

Coordination Games With Quantum Information

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Abstract

The paper discusses coordination games with remote players that have access to an entangled quantum state. It shows that the entangled state cannot be used by players for communicating information, but that in certain games it can be used for improving coordination of actions. A necessary condition is provided that helps to determine when an entangled quantum state can be useful for improving coordination.

1 Introduction

Progress in quantum technology has recently stimulated a surprising development in game theory. In this development game theory is applied to conflict situations with the outcome that depends both on participants' actions and on results of measurements of a quantum state. These conflict situations has been named quantum games.¹

Quantum games occur in two types: games with quantum strategies and games with quantum information. In games with quantum strategies actions of players are operations on a quantum state. A particular emphasis is placed on

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¹See seminal papers by Meyer (1999) and Eisert et al. (1999), and further development in Johnson (2001), Benjamin and Hayden (2001), Kay et al. (2001), Du et al. (2002), Lee and Johnson (2003a), Lee and Johnson (2003b), and Landsburg (2004).

quantizations of classical games, which often result in unexpected outcomes like greater degree of cooperation in quantizations of prisoners' dilemma.

A different type consists of games with quantum information. These are games with classical actions, in which players have an access to a shared quantum state. This quantum state can either encode some useful information or be used purely for communication and coordination between players. Its potential usefulness for communication is due to a property of quantum states that is called entanglement. Entanglement allows measurements on two remote particles to exhibit a correlated behavior, which cannot be reproduced using classical correlated random variables.

This paper considers a game with quantum information, in which players need to coordinate their actions depending on the states of nature that they privately observe. The players are remote and cannot communicate using classical channels. They, however, share an entangled quantum state and are allowed to measure it. The main questions are whether the players can communicate information about their states of nature and whether they can use the shared quantum state to coordinate their actions.

It turns out that the players are not able to communicate their private information but that for certain games they can improve coordination of their actions. A necessary condition for this to be possible is that the game should be truly coordination game, that is, that the payoff of the game should non-trivially depend on both players' states of nature and on both player's actions.

The rest of the paper is organized as follows: Section 2 provides necessary background from quantum mechanics and formulates the "no-signalling" theorem showing that information cannot be transmitted using an entangled quantum state. Section 3 introduces a coordination game with quantum information and defines concepts of entangled and classically generated signals. Section 4 exhibits an example that shows that entangled signals can be useful for coordination and describes a necessary condition for their usefulness. And Section 5 concludes.

2 Basics of quantum mechanics

According to quantum mechanics, a quantum state is completely described by a density matrix. A *density matrix* is a non-negative operator in a complex Hilbert space that has unit trace. A state with a rank-one density matrix is called *pure state*. Any state can be represented as a statistical ensemble of pure states, that is, its density matrix can be represented as a linear convex combination of rank-one operators.²

Example 1 Qubit

Qubit is a quantum system described by a density matrix in the two-dimensional Hilbert space. The density matrix can be conveniently written as a linear combination of the Pauli matrices:

$$\rho = \frac{1}{2}(I + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3), \quad (1)$$

where a_i are real numbers, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

The condition that the density matrix is non-negative restricts the coefficients: $\sum |a_i|^2 \leq 1$. Therefore the totality of density matrices corresponds to points inside the unit sphere in 3-dimensional real space. The pure states corresponds to points on the border of this sphere.

One way to construct a complex system is to join two simpler systems together. Here is where phenomenon of *entanglement* arises.

Any joint system of two quantum states can be represented by a density matrix in the product of the Hilbert spaces where each of the states lives. The simplest possible joint systems are described by Kronecker products of the density matrices of the parts. We can also take the statistical ensembles of these joint systems. The density matrix of these more complex systems is a linear

²Good sources of information about modern formulation of quantum mechanics are Peres (1995), Preskill (1999), and Barndorff-Nielsen et al. (2003).

convex combination of the product states. However, some density matrices in the product space cannot be represented in this way. These matrices are called *entangled*. The surprising fact is that the concept of entanglement does not refer to the concept of physical distance so the parts of an entangled states can be very remote. A measurement on one part of the state can instantaneously change our knowledge about the state of the other part. This suggests a question of whether entangled states can be used for communication.

To address this question, let us first explain how measurements of the quantum states are described. A *measurement* with a finite number of outcomes is represented by a collection of non-negative operators M_i that adds up to the identity operator. The probability of outcome i is given by $p_i = \text{tr}(M_i\rho)$. In the case with a continuous set of outcomes R we have a family of non-negative operators $M(x)$ such that

$$\int_R M(x)dx = I. \quad (3)$$

The probability density of outcome x is then given by

$$p(x) = \text{tr}(M(x)\rho). \quad (4)$$

This continuous family of operators is often called *Probability Operator-Valued Measure (POVM)*.

