Nash Equilibria of Games with a Continuum of Players*

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Abstract

We characterize Nash equilibria of games with a continuum of players (Mas-Colell (1984)) in terms of approximate equilibria of large finite games. For the concept of $(\varepsilon, \varepsilon)$ – equilibrium — in which the fraction of players not ε – optimizing is less than ε — we show that a strategy is a Nash equilibrium in a game with a continuum of players if and only if there exists a sequence of finite games such that its restriction is an $(\varepsilon_n, \varepsilon_n)$ – equilibria, with ε_n converging to zero. The same holds for ε – equilibrium — in which almost all players are ε – optimizing — provided that either players' payoff functions are equicontinuous or players' action space is finite.

Furthermore, we give conditions under which the above results hold for all approximating sequences of games. In our characterizations, a sequence of finite games approaches the continuum game in the sense that the number of players converges to infinity and the distribution of characteristics and actions in the finite games converges to that of the continuum game. These results render approximate equilibria of large finite economies as an alternative way of obtaining strategic insignificance.

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1 Introduction

Many economic situations involve a large number of participants, each of which has a negligible influence on the aggregate outcome. As Aumann (1964) has convincingly argued, the ideal situation of strategic insignificance can only be obtained in models featuring a continuum of agents. Likewise, equilibrium concepts that depend on the idea of strategic insignificance make good sense only in those models. This makes models with a continuum of agents an appealing framework for economic analysis.

Of course, real economies have a finite number of agents. Hence, conclusions obtained by studying economies with a continuum of agents will, typically, hold only approximately for real, finite economies. Moreover, models with a finite number of agents are more intuitive and therefore easier to understand than models with a continuum of agents. The same is true regarding equilibria of those models. Thus, we ask: can we relate equilibria of models with a continuum of agents with the more intuitive notion of equilibria of finite models? Can we make precise what it means for a conclusion obtained in a model with a continuum of players to hold approximately in finite models?

We answer these questions for normal form games in which the payoff of each player depends on his choice and on the distribution of actions (see Mas-Colell (1984)). This is done by providing a complete characterization of Nash equilibria of games with a continuum of players in terms of approximate equilibria of games with a finite number of players.

Our first characterization result uses the notion of $(\varepsilon, \varepsilon)$ – equilibrium, defined by requiring the fraction of players which are not ε – optimizing to be less than ε. It shows that a strategy is a Nash equilibrium in a game with a continuum of players if and only if there exists a sequence of finite games such that its restriction is an $(\varepsilon_n, \varepsilon_n)$ – equilibria, with ε_n converging to zero. Thus, Nash equilibria of games with a continuum of players are exactly the strategies that are approximate equilibria in some games obtained from the original one by selecting a finite number of players.

This relation can be strengthened in equicontinuous games, as our second and third characterization results show: in those games, the above characterization holds for ε – equilibrium, defined as usual by requiring almost all players to be ε – optimizing.[1] Furthermore, the above characterization in

[1] This result also holds without equicontinuous payoff functions provided that the action

terms of $(\varepsilon, \varepsilon)$ – equilibrium holds for all approximating sequences of finite games, and every Nash equilibrium can be changed in a set of measure zero to obtain another Nash equilibrium for which the same holds for ε – equilibrium.

One implication of these results is that they give us a sense in which approximate equilibria of large finite games is an alternative way of obtaining strategic insignificance. This is so because they converge to Nash equilibria of games with a continuum of players, in which players are indeed insignificant. Furthermore, they give us a precise way of stating that properties of Nash equilibria of games with a continuum of players hold approximately in similar finite games.

Intuitively, a game with a finite number of players is similar to a game with a continuum of players if it can approximately describe the same strategic situation as the continuum game. We say that a sequence of finite games approximates the strategic situation described by the given strategy in the game with a continuum of agents if both the number of players in the finite games converges to infinity and the distribution of characteristics and actions in the finite games converges to that of the continuum game.

Since we obtain a complete characterization of equilibria, we can in fact interpret this approximation as convergence of the strategic situation in the finite games to the one in the continuum game. Furthermore, this notion of convergence allow us to define Nash equilibria of games with a continuum of players as the limit points of $(\varepsilon, \varepsilon)$ – equilibria of games with finitely many players, with ε converging to zero, a principle defended by Fudenberg and Levine (1986). Alternatively, we can take Fudenberg and Levine (1986)'s principle as a criterion for defining an appropriate notion of approximate equilibria. In this case, our results imply that, for games with a continuum of players, such notion is $(\varepsilon, \varepsilon)$ – equilibrium.

Aside from these conceptual aspects, our characterization results also have practical implications. In particular, they can make games with a continuum of players accessible to researchers that are not familiar with the measure theoretical tools needed to analyze them. This is especially the case for games with a finite number of different payoff functions and possible actions: for such games, the only tools needed are the usual notions of convergence in the real line and approximate equilibrium in a finite normal form game.

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space is finite.

2 Related Literature

A related question is whether the limit of a converging sequence of equilibria of finite economies is an equilibrium in the continuum economy. This was shown to be the case by Hildenbrand and Mertens (1972) for pure exchange economies, by Dubey, Mas-Colell, and Shubik (1980) for strategic market games, and by Green (1984) for normal form games. Regarding the other direction of our characterization results, Mas-Colell (1983) and Novshek and Sonnenschein (1983) have shown that regular Walrasian equilibria of a continuum economy can be approximated by the noncooperative Cournot equilibria for the tail of any approximating sequence of finite economies.

Some papers have shown that large finite games have properties that are approximate versions of those of continuum economies. For example, **?**), Rashid (1983) and Wooders, Selten, and Cartwright (2002) showed that all Nash equilibria of large finite games in a certain class can be approximately purified; this result clearly parallels Schmeidler (1973)'s Theorem 2, asserting that any game with a continuum of players of the same class has a Nash equilibrium in which almost all players play a pure strategy.[2]

Similar characterizations are presented in Carmona (2003b) and in Carmona (2003a). In the first paper, we consider a more specialized framework in which each player's payoff functions depend only on his action and on the average choice of the others. There we obtain similar characterization results for different notions of approximation of games, which can be thought of as alternative ways of describing the convergence of the economic situation. In the second, we use tools similar to those used here to characterize Nash equilibrium distributions of games with a continuum of players in terms of symmetric, approximate equilibrium distributions with finite support of similar continuum-of-players games.

Finally, our limit results suggest a natural way to define a refinement of Nash equilibrium for games with a continuum of players: we say that a strategy is a limit equilibrium if it is the limit point in the above sense of a sequence of equilibria in finite games. The concept of limit equilibrium is defined in Carmona (2004a), where some of its properties are investigated. Also, in Barlo and Carmona (2004) we propose a refinement of Nash equilibria which is similar to limit equilibria, and can also be used to discard those Nash equilibria that are an artifact of the continuum construction.

[2]See Carmona (2004c) for a correct statement of Rashid (1983)'s Theorem.

3 Large Games

Let A be a non-empty, compact metric space of *actions* and $\mathcal{M}$ be the set of Borel probability measures on A endowed with the weak convergence topology. By Parthasarathy (1967, Theorem II.6.4), it follows that $\mathcal{M}$ is a compact metric space. We use the following notation: we write $\mu_n \Rightarrow \mu$ whenever $\{\mu_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ converges to μ and ρ denote the Prohorov metric on $\mathcal{M}$, which is known to metricize the weak convergence topology. We let d_A denote the metric on A.

Let $\mathcal{U}$ denote the space of continuous utility functions $u : A \times \mathcal{M} \to \mathbb{R}$ endowed with the supremum norm. The set $\mathcal{U}$ represents the space of *players' characteristics*; it is a complete, separable metric space.

A *game with a continuum of players* is characterized by a measurable function $U : [0,1] \to \mathcal{U}$, where the unit interval $[0,1]$ is endowed with the Lebesgue measure λ on the Lebesgue measurable sets and represents the set of *players*. We represent such game by $G = (([0,1], \lambda), U, A)$.

A *game with a finite number of players* is characterized by a function $U : T \to \mathcal{U}$, where T is a finite subset of $[0,1]$. The set T represents the set of players and it is endowed with the uniform measure ν: if T has N elements, then the measure ν on T satisfies $\nu(\{t\}) = 1/N$ for all $t \in T$. We represent such game by $G = ((T, \nu), U, A)$.

We are especially interested in games with a finite number of players that are derived from a given game $G = (([0,1], \lambda), U, A)$ with a continuum of players. Those are games $H = ((T, \nu), U_{|T}, A)$ where $U_{|T}$ denotes the restriction of U to T.

In all the cases above, a game is defined as a measurable function from a measure space of players into $\mathcal{U}$. Although we will focus exclusively on the particular cases mentioned above, we present the following definition in this general case.

Let $(X, \mathcal{X}, \mu)$ be a measure space and $G = ((X, \mathcal{X}, \mu), U, A)$ be a game. A strategy is a measurable function $f : X \to A$. Given a strategy f, $y \in A$, and $t \in T$, let $f \setminus_t y$ denote the strategy g defined by $g(t) = y$, and $g(\tilde{t}) = f(\tilde{t})$, for all $\tilde{t} \neq t$.

For any $\varepsilon \geq 0$ and strategy f let

$$E(f, \varepsilon, \mu) = \big\{t \in \operatorname{supp}(\mu) : U(t)(f(t), \mu \circ f^{-1}) \geq U(t)(a, \mu \circ (f \setminus_t a)^{-1}) - \varepsilon \text{ for all } a \in A\big\}. \tag{1}$$

The set $E(f, \varepsilon, \mu)$ is the set of players in the support of μ that are within ε

of their best response by playing according to f. When $\varepsilon = 0$, we will write $E(f, \mu)$ instead of $E(f, 0, \mu)$.

Lemma 1 *Let $\varepsilon \geq 0$, a game $G = (([0,1], \lambda), U, A)$, and a strategy f be given. Then, $E(f, \varepsilon, \lambda)$ is measurable.*

It is clear that $E(f, \varepsilon, \mu)$ is measurable when μ has finite support. Hence, $\mu(E(f, \varepsilon, \mu))$ is well-defined both when $\mu = \lambda$, and when μ has finite support. We then say that f is an (ε, δ) – *equilibrium of a game* G if $\mu(E(f, \varepsilon, \mu)) \geq 1 - \delta$. Thus, in an (ε, δ) – equilibrium, all but a small fraction of players are close to their optimum by choosing according to f. A strategy f is an ε – *equilibrium of a game* G if $\mu(E(f, \varepsilon, \mu)) = 1$, i.e., if it is an (ε, δ) – equilibrium for $\delta = 0$. A strategy f is a *Nash equilibrium of* G if f is an ε – equilibrium of G for $\varepsilon = 0$.

