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Some Notes on Learning in Games with Strategic Complementarities

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Abstract Fictitious play is the classical myopic learning process, and games with strategic complementarities are an important class of games including many economic applications. Knowledge about convergence properties of fictitious play in this class of games is scarce, however. Beyond dominance solvable games, global convergence has only been established for games with strategic complementarities and diminishing marginal returns (Krishna, 1992, HBS Working Paper 92-073). This result is known to depend critically on the assumption of a tie-breaking rule. We show that restricting the analysis to nondegenerate games allows us to drop this assumption. More importantly, an ordinal version of strategic complementarities turns out to suffice. As a byproduct, we also obtain global convergence in generalized ordinal potential games with diminishing marginal returns.

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1 Introduction

The idea of *fictitious play* (FP) is over half a century old. It was originally introduced by Brown (1951) as an algorithm to calculate the value of a zero-sum game. Apart from this, fictitious play is the prime example of myopic belief learning. Recently, fictitious play has also aroused interest as an optimization heuristic (Garcia et al., 2000, Lambert et al., 2002). In a fictitious play process two players are engaged in the repeated play of a bimatrix game. Each player believes that her opponent plays a stationary mixed strategy. After an arbitrary initial move, in each round, she uses the empirical distribution of pure strategies played by the opponent as her belief and responds with a pure strategy that maximizes her expected payoff given this belief, i.e., with a *myopic best response*. We say that an FP process approaches equilibrium, if the sequence of beliefs converges to the set of Nash equilibria of the game. A game is said to have the *fictitious play property* (FPP), if every FP process approaches equilibrium in this game.

It is well known that not every game has the FPP. Shapley (1964) demonstrated this with an example of a 3×3 game, where the beliefs converge to a limit cycle. Most of the research concerned with fictitious play tried to identify classes of games with the FPP. The most often cited convergence results[1] are those for zero-sum games (Robinson, 1951), 2×2 games (Miyazawa, 1961), games with strategic complementarities and a unique equilibrium (Milgrom and Roberts, 1991), games with strategic complementarities and diminishing marginal returns (Krishna, 1992), and weighted potential games (Monderer and Shapley, 1996a, 1996b).

The results of Miyazawa and Krishna are subject to a technical constraint: both use a particular tie-breaking rule. With the original definition of FP, without the assumption of a tie-breaking rule, their results need not hold, at least for degenerate games. This has been demonstrated by Monderer and Sela (1996) with an example of a 2×2 game with strategic complementarities and diminishing marginal returns, for which FP need not converge.[2] In their example, it is a degeneracy of the game which permits nonconvergent FP processes. Subsequently, Monderer and Shapley (1996a) showed that without assuming a tie-breaking rule, one must restrict the analysis to nondegenerate games in order to save Miyazawa's result. Specifically, they proved that every 2×2 game satisfying a particular nondegeneracy assumption has the FPP. This result was later extended to $2 \times n$ games by