t arrives in $[H_{*,1}, H_1^*]$, invest in project 1. In this case, the inaction region consists of a conventional region $(0, H_{20})$ adjacent to 0, and two Buridan zones $(H_{*,2}, H_{*,1})$ and (H_1^*, H_2^*). Observe that now, investment in project 2 requires either relatively low or relatively high realization of the spot price. Investment in project 1 is optimal at moderately high values of the spot price. The intuition behind this result is that for lower realizations of the spot price, project 1 is more costly to implement than project 2. At higher realizations of the spot price, project 2 becomes more attractive than project 1 in terms of profit flows.

The calculation of the boundary points and the value function of investment opportunity, $V(P)$, is similar in all cases. We consider the cases I (b) and I (c).

4 The case of one Buridan zone

4.1 General analysis

Case I (b) is observed iff $V^1_{\rm opt}(H_{10}) > V^2_{\rm opt}(H_{10})$, that is,

$$I_1 \cdot [\kappa_q^+(\alpha_1) - 1] > I_2 \cdot [\kappa_q^+(\alpha_2) - 1](H_{10}/H_{20})^{\beta^+}.$$

The equivalent condition is

$$\left(\frac{H_{10}}{H_{20}}\right)^{\beta^+} < \frac{I_1(\kappa_q^+(\alpha_1)-1)}{I_2(\kappa_q^+(\alpha_2)-1)}.$$

For simplicity of exposition assume that $K_1 = K_2 = K$. Then, using (3.9), we obtain a condition

$$K^{1/\alpha_1 - 1/\alpha_2}\frac{\kappa_q^-(\alpha_2)^{1/\alpha_2}}{\kappa_q^-(\alpha_1)^{1/\alpha_1}} < \frac{I_1}{I_2}\left(\frac{\kappa_q^+(\alpha_1)-1}{\kappa_q^+(\alpha_2)-1}\right)^{1/\beta^+}.$$

Finally, the upper bound for K is:

$$K < K_{\rm up} := \left\{\frac{\kappa_q^-(\alpha_2)^{\alpha_1}}{\kappa_q^-(\alpha_1)^{\alpha_2}}\left(\frac{I_1}{I_2}\right)^{\alpha_1\alpha_2}\left[\frac{\kappa_q^+(\alpha_1)-1}{\kappa_q^+(\alpha_2)-1}\right]^{\alpha_1\alpha_2/\beta^+}\right\}^{1/(\alpha_2-\alpha_1)}. \quad (4.1)$$

In Case I (c), we obtain condition (4.1) with the equality.

4.2 Explicit formulas

4.2.1 Reduction to a system of algebraic equations

In this subsection, explain how to find the boundaries H_1 and H_2 of the Buridan zone. Let $K < K_{\rm up}$. Then the Buridan zone exists. Let $V(P)$ be the value of the investment opportunity with the possibility of a choice between the two projects. In the Buridan zone (H_1, H_2), which is a part of the inaction region, the value function V is of the form

$$V(P) = A_+P^{\beta^+} + A_-P^{\beta^-}, \quad (4.2)$$

and at the boundary points $P = H_j, j = 1, 2$, V satisfies the value-matching and smooth pasting conditions (see (3.10)):

$$\begin{aligned} V(H_j) &= I_j[\kappa_q^+(\alpha_j)(H_j/H_{j0})^{\alpha_j} - 1]; & (4.3)\\ V'(H_j) &= \alpha_j I_j \kappa_q^+(\alpha_j)(H_j/H_{j0})^{\alpha_j}{H_j}^{-1}. & (4.4)\end{aligned}$$

Substitute (4.2) into (4.3)-(4.4), and multiply (4.4) by H_j:

$$\begin{aligned} A_+{H_j}^{\beta^+} + A_-{H_j}^{\beta^-} &= I_j[\kappa_q^+(\alpha_j)(H_j/H_{j0})^{\alpha_j} - 1]; & (4.5)\\ \beta^+A_+{H_j}^{\beta^+} + \beta^-A_-{H_j}^{\beta^-} &= \alpha_j I_j \kappa_q^+(\alpha_j)(H_j/H_{j0})^{\alpha_j}. & (4.6)\end{aligned}$$

The system (4.5)-(4.6) is a system of four equations in four unknowns $A_\pm$, and $H_{1,2}$. Below we demonstrate that it is possible to reduce the solution of the above system to finding the smallest of two zeros of a concave function.