4 Equilibrium Distributions

4.1 Games with a continuum of players

Instead of defining a game as a measurable function from players into characteristics, we could have started by describing the game as a probability measure μ on $\mathcal{U}$ as in Mas-Colell (1984). For our purpose, equilibrium distributions provide a useful device for studying properties of equilibria in games with a continuum of players.

Given a Borel probability measure τ on $\mathcal{U} \times A$, we denote by $\tau_{\mathcal{U}}$ and τ_A the marginals of τ on $\mathcal{U}$ and A respectively. The expression $u(a, \tau) \geq u(A, \tau)$ means $u(a, \tau) \geq u(a', \tau)$ for all $a' \in A$.

Given a game μ, a Borel probability measure τ on $\mathcal{U} \times A$ is an *equilibrium distribution for* μ if

1. $\tau_{\mathcal{U}} = \mu$, and

2. $\tau(\{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(A, \tau_A)\}) = 1$.

We will use the following notation: $B_\tau = \{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(A, \tau_A)\}$. Note that B_τ is closed, and so a Borel set; hence $\tau(B_\tau)$ is well defined. Also, if (u, a) belong to B_τ, then a maximizes the function $\tilde{a} \mapsto u(\tilde{a}, \tau_A)$. Thus, we are implicitly assuming that the choice of any player does not affect the distribution of actions. It is in this sense that the notions of this section describe a game with a continuum of players.

Any game $G = (([0,1], \lambda), U, A)$ and strategy f induces a Borel probability measure τ on $\mathcal{U} \times A$ by the formula $\tau = \lambda \circ (U, f)^{-1}$. Furthermore, as the next lemma shows, if f is a Nash equilibrium of G, then $\tau = \lambda \circ (U, f)^{-1}$ is an equilibrium distribution of $\lambda \circ U^{-1}$; conversely, if τ is an equilibrium distribution and $\tau = \lambda \circ (U, f)^{-1}$, then f is a Nash equilibrium of G.

Lemma 2 *A strategy f is a Nash equilibrium of a game $G = (([0,1], \lambda), U, A)$ if and only if $\tau = \lambda \circ (U, f)^{-1}$ is an equilibrium distribution of $\lambda \circ U^{-1}$.*

4.2 Games with a finite number of players

As for games with a continuum of players, any game with a finite number of players together with a strategy also induces a Borel probability measure on $\mathcal{U} \times A$, again by the formula $\tau = \nu \circ (U, f)^{-1}$. However, in such a game, the choice of a single player has an affect on the distribution of actions and the definition of an equilibrium distribution needs to be adapted accordingly.

Let $G = ((T, \nu), U, A)$ be a game with a finite number of players, f a strategy, $\tau = \nu \circ (U, f)^{-1}$ and $\varepsilon, \delta \geq 0$. Then τ is an (ε, δ) – equilibrium distribution of $\nu \circ U^{-1}$ if $\tau(\{(u,a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(\bar{a}, \tau_A^{u,a,\bar{a}}) - \varepsilon$ for all $\bar{a} \in A\}) \geq 1 - \delta$, where $\tau_A^{u,a,\bar{a}} = \nu \circ g^{-1}$, g is defined by $g(\bar{t}) = \bar{a}$, and $g(t) = f(t)$ for all $t \neq \bar{t}$ and $\bar{t} \in T$ is such that $(U(\bar{t}), f(\bar{t})) = (u, a)$.

Note first that the distribution $\tau_A^{u,a,\bar{a}}$ is independent of the choice of $\bar{t}$. This is the distribution on the action space A that will arise if one player with characteristic u and playing a deviates and plays $\bar{a}$. In fact, we can simply define $\tau_A^{u,a,\bar{a}}$ as the marginal on A of $\tau^{u,a,\bar{a}}$, which is defined from τ as follows: $\tau^{u,a,\bar{a}}(\{(u,a)\}) = \tau(\{(u,a)\}) - 1/|T|$, $\tau^{u,a,\bar{a}}(\{(u,\bar{a})\}) = \tau(\{(u,\bar{a})\}) + 1/|T|$ and $\tau^{u,a,\bar{a}}(\{(\tilde{u},\tilde{a})\}) = \tau(\{(\tilde{u},\tilde{a})\})$ for all $(\tilde{u},\tilde{a})$ in the support of τ that are different from (u,a) and from $(u,\bar{a})$.[3] Note that this definition allows us to define an equilibrium distribution of a finite game without the explicit knowledge of a strategy, which will be useful later on.

The following lemma show that for large finite games $\tau_A^{u,a,\bar{a}}$ is close to τ_A.

[3]Note that since τ has finite support, so will $\tau^{u,a,\bar{a}}$; because of this, it is enough to define it on those points.

Lemma 3 *Let $G = ((T, \nu), U, A)$ be a game with a finite number of players and f a strategy. If g is another strategy that differs from f in at most one point, then*

$$\rho(\nu \circ f^{-1}, \nu \circ g^{-1}) \leq \frac{1}{|T|}.$$

Note also that if the game G had a continuum of players then $\tau_A^{u,a,\bar{a}} = \tau_A$ for all $u, a, \bar{a}$ and so in fact this definition coincides with the one given before. Similar to that case we have:

Lemma 4 *Let $\varepsilon \geq 0$ and $\delta \geq 0$. Then, a strategy f is an (ε, δ) – equilibrium of a game $G = ((T, \nu), U, A)$ if and only if $\tau = \nu \circ (U, f)^{-1}$ is an (ε, δ) – equilibrium distribution of $\nu \circ U^{-1}$.*

We will use the following notation: $B_\tau^\varepsilon = \{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(\bar{a}, \tau_A^{u,a,\bar{a}}) - \varepsilon$ for all $\bar{a} \in A\}$, where $\tau_A^{u,a,\bar{a}}$ is as before.

5 Characterizations of Nash Equilibria

Our results relate Nash equilibria of any game G with a continuum of players with ε – equilibria and $(\varepsilon, \varepsilon)$ – equilibria of games with a finite number of players that approximate G. Thus, in order to proceed, we will next present our notions of approximation for games and for equilibria.

Regarding the approximation of a game with a continuum of players by finite games, we will use the following notion: Let $G = (([0,1], \lambda), U, A)$ be a game with a continuum of players and $f : [0,1] \to A$ a strategy. A sequence $\{\nu_n\}_{n=1}^\infty$ of measures is an *approximating sequence of G at f* if

1. ν_n is the uniform measure on T_n, a finite subset of $[0,1]$,
2. $\nu_n \circ (U_{|T_n}, f_{|T_n})^{-1} \Rightarrow \lambda \circ (U, f)^{-1}$, and
3. $|T_n| \to \infty$.

This definition says that when the restriction of f is used in the sequence of games $G_n = ((T_n, \nu_n), U_{|T_n}, A)$ with a finite number of players, which eventually becomes arbitrarily large, it generates a sequences of distributions of characteristics and actions that converges to the one induced by f in G.

Therefore, for large n, the pair $(G_n, f_{|T_n})$ approximately describes the same strategic situation described by the pair (G, f).

We now define our first notion of approximation for equilibria. We say that f *can be approximated in* ε *– equilibrium* if there exist an approximating sequence $\{\nu_n\}_{n=1}^{\infty}$ of G at f and a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers such that:

1. $\varepsilon_n \longrightarrow 0$, and
2. $f_{|T_n}$ is an ε_n – equilibrium of $G_n = ((T_n, \nu_n), U_{|T_n}, A)$ for all $n \in \mathbb{N}$.

Similarly, f *can be approximated in* $(\varepsilon, \varepsilon)$ *– equilibrium* if there exist an approximating sequence $\{\nu_n\}_{n=1}^{\infty}$ of G at f and a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers such that:

1. $\varepsilon_n \longrightarrow 0$, and
2. $f_{|T_n}$ is an $(\varepsilon_n, \varepsilon_n)$ – equilibrium of $G_n = ((T_n, \nu_n), U_{|T_n}, A)$ for all $n \in \mathbb{N}$.

Our first characterization result establishes the equivalence between Nash equilibria and approximation in $(\varepsilon, \varepsilon)$ – equilibrium.

Theorem 1 *Let $G = (([0,1], \lambda), U, A)$ be a game with a continuum of players and f be a strategy. Then, the following conditions are equivalent:*

1. *f is a Nash equilibrium of G.*
2. *f can be approximated in $(\varepsilon, \varepsilon)$ – equilibrium.*

Theorem 1 provides a natural interpretation of Nash equilibria of games with a continuum of players: if f is such a strategy, then we can find a finite game, similar to the original continuum one, in which f is close to being a Nash equilibrium. Conversely, if a strategy f can be made as close to being a Nash equilibrium as we want in some finite game similar to the original continuum one, then f will be a Nash equilibrium of the continuum game. Given this equivalence, it is quite natural that approximate equilibria of large finite games have approximately the same properties of Nash equilibria of games with a continuum of players, as has been shown by many authors.

This result also confirms Fudenberg and Levine (1986)'s conclusion on the appropriate definition of equilibria in games that are defined as limits. As in their paper, Theorem 1 shows that in order to describe all equilibria of a game with a continuum of players it is necessary to take limits, not only of equilibria of converging finite games, but of $(\varepsilon, \varepsilon)$ – equilibria with ε converging to zero. This suggests that the appropriate notion of approximate equilibria for games with a continuum of players is that of $(\varepsilon, \varepsilon)$ – equilibrium.

However, note that Fudenberg and Levine (1986) study games with a finite number of players. This implies that the family of players' payoff functions forms an equicontinuous family. As Theorem 2 below shows, for equicontinuous games we can indeed characterize all Nash equilibria by using ε – equilibria as they did. Therefore, Theorem 2 is the analog of their result in our framework. Interestingly, its conclusion also holds when A is finite, regardless of whether or not players' payoff functions form an equicontinuous family.

Theorem 2 *Let $G = (([0,1], \lambda), U, A)$ be a game with a continuum of players and f a strategy. If either $U([0,1])$ is equicontinuous or A is finite, then the following conditions are equivalent:*

1. *f is a Nash equilibrium of G.*
2. *f can be approximated in ε – equilibrium.*

An important implication of our two characterization results is that they allow us to determine whether a given strategy is a Nash equilibrium of a game with a continuum of agents without necessarily having to deal with the technical difficulties involved in such games. Consider, for instance, a game with a continuum of agents in which there is a finite number of actions and a finite number of possible payoff functions, a typical assumption in applications. In this case, all the tools we need to analyze such a game are standard: we need to determine what the minimal ε is that makes a given strategy an ε – equilibrium in a finite normal form game and we need to guarantee that $\nu_n \circ (U_{|T_n}, A)^{-1}(\{(u,a)\})$ converges to $\lambda \circ (U, A)^{-1}(\{(u,a)\})$ (in $\mathbb{R}$) for all pairs (u, a) in $U([0,1]) \times A$.