If the system consists of two parts and measurements on the parts are described by $M(x)$ and $N(y)$, the joint measurement is represented by the product $M(x) \otimes N(y)$. The outcomes x and y of the joint measurement are in general correlated. Can we use the correlations for communication of information? The answer is “No,” which can be seen from the following result.

Suppose that a researcher, Alice, performs one of two possible measurements, $M^{(1)}$ or $M^{(2)}$, at her location, and that another researcher, Bob, performs measurement N at his location. The measurements are performed on an entangled state described by matrix ρ in the product Hilbert space $H_1 \otimes H_2$. Let for simplicity the measurements have finite sets of outcomes. Then the joint measurement is represented by product matrices: $M_i \otimes N_j$.

The task of Bob is to determine which measurement, $M^{(1)}$ or $M^{(2)}$, Alice performed. If this were possible, Alice could send information to Bob by choosing either $M^{(1)}$ or $M^{(2)}$. However, this is not possible as the following theorem shows.³

Theorem 2 *Probability of an outcome of measurement N is independent of the choice of measurement M . For any j :*

$$\sum_i \text{tr}(\rho M_i \otimes N_j) = \text{tr}[\rho_2 N_j], \quad (5)$$

where $\rho_2 = (\text{tr}_{H_1} \rho)$ is a partial trace of ρ over H_1 .

Proof: Any state is a linear combination of rank-one projectors, and because of linearity it is enough to prove the theorem for the rank-one projectors. So assume that⁴

$$\rho = \left(\sum a_{ij} |e_i f_j\rangle \right) \left(\sum \bar{a}_{ij} \langle e_i f_j| \right), \quad (6)$$

where $|e_i f_j\rangle$ is a basis of $H_1 \otimes H_2$. Then

$$\begin{aligned} \sum_s \text{tr}(\rho M_s \otimes N_t) &= \sum_s \sum_{ijkl} a_{ij} \bar{a}_{kl} \text{tr}(M_s |e_i\rangle \langle e_k|) \text{tr}(N_t |f_j\rangle \langle f_l|) \\ &= \sum_{ijkl} a_{ij} \bar{a}_{kl} \langle e_k | e_i \rangle \langle f_l | N_t | f_j \rangle. \end{aligned} \quad (7)$$

This sum clearly does not depend on M . For arbitrary ρ , it can be easily evaluated by substituting $M_s = I$:

$$\sum_i \text{tr}(\rho I \otimes N_j) = \text{tr}[\rho_2 N_j]. \quad (8)$$

QED.

³This theorem is often called “no-signalling theorem” (Eberhard (1978), Ghirardi et al. (1980), and Bussey (1982)). The version presented here is slightly different in that it uses the language of operator families to describe measurements.

⁴For convenience, we use the Dirac ket-bra notation: the elements of the Hilbert space are denoted as $|e\rangle$, and the linear functionals on the Hilbert space are denoted as $\langle e|$. In particular, $|e_0\rangle \langle e_0|$ is the orthogonal projector on $|e_0\rangle$. We also write $|e_i f_j\rangle$ instead of $|e_i\rangle \otimes |f_j\rangle$.

3 Coordination Game with Quantum Information

Suppose player A observe a random state of nature φ and player B observe state of nature ψ . (Here we use the word state in its classical game-theoretical sense. No confusion should arise with quantum states.) We assume that states of nature φ and ψ are independent, A cannot observe ψ , and B cannot observe φ . The payoffs to players depend both on players' actions, a and b , and on realization of states. We assume that players maximize the joint payoff that they can achieve by coordinating their actions, and we will call this game the *coordination game*.

In case of no communication the strategy of a player can depend only on his own state. For example, player A's typical strategy is: Play action a with probability $p_a(\varphi)$. Player B plays action b with probability $q_b(\psi)$. A slightly more general situation is when both players observe a random variable x , which is independent of φ and ψ . Then the players can condition their action on this variable and the strategies are characterized by probabilities $p_a(\varphi, x)$ and $q_b(\psi, x)$.

In quantum case the players can perform measurements (which may depend on φ and ψ) of a shared entangled quantum state and observe outcomes s and t . Consequently, they can condition their strategies on these outcomes. The corresponding probability functions are $p_a(\varphi, s)$ and $q_b(\psi, t)$. The essential difference with the classical case is that s and t are not necessarily independent of φ and ψ .

Theorem 2 shows that entangled states cannot be used for communication of information. However, this result does not rule out the possibility that entangled states can be used for coordination purposes. Thus the question is whether there exist such games, in which measurements of an entangled quantum state can enhance the joint payoff. A weaker question is whether there is a couple of random variables s and t that cannot be used for communication but that can

increase the payoff in a coordination game.