4.2.2 Reduction to an algebraic equation

Multiply (4.5) by β^+, subtract (4.6) from (4.5):

$$(\beta^+ - \beta^-)A_-{H_j}^{\beta^-} = (\beta^+ - \alpha_j)I_j\kappa_q^+(\alpha_j)(H_j/H_{j0})^{\alpha_j} + \beta^+ I_j,$$

and divide by β^+. Since $(\beta^+ - z)/\beta^+ = \kappa_q^+(z)^{-1}$, the result is

$$\kappa_q^+(\beta^-)^{-1}A_-{H_j}^{\beta^-} = I_j[(H_j/H_{j0})^{\alpha_j} - 1]. \tag{4.7}$$

Multiply (4.5) by β^-, subtract (4.6) from (4.5):

$$(\beta^- - \beta^+)A_+{H_j}^{\beta^+} = (\beta^- - \alpha_j)I_j\kappa_q^+(\alpha_j)(H_j/H_{j0})^{\alpha_j} + \beta^- I_j,$$

and divide by β^-. Since $(\beta^- - z)/\beta^- = \kappa_q^-(z)^{-1}$, the result is

$$\kappa_q^-(\beta^+)^{-1}A_+{H_j}^{\beta^+} = I_j[(\kappa_q^+(\alpha_j)/\kappa_q^-(\alpha_j))(H_j/H_{j0})^{\alpha_j} - 1]. \tag{4.8}$$

Note that (4.7) is, in fact, a system of two equations, which can be used to exclude A_- and express $H = H_2/H_1$ as follows:

$$H^{\beta^-} = R\frac{(H_2/H_{20})^{\alpha_2} - 1}{(H_1/H_{10})^{\alpha_1} - 1}, \tag{4.9}$$

where $R = I_2/I_1$. Similarly, from (4.8), we find

$$H^{\beta^+} = R\frac{(\kappa_q^+(\alpha_2)/\kappa_q^-(\alpha_2))(H_2/H_{20})^{\alpha_2} - 1}{(\kappa_q^+(\alpha_1)/\kappa_q^-(\alpha_1))(H_1/H_{10})^{\alpha_1} - 1}. \tag{4.10}$$

To simplify the calculations and answers below, we assume that $I_1 = I_2 = I$; in the general case $I_2 \geq I_1$, the equations below will contain an additional parameter $R = I_2/I_1$. Set $B_j = \kappa_q^-(\alpha_j)(H_j/H_{j0})^{\alpha_j} = K^{-1}{H_j}^{\alpha_j}$. The last equality holds on the strength of (3.9). For H fixed, we can regard (4.9)-(4.10) as a linear system with the unknowns (B_1, B_2). We rewrite (4.9)-(4.10), first, as

$$\begin{aligned} H^{-\beta^-} &= \frac{\kappa_q^-(\alpha_1)B_1 - 1}{\kappa_q^-(\alpha_2)B_2 - 1}, \\ H^{-\beta^+} &= \frac{\kappa_q^+(\alpha_1)B_1 - 1}{\kappa_q^+(\alpha_2)B_2 - 1}, \end{aligned}$$

and then as

$$\begin{aligned} \kappa_q^-(\alpha_1)H^{\beta^-}B_1 - \kappa_q^-(\alpha_2)B_2 &= H^{\beta^-} - 1, \\ \kappa_q^+(\alpha_1)H^{\beta^+}B_1 - \kappa_q^+(\alpha_2)B_2 &= H^{\beta^+} - 1. \end{aligned}$$

Applying Cramer's rule, we obtain

$$B_j = B_j(H) = \frac{\Delta_j(H)}{\Delta(H)}, \tag{4.11}$$

where

$$\Delta(H) = H^{\beta^+}\kappa_q^+(\alpha_1)\kappa_q^-(\alpha_2) - H^{\beta^-}\kappa_q^+(\alpha_2)\kappa_q^-(\alpha_1), \tag{4.12}$$

$$\Delta_1(H) = (H^{\beta^+} - 1)\kappa_q^-(\alpha_2) + (1 - H^{\beta^-})\kappa_q^+(\alpha_2), \tag{4.13}$$

$$\Delta_2(H) = (H^{\beta^+} - 1)H^{\beta^-}\kappa_q^-(\alpha_1) + (1 - H^{\beta^-})H^{\beta^+}\kappa_q^+(\alpha_1). \tag{4.14}$$