We illustrate the above comment with the following simple example. Let $G = (([0,1], \lambda), U, A)$ be described by: $A = \{a, b\}$ and $U(t) = u$ for all $t \in [0,1]$, where $u(a, \mu) = \mu(\{a\})$ and $u(b, \tau) = 1 - \mu(\{a\})$. It is clear that the only equilibrium distributions of this game are τ_1, τ_2 and τ_3 satisfying $\tau_1(\{(u,a)\}) = 1/2$, $\tau_2(\{(u,a)\}) = 1$ and $\tau_3(\{(u,a)\}) = 0$, and where $\tau_i(\{(u,b)\}) = 1 - \tau_i(\{(u,a)\})$ for $i = 1, 2, 3$. Thus, a strategy f defined by

$$f(t) = \begin{cases} a & \text{if } 0 \leq t \leq \frac{1}{2}, \\ b & \text{otherwise} \end{cases} \tag{2}$$

is a Nash equilibrium. This fact can be inferred by Theorem 1 as follows: for each $n \in \mathbb{N}$, let $t_n^1 = 1/2 - 1/(2n)$, $t_n^2 = 1/2 + 1/(2n)$, $T_n^1 = \{t_1^1, \ldots, t_n^1\}$, $T_n^2 = \{t_1^2, \ldots, t_n^2\}$ and $T_n = T_n^1 \cup T_n^2$. Letting $\tau_n = \nu_n \circ (U_{|T_n}, f_{|T_n})^{-1}$, we have that $\tau_n(\{(u,a)\}) = \tau_n(\{(u,b)\}) = 1/2$ and so, obviously, $\tau_n \Rightarrow \tau$. For $t \in T_n^1$, we have $u(f(t), \tau_{A,n}) = 1/2$ if player t plays $f(t)$; if she chooses b, then she changes the distribution of actions to $\tau_{A,n}^{u,a,b}(\{a\}) = 1/2 - 1/(2n)$, thus receiving $u(b, \tau_{A,n}^{u,a,b}) = 1/2 + 1/(2n)$. Defining $\varepsilon_n = 1/2n$, we conclude that player t is ε_n-optimizing. Since a similar result holds for any $t \in T_n^2$, it follows that $f_{|T_n}$ is an ε_n – equilibrium of $G_n = ((T_n, \nu_n), U_{|T_n}, A)$ for all $n \in \mathbb{N}$. Finally, since $\varepsilon_n \to 0$, $\tau_n \Rightarrow \tau$ and $|T_n| \to \infty$, then f is an equilibrium of G.

We showed in Theorem 1 that for any Nash equilibrium f we can find a sequence of finite games such that f is an approximate equilibrium in those games. The following question arises naturally: if we are given a Nash equilibrium f and arbitrary approximating sequence of G at f, when is it the case that f is a ε_n – equilibrium of the approximating games with $\varepsilon_n \to 0$? There are essentially two difficulties with this question: first, players' characteristics may be too diverse; second, some players that are not optimizing in the limit game by playing according to f, may be players in all finite games, making it impossible to be an ε – equilibrium. We can solve the first problem by adding an equicontinuity assumption; we solve the second by replacing f by an equivalent strategy.

Let K be a subset of $\mathcal{U}$. Then K is equicontinuous if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|u(a,\tau) - u(b,\mu)| < \varepsilon$$

whenever $\max\{d_A(a,b), \rho(\tau,\mu)\} < \delta$, $a, b \in A$, $\tau, \mu \in \mathcal{M}$ and $u \in K$ (see Rudin (1976, p. 156)). In our framework, equicontinuity can be interpreted as placing "a bound on the diversity of payoffs," as pointed out by Khan, Rath, and Sun (1997).

For any strategy f and $t \in [0,1]$ in a game $G = (([0,1], \lambda), U, A)$, let $\beta(t,f) = \{a \in A : U(t)(a, \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})\}$ be the set of player t's best replies to f. Define a strategy f^* by letting $f^*(t) = f(t)$ if $f(t) \in \beta(t,f)$ and by letting $f^*(t) \in \beta(t,f)$ otherwise. That is, f^* is defined by changing f only for those players that are not optimizing. Let F_f^* be the set of all functions f^* defined in this way. Note that if $f^* \in F_f^*$ and if f is a Nash equilibrium of G, it follows that $f = f^*$ almost everywhere and so f^* is a

Nash equilibrium of G as well. Also, note that if $f^* \in F_f^*$ then $F_{f^*}^* = \{f^*\}$; furthermore, if an equilibrium strategy exists (see Khan, Rath, and Sun (1997)), then there exists a Nash equilibrium f such that $F_f^* = \{f\}$.

Let $G = (([0,1],\lambda),U,A)$ be a game with a continuum of players and f a strategy. We say that f *can be strongly approximated in* ε *– equilibrium* if for all approximating sequences $\{\nu_n\}_{n=1}^{\infty}$ of G at f there exists a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers such that

1. $\varepsilon_n \to 0$ and
2. $f_{|T_n}$ is an ε_n – equilibrium of $G_n = ((T_n,\nu_n),U_{|T_n},A)$ for all $n \in \mathbb{N}$.

Similarly, f *can be strongly approximated in* $(\varepsilon,\varepsilon)$ *– equilibrium* if for all approximating sequences $\{\nu_n\}_{n=1}^{\infty}$ of G at f there exists a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive real numbers such that

1. $\varepsilon_n \to 0$ and
2. $f_{|T_n}$ is an $(\varepsilon_n,\varepsilon_n)$ – equilibrium of $G_n = ((T_n,\nu_n),U_{|T_n},A)$ for all $n \in \mathbb{N}$.

Theorem 3 gives a second characterization of Nash equilibria in games with equicontinuous payoff functions.

Theorem 3 *Let* $G = (([0,1],\lambda),U,A)$ *be a game with a continuum of players such that* $U([0,1])$ *is equicontinuous and* f *a strategy. Then, the following conditions are equivalent:*

1. f *is a Nash equilibrium of* G.
2. f^* *can be strongly approximated in* ε *– equilibrium, for all* $f^* \in F_f^*$.
3. f *can be strongly approximated in* $(\varepsilon,\varepsilon)$ *– equilibrium.*

Theorem 3 strengthens the idea that games with a continuum of players can be useful in order to infer properties about large finite games. In fact, for equicontinuous games, any property that a Nash equilibrium has, will hold in approximate equilibrium in all close finite games. Carmona (2004b) explored this idea by showing that all sufficiently large finite games, with payoff functions belonging to an equicontinuous family, have approximate equilibria in which a large fraction of players play a pure strategy.

We illustrate how Theorem 3 can be used to show that a strategy is not a Nash equilibrium. In the example following Theorem 1, let strategy g be defined by

$$g(t) = \begin{cases} a & \text{if } 0 \leq t \leq \frac{1}{4}, \\ b & \text{otherwise} \end{cases} \tag{3}$$

One easily sees that g is a not a Nash equilibrium. This fact can also be inferred via Theorem 3 as follows: for each $n \in \mathbb{N}$, let $t_n^1 = 1/4 - 1/(4n)$, $t_n^2 = 1/4 + 1/(4n)$, $T_n^1 = \{t_1^1, \ldots, t_n^1\}$, $T_n^2 = \{t_1^2, \ldots, t_{3n}^2\}$ and $T_n = T_n^1 \cup T_n^2$. Letting $\mu_n = \nu_n \circ (U_{|T_n}, f_{|T_n})^{-1}$, we have that $\mu_n(\{(u,a)\}) = 1/4$ and $\mu_n(\{(u,b)\}) = 3/4$; obviously, $\mu_n \Rightarrow \mu$, where $\mu = \lambda \circ (U,g)^{-1}$. For $t \in T_n^1$, we have $u(f(t), \mu_{A,n}) = 1/4$ if player t plays $f(t)$; if he chooses b, then he changes the distribution of actions to $\mu_{A,n}^{u,a,b}(\{a\}) = 1/4 - 1/(4n)$, thus receiving $u(b, \mu_{A,n}^{u,a,b}) = 3/4 + 1/(4n)$. Since $\nu_n(\{t \in T_n : u(f(t), \mu_{A,n}) < u(\bar{a}, \mu_{A,n}^{a,\bar{a}}) - 1/2 \text{ for } \bar{a} \neq a\} \geq 1/4$ it follows that $g_{|T_n}$ is not a $(1/8, 1/8)$ – equilibrium of G_n for all $n \in \mathbb{N}$. Hence, g is not a Nash equilibrium of G.

Note that in contrast to what happens with ε – equilibrium, we do not need to replace f by an equivalent strategy in Theorem 3 when we consider $(\varepsilon, \varepsilon)$ – equilibrium. Therefore, if we do replace f by an equivalent strategy, we can expect to characterize Nash equilibria of games with a continuum of players with strong approximation in $(\varepsilon, \varepsilon)$ – equilibrium under weaker assumptions than those of Theorem 3. Theorem 4 below shows that this is the case.

In order to state Theorem 4 we need the following definitions: A subset X of $\mathcal{U}$ is *locally equicontinuous* if every point $x \in X$ has an equicontinuous neighborhood in its relative topology. It is *relatively locally equicontinuous* if $\overline{X}$ (i.e., its closure in $\mathcal{U}$) is locally equicontinuous.

Theorem 4 *Let $G = (([0,1], \lambda), U, A)$ be a game with a continuum of players such that $U([0,1])$ is relatively locally equicontinuous and f a strategy. Then, the following conditions are equivalent:*

1. *f is a Nash equilibrium of G.*

2. *f^* can be strongly approximated in $(\varepsilon, \varepsilon)$ – equilibrium, for all $f^* \in F_f^*$.*

Note that if $U([0,1])$ is equicontinuous, then $\overline{U([0,1])}$ is also equicontinuous, and locally so. Therefore, the assumption of Theorem 4 is weaker than that of Theorem 3.

6 Concluding Remarks

The main objective of this paper is to relate equilibria of games with a continuum of players with equilibria of games with a finite number of players. This is done by characterizing Nash equilibria in terms of approximate equilibria of games with a finite number of players.

Our characterization results show that approximate equilibria of finite games provides an alternative way for obtaining strategic insignificance of players, which is the main motivation of games with a continuum of players. In this way, they render as natural all the results that show that approximate equilibria of finite games have the same, or approximately the same, properties as equilibria of continuum games.

A Appendix

A.1 Lemmata

In this section, we present some technical results that we use and for which we were unable to find a reference. Lemma 5 below extends the well-known fact that the set of measures with finite support is dense in the set of all Borel measures in a separable metric space (see Parthasarathy (1967, Theorem II.6.3)).

