An Algorithm for Stable and Equitable Coalition Structures with Public Goods

Fan-chin Kung*

Department of Economics and Finance, City University of Hong Kong

Institute of Economics, Academia Sinica, Taiwan

kungfc@cityu.edu.hk

May 2005

Abstract

We study the formation of coalitions that provide public goods to members. Individuals are linked on a tree graph and those with similar preferences are connected on the tree. We present a well defined solution that selects envy-free allocations from the core.

JEL Classification: C71, D63, D71, H41.
Keywords: Coalition formation, core, envy-free, public goods.

1. Introduction

Individuals in a society face many collective decisions: for example, how much tax to levy, how much to spend on local public schools, and whether to build a community center. Collective decisions are carried out in groups where we will call coalitions.

*I am very grateful to Myrna Wooders, an associate editor, and two anonymous referees for their comments. I would like to thanks Marcus Berliant, Paul Rothstein, Norman Schofield, and William Thomson for helpful discussions, and participants at the Public Choice Society Meeting 2000 in Charleston, the 2001 North American Winter Meeting of the Econometric Society in New Orleans, and the 2004 Meeting of the Association for Public Economic Theory in Beijing for comments.

People form coalitions to decide on the provision of public goods: A coalition is endowed with a feasible set of alternatives. Each individual joins one and only one coalition, and each coalition chooses an alternative from its feasible set. A partition of individuals is a coalition structure.

The notion of a Tiebout equilibrium arises from Tiebout's (1965) seminal paper. A Tiebout equilibrium is a state of the economy where no individual wants to move to another jurisdiction; cf. Tiebout (1956), Westhoff (1977), and Bewley (1981). Other authors have considered the relationships between the core and Tiebout equilibrium in economies with local public goods when agents are allowed to endogenously form jurisdictions (or clubs) for the purposes of collective consumption within jurisdictions. In the presence of local public goods, the core consists of feasible states of the economy with the property that no group of agents can improve, by forming new jurisdictions and/or providing different feasible allocations of goods to the group members. Two early papers showing equivalence of outcomes from these two sorts of solution concepts with anonymous crowding are Wooders (1978, 1980). More recent contributions allow differential crowding and separate the taste types of individuals from their crowding types (their external effects on others); cf. Conley and Wooders (1997), Ellickson *et al* (1999), and demonstrate that the equivalence of the core and price taking equilibrium still obtains. A review of this literature appears in Conley and Smith (2005).

Guesnerie and Oddou (1981), and Greenberg and Weber (1986) study the core in cooperative games with local public goods and endogenous jurisdiction formation. Greenberg and Weber (1993) and Demange (1994) study a stronger notion of stability: the intersection of the core and Tiebout equilibria. This intersection is shown to be nonempty when preferences are single-peaked on a line in the former and when preferences are intermediate on a tree graph in the latter. Coalition feasible sets are assumed to be monotonic in both. See Demange and Wooders (2005) for further discussion of these models and additional references.

We study a public goods provision game where individuals are linked on a tree graph and their preferences have *connected support*. This means that for any pair of alternatives the set of individuals who strictly prefer one alternative to the other is connected on the tree. An implication is that people with the same preferences are connected. We also assume the feasible sets of coalitions to be monotonic; as new members join, more alternatives become feasible to a coalition . The question is: can we derive allocations that are both stable and equitable? We use the core as our stability concept and *envy-freeness* as the equity standard. An allocation is envy-free if no individual wants to switch places with another.

Our result relates closely to the following articles. Using Scarf's (1967) balancedness condition, Kaneko and Wooders (1982) develop conditions guaranteeing the nonemptiness of the core independent of the payoff function in partitioning games. In related research, Le Breton, Owen and Weber (1992) study communication games on graphs where only connected coalitions are effective. Demange (1994) uses the same setting with public goods as ours and shows nonemptiness of the core using Scarf's theorem. We present an algorithm, called the *hierarchical dictator* solution, to solve for the core and to select envy-free allocations from the core. This solution also serves as a constructive proof; Scarf's balancedness condition is not required. We use a preference restriction that is weaker than intermediate preferences used in Demange (1994). The idea of the solution is to utilize the natural tree order associated with a root and set up a decision hierarchy. Individuals are assigned ranks and make decisions according to their ranks. Each individual acts as a dictator for individuals on the subtree originating with her. The dictator chooses an allocation for individuals on the subtree assuring them of the welfare that they could obtain when they become dictators.