Since $\beta^+ > 0 > \beta^-$, we have $\Delta_1(H) > 0, \Delta_2(H) > 0$ for all $H > 1$, therefore we must have $\Delta = \Delta(H) > 0$. However, direct calculations show that

$$\Delta(1) = \frac{\beta^+\beta^-(\beta^+ - \beta^-)(\alpha_2 - \alpha_1)}{(\beta^+ - \alpha_2)(\alpha_1 - \beta^-)(\beta^+ - \alpha_1)(\alpha_2 - \beta^-)} < 0,$$

therefore, if Buridan zone exist, then

$$H > H_{\min} = \left[\frac{\kappa_q^+(\alpha_2)\kappa_q^-(\alpha_1)}{\kappa_q^+(\alpha_1)\kappa_q^-(\alpha_2)}\right]^{1/(\beta^+ - \beta^-)} (> 1). \tag{4.15}$$

Since $B_j = K^{-1}H_j{}^{\alpha_j}$, we have

$$H_j = \left(K \cdot \frac{\Delta_j(H)}{\Delta(H)}\right)^{1/\alpha_j}, \tag{4.16}$$

and dividing H_2 by H_1, we obtain the equation for H:

$$K^{1/\alpha_1 - 1/\alpha_2} H = F(H), \tag{4.17}$$

where

$$F(H) = \Delta_1(H)^{-1/\alpha_1}\Delta_2(H)^{1/\alpha_2}\Delta(H)^{1/\alpha_1 - 1/\alpha_2}.$$

As $H \downarrow H_{\min}$, $\Delta(H) \to 0$ but $\Delta_j(H)$ remain bounded away from 0. Taking into account that $1/\alpha_1 - 1/\alpha_2 > 0$, we conclude that $F(H) \to 0$ as $H \downarrow H_{\min}$. As $H \to +\infty$, all three functions $\Delta_j(H)H^{-\beta^+}$, $j = 1, 2$, and $\Delta(H)H^{-\beta^+}$ have positive finite limits, therefore the limit $F(+\infty)$ exists, and it is positive and finite. These two properties of function F and its continuity imply that there exists $K^* > 0$ such that for all $K > K^*$, equation (4.17) has no solutions, for $K \le K^*$, it has solutions, and moreover, for $K < K^*$, there must exist at least two solutions. The first two properties agree with the general analysis made in the previous section: Buridan zone must exist for $K \le K_{\text{up}}$. In all numerical examples which we considered, $K_{\text{up}} < K^*$, therefore we have at least two solutions.

4.2.3 Choice of the solution

To distinguish the correct root, we need the next property of F: F is concave, which we were unable to prove analytically due to the complicated structure of F, but verified in many numerical experiments. If F is concave and there are more than one solution, then there are exactly two solutions: $H^- < H^+$. Suppose

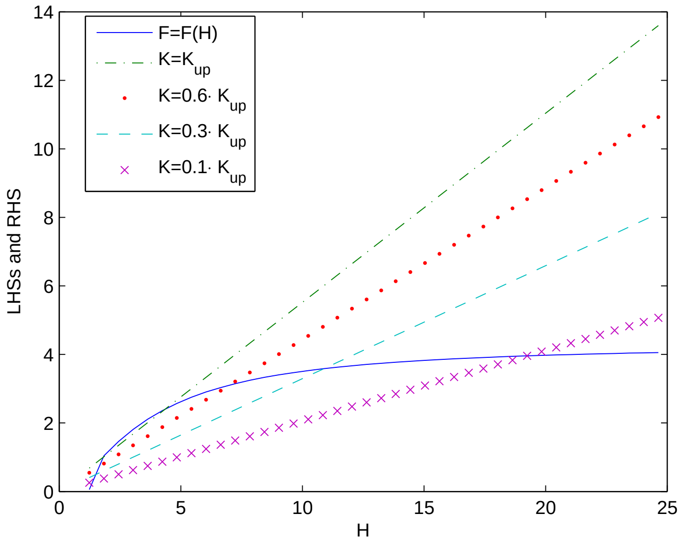

Figure 6: The LHS and RHS of (4.17). Parameters: $q = 0.04; \sigma^2 = 0.04; \alpha_1 = 0.7; \alpha_2 = 1; \beta^+ = 1.5$.

that we have fixed all the parameters but the cost; equivalently, in (4.17), all the parameters are fixed but K. As I (equivalently, K) increases, $H^-(K)$ increases and $H^+(K)$ decreases. But as the cost increases, the entrepreneur must wait longer, therefore the lower boundary of the Buridan zone should decrease, and the upper one should increase. Hence, the ratio of these boundaries, $H = H(K)$, must increase. It follows that the smallest solution of (4.17) determines the Buridan zone.