Lemma 5 *Let X be a separable metric space, $\mu \in \mathcal{M}(X)$ and $K \subseteq \mathrm{supp}(\mu)$ be compact. If $\mu = \lambda \circ h^{-1}$, where $h : [0,1] \to X$ is measurable, and λ is the Lebesgue measure on $[0,1]$, then there exists a sequence $\{\mu_n\}$ in $\mathcal{M}(X)$ such that*

1. $\mathrm{supp}(\mu_n) \subseteq \mathrm{supp}(\mu)$ *for all* $n \in \mathbb{N}$,
2. $\mu_n \Rightarrow \mu$,
3. $\lim_n \mu_n(K) = \mu(K)$
4. *for all* $n \in \mathbb{N}$, $\mu_n = \nu_n \circ h^{-1}_{|T_n}$ *where* T_n *is a finite subset of* $[0,1]$ *and* ν_n *is the uniform measure on* T_n *and*
5. $|T_n| \to \infty$.

Proof. Let $n \in \mathbb{N}$. Since K is compact, then we can write $K = K \cap \left(\cup_{j=1}^{J_n} B_{1/2n}(y_{n,j})\right)$ for some $y_{n,j} \in K$, $j = 1, \ldots, J_n$. Hence, we can write $K = \cup_{j=1}^{J_n} B_{n,j}$, where $B_{n,1} = K \cap B_{1/2n}(y_{n,1})$, and

$$B_{n,j} = K \cap \left(B_{1/2n}(y_{n,j}) \setminus \cup_{i=1}^{j-1} B_{1/2n}(y_{n,i})\right).$$

Thus, $\{B_{n,j}\}_j$ is a disjoint collection, each of its members being a Borel set with a diameter no greater than $1/n$.

Since X is separable, we can write $\mathrm{supp}(\mu) \setminus K = \cup_{i=1}^{\infty} A_{n,i}$ where $\{A_{n,i}\}$ is a disjoint collection, each of its members being a Borel set with a diameter no greater than $1/n$.

Let $I_n \in \mathbb{N}$ be such that $\sum_{i=I_n}^{\infty} \mu(A_{n,i}) < 1/n$. Let $\{\tilde{q}_{n,j}\}_{j=1}^{J_n} \subset \mathbb{Q}_+$ and $\{\tilde{p}_{n,i}\}_{i=1}^{I_n} \subset \mathbb{Q}_+$ be such that

$$\begin{aligned}
&|\tilde{q}_{n,j} - \mu(B_{n,j})| < 1/(nJ_n), j = 1, \ldots, J_n, \\
&|\tilde{p}_{n,i} - \mu(A_{n,i})| < 1/(nI_n), i = 1, \ldots, I_n - 1, \text{ and} \\
&\sum_{j=1}^{J_n} \tilde{q}_{n,j} + \sum_{i=1}^{I_n} \tilde{p}_{n,i} = 1.
\end{aligned} \tag{4}$$

Also, if $\mu(B_{n,j}) = 0, j = 1, \ldots, J_n$, let $\tilde{q}_{n,j} = 0$; if $\mu(A_{n,i}) = 0, i = 1, \ldots, I_n - 1$, let $\tilde{p}_{n,i} = 0$, and if $\sum_{i=I_n}^{\infty} \mu(A_{n,i}) = 0$ let $\tilde{p}_{n,I_n} = 0$. We remark that the above construction can always be done: if there is just one such set with positive measure this is clear, as its measure will be 1, a rational number. If there are $k > 1$ such sets with positive measure, then approximate the measure of $k - 1$ of those sets by rational points in a way that the rational number is the smallest of the two and their difference is smaller than $\zeta > 0$, where $\zeta(I_n + J_n) < \min\{1/(nI_n), 1/(nJ_n)\}$, and set the rational approximation for the remaining set using the formula $\sum_{j=1}^{J_n} \tilde{q}_{n,j} + \sum_{i=1}^{I_n} \tilde{p}_{n,i} = 1$.

Since $\sum_{i=I_n}^{\infty} \mu(A_{n,i}) = 1 - \sum_{j=1}^{J_n} \mu(B_{n,j}) - \sum_{i=1}^{I_n - 1} \mu(A_{n,i})$, it follows that $|\tilde{p}_{n,I_n} - \sum_{i=I_n}^{\infty} \mu(A_{n,i})| < 2/n$ and that $\sum_{j=1}^{J_n} \tilde{q}_{n,j} + \sum_{i=1}^{I_n} \tilde{p}_{n,i} = 1$. Furthermore, there exists $N_n \in \mathbb{N}$, $\{q_{n,j}\}_{j=1}^{J_n} \subset \mathbb{N}$ and $\{p_{n,i}\}_{i=1}^{I_n} \subset \mathbb{N}$ such that $\tilde{q}_{n,j} = q_{n,j}/N_n$, $j = 1, \ldots, J_n$ and $\tilde{p}_{n,i} = p_{n,i}/N_n$, $i = 1, \ldots, I_n$. We may assume that $N_n \geq n$.

Let $i \in \{1, \ldots, I_n - 1\}$. Since $\mu(A_{n,i}) = \lambda(\{t \in [0,1] : h(t) \in A_{n,i}\})$, select $p_{n,i}$ points from $\{t \in [0,1] : h(t) \in A_{n,i}\}$. By the above convention, $p_{n,i} > 0$ implies $\mu(A_{n,i}) > 0$, and so we can indeed select such points from $\{t \in [0,1] : h(t) \in A_{n,i}\}$. Similarly, select p_{n,I_n} points from $\{t \in [0,1] : h(t) \in$

$\cup_{i=I_n}^{\infty} A_{n,i}\}$ and $q_{n,j}$ points from $\{t \in [0,1] : h(t) \in B_{n,j}\}$, $j = 1, \ldots, J_n$. Let $T_n = \{t_l^n\}_{l=1}^{N_n}$ denote this collection of points from $[0,1]$ and let $\mu_n = \nu_n \circ h_{|T_n}^{-1}$. By construction, it follows that $\operatorname{supp}(\mu_n) \subseteq \operatorname{supp}(\mu)$.

Since $|T_n| = N_n \geq n$, it follows that $|T_n| \to \infty$. We claim that $\lim_n \mu_n(K) = \mu(K)$. We have that

$$\begin{aligned} \mu_n(K) &= \nu_n \circ h_{|T_n}^{-1}(K) \\ &= \nu_n(\{t \in T_n : h(t) \in K\}) \\ &= \sum_{j=1}^{J_n} \nu_n(\{t \in T_n : h(t) \in B_{n,j}\}) \\ &= \sum_{j=1}^{J_n} \frac{q_{n,j}}{N_n} = \sum_{j=1}^{J_n} \tilde{q}_{n,j}. \end{aligned} \tag{5}$$

Hence, $|\mu_n(K) - \mu(K)| \leq \sum_{j=1}^{J_n} |\tilde{q}_{n,j} - \mu(B_{n,j})| < 1/n \to 0$ as $n \to \infty$.

Finally, we show that $\mu_n \Rightarrow \mu$. Let g be a bounded, uniformly continuous real-valued function on X, and let g be bounded by M. Since

$$\begin{aligned} \left| \int g \mathrm{d}\mu_n - \int g \mathrm{d}\mu \right| \leq & \left| \sum_{j=1}^{J_n} \left(\int_{B_{j,n}} g \mathrm{d}\mu_n - \int_{B_{j,n}} g \mathrm{d}\mu \right) \right| \\ & + \left| \sum_{i=1}^{I_n - 1} \left(\int_{A_{i,n}} g \mathrm{d}\mu_n - \int_{A_{i,n}} g \mathrm{d}\mu \right) \right| \\ & + \left| \sum_{i=I_n}^{\infty} \left(\int_{A_{i,n}} g \mathrm{d}\mu_n - \int_{A_{i,n}} g \mathrm{d}\mu \right) \right|, \end{aligned} \tag{6}$$

it is enough to show that each of the three terms on the right side of the above inequality converges to zero as n converges to infinity.

We have that $\left| \sum_{i=I_n}^{\infty} \int_{A_{i,n}} g \mathrm{d}\mu_n \right| \leq M \sum_{i=I_n}^{\infty} \mu_n(A_{i,n}) = M \mu_n(\cup_{i=I_n}^{\infty} A_{i,n}) = M \tilde{p}_{n,i} < M \left(\sum_{i=I_n}^{\infty} \mu(A_{i,n}) + 2/n \right) < 3M/n \to 0$ as $n \to \infty$. Similarly, $\left| \sum_{i=I_n}^{\infty} \int_{A_{i,n}} g \mathrm{d}\mu \right| \leq M \sum_{i=I_n}^{\infty} \mu(A_{i,n}) < M/n \to 0$ as $n \to \infty$. Hence

$$\left| \sum_{i=I_n}^{\infty} \left(\int_{A_{i,n}} g \mathrm{d}\mu_n - \int_{A_{i,n}} g \mathrm{d}\mu \right) \right| \to 0$$

as $n \to \infty$.

Let $\alpha_{n,j} = \inf_{x \in B_{n,j}} g(x)$ and $\beta_{n,j} = \sup_{x \in B_{n,j}} g(x)$, $j = 1, \ldots, J_n$. Also, let $\alpha_{n,i} = \inf_{x \in A_{n,i}} g(x)$ and $\beta_{n,i} = \sup_{x \in A_{n,i}} g(x)$, $i = 1, \ldots, I_n - 1$. Since g is uniformly continuous, and the diameters of $A_{n,i}$, and $B_{n,j}$ converge to zero as n converges to infinity uniformly on i and j respectively, it follows that $\sup_i(\beta_{n,i} - \alpha_{n,i})$ and $\sup_j(\beta_{n,j} - \alpha_{n,j})$ converge to zero as n converges to infinity.