Section 2 introduces the model. Section 3 presents the hierarchical dictator solution. Section 4 shows that the solution selects envy-free allocations from the core. Section 5 concludes.

2. Formation of Public Coalitions

Let N denote the set of all individuals in a society. Let $X \subset \mathbb{R}^m$ denote the set of potential alternatives. Each individual $i \in N$ has preferences, given by a weak order R_i, over X. Let P_i denote strict preference and let I_i denote indifference. The preference profile of individuals in set N is denoted by $R = \{R_i\}_{i \in N}$. A *coalition* is a subset $S \in 2^N$ (the power set of N). The set of feasible alternatives of coalition S is denoted by $\phi(S)$. The correspondence $\phi : 2^N \to 2^X$ is called a feasibility correspondence. (Note that a coalition may have an empty feasible set. However, individuals cannot form a coalition with an empty feasible set.) To eliminate triviality, we further assume that there exists $S \in 2^N$ such that $\phi(S) \neq \varnothing$. A *society* is a list $\mathcal{S} = (N, X, R, \phi)$. We make the following assumptions: X is a closed set, R_i is continuous in X, ϕ is compact-valued, and N is a finite set.[1] Each individual joins one and only one coalition. A coalition chooses an alternative from its feasible set. A *coalition structure* C in society $\mathcal{S}$ is a partition of N, where $C \subset 2^N$ and (i) $S \cap S' = \varnothing$ for all $S, S' \in C$, $S \neq S'$; (ii) $\cup_{S \in C} S = N$; (iii) $\phi(S) \neq \varnothing$ for all $S \in C$. An *allocation* a is a mapping $a : N \to X$, which assigns alternative $a(i)$ to individual i. Allocation a is *feasible* if there is a coalition structure and a list of alternatives $\left(C, \{x_S\}_{S \in C}\right)$ with $x_S \in \phi(S)$ for all $S \in C$ such that $a(i) = x_S$ for all $i \in S$ and all $S \in C$.

Moreover, there is given a tree graph G on N. A *path* p in G is a sequence of distinct edges $\{i_0 i_1, i_1 i_2, ..., i_{k-1} i_k\}$; we also denote a path by $p(i_0, i_k)$. A path is of *length* k if it contains k distinct edges.

We restrict ourselves to societies with the following two properties.

- The feasibility correspondence ϕ is *monotonic*: $\phi(S) \subseteq \phi(S')$ for all $S, S' \in 2^N$ with $S \subset S'$.

- The preference profile R has *connected support* on tree G: for any pair $x, y \in X$, the set $\{i \in N \mid x P_i y\}$ is connected on G.

Connected support requires that the set of individuals who strictly prefer one alternative to the other in any pair is connected. For example, in many models with public goods, if people are ranked by income (a linear tree), the set of individuals who strictly prefer more public goods is connected. Connected support is weaker than *intermediate preferences* (Grandmont 1978 and Demange 1994) which requires, in addition, that the set of individuals with the same weak preference over any pair of alternatives is also connected. Based on the idea that individuals who are indifferent should not play a role in making collective decisions, connected support does not restrict their positions. When applied to a linear order, intermediate preferences is equivalent to *single-crossing* and *order restriction* (Rothstein 1990), which are commonly used in the literature (see Gans and Smart 1996 and Kung 2002 for

[1] These assumptions can be replaced with "X and N are both finite sets". All results still hold.

the equivalence results). The following example illustrates the difference between connected support and intermediate preferences.