More precisely, let us introduce the following definitions. The random variables φ , ψ , s , and t are *disjoint* if

$$\begin{aligned}\Pr\{\psi|\varphi, s\} &= \Pr\{\psi|\varphi\}, \\ \Pr\{\varphi|\psi, t\} &= \Pr\{\varphi|\psi\}.\end{aligned}\tag{9}$$

In other words, signals s and t do not provide additional information about ψ and φ respectively. The random variables φ , ψ , s , and t are *classically generated* if there exists such a random variable x independent from φ and ψ that the following equality holds for conditional distributions:

$$p(s, t|x, \varphi, \psi) = p(s|x, \varphi)p(t|x, \psi).\tag{10}$$

This means that the pairs of random variables (s, φ) and (t, ψ) are independent conditionally on x . For example, the variables are classically generated if s and t can be represented in the following form:

$$\begin{aligned}s &= s(\varphi, x), \\ t &= t(\psi, x)\end{aligned}\tag{11}$$

for some random variable x independent from φ and ψ . Classically generated signals are necessarily disjoint.

A quadruple of random variables φ , ψ , s , and t is *entangled* if they are disjoint and cannot be classically generated. Abusing notation we will call signals s and t entangled keeping variables φ and ψ in the background. We can think about φ and ψ as the configurations of the measurement apparatuses, and s and t as the outcomes of the measurements. The famous non-locality theorem by Bell (see for a precise formulation Clauser et al. (1969)) can be interpreted as saying that the outcomes of measurements of an entangled quantum state are entangled in the sense of our definition. However, it is worth noting that not every quadruple of entangled signals can be realized by measurements of an entangled quantum state.

4 Using Entangled Signals for Coordination

It is surprising but the entangled signals – although useless for communication – can be successfully used for increasing payoff in a coordination game. Here is a modification of an example due to Cleve et al. (2004) that shows that measurements of an entangled quantum state can help in increasing the game payoff.

This example uses only two states per player, which we will take as $\varphi \in \{0, \pi/4\}$ and $\psi \in \{-\pi/8, \pi/8\}$. The players have two actions: 0 and 1 and their task is to play the opposite actions unless $\varphi = \pi/4$ and $\psi = -\pi/8$, in which case they should play the same action. If they play correctly, then they win and get 1, otherwise they lose and get zero.

The maximal expected classical payoff is $3/4$, which is reached by the following strategy: Player 1 always plays 0, player 2 always plays 1. What is the optimal quantum strategy?

Suppose that the players share an entangled state with the density matrix, which is the projector on the following vector:

$$\eta = \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle),$$

where $|0\rangle$ and $|1\rangle$ are basis vectors in a two-dimensional Hilbert space. First, let us calculate the probabilities of outcomes if the first player measures the state by projecting it on two orthogonal vectors

$$\begin{aligned} m_0(\theta_1) &= \cos \theta_1 |0\rangle + \sin \theta_1 |1\rangle, \\ m_1(\theta_1) &= -\sin \theta_1 |0\rangle + \cos \theta_1 |1\rangle, \end{aligned}$$

and the second player measures the state by projecting it on

$$\begin{aligned} n_0(\theta_2) &= \cos \theta_2 |0\rangle + \sin \theta_2 |1\rangle, \\ n_1(\theta_2) &= -\sin \theta_2 |0\rangle + \cos \theta_2 |1\rangle. \end{aligned}$$

(It is easy to see that projectors on a complete system of orthogonal vectors form a collection of measurement operators as was defined above.)

For example, the probability of outcome 00 is

$$\begin{aligned}
 p_{00} &= \text{tr} \{ (|m_0\rangle \langle m_0| \otimes |n_0\rangle \langle n_0|) |\eta\rangle \langle \eta| \} \\
 &= \frac{1}{2} (\cos \theta_1 \sin \theta_2 - \sin \theta_1 \cos \theta_2)^2 \\
 &= \frac{1}{2} \sin^2 (\theta_2 - \theta_1).
 \end{aligned}$$

Computing similarly all other probabilities we have the following table:

$$\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sin^2 (\theta_2 - \theta_1) & \cos^2 (\theta_2 - \theta_1) \\ \cos^2 (\theta_2 - \theta_1) & \sin^2 (\theta_2 - \theta_1) \end{pmatrix}.$$

Now, suppose that the first and second players use measurements with parameters $\theta_1 = \varphi$ and $\theta_2 = \psi$ respectively. Let the players play the outcome they observed. Then they play the opposite actions with probability $\cos^2(\pi/8)$ unless $\varphi = \pi/4$ and $\psi = -\pi/8$, in which case they will play it with probability $\cos^2(3\pi/8) = \sin^2(\pi/8)$. Therefore the probability to win is $1/4 \{3 \cos^2(\pi/8) + (1 - \sin^2(\pi/8))\} = \cos^2(\pi/8) > 3/4$.