Let $j \in \{1, \ldots, J_n\}$, and let $\{x_{n,j}^m\}_{m=1}^{q_{n,j}} = h(T_n) \cap B_{n,j}$. Then

$$\int_{B_{n,j}} g \mathrm{d}\mu_n = \sum_{m=1}^{q_{n,j}} \frac{g(x_{n,j}^m)}{N_n}. \tag{7}$$

We have that

$$\begin{aligned} \int_{B_{n,j}} g \mathrm{d}\mu - \sum_{m=1}^{q_{n,j}} \frac{g(x_{n,j}^m)}{N_n} &\leq \beta_{n,j}\mu(B_{n,j}) - \alpha_{n,j}\frac{q_{n,j}}{N_n} \\ &= \mu(B_{n,j})(\beta_{n,j} - \alpha_{n,j}) + \alpha_{n,j}(\mu(B_{n,j}) - \tilde{q}_{n,j}) \\ &\leq \mu(B_{n,j}) \sup_{j'}(\beta_{n,j'} - \alpha_{n,j'}) + M \left|\mu(B_{n,j}) - \tilde{q}_{n,j}\right|. \end{aligned} \tag{8}$$

Similarly,

$$\sum_{m=1}^{q_{n,j}} \frac{g(x_{n,j}^m)}{N_n} - \int_{B_{n,j}} g \mathrm{d}\mu \leq \mu(B_{n,j}) \sup_{j'}(\beta_{n,j'} - \alpha_{n,j'}) + M \left|\mu(B_{n,j}) - \tilde{q}_{n,j}\right|.$$

Thus,

$$\left|\int_{B_{j,n}} g \mathrm{d}\mu_n - \int_{B_{j,n}} g \mathrm{d}\mu\right| < \mu(B_{n,j}) \sup_{j'}(\beta_{n,j'} - \alpha_{n,j'}) + \frac{M}{nJ_n},$$

and so

$$\left|\sum_{j=1}^{J_n} \left(\int_{B_{j,n}} g \mathrm{d}\mu_n - \int_{B_{j,n}} g \mathrm{d}\mu\right)\right| < \sup_j(\beta_{n,j} - \alpha_{n,j}) + \frac{M}{n} \to 0 \tag{9}$$

as $n \to \infty$.

Similarly, we can show that

$$\left|\sum_{i=1}^{I_n - 1} \left(\int_{A_{i,n}} g \mathrm{d}\mu_n - \int_{A_{i,n}} g \mathrm{d}\mu\right)\right| < \sup_i(\beta_{n,i} - \alpha_{n,i}) + \frac{M}{n} \to 0$$

as $n \to \infty$. Hence, we conclude that $\left|\int g \mathrm{d}\mu_n - \int g \mathrm{d}\mu\right| \to 0$ as $n \to \infty$ and, since g is arbitrary, that $\mu_n \Rightarrow \mu$. ■

The following lemma shows that the convergence of a sequence of measures can be studied with respect to any closed set containing both their support and the support of the limit measure.

Lemma 6 *Let X be a metric space and $\mu, \mu_n \in \mathcal{M}(X)$ for all $n \in \mathbb{N}$. Let C be a closed subset of X satisfying* $\mathrm{supp}(\mu_n), \mathrm{supp}(\mu) \subseteq C$. *Then, $\mu_n \Rightarrow \mu$ in C if and only if $\mu_n \Rightarrow \mu$ in X.*

Proof. (Necessity) Let $h : X \to \mathbb{R}$ be bounded and continuous. Then $h_{|C}$ is also bounded and continuous, and so

$$\int_C h \mathrm{d}\mu_n \to \int_C h \mathrm{d}\mu.$$

Thus,

$$\int_X h \mathrm{d}\mu_n = \int_C h \mathrm{d}\mu_n \to \int_C h \mathrm{d}\mu = \int_X h \mathrm{d}\mu.$$

It follows that $\mu_n \Rightarrow \mu$ in X.

(Sufficiency) Let $h : C \to \mathbb{R}$ be bounded and continuous. Then by the Tietze-Urysohn extension theorem, there exists a bounded, continuous function $h^* : X \to \mathbb{R}$ such that $h^*(x) = h(x)$ for all $x \in C$. Therefore,

$$\int_X h^* \mathrm{d}\mu_n \to \int_X h^* \mathrm{d}\mu,$$

and

$$\int_X h^* \mathrm{d}\mu = \int_C h^* \mathrm{d}\mu = \int_C h \mathrm{d}\mu$$

(and similarly, $\int_X h^* \mathrm{d}\mu_n = \int_C h \mathrm{d}\mu_n$ for all $n \in \mathbb{N}$). Therefore, $\int_C h \mathrm{d}\mu_n \to \int_C h \mathrm{d}\mu$. It follows that $\mu_n \Rightarrow \mu$ in C. ■

Lemma 7 shows that, for subsets of $\mathcal{U}$, local compactness is equivalent to local equicontinuity.

Lemma 7 *Let X be a subset of $\mathcal{U}$. Then, X is locally compact if and only if X is locally equicontinuous.*

Proof. (Necessity) Let $x \in X$ and let K be a compact neighborhood of x. Then K is equicontinuous, and so X is locally equicontinuous.

(Sufficiency) Let $x \in X$ and let K be an equicontinuous neighborhood of x. Since K is a neighborhood of x in X, there exists an open set V such that $x \in V \subseteq K$. Then, $V = X \cap O$, with O open in $\mathcal{U}$, and so there exists $\delta > 0$ such that $X \cap B_\delta(x) \subseteq K$. This implies that the closure of $X \cap B_\delta(x)$ in X is contained in the closure of K in X and so is equicontinuous. Furthermore, it equals $X \cap \overline{B_\delta(x)}$ (see Kelley (1955, Theorem 16 (c))), and it is therefore also bounded. Since it is obviously closed in X, the closure of $X \cap B_\delta(x)$ in X is compact. Thus, X is locally compact. ■

A.2 Proofs

In this section we present the proofs of all the results stated in the main text.

Proof of Lemma 1. Let $\{r_k\}_{k=1}^{\infty} \subseteq A$ be dense in A. Then, the continuity of $U(t)$ for all $t \in [0,1]$, implies that

$$E(f, \varepsilon, \lambda) = \bigcap_k \{t \in [0,1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\}. \quad (10)$$

Hence, it is enough to show that $\{t \in [0,1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\}$ is measurable for all k.

Let $k \in \mathbb{N}$, and $\eta > 0$. Let $\hat{u} : [0,1] \times A \to \mathbb{R}$ be defined by $(t,a) \mapsto U(t)(a, \lambda \circ f^{-1})$ and, for each $\tau \in \mathcal{M}$, let $\pi_\tau : \mathcal{U} \times A \to \mathbb{R}$ be defined by $\pi_\tau(u,a) = u(a,\tau)$. Since π_τ is continuous for all τ, and $\hat{u} = \pi_{\lambda \circ f^{-1}} \circ (U, i)$, where $i : A \to A$ denotes the identity function on A, it follows that $\hat{u}$ is measurable. By changing it in a set of measure zero, we may assume that it is Borel measurable; similarly, assume that f is Borel measurable. Then, the functions $t \mapsto U(t)(f(t), \lambda \circ f^{-1})$ and $t \mapsto U(t)(r_k, \lambda \circ f^{-1})$ are Borel measurable in F since they equal $\hat{u} \circ h$, and $\hat{u} \circ g$ respectively, where $h(t) = (t, f(t))$, and $g(t) = (t, r_k)$ are Borel measurable. Thus, by Lusin's Theorem, let $C \subseteq [0,1]$ be a compact set, $\lambda([0,1] \setminus C) < \eta$, be such that $t \mapsto U(t)(f(t), \lambda \circ f^{-1})$ and $t \mapsto U(t)(r_k, \lambda \circ f^{-1})$ are continuous in C. Since

$$\begin{aligned} &\{t \in [0,1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\} \setminus \\ &\{t \in C : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\} \\ &\subseteq [0,1] \setminus C, \end{aligned} \quad (11)$$

the outer measure of $\{t \in [0,1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\} \setminus \{t \in C : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\}$ is smaller than η. Hence, it is enough to show that $\{t \in C : u(t, f(t), \lambda \circ f^{-1}) \geq u(t, r_k, \lambda \circ f^{-1}) - \varepsilon\}$ is closed (see Wheeden and Zygmund (1977, Lemma 3.22)). This follows easily from the fact that both $t \mapsto U(t)(f(t), \lambda \circ f^{-1})$ and $t \mapsto U(t)(r_k, \lambda \circ f^{-1})$ are continuous in C. ■

Proof of Lemma 2. For notational convenience let $h = (U, f)$. We have

$$\begin{aligned} & h^{-1}(B_\tau) = \\ & \{t \in [0,1] : (U(t), f(t)) \in B_\tau\} = \\ & \{t \in [0,1] : U(t)(f(t), \tau_A) \geq U(t)(A, \tau_A)\} = \\ & \{t \in [0,1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})\}. \end{aligned} \tag{12}$$

Hence, τ is an equilibrium distribution if and only if $\tau(\{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(A, \tau_A)\}) = 1$ if and only if $\lambda(h^{-1}(B_\tau)) = 1$ if and only if $\lambda(\{t \in [0,1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})\}) = 1$ if and only if f is a Nash equilibrium. ■

Proof of Lemma 3. Let $\tau = \nu \circ f^{-1}$ and $\tau' = \nu \circ g^{-1}$. We have that

$$\tau(\{a\}) = \frac{\{t \in T : f(t) = a\}}{|T|}$$

for all $a \in A$, and similarly for τ'. Since there is just one $t \in T$ such that g and f differ, there are only two points in A such that $\tau(\{a\})$ differs from $\tau'(\{a\})$. Note that in that case $|\tau(\{a\}) - \tau'(\{a\})| = 1/|T|$. Denoting these points a' and a'' we see that for any Borel subset E of A $|\tau(E) - \tau'(E)| = 0$ if either a' and a'' belong to E or a' and a'' belong to E^c, while $|\tau(E) - \tau'(E)| = 1/|T|$ otherwise. This implies $\rho(\tau, \tau') \leq 1/|T|$. ■

Proof of Lemma 4. For notational convenience let $h = (U, f)$. We have

$$\begin{aligned} & h^{-1}(B^\varepsilon_\tau) = \\ & \{t \in T : (U(t), f(t)) \in B^\varepsilon_\tau\} = \\ & \{t \in T : U(t)(f(t), \tau_A) \geq U(t)(\bar{a}, \tau_A^{u,a,\bar{a}}) - \varepsilon \text{ for all } \bar{a}\} = \\ & \{t \in T : U(t)(f(t), \nu \circ f^{-1}) \geq U(t)(\bar{a}, \nu \circ (f \setminus_t \bar{a})^{-1}) - \varepsilon \text{ for all } \bar{a}\}. \end{aligned} \tag{13}$$

Hence, τ is an (ε, δ) – equilibrium distribution if and only if $\tau(B^\varepsilon_\tau) \geq 1 - \delta$ if and only if $\nu(h^{-1}(B^\varepsilon_\tau)) \geq 1 - \delta$ if and only if $\nu(\{t \in T : U(t)(f(t), \nu \circ f^{-1}) \geq$

$U(t)(\bar{a}, \nu \circ (f \setminus_t \bar{a})^{-1}) - \varepsilon$ for all $\bar{a}\}) \geq 1 - \delta$ if and only if f is an (ε, δ) – equilibrium. ■

Proof of Theorem 1. (Necessity) Let $\tau = \lambda \circ (U, f)^{-1}$; by Lemma 2, τ is an equilibrium distribution on $\mathcal{U} \times A$. For notational convenience, let $h = (U, f)$. Let $n \in \mathbb{N}$ and define $\varepsilon_n = 1/n$. Since $\mathcal{U}$ is a complete separable metric space, and A is compact, it follows that τ is tight by Parthasarathy (**?**, Theorem II.3.2), as $\mathcal{U} \times A$ is also a complete separable metric space.