Example 1. There are five people linked according to their labels 1 to 5 with two preference profiles.

$$R = \begin{array}{c} xI_1yP_1z \\ xP_2yP_2z \\ xI_3yP_3z \\ zP_4yP_4x \\ zP_5xI_5y \end{array}, \quad R' = \begin{array}{c} xP_1yP_1z \\ xI_2yP_2z \\ xI_3yP_3z \\ zP_4xI_4y \\ zP_5yP_5x \end{array}.$$

R has consecutive support but does not satisfy intermediate preferences; R' satisfies intermediate preferences. The difference lies in those who are indifferent between x, y. Intermediate preferences requires indifferent individuals to be in-between those with strict preferences, while this is relaxed in consecutive support. ∎

Take individual $r \in N$ as the root of G; we call r the first level individual. The rooted tree G^r generates a decision hierarchy as follows: The distance between individual $i \in N$ to r is $\delta(r,i) = k$ if r and i are linked by a path of length k (this path is unique since it is on a tree). We say that i is of level $k+1$. Since N is finite, there exists an individual who has the maximum distance from r. Let $\bar{k} = \max_{i \in N} \delta(r,i)$. Let N^i be the subtree originated from i that contains i's lower level individuals. Individual i is a "higher-up" of j $(i \neq j)$ if i is on the path linking j and r, or equivalently if $N^j \subset N^i$. The next lemma shows that the subtrees originated from distinct vertices i and j are disjoint if i, j are of the same level.

Lemma 1. *$N^i \cap N^j = \varnothing$ if i, j are of the same level and $i \neq j$.*

Proof. Note that any two vertices are linked by a unique path on a tree. Suppose $h \in N^i \cap N^j \neq \varnothing$. Then, $p(i,h) \cup p(j,h)$ contains a path linking i and j that only consists of individuals of levels lower than l, except for i, j. Also, $p(i,r) \cup p(j,r)$ contains a path linking i and j that consists of individuals of levels no lower than l. Apparently, we have found two distinct paths linking i, j; a contradiction to the tree structure. ∎

3. The Hierarchical Dictator Solution

The hierarchical dictator solution is defined by a recursive algorithm.[2] Take a decision hierarchy G^r. Each individual is assigned an "admissible set" of coalition-alternative pairs. This set is obtained by allowing each individual to be a dictator for all individuals on her subtree. The dictator can form a connected coalition on her subtree that affords one of her most preferred alternatives, given that all members of the chosen coalition are no worse off than when they were dictators themselves. The admissible set of i acting as a dictator for N^i is defined as follows.

Add, temporarily, a common worst alternative $\underline{z}$ to every individual's preferences; $xP_i\underline{z}$ for all $x \in X$ for all $i \in N$. We will show that $\underline{z}$ does not play a role in the final allocation later. Let $\bar{X} = X \cup \{\underline{z}\}$ and $\bar{\phi}(S) = \phi(S) \cup \{\underline{z}\}$ for all $S \in 2^N$. A coalition-alternative pair (S, x) is admissible to i if (i) S is a connected coalition in N^i that contains i, (ii) x is feasible to S, (iii) for all other members of S, x is at least as good as any alternative in their admissible sets, and (iv) x is one of i's most preferred alternatives among all (S, x) that satisfy i, ii, iii; moreover, we require that (v) there is no $S' \supset S$ such that i to iv are satisfied and $x \in \phi(S')$. Formally, let A^i denote the set of admissible pairs for individual i.

First, A^i is defined for the following two cases.
Case 1: $N^i = \{i\}$. Let $A^i = \left\{(\{i\}, x) \mid x \in \bar{\phi}(i) \text{ and } xR_ix', \forall x' \in \bar{\phi}(i)\right\}$.
Case 2: $N^i \neq \{i\}$ and A^j is defined for all $j \in N^i \backslash i$. Let $Z^j = \left\{x \in \bar{X} \mid \exists S \text{ s.t. } (S, x) \in A^j\right\}$ be the set of j's admissible alternatives. Let $\tilde{C}^i$ be the collection of all connected subsets of N^i containing i.

$$\tilde{C}^i = \left\{S \in 2^{N^i} \mid i \in S,\, S \text{ is connected}\right\}.$$

Let

$$B^i = \left\{(S, x) \in 2^{N^i} \times \bar{X} \mid S \in \tilde{C}^i,\, x \in \bar{\phi}(S) \text{ and } xR_jy\, \forall y \in Z^j\, \forall j \in S\backslash i\right\}.$$

[2] This algorithm is based on a chapter of my dissertation, Kung (2002). Similar algorithms that utilize a tree hierarchy can be found in Demange (2004) where only connected coalitions that are connected on a hierarchy can form.