It turns out that this is the maximal probability of win in this game achievable by quantum strategies.

Does sharing a quantum state always increase the maximal expected payoff in coordination games? The answer is “No”. In some games sharing a quantum state does not help. Then what are conditions that make the sharing helpful? One necessary condition is that the payoff must depend on the states of nature of both players.

We need some preliminary definitions to formulate the theorem. First, let us for simplicity identify the actions of players with their signals: The players simply play the signal that they obtained. It can be shown that every coordination game can be cast in this form. Let the distribution of states of nature φ and ψ be $p(\varphi)p(\psi)$, where $p(\varphi)$ and $p(\psi)$ are marginal distributions of φ and

ψ .⁵ Let us also use notation

$$p(s, t, \varphi) = \sum_{\psi} p(s, t, \varphi, \psi). \quad (12)$$

We will call signals s and t state-consistent if the φ - ψ marginal of the distribution $p(s, t, \varphi, \psi)$ coincide with $p(\varphi, \psi)$. Then the following theorem holds:

Theorem 3 *Assume that payoff in a game depends only the first player's state of nature: $\pi = \pi(s, t; \varphi)$. Then the maximal expected payoff when players use state-consistent entangled signals coincides with the maximal expected payoff when players use certain state-consistent classically-generated signals.*

Proof: Let the distribution of state-consistent entangled signals that maximize payoff be $p(s, t, \varphi, \psi)$. Consider the following distribution:

$$\tilde{p}(s, t, \varphi, \psi) = p(s, t, \varphi)p(\psi), \quad (13)$$

which is a product of marginal distributions for (s, t, φ) and ψ . It is clear that this new distribution is state consistent: The marginal distribution $\tilde{p}(\varphi, \psi)$ is the same as $p(\varphi, \psi)$. Moreover, the marginal distribution $\tilde{p}(s, t, \varphi)$ is the same as $p(s, t, \varphi)$, which implies that the expected payoff based on the strategy that uses signals distributed according to $\tilde{p}(s, t, \varphi, \psi)$ is the same as that of strategy that uses signals distributed according to $p(s, t, \varphi, \psi)$. Indeed,

$$\sum_{\varphi, \psi} \pi(s, t; \varphi)p(s, t, \varphi, \psi) = \sum_{\varphi} \pi(s, t; \varphi)p(s, t, \varphi) \quad (14)$$

$$= \sum_{\varphi, \psi} \pi(s, t; \varphi)\tilde{p}(s, t, \varphi, \psi). \quad (15)$$

We will prove the theorem by showing that the signals with distribution $\tilde{p}(s, t, \varphi, \psi)$ can be classically generated.

We need a lemma that claims that conditional distribution of the second player's signal does not depend on the first player's state.

⁵Here p simply stands for probability distribution. In particular it may denote two different distribution functions for φ and ψ . A more precise but cumbersome notation would be $p_{\varphi}(x)$ and $p_{\psi}(y)$.

Lemma 4 *For entangled signals, the following equality holds for any φ_1 and φ_2 : $p(t|\varphi_1) = p(t|\varphi_2)$.*

Proof: By definition of entangled signals (9) we have $p(\varphi|t, \psi) = p(\varphi|\psi)$. Consequently $p(\varphi|t) = p(\varphi)$, and

$$p(t|\varphi) = \frac{p(t, \varphi)}{p(\varphi)} = \frac{p(\varphi|t)}{p(\varphi)}p(t) = p(t). \quad (16)$$

The last term in this expression does not depend on φ . QED.

Corollary 5 *If (s, t, φ, ψ) is distributed according to $\tilde{p}(s, t, \varphi, \psi)$ then t is independent of φ and ψ .*

Proof:

$$\tilde{p}(s, t, \varphi, \psi) = p(s, t, \varphi)p(\psi) \quad (17)$$

$$= p(s|t, \varphi)p(t|\varphi)p(\varphi)p(\psi) \quad (18)$$

$$= p(s|t, \varphi)p(t)p(\varphi)p(\psi). \quad (19)$$

Summing over s we get

$$\tilde{p}(t, \varphi, \psi) = p(t)p(\varphi, \psi). \quad (20)$$

QED.

Because of the Corollary, we can take t as x in the definition of the classically generated variables (10) and we then get:

$$\tilde{p}(s, t|t, \varphi, \psi) = p(s|t, \varphi)p(t|t, \psi) \quad (21)$$

as required in this definition. QED.

5 Conclusion

We discussed the value of quantum entanglement for coordination games with remote players. We showed that quantum entanglement cannot be used for communicating information, but that it can be useful for coordination purposes. We

also introduced the concept of entangled signals which is a weaker concept than the concept of quantum entanglement but captures some of its properties. This new concept may be helpful in analyzing the properties of quantum entanglement.

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