Since B_τ is closed, and so a Borel set, let $K_n \subseteq B_\tau$ be compact, and satisfy $\tau(B_\tau \setminus K_n) < 1/2n$. Since τ is an equilibrium distribution, it follows that $\tau(B_\tau) = 1$, and so $\tau(K_n) > 1 - 1/2n$. If π denotes the projection of $\mathcal{U} \times A$ into $\mathcal{U}$, then $\pi(K_n)$ is compact, and $K_n \subseteq \pi(K_n) \times A$. In particular, $\pi(K_n)$ is equicontinuous by the Ascoli-Arzela Theorem since A and $\mathcal{M}$ are both compact metric spaces. Furthermore, denoting $C_n = \pi(K_n) \times A$, it follows that $C_n \cap B_\tau$ is compact and $\tau(C_n \cap B_\tau) \geq \tau(K_n \cap B_\tau) = \tau(K_n) > 1 - 1/2n$.

Let $\delta_n > 0$ be such that $d((a, \mu), (b, \nu)) := \max\{d_A(a, b), \rho(\mu, \nu)\} < \delta_n$ implies that $|u(a, \mu) - u(b, \nu)| < 1/4n$ for all $u \in \pi(K_n)$. By Lemma 5, there exists a sequence $\{\mu_j\}$ such that $\mu_j \Rightarrow \tau$, $\lim_j \mu_j(C_n \cap B_\tau) = \tau(C_n \cap B_\tau)$, $\mu_j = \nu_j \circ h_{T_j}^{-1}$ where ν_j is the uniform measure on some finite set $T_j \subset [0, 1]$, and $|T_j| \to \infty$. Hence, $\mu_{A,j} \Rightarrow \tau_A$, and let $J_n \in \mathbb{N}$ be such that $\rho(\mu_{A,J_n}, \tau_A) < \delta_n$, $|\mu_{J_n}(C_n \cap B_\tau) - \tau(C_n \cap B_\tau)| < 1/2n$ $\rho(\mu_{J_n}, \tau) < 1/n$ and $1/|T_{J_n}| < \delta_n$. Define $\tau_n = \mu_{J_n}$, $T_n = T_{J_n}$ and $\nu_n = \nu_{J_n}$.

By construction of $\{\tau_n\}_n$ we have $\tau_n \Rightarrow \tau$, and that, for every $n \in \mathbb{N}$, $\rho(\tau_n, \tau) < 1/n$, $\rho(\tau_{A,n}, \tau_A) < \delta_n$, $|\tau_n(C_n \cap B_\tau) - \tau(C_n \cap B_\tau)| < 1/2n$, $1/|T_n| < \delta_n$ and $\tau_n = \nu_n \circ h_{|T_n}^{-1}$ where T_n is a finite subset of $[0, 1]$ and ν_n is the uniform measure on T_n.

We have that $C_n \cap B_\tau \subseteq C_n \cap B_{\tau_n}^{1/n}$, since if $(u, a) \in C_n \cap B_\tau$ and $\bar{a} \in A$ then $u(a, \tau_{A,n}) > u(a, \tau_A) - 1/4n \geq u(\bar{a}, \tau_A) - 1/4n > u(\bar{a}, \tau_{A,n}) - 1/2n > u(\bar{a}, \tau_{A,n}^{u,a,\bar{a}}) - 1/n$ since $\rho(\tau_{A,n}, \tau_A) < \delta_n$ and $\rho(\tau_{A,n}, \tau_{A,n}^{u,a,\bar{a}}) \leq 1/|T_n| < \delta_n$ by Lemma 3. So

$$\tau_n(B_{\tau_n}^{1/n}) \geq \tau_n(C_n \cap B_{\tau_n}^{1/n}) \geq \tau_n(C_n \cap B_\tau) > \tau(C_n \cap B_\tau) - 1/2n > 1 - 1/n. \quad (14)$$

Hence, $\tau_n = \nu_n \circ h_{|T_n}^{-1} = \nu_n \circ (U_{|T_n}, f_{|T_n})^{-1}$ is an ε_n-equilibrium distribution of the game $\tau_{\mathcal{U},n} = \nu_n \circ U_{|T_n}^{-1}$. By Lemma 4, then $f_{|T_n}$ is an ε_n–equilibrium of $G_n = ((T_n, \nu_n), U_{|T_n})$.

(Sufficiency) Let $\tau = \lambda \circ (U, f)^{-1}$ and let $\tau_n \Rightarrow \tau$, where $\tau_n = \nu_n \circ (U, f)_{|T_n}^{-1}$ is an $(\varepsilon_n, \varepsilon_n)$ – equilibrium distribution, $\varepsilon_n \to 0$ and $|T_n| \to \infty$. Then

$\tau_{A,n} \Rightarrow \tau_A$; so, taking a subsequence if necessary, we may assume that $\varepsilon_n \searrow 0$, $\rho(\tau_A, \tau_{A,n}) < 1/2n$ and $\rho(\tau_{A,n}, \tau_{A,n}^{u,a,\bar{a}}) < 1/2n$ for every $u \in \mathcal{U}$ and $a, \bar{a} \in A$; the second inequality is obtained via Lemma 3 by taking $1/|T_n| < 1/2n$. Clearly, we have $\rho(\tau_A, \tau_{A,n}^{u,a,\bar{a}}) < 1/n$ for every $u \in \mathcal{U}$ and $a, \bar{a} \in A$.

Define, for each $u \in \mathcal{U}$,

$$\beta_n(u) = \sup_{a \in A, \nu \in \mathcal{M}} \{|u(a,\nu) - u(a,\tau_A)| : \rho(\nu, \tau_A) < 1/n\}.$$

Since u is continuous on $A \times \mathcal{M}$, which is compact, it follows that u is uniformly continuous. Thus, $\beta_n(u) \searrow 0$ as $n \to \infty$. We claim that β_n is continuous in $\mathcal{U}$.

Let $\eta > 0$. Define $\delta < \eta/2$. Then if $||u - v|| < \delta$, we have for any $a \in A$, and $\nu \in \mathcal{M}$ such that $\rho(\nu, \tau_A) < 1/n$

$$\begin{aligned} |v(a,\nu) - v(a,\tau_A)| \le |v(a,\nu) - u(a,\nu)| + |u(a,\nu) + u(a,\tau_A)| + \\ + |v(a,\tau_A) - u(a,\tau_A)| < \delta + \beta_n(u) + \delta, \end{aligned} \tag{15}$$

and so $\beta_n(v) \le 2\delta + \beta_n(u) < \eta + \beta_n(u)$. By symmetry, $\beta_n(u) < \eta + \beta_n(v)$, and so $|\beta_n(u) - \beta_n(v)| < \eta$. Hence, β_n is continuous as claimed.

Given the definition of β_n, we have that $B_{\tau_n}^{\varepsilon_n} \subseteq D_n := \{(u,a) : u(a,\tau_A) \ge u(A,\tau_A) - \varepsilon_n - 2\beta_n(u)\}$. Since β_n is continuous, we see that D_n is closed, and so Borel measurable. Thus, $\tau_n(D_n) \ge 1 - \varepsilon_n$. Also, $D_n \searrow B_\tau$.

Let $n \in \mathbb{N}$ be given. Then, if $k \ge n$, it follows that $\tau_k(D_n) \ge \tau_k(D_k) \ge 1 - \varepsilon_k \ge 1 - \varepsilon_n$, and so $\tau(D_n) \ge \limsup_j \tau_j(D_n) \ge 1 - \varepsilon_n$. Hence, $\tau(B_\tau) = \lim_n \tau(D_n) = 1$. Therefore, $\tau = \lambda \circ (U,f)^{-1}$ is an equilibrium distribution of $\lambda \circ U^{-1}$ and so f is an equilibrium of G. ■

We will prove Theorem 3 before Theorem 2.

Proof of Theorem 3. It follows from Theorem 1 that 3 implies 1. Similarly, if condition 2 holds, it follows again from Theorem 1 that f^* is a Nash equilibrium of G. Since $f^* = f$ almost everywhere, then f is a Nash equilibrium of G. That is, 2 implies 1.

We turn to the proof that 1 implies 2. Let $\tau = \lambda \circ (U, f^*)^{-1}$ and let $\tau_n = \nu_n \circ (U_{|T_n}, f^*_{|T_n})^{-1}$. Define

$$\alpha_n(u) = \sup_{a \in A, \mu, \phi \in \mathcal{M}} \{|u(a,\mu) - u(a,\phi)| : \rho(\mu,\phi) \le \rho(\tau_{A,n}, \tau_A)\},$$

and $\alpha_n = \sup_{u \in U([0,1])} \alpha_n(u)$; similarly, define

$$\gamma_n(u) = \sup_{a \in A, \mu, \phi \in \mathcal{M}} \{|u(a,\mu) - u(a,\phi)| : \rho(\mu,\phi) \leq \rho(\tau_{A,n}, \tau_{A,n}^{u,a,\bar{a}})\},$$

and $\gamma_n = \sup_{u \in U([0,1])} \gamma_n(u)$.

As in the proof of Theorem 1 we can show that $B_\tau \subseteq B_{\tau_n}^{\varepsilon_n}$ if we define $\varepsilon_n = 2\alpha_n + \gamma_n$. Since $U([0,1])$ is equicontinuous and $\lim_{n\to\infty} \rho(\tau_{A,n}, \tau_A) = \rho(\tau_{A,n}, \tau_{A,n}^{u,a,\bar{a}}) = 0$, it follows that $\varepsilon_n \to 0$. Finally, note that $(U, f^*)([0,1]) \subseteq B_\tau$ since all players are optimizing by choosing according to f^*; hence, $\tau_n(B_{\tau_n}^{\varepsilon_n}) = 1$, and so $f^*_{|T_n}$ is an ε_n – equilibrium of G_n.

Finally, we show that 1 implies 3. For each $n \in \mathbb{N}$, we define $\gamma_n = \inf\{\varepsilon \geq 0 : f_{|T_n}$ is an $(\varepsilon, \varepsilon)$ – equilibrium of $G_n\}$. Note that the set $\{\varepsilon \geq 0 : f_{|T_n}$ is an $(\varepsilon, \varepsilon)$ – equilibrium of $G_n\}$ is nonempty since if $B > 0$ is such that u is bounded by B for all $u \in U(T_n)$, then $f_{|T_n}$ is a $2B$ – equilibrium of G_n and so a $(2B, 2B)$ – equilibrium of G_n. Define $\varepsilon_n = \gamma_n + 1/n$. Thus, it is enough to show that $\gamma_n \to 0$.