So, $(S,x) \in B^i$ if (i) $S \subset N^i$ is connected and S contains i, (ii) x is feasible to S, and (iii) for all other members h, x is as good as any admissible alternative in A^h.

Let $W^i \subseteq B^i$ be the set of coalition-alternative pairs with i's most preferred alternatives in B^i.

$$W^i = \left\{ (S,x) \in B^i \mid xR_ix', \forall (S',x') \in B^i \right\}.$$

And, finally,

$$A^i = \left\{ (S,x) \in W^i \mid \nexists (S',x) \in W^i \text{ s.t. } S \subset S', S \neq S' \right\}.$$

A^i consists of pairs $(S,x) \in W^i$ such that S is maximal for x.

The next lemma shows that the admissible set is well-defined in these two cases.

Lemma 2. *In cases 1 and 2, $A^i \neq \varnothing$ for all $i \in N$.*

Proof. In case 1, $A^i \neq \varnothing$ since a maximizer is guaranteed by the continuity of R_i and the non-emptiness and compactness of $\bar{\phi}(i)$. In case 2, note that $B^i \supseteq \left\{ (\{i\}, x) \mid x \in \bar{\phi}(i) \text{ and } xR_ix', \forall x' \in \bar{\phi}(i) \right\} \neq \varnothing$ since i can always form a one-person coalition. Let $O^i = \left\{ x \in \bar{X} \mid \exists S \text{ s.t. } (S,x) \in B^i \right\}$. Then $W^i \neq \varnothing$ if R_i has a maximizer in O^i. Since R_i is continuous and $O^i \neq \varnothing$ by $B^i \neq \varnothing$, we have to show that O^i is compact. Let $R_i(x) = \left\{ y \in \bar{X} \mid yR_ix \right\}$ be the upper contour set of R_i in $\bar{X}$ at x. Note that

$$O^i = \cup_{S \in \tilde{C}^i} \left(\cap_{x \in Z^h, h \in S \setminus i} R_h(x) \cap \bar{\phi}(S) \right).$$

All $R_i(.)$ are closed, all $\bar{\phi}(S)$ are compact, and all $S \in \tilde{C}^i$ and all $\tilde{C}^i$ are finite. Thus, O^i is compact. So, $W^i \neq \varnothing$. Taking maximal coalitions among $(S,x) \in W^i$, $A^i \neq \varnothing$. ∎

A^i is defined recursively starting from the lowest-level individuals and then move one level up at a time until r. For every individual, either Case 1 or Case 2 applies .

In the following, we construct an allocation using the admissible sets. Given a collection of admissible pairs $\{(S^i, x^i)\}_{i \in N}$ such that $(S^i, x^i) \in A^i$ for all $i \in N$, we assign coalitions sequentially starting from r. First, S^r forms with alternative x^r. Let $L^0(r) = \{r\}$, and

$$L^1(r) = \left\{ j \in N \backslash S^r \mid \nexists h \in N \backslash S^r \; s.t. \, N^j \subset N^h \right\}.$$

$L^1(r)$ is the set of individuals without high-ups in N after deleting coalition S^r. Next, each S^i forms with x^i for all $i \in L^1(r)$.

Suppose $L^k(r)$ is determined for $k = 0, ..., m-1$. Let $\hat{S}(m-1) = \cup_{i \in \cup_{k=0}^{m-1} L^k(r)} S^i$. This is the union of all connected coalitions that have been assigned so far. Let

$$L^m(r) = \left\{ j \in N \backslash \hat{S}(m-1) \mid \nexists h \in N^i \backslash \hat{S}(m-1) \;\; s.t. \, N^j \subset N^h \right\};$$