In Figure 6, we plot the RHS and LHS of (4.17), for several values of K, starting from $K = K_{\mathrm{up}}$. The other 3 straight lines correspond to $K = 0.6 \cdot K_{\mathrm{up}}; K = 0.3 \cdot K_{\mathrm{up}}; K = 0.1 \cdot K_{\mathrm{up}}$, respectively. Similar pictures are observed for all parameter's values which we considered.

4.2.4 Explicit formula for the value function

Using (4.7)–(4.8) and (3.9), we define

$$\begin{aligned} A_{--} &:= A_- {H_1}^{\beta^-}/I = \kappa_q^+(\beta^-)[(H_1/H_{10})^{\alpha_1} - 1], \\ A_{++} &:= A_+ {H_1}^{\beta^+}/I = \kappa_q^-(\beta^+)[(\kappa_q^+(\alpha_1)/\kappa_q^-(\alpha_1))(H_1/H_{10})^{\alpha_1} - 1], \end{aligned}$$

and finally, calculate the value function of the firm in Buridan zone:

$$V(P) = A_+ P^{\beta^+} + A_- P^{\beta^-} = I \cdot [A_{++} \cdot (P/H_1)^{\beta^+} + A_{--} \cdot (P/H_1)^{\beta^-}].$$

To summarize, if $K \le K_{\mathrm{up}}$, then the optimal investment rule is: invest the first time $P_t \in [H_{10}, H_1]$ or $P_t \ge H_2$. In the former case, the investment is made in

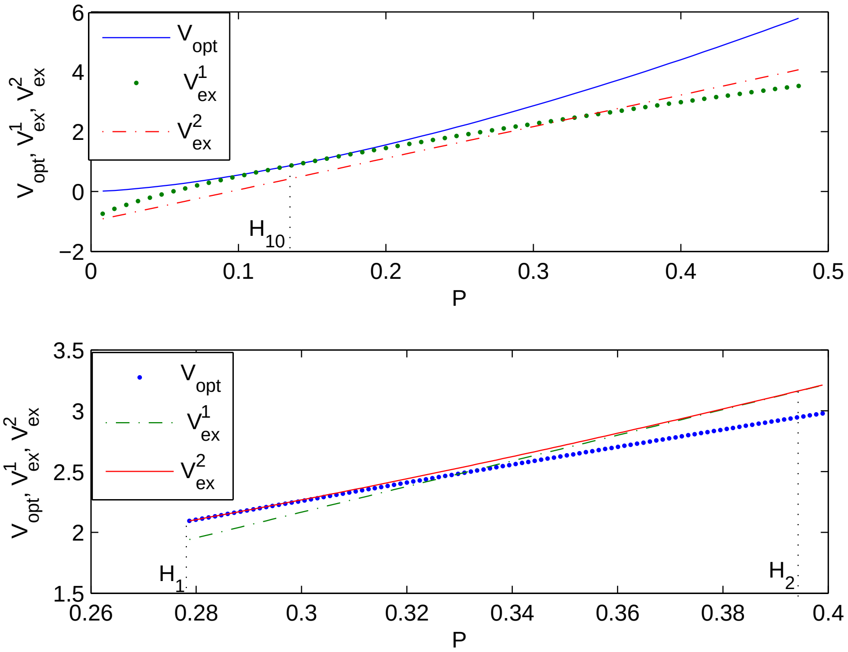

Figure 7: Curves $V_*(P) = D_1(P/H_{10})^{\beta^+}, V^1_{\rm ex}(P), V^2_{\rm ex}(P)$ in Case I (b). Upper panel: inaction region adjacent to zero. Lower panel: the Buridan zone. Parameters: $q = 0.04; \sigma^2 = 0.04; \alpha_1 = 0.7; \alpha_2 = 1; \beta^+ = 1.5; I = 1; K = 0.65 \cdot K_{\rm up}$. Thresholds: $H_{10} = 0.1362, H_1 = 0.2787, H_2 = 0.3989$.