Let $\eta > 0$ be given. Denote $\tau_n = \nu_n \circ (U_{|T_n}, f_{T_n})^{-1}$ and $\tau = \lambda \circ (U, f)^{-1}$. Let $\delta > 0$ be such that $d((a,\mu),(b,\nu)) := \max\{d_A(a,b), \rho(\mu,\nu)\} < \delta$ implies that $|u(a,\mu) - u(b,\nu)| < \eta/5$ for all $u \in U([0,1])$. Since A is compact, there exists $\{a_i\}_{i=1}^I$ such that $A \subseteq \cup_{i=1}^I B_\delta(a_i)$.

Define V_i, $i = 1, \ldots, I$, by

$$V_i = \left\{(u,a) \in \mathcal{U} \times A : u(a,\tau_A) > u(a_i,\tau_A) - \frac{\eta}{5}\right\}.$$

One easily sees that V_i is open and $B_\tau \subseteq V_i$. Therefore, $\cap_{i=1}^I V_i$ is open and that $B_\tau \subseteq \cap_{i=1}^I V_i$. In particular, $\tau(\cap_{i=1}^I V_i) \leq \liminf_n \tau_n(\cap_{i=1}^I V_i)$.

Let $N \in \mathbb{N}$ be such that $n \geq N$ implies that $1/|T_n| < \delta$, $\rho(\tau_{A,n}, \tau_A) < \delta$ and $\tau_n(\cap_{i=1}^I V_i) \geq \tau(\cap_{i=1}^I V_i) - \eta$.

Hence, if $n \geq N$, it follows that $\cap_{i=1}^I V_i \subseteq B_{\tau_n}^\eta$: if $(u,a) \in \cap_{i=1}^I V_i$, let $\bar{a} \in A$ be arbitrary and let a_i be such that $\bar{a} \in B_\delta(a_i)$; then

$$\begin{aligned} u(a,\tau_{A,n}) &> u(a,\tau_A) - \frac{\eta}{5} \\ &> u(a_i,\tau_A) - \frac{2\eta}{5} \\ &> u(\bar{a},\tau_A) - \frac{3\eta}{5} \\ &> u(\bar{a},\tau_{A,n}) - \frac{4\eta}{5} \\ &> u(\bar{a},\tau_{A,n}^{u,a,\bar{a}}) - \eta. \end{aligned} \tag{16}$$

Therefore, $(u,a) \in B^{\eta}_{\tau_n}$. Hence,

$$\tau_n(B^{\eta}_{\tau_n}) \geq \tau_n(\cap_{i=1}^{I} V_i) \geq \tau(\cap_{i=1}^{I} V_i) - \eta \geq \tau(B_\tau) - \eta = 1 - \eta, \qquad (17)$$

and $f_{|T_n}$ is an (η, η) – equilibrium of G_n. This implies that $\gamma_n \leq \eta$ and, since η is arbitrary, that $\gamma_n \to 0$. ■

Proof of Theorem 2. It follows from Theorem 1 that 2 implies 1 in both cases.

We show next that 1 implies 2 when $U([0,1])$ is equicontinuous. Let $\tau = \lambda \circ (U, f)^{-1}$ and let $\{\tau_n\}_{n=1}^{\infty}$ be as in Lemma 5 (i.e., $\tau_n = \nu_n \circ (U_{|T_n}, f_{|T_n})^{-1}$). In particular, $\text{supp}(\tau_n) \subseteq \text{supp}(\tau) \subseteq B_\tau$ for all $n \in \mathbb{N}$. Hence, if $f^* \in F_f^*$, it follows that $f^*_{|T_n} = f_{|T_n}$, which implies that $\tau_n = \nu_n \circ (U_{|T_n}, f^*_{|T_n})^{-1}$. Also, we have that $\lambda \circ (U, f^*)^{-1} = \tau$. Thus, it follows by Theorem 3 that there exists $\{\varepsilon_n\}_{n=1}^{\infty}$, with $\varepsilon_n \to 0$, such that $f_{|T_n}$ is an ε_n – equilibrium of G_n. This implies 2.

Finally we show that 1 implies 2 when A is finite. Let $\{a_1, \ldots, a_I\} = \text{supp}(\tau_A)$. For each $1 \leq i \leq I$, let $A_i = B_\tau \cap (\mathcal{U} \times \{a_i\})$.

Let $n \in \mathbb{N}$ and $1 \leq i \leq I$. Then, by Parthasarathy (1967, Theorem II.3.1), there exists a compact set $K_i^n \subseteq A_i$ such that $\tau(A_i \setminus K_i^n) < 1/n$. Then, $K_i^n = \pi(K_i^n) \times \{a_i\}$ and $\pi(K_i^n)$ is a compact subset of $\mathcal{U}$. Therefore, there exists $\delta_n > 0$ such that if $\rho(\mu, \nu) < \delta_n$ implies that $|u(a, \mu) - u(a, \nu)| < 1/4n$ for all $a \in A$ and all $u \in \cup_{i=1}^{I} \pi(K_i^n)$.

Claim 1 *There exists a sequence* $\{\tau_n\}$ *such that*

1. $\tau_n \Rightarrow \tau$,
2. $\tau_n(\cup_{i=1}^{I} K_i^n) = 1$ *for all* $n \in \mathbb{N}$,
3. $\rho(\tau_{A,n}, \tau_A) < \delta_n$ *for all* $n \in \mathbb{N}$,
4. *for all* $n \in \mathbb{N}$, $\tau_n = \nu_n \circ (U_{|T_n}, f_{|T_n})^{-1}$, *where* T_n *is a finite subset of* $[0,1]$ *and* ν_n *is the uniform measure on* T_n,
5. $|T_n| \to \infty$ *and*
6. $1/|T_n| < \delta_n$ *for all* $n \in \mathbb{N}$.

Proof of Claim 1. Since K_i^n is compact, then we can write $K_i^n = \cup_{j=1}^{J_i^n} B_{i,j}^n$, where $\{B_{i,j}^n\}_j$ is a disjoint collection, each of its members being a Borel set with a diameter no greater than $1/n$. We make the convention that $\tau(B_{i,1}^n) \geq \tau(B_{i,j}^n)$ for all $j = 2, \ldots, J_i^n$. Furthermore, we assume that n is large enough to imply that $\tau(K_i^n) > 0$ and so that $\tau(B_{i,1}^n) > 0$.

Let $\eta < \min\{1/n, \delta_n/2|A|\}$ and $J_n = \sum_i J_i^n$. Let $\tilde{q}_{i,j}^n \in \mathbb{Q}_+$, $1 \leq i \leq I$ and $1 \leq j \leq J_i^n$, be such that:

$$\sum_{i=1}^{I} \sum_{j=1}^{J_i^n} \tilde{q}_{n,j} = 1; \tag{18}$$

For all $i \geq 2$,

$$\begin{aligned} &|\tilde{q}_{i,j}^n - \tau(B_{i,j}^n)| < \eta/J_n,\ j = 2, \ldots, J_i^n, \\ &|\tilde{q}_{i,1}^n - (\tau(B_{i,1}^n) + \tau(A_i \setminus K_i^n))| < \eta/J_n,\ \text{and} \\ &\tilde{q}_{i,j}^n = 0 \text{ if } \tau(B_{i,j}^n) = 0,\ j = 2, \ldots, J_i^n. \end{aligned} \tag{19}$$

And for $i = 1$,

$$|\tilde{q}_{1,j}^n - \tau(B_{1,j}^n)| < \eta/J_n,\ j = 2, \ldots, J_1^n. \tag{20}$$

We remark that the above construction can always be made: choose $\tilde{q}_{i,j}^n \leq \tau(B_{i,j}^n)$ for all $i \geq 2$ and $j \geq 2$, $\tilde{q}_{i,1}^n \leq \tau(B_{i,1}^n) + \tau(A_i \setminus K_i^n)$ for all $i \geq 2$ and set $\tilde{q}_{1,1}^n$ using the formula $\sum_{i=1}^{I} \sum_{j=1}^{J_i^n} \tilde{q}_{n,j} = 1$.

Since

$$\sum_{i=1}^{I} \left[\tau(A_i \setminus K_i^n) + \sum_{j=1}^{J_i^n} \tau(B_{i,j}^n) \right] = \sum_{i=1}^{I} \tau(A_i) = \tau(B_\tau) = 1, \tag{21}$$

it follows that

$$\begin{aligned} &|\tilde{q}_{1,1}^n - (\tau(B_{1,1}^n) + \tau(A_1 \setminus K_1^n))| = \\ &\left| 1 - \sum_{(i,j)\neq(1,1)} \tilde{q}_{i,j}^n - 1 + \sum_{i\geq 2} \tau(A_i \setminus K_1^n) + \sum_{(i,j)\neq(1,1)} \tau(B_{i,j}^n) \right| < \eta. \end{aligned} \tag{22}$$

Furthermore, for all $1 \leq i \leq I$,

$$\left|\sum_{j=1}^{J_i^n} \tilde{q}_{i,j}^n - \tau(A_i)\right| \leq \sum_{j=2}^{J_i^n} |\tilde{q}_{i,j}^n - \tau(B_{i,j}^n)| + |\tilde{q}_{i,1}^n - (\tau(B_{i,1}^n) + \tau(A_i \setminus K_i^n))| < 2\eta. \tag{23}$$

This inequality will be used to show that $\rho(\tau_{A,n}, \tau_A) < \delta_n$.

Let $N_n \in \mathbb{N}$ and $\{q_{n,j}\}_{j=1}^{J_n} \subset \mathbb{N}$ such that $\tilde{q}_{n,j} = q_{n,j}/N_n$, $j = 1, \ldots, J_n$. We may assume that $N_n > \max\{n, 1/\delta_n\}$.

Let $i \in \{1, \ldots, I\}$, and $j \in \{1, \ldots, J_i^n\}$. If $j > 1$, and since $\tau(B_{i,j}^n) = \lambda(\{t \in [0,1] : h(t) \in B_{i,j}^n\})$, select $q_{i,j}^n$ points from $\{t \in [0,1] : h(t) \in B_{i,j}^n\}$; if $j = 1$, select $q_{i,j}^n$ points from $\{t \in [0,1] : h(t) \in B_{i,1}^n\}$. By the above convention, $q_{i,j}^n > 0$ implies $\tau(B_{i,j}^n) > 0$, and so we can indeed select such points from $\{t \in [0,1] : h(t) \in B_{i,j}^n\}$. Since, by convention, $\tau(B_{i,1}^n) \geq \tau(B_{i,j}^n)$ for all $j = 2, \ldots, J_i^n$ and $\tau(K_i^n) > 0$, then $\tau(B_{i,1}^n) > 0$ and we can in fact select $q_{i,1}^n$ points from $\{t \in [0,1] : h(t) \in B_{i,1}^n\}$. Let T_n denote this collection of points from $[0,1]$ and let $\tau_n = \nu_n \circ h_{|T_n}^{-1}$.