$L^m(r)$ is the set of individuals without higher-ups in N after deleting all S^i for all $i \in L^k(r)$ for all $k = 0, ..., m-1$. There is an integer $\bar{l} \leq \bar{k} - 1$ such that $L^{\bar{l}+1}(r) = \varnothing$ since N is finite. Then, each S^i forms with x^i for all $i \in L^m(r)$. Assign coalitions this way up to $L^{\bar{l}}(r)$. Note that all coalitions that have formed are connected. Let $\overline{L} = \cup_{k=0,...,\bar{l}} L^k(r)$. Thus, $\{S^i\}_{i \in \overline{L}}$ is a partition of N. The collection of pairs $\{(S^i, x^i)\}_{i \in \overline{L}}$ constitute an allocation $a^r_{\{(S^i, x^i)\}_{i \in N}}$ (we denote it with a^r for simplicity). Each pair $(S^i, x^i) \in A^i$ is constructed as if i were a benevolent dictator who assures each member a guaranteed welfare level. This welfare level is the satisfaction the member could achieve from her admissible sets. This benevolence is carried over to the final allocation a^r; everyone is guaranteed a welfare level no less than what she could enjoy from the admissible set.

Lemma 3. *$a^r(i) \, R_i x$ for all $x \in Z^i$ for all $i \in N$.*

Proof. For all $i \in N$, either $i \in \overline{L}$ and $a^r(i) = x^i I_i x$ for all $x \in Z^i$, or $i \notin \overline{L}$ and $i \in S^j$ for some $j \in \overline{L}$ and $a^r(i) = x^j R_i x$ for all $x \in Z^h$. ∎

Next, we show that the resulting allocation does not involve the added alternative $\underline{z}$.

Lemma 4. *$a^r(i) \neq \underline{z}$ for all $i \in N$.*

Proof. Suppose coalition S consumes $\underline{z}$. Also, suppose coalition T is adjacent to S and consumes alternative $x \neq \underline{z}$. (Two connected subsets $S, T \subset N$ are *adjacent* on G if there exists an edge ij such that $i \in S$, $j \in T$.) Note that there is only one edge ij such that $i \in S$, $j \in T$. First, suppose j is of a lower level than i. Thus, j has no higher-ups in T and $T \subset N^j$. By monotonicity, $x \in \bar{\phi}(T \cup i)$. By construction, $xR_h y$ for all $y \in Z^h$ for all $h \in T \subset N^j$. Therefore, $(T \cup i, x) \in B^i$. This implies $x^i R_i x P_i \underline{z} = a^r(i)$; a contradiction to Lemma 3.

Second, suppose j is of a higher level than i. Suppose g is the highest level individual in T (there is a cycle if g is not unique). Thus, $(T, x) = (S^g, x^g) \in A^g$. By monotonicity, $x \in \bar{\phi}(T \cup S)$. Note that by construction $xR_h x^h$ for all $h \in T \backslash g$ and $xP_h \underline{z} R_h x^h$ for all $h \in S$. This implies $(S \cup T, x) \in W^g$ and $(T, x) \notin A^g$; a contradiction.

So, every T adjacent to S must consume $\underline{z}$. Note that every coalition in allocation a^r is adjacent to another. If $a^r(i) = \underline{z}$ for some $i \in N$, then $a^r(i) = \underline{z}$ for all $i \in N$. Note that there exists $S \in 2^N$ such that $\phi(S) \neq \varnothing$. So, there exists $x \in \bar{\phi}(N)$ such that $xP_r \underline{z}$. Moreover, $xP_i \underline{z} R_i x^i$ for all $i \in N \backslash r$. Thus, $(N, x) \in W^r$. This is a contradiction. ∎

Definition 1. The *r-hierarchical dictator solution* for society $\mathcal{S}$ is the collection of all potential allocations a^r constructed above;

$$\mathcal{H}^r(\mathcal{S}) = \left\{ a^r_{\{(S^i, x^i)\}_{i \in N}} \mid \left(S^i, x^i\right) \in A^i \text{ for all } i \in N \right\}.$$

The *hierarchical dictator solution* is

$$\mathcal{H}(\mathcal{S}) = \cup_{r \in N} \mathcal{H}^r(\mathcal{S}).$$

It is straightforward from Lemma 2 that the solution is well-defined.

Proposition 1. $\mathcal{H}^r(\mathcal{S}) \neq \varnothing$.

The following example illustrates the algorithm in a simple society with three individuals and three alternatives.