the first project, and in the latter case - in the second one. The value function of the investment opportunity is given by

$$V(P) = I \begin{cases} (\kappa^+_q(\alpha_1) - 1)(P/H_{10})^{\beta^+}, & 0 < P < H_{10}; \\ \kappa^+_q(\alpha_1)(P/H_{10})^{\alpha_1} - 1, & H_{10} \le P \le H_1; \\ A_{++} \cdot (P/H_1)^{\beta^+} + A_{--} \cdot (P/H_1)^{\beta^-}, & H_1 < P < H_2; \\ \kappa^+_q(\alpha_2)(P/H_{20})^{\alpha_2} - 1, & H_2 \le P. \end{cases}$$

Finally, if $K > K_{\rm up}$, then the investment should be made in the second project, the first time $P \ge H_2$, and the value function is the same as if the first investment opportunity did not exist at all.

4.3 Numerical examples

Figures 7 and 8 illustrate Cases I (b), and I (c), respectively. The upper panel in Figure 7 blows up the inaction region adjacent to zero. The lower panel shows the value functions in the Buridan zone. Given q and σ, the choice of the positive root β^+ uniquely defines β^- and b, therefore we may use $\alpha_1, \alpha_2, q, \sigma$

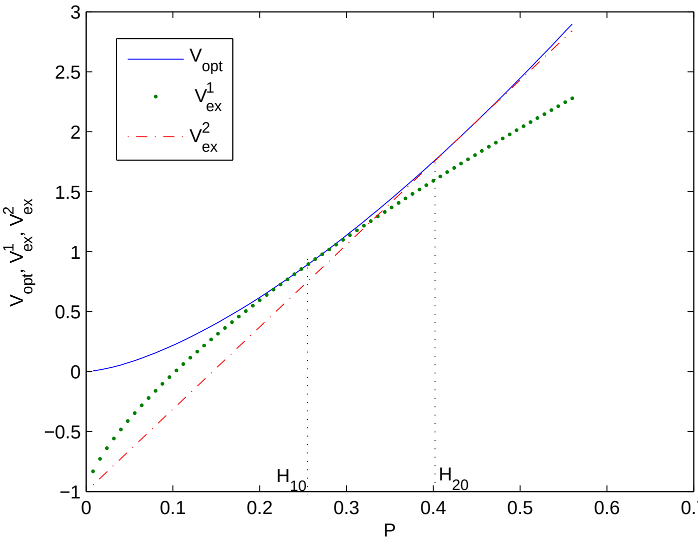

Figure 8: Curves $V_*(P) = D_0 P^{\beta^+}, V^1_{\rm ex}(P), V^2_{\rm ex}(P)$ in Case I (c). Parameters: $q = 0.04; \sigma^2 = 0.04; \alpha_1 = 0.7; \alpha_2 = 1; \beta^+ = 1.5; I = 1; K = K_{\rm up}$.

and β^+ as free parameters instead of $\alpha_1, \alpha_2, q, \sigma$ and b, which is convenient because we can satisfy the requirement $\beta^- < 0 < \alpha_1 < \alpha_2 < \beta^+$ automatically. These parameters being fixed, we can study the existence of the Buridan zone by playing with one additional parameter, K. Recall that if q and $G_1 = G_2$ are fixed, then K is proportional to the cost of investment, and the comparative statics w.r.t. K is equivalent to the one w.r.t. I. In Figure 9, we plot the investment thresholds as functions of K. It is seen that as K increases, both inaction subregions grow. At $K = K_{\rm up}$, the right boundary of the Buridan zone becomes the investment threshold for project 2, and the left boundary coincides with the investment threshold for project 1. So, at $K = K_{\rm up}$, only a point separates the two subregions. This point can be removed, and then the investment strategy is to disregard project 1. For $K > K_{\rm up}$, the Buridan zone disappears. Notice that as $K \to 0$, i.e., the cost of investment vanishes, the inaction region adjacent to 0 vanishes in the limit, but the Buridan zone does not.

The option value in the Buridan zone is not very large, however it is not negligible, and for realistic parameter values, the relative option value $(V_{\rm opt}(P) - \max\{V^1_{\rm ex}(P), V^2_{\rm ex}(P)\})/V_{\rm opt}(P)$, is of order 1-2 percent or even more, and it increases with K (equivalently, with the investment cost), as Figure 10 shows.