Since $|T_n| = N_n > \max\{n, \delta_n\}$, it follows that $|T_n| \to \infty$ and $1/|T_n| < \delta_n$. Also, by construction, we have that $\tau_n(\cup_{i=1}^I K_i^n) = 1$. Furthermore,

$$|\tau_{A,n}(\{a\}) - \tau_A(\{a\})| < \frac{\delta_n}{|A|}, \tag{24}$$

for all $a \in A$. This is clear if $a \notin \text{supp}(\tau_A)$, since then both terms are zero. If $a = a_i$, $1 \leq i \leq I$, then $\tau_{A,n}(\{a\}) = \sum_{j=1}^{J_i^n} \tilde{q}_{i,j}^n$ and $\tau_A(\{a\}) = \tau(A_i)$; hence, 24 follows from 23. Since

$$\rho(\tau_{A,n}, \tau_A) \leq |A| \max_{a \in A} |\tau_{A,n}(\{a\}) - \tau_A(\{a\})|, \tag{25}$$

it follows that $\rho(\tau_{A,n}, \tau_A) < \delta_n$.

Finally, we show that $\tau_n \Rightarrow \tau$. Let g be a bounded, uniformly continuous real-valued function on $\mathcal{U} \times A$, and let g be bounded by M. To show that $\int g \mathrm{d}\tau_n \to \int g \mathrm{d}\tau$, it is enough to show that

$$\int_{A_i} g \mathrm{d}\tau_n \to \int_{A_i} g \mathrm{d}\tau, \tag{26}$$

for all $i = 1, \dots, I$.

Let $1 \le i \le I$. Since

$$\left|\int_{A_i} g \mathrm{d}\tau_n - \int_{A_i} g \mathrm{d}\tau\right| \le \left|\int_{K_i^n} g \mathrm{d}\tau_n - \int_{K_i^n} g \mathrm{d}\tau\right| + \left|\int_{A_i \setminus K_i^n} g \mathrm{d}\tau_n - \int_{A_i \setminus K_i^n} g \mathrm{d}\tau\right|, \tag{27}$$

it is enough to show that each of the two terms on the right side of the above inequality converges to zero as n converges to infinity.

We have that $\left|\int_{A_i \setminus K_i^n} g \mathrm{d}\tau_n - \int_{A_i \setminus K_i^n} g \mathrm{d}\tau\right| = \left|\int_{A_i \setminus K_i^n} g \mathrm{d}\tau\right| < M/n \to 0$ as $n \to \infty$.

Let $\alpha_{i,j}^n = \inf_{x \in B_{i,j}^n} g(x)$ and $\beta_{i,j}^n = \sup_{x \in B_{i,j}^n} g(x)$, $j = 1, \dots, J_i^n$. Also, let $\gamma_n = \sup_{1 \le i \le I, 1 \le j \le J_i^n} (\beta_{i,j}^n - \alpha_{i,j}^n)$. Since g is uniformly continuous, and the diameters of $B_{i,j}^n$ converge to zero as n converges to infinity uniformly on i and j, it follows that γ_n converges to zero as n converges to infinity.

As in the proof of Lemma 5, we can show that

$$\left|\int_{B_{i,j}^n} g \mathrm{d}\tau_n - \int_{B_{i,j}^n} g \mathrm{d}\tau\right| \tau(B_{i,j}^n)\gamma_n + M\left|\tau(B_{i,j}^n) - \tilde{q}_{i,j}^n\right| \tag{28}$$

for all $j = 1, \dots, J_i^n$. Thus, for $j = 2, \dots, J_i^n$,

$$\left|\int_{B_{i,j}^n} g \mathrm{d}\tau_n - \int_{B_{i,j}^n} g \mathrm{d}\tau\right| < \tau(B_{i,j}^n)\gamma_n + \frac{M}{nJ_n}.$$

Also, it follows that

$$\begin{aligned} &\left|\int_{B_{i,1}^n} g \mathrm{d}\tau_n - \int_{B_{i,1}^n} g \mathrm{d}\tau\right| \le \tau(B_{i,1}^n)\gamma_n + M\left|\tau(B_{i,1}^n) - \tilde{q}_{i,1}^n\right| \le \\ &\le \tau(B_{i,j}^n)\gamma_n + M\left(\left|\tau(B_{i,j}^n) + \tau(A_i \setminus K_i^n) - \tilde{q}_{i,j}^n\right| + \left|\tau(A_i \setminus K_i^n)\right|\right) < \\ &\tau(B_{i,j}^n)\gamma_n + M\left(\eta + \frac{1}{n}\right) < \tau(B_{i,j}^n)\gamma_n + \frac{2M}{n}. \end{aligned} \tag{29}$$

So,

$$\left|\sum_{j=1}^{J_n}\left(\int_{B_{n,j}} g \mathrm{d}\tau_n - \int_{B_{n,j}} g \mathrm{d}\tau\right)\right| < \gamma_n + \frac{3M}{n} \to 0 \tag{30}$$

as $n \to \infty$. Hence, we conclude that $\left|\int g \mathrm{d}\tau_n - \int g \mathrm{d}\tau\right| \to 0$ as $n \to \infty$ and, since g is arbitrary, that $\tau_n \Rightarrow \tau$. ■

Let $\{\tau_n\}$ be as in Claim 1 and $K_n = \cup_{i=1}^{I} K_i^n$. Since $K_n \subseteq B_\tau$ it follows that $K_n \subseteq B_{\tau_n}^{1/n}$: if $(u,a) \in K_n$ and $\bar{a} \in A$ then $u \in \cup_{i=1}^{I} \pi(K_i^n)$ and so

$$\begin{aligned} &u(a,\tau_{A,n}) > u(a,\tau_A) - 1/4n \geq u(\bar{a},\tau_A) - 1/4n > \\ &u(\bar{a},\tau_{A,n}) - 1/2n > u(\bar{a},\tau_{A,n}^{u,a,\bar{a}}) - 1/n, \end{aligned} \tag{31}$$

since $\rho(\tau_{A,n},\tau_A) < \delta_n$ and $\rho(\tau_{A,n},\tau_{A,n}^{u,a,\bar{a}}) \leq 1/|T_n| < \delta_n$.

Since $\tau_n(K_n) = 1$, it follows that $\tau_n(B_{\tau_n}^{1/n}) = 1$ and $\tau_n = \nu_n \circ (U_{|T_n}, f_{|T_n})^{-1}$ is an ε_n – equilibrium distribution of the game $\tau_{\mathcal{U},n} = \nu_n \circ U_{|T_n}^{-1}$. By Lemma 4, then $f_{|T_n}$ is an ε_n – equilibrium of $G_n = ((T_n,\nu_n), U_{|T_n})$. ■

Proof of Theorem 4. The proof that 2 implies 1 is similar to the one in Theorem 3.

We now show that 1 implies 2. By Lemma 7, it follows that $\overline{U([0,1])}$ is locally compact. Then, $\overline{(U,f^*)([0,1])}$ is locally compact since it is a closed subset of $\overline{U([0,1])} \times A$, which is locally compact since A is a compact metric space.

Define $\tau = \lambda \circ (U,f^*)^{-1}$ and $C = \overline{(U,f^*)([0,1])}$. Then, $\tau(C) = 1$ and also $C \subseteq B_\tau$ since $f^* \in F_f^*$. Then, since C is locally compact and τ is tight, there exist sequences $\{K_j\}_{j=1}^{\infty}$ and $\{V_j\}_{j=1}^{\infty}$ satisfying for all $j \in \mathbb{N}$:

1. $K_j \subseteq C$ is compact,
2. $\tau(K_j) > 1 - 1/j$,
3. $V_j \subseteq C$ is open in C and
4. $K_j \subseteq V_j \subseteq K_{j+1}$.

For each $n \in \mathbb{N}$, we define $\varepsilon_n = \inf\{\varepsilon \geq 0 : f_{|T_n}^*$ is an $(\varepsilon,\varepsilon)$ – equilibrium of $G_n\}$. Note that the set $\{\varepsilon \geq 0 : f_{|T_n}^*$ is an $(\varepsilon,\varepsilon)$ – equilibrium of $G_n\}$ is nonempty since if $B > 0$ is such that u is bounded by B for all $u \in U(T_n)$, then $f_{|T_n}^*$ is a $2B$ – equilibrium of G_n. Thus, it is enough to show that $\varepsilon_n \to 0$.

Let $\eta > 0$ be given. Denote $\tau_n = \nu_n \circ (U_{|T_n}, f_{T_n}^*)^{-1}$. Let $J \in \mathbb{N}$ be such that $1/J < \eta/2$, which then implies that $\tau(V_J) > 1 - 1/J > 1 - \eta/2$. Let π denote the projection of $\mathcal{U} \times A$ onto $\mathcal{U}$. Clearly, $\pi(V_J) \subseteq \pi(K_{J+1})$, and since $\pi(K_{J+1})$ is compact in $\mathcal{U}$, it follows that $\pi(K_{J+1})$ is equicontinuous, and so is

$\pi(V_J)$. Let $\delta > 0$ be such that $d((a,\mu),(b,\nu)) := \max\{d_A(a,b), \rho(\mu,\nu)\} < \delta$ implies that $|u(a,\mu) - u(b,\nu)| < \eta/4$ for all $u \in \pi(V_J)$.

Since both $\text{supp}(\tau_n)$ and $\text{supp}(\tau_n)$ are contained in C, it follows from Lemma 6 that $\tau_n \Rightarrow \tau$ in C. Because V_J is open C, then $\tau(V_J) \leq \liminf_n \tau_n(V_J)$. As a result, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $1/|T_n| < \delta$, $\rho(\tau_{A,n}, \tau_A) < \delta$ and $\tau_n(V_J) \geq \tau(V_J) - \eta/2$.

Since $V_J \subseteq B_\tau$, one easily shows that $V_J \subseteq B^\eta_{\tau_n}$ if $n \geq N$. Hence,

$$\tau_n(B^\eta_{\tau_n}) \geq \tau_n(B^\eta_{\tau_n} \cap V_J) = \tau_n(V_J) \geq \tau(V_J) - \eta/2 > 1 - \eta, \tag{32}$$

and $f^*_{|T_n}$ is an η – equilibrium of G_n. This implies that $\varepsilon_n \leq \eta$ and, since η is arbitrary, that $\varepsilon_n \to 0$. ■

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