Example 2. Consider society $\mathcal{S} = (N, X, R, \phi)$ where $N = \{1\ 2\ 3\}$, $X = \{x\ y\ z\}$, $\phi(1) = \{y\}$, $\phi(2) = \{z\}$, $\phi(3) = \{z\}$, $\phi(1\ 2) = \{y\ z\}$, $\phi(2\ 3) = X$, $\phi(1\ 3) = \{y\ z\}$, and $\phi(N) = X$. Individuals are linked according to their labels $1 - 2 - 3$ and their preferences are the following:

$$\begin{array}{l} xP_1yP_1z \\ yP_2zP_2x \\ zP_3yP_3x \end{array} .$$

The preference profile has connected support and ϕ is monotonic. Let 1 be the root individual for example. We construct the admissible sets starting at level 3. Individual 3 can only form a one-person coalition, so $A^3 = \{(3, z)\}$. Individual 2 can form $\{2\}$ with z or $\{2\ 3\}$ with z. (x, y are also feasible but to keep 3 as well off as consuming z, 2 cannot choose x or y.) Since $\{2\ 3\}$ is maximal, $A^2 = \{(2\ 3\,, z)\}$. Last, 1 can form $\{1\}$ with y, $\{1\ 2\}$ with y, or $\{1\ 2\ 3\}$ with z. Since y is more preferred, $A^1 = \{(1\ 2, y)\}$. We pick $(S^1, x^1) = (1\ 2, y)$, thus $L(1) = \{3\}$ and $(S^3, x^3) = (3, z)$. The coalitions $\{1\ 2\}, \{3\}$ with alternatives y, z respectively constitute an allocation a^r. This is the only allocation in $\mathcal{H}^1(\mathcal{S})$. $\mathcal{H}^2(\mathcal{S})$ and $\mathcal{H}^3(\mathcal{S})$ can be constructed in the same way. ∎

A single-valued solution is sometimes desired. It is, however, difficult to find a natural selection of an admissible pair from A^i. There are two sources of multiplicity. First, there may be multiple coalitions in A^i. Unless the tree is linear and r is one of the end vertices, the coalitions in A^i do not satisfy inclusion relationship. Second, even if we can select a largest coalition in A^i, this coalition may support two alternatives that are indifferent to i. It seems only a predetermined linear order $>$ on X can break the ties. The admissible sets can be further refined to

$$\hat{A}^i = \left\{(S, x) \in A^i \mid x > x',\ \forall (S', x') \in A^i\right\},$$

and the new solution $\mathcal{H}^r(\mathcal{S})$ is single-valued. Another special case is when there is no indifference, then, every $A^i = W^i$ is a singleton.

4. Stability and Equity

Definition 2. A feasible allocation a in society $\mathcal{S}$ is in the *core* if there is no coalition $S \in 2^N$ that blocks it. A coalition $S \in 2^N$ *blocks* allocation a if there is an alternative $x \in \phi(S)$ such that $xP_i a(i)$ for all $i \in S$.

Definition 3. A feasible allocation a in society $\mathcal{S}$ is *envy-free* if $a(i) R_i a(j)$ for all $j \neq i$ for all $i, j \in N$.

Envy-free is also defined as a version of Tiebout equilibrium in the literature. It is combined with the core to create a stronger stability concept. The core allows a blocking coalition to form only if "every" member can be better off. This means that a coalition can exclude members, since one cannot join a coalition if her arrival makes others worse off. On the other hand, Tiebout equilibrium allows an individual to join another coalition freely and a coalition cannot exclude individual members. It is possible to make existing members worse off when joining a coalition. We found it interesting to use envy-free as an equity criterion, in accordance with the fair allocation literature (see Foley 1967 and Tadenuma and Thomson 1995).

Theorem 1. *For any society $\mathcal{S}$ satisfying monotonicity and connected support, if allocation $a \in \mathcal{H}(\mathcal{S})$, then it belongs to the core and is envy-free.*

Proof. Suppose allocation $a^r \in \mathcal{H}(\mathcal{S})$ is constructed on G^r with admissible pairs $\{(S^i, x^i)\}_{i \in N}$. The proof is composed of the following three lemmas.

Lemma 5. *If S and T are two adjacent coalitions in allocation a^r and they have a linking edge ij such that $i \in S$ and $j \in T$, then*

$$a^r(i) R_h a^r(j) \text{ for all } h \in M(ij),$$
$$a^r(j) R_h a^r(i) \text{ for all } h \in M(ji),$$

where $M(ij) = \{h \in N \mid ij \notin p(h, i)\}$.