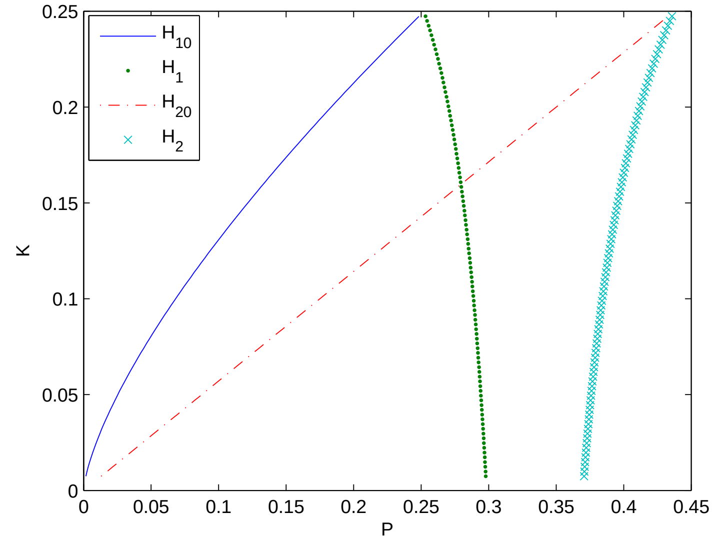

Figure 9: Thresholds H_{j0} and the boundary points of the Buridan zone. Parameters: $q = 0.04; \sigma^2 = 0.04; \alpha_1 = 0.7; \alpha_2 = 1; \beta^+ = 1.5$.

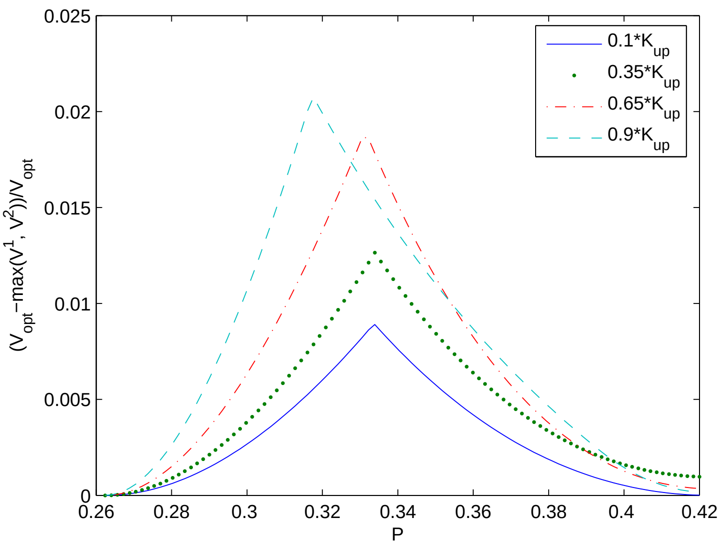

Figure 10: The relative option value in the Buridan zone. Parameters: $q = 0.04; \sigma^2 = 0.04; \alpha_1 = 0.7; \alpha_2 = 1; \beta^+ = 1.5$.

5 When Buridan's ass jumps

Using a simple perturbation argument, it can be shown that the results of the paper hold for the Brownian motion with embedded jumps provided the intensity of the latter is sufficiently small; the location of the boundary points of the action subregions will change insignificantly. The interesting part concerns the situation when at time 0, P_0 is in the inaction subregion adjacent to 0. In the absence of jumps, one may argue that all the study of Buridan zones is irrelevant: the price will never reach them, anyway. However, if the jumps are possible, the price can jump into one of the Buridan zones, and as the analysis above showed, the margin between the standard inaction region adjacent to 0, and a Buridan zone can be very small indeed, hence the probability of the price jump into the Buridan zone is non-negligible.

6 Conclusion

In the real options theory, delays are natural when at least partially irreversible decision has to be made under uncertainty. The conventional perception in the latter theory is that both irreversibility and uncertainty play their role in delays, so that if investment costs are negligible, there is no reason to wait. Our model demonstrates that this is not necessarily the case if at least two investment projects are available. Multiple options not only increase the barrier which the underlying stochastic variable has to reach in order investment became optimal, but cause the investor to be inactive even when the cost of investment is vanishing. We conclude that with a menu of options, uncertainty alone may be the reason for the delay.

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