Proof. Without loss of generality, suppose i is of a higher level than j. Note that $T \subset N^i$, $T \cup i \in \tilde{C}^i$, and $a^r(i) R_i x^i$. First, $a^r(j) \in \phi(T \cup i)$ by monotonicity, and $(T \cup i, a^r(j)) \in B^i$. Thus, $a^r(i) R_i a^r(j)$ by construction. Second, suppose

$a^r(i) R_j a^r(j)$; then $a^r(i) R_j x^j$ by construction. By monotonicity, $a^r(i) \in \phi(S \cup j)$. Let g be the highest level individual in S. Note that $T \subset N^g$. Therefore, $(S \cup j, a^r(i)) \in W^g$ which means $(S, a^r(i)) \notin A^g$; a contradiction. So, $a^r(j) P_j a^r(i)$. Third, suppose $a^r(j) R_h a^r(i)$ for all $h \in S$. Then, $a^r(j) R_h x^h$ for all $h \in S$. By monotonicity, $a^r(j) \in \phi(S \cup T)$. Therefore, $(S \cup T, a^r(j)) \in W^g$, which means $(T, a^r(j)) \notin A^g$; a contradiction. So, there is $h \in S$ such that $a^r(i) P_h a^r(j)$. Finally, by connected support, there is no $h \in M(ij)$ such that $a^r(j) P_h a^r(i)$, and no $h \in M(ji)$ such that $a^r(i) P_h a^r(j)$. ∎

Lemma 6. *$a^r(i) R_i a^r(j)$ for all $i, j \in N$.*

Proof. Each pair i, j are linked by a unique path that passes through adjacent connected coalitions. Let $p(i,j) = \{i_0 i_1, i_1 i_2, ..., i_{k-1} i_k\}$ and $i_0 = i$, $i_k = j$. For all $m = 1, ..., k$, either i_{m-1} and i_m belong to the same coalition and $a^r(i_{m-1}) = a^r(i_m)$, or $i_{m-1} i_m$ links two adjacent coalitions, $i \in M(i_{m-1} i_m)$ and $a^r(i_{m-1}) R_i a^r(i_m)$ by Lemma 5. So, $a^r(i_0) R_i a^r(i_k)$. ∎

Lemma 7. *There exists no $S \in 2^N$ that blocks a^r.*

Proof. Suppose S blocks a^r with x and S is not connected. Let T be the minimal connected set containing S. That is, $T \supseteq S$, $T \in 2^N$ and there is no connected $T' \in 2^N$ such that $T' \subset T$, $T' \supseteq S$. For any $h \in T \backslash S$, we can find $i, j \in S$ such that h is on the path linking i and j. We have $x P_i a^r(i) R_i a^r(h)$ and $x P_j a^r(j) R_j a^r(h)$ by Lemma 6 and that S blocks a^r. Connected support implies $x P_h a^r(h)$. Hence, $x P_h a^r(h)$ for all $h \in T$, and $x \in \phi(T)$ by monotonicity. This means that T blocks a^r with x as well. Therefore, we have $x P_h a^r(h) R_h x^h$ for all $h \in T$. Suppose g is the highest level individual in T, then $T \in N^g$ and $(T, x) \in B^g$. This contradicts with $x P_g x^g$.

If the blocking coalition S is connected, the second half of the proof applies. ∎

Since $\mathcal{H}(\mathcal{S}) \neq \varnothing$, our result means that the intersection of the core and the set of envy-free allocations is nonempty for any society satisfying connected support and monotonicity.

5. Conclusion

We study the endogenous formation of coalitions that provide public goods to members. Allocations that are both stable and equitable exist under the following two assumptions. First, coalitions can afford more as new members join. Second, people are linked on a tree graph and the set of individuals with the same strict preferences over a pair of alternatives is connected. The most common example of a tree graph used in the literature is the linear order. For example, people are ranked by income or taste in many models with public goods. We present a well-defined algorithm to construct envy-free allocations that also belong to the core. This solution can be further motivated by a noncooperative story of how coalitions form, based on the following observation: Policies are usually advocated by a few leaders. Others, evaluating the proposals, decide whether to follow the leaders or not. This suggests that policy leaders, who announce public alternatives, may serve the roles of benevolent dictators and initiate the formation of coalitions. In the process, a dictator should give coalition members enough welfare so that they will not break off and form a new coalition them selves.

References

[1] Bewley, T. (1981) "A Critique of Tiebout's Theory of Local Public Expenditures" *Econometrica*, Vol. 49 pp. 713-741.

[2] Conley, J. P. and M. H. Wooders (1997) "Equivalence of the Core and Competitive Equilibrium in a Tiebout Economy with Crowding Types" *Journal of Urban Economics*, Vol. 41 pp. 421-40.

[3] Conley, J. P. and S. Smith (2005) "Coalitions and Clubs: Tiebout Equilibrium in Large Economies" in *Group Formation in Economics: Networks, Clubs, and Coalitions* by G. Demange and M. H. Wooders, Eds, Cambridge University Press.

[4] Demange, G. (1994) "Intermediate Preferences and Stable Coalition Structures" *Journal of Mathematical Economics*, Vol. 23 pp. 45-58.

[5] ———— (2004) "On Group Stability in Hierarchies and Networks" *Journal of Political Economy*, Vol. 112 pp. 754-778.

[6] Demange, G. and M. H. Wooders, Eds (2005) *Group Formation in Economics: Networks, Clubs, and Coalitions*, Cambridge University Press.

[7] Ellickson, B., B. Grodal, S., Scotchmer and W. Zame (1999) "Clubs and the Market" *Econometrica*, Vol. 67 pp. 1185-1217.

[8] Foley, D. (1967) "Resource Allocation and the Public Sector" *Yale Economic Essays*, Vol. 7 pp. 45-98.

[9] Gans, J. S. and M. Smart (1996) "Majority Voting with Single-Crossing Preferences" *Journal of Public Economics*, Vol. 59 pp. 219-237.

[10] Grandmont, J-M. (1978) "Intermediate Preferences and the Majority Rule" *Econometrica*, Vol. 46 pp. 317-350.

[11] Greenberg, J. and S. Weber (1986) "Strong Tiebout Equilibrium under Restricted Preferences Domain" *Journal of Economic Theory*, Vol. 38 pp. 101-117.

[12] ———————— (1993) "Stable Coalition Structure with a Unidimensional Set of Alternatives" *Journal of Economic Theory*, Vol. 60 pp. 62-82.

[13] Guesnerie, R. and C. Oddou (1981) "Second Best Taxation as a Game" *Journal of Economic Theory*, Vol. 25 pp. 67-91.

[14] Kaneko, M. and M. H. Wooders (1982) "Cores of Partitioning Games" *Mathematical Social Sciences*, Vol. 3 pp. 313-327.

[15] Kung, F. (2002) "Formation of Collective Decision-Making Units: Stability and a Solution" mimeo, Washington University.

[16] Le Breton, M., G. Owen and S. Weber (1992) "Strongly Balanced Cooperative Games," *International Journal of Game Theory*, Vol. 20 pp. 419-427.

[17] Rothstein, P. (1990) "Order Restricted Preferences and the Majority Rule" *Social Choice and Welfare*, Vol. 7 pp. 331-342.

[18] Scarf H. E. (1967) "The Core of an N-Person Game" *Econometrica*, Vol. 35 pp. 50-69.

[19] Tadenuma, K. and W. Thomson (1995) "Refinements of the No-Envy Solution in Economies with Indivisible Goods" *Theory and Decision*, Vol. 39 pp. 189-206.

[20] Tiebout, C. (1956) "A pure Theory of Local Expenditures" *Journal of Political Economy*, Vol. 64 pp. 416-424.

[21] Westhoff, F. (1977) "Existence of Equilibria in Economies with a Local Public Good" *Journal of Economic Theory*, Vol. 14 pp. 84-112.

[22] Wooders, M. (1978) "Equilibria, the Core, and Jurisdiction Structures in Economies with a Local Public Good" *Journal of Economic Theory*, Vol. 18 pp. 328-348.

[23] ———— (1980) "The Tiebout Hypothesis: Near Optimality in Local Public Good Economies" *Econometrica*, Vol. 48 pp. 1467-1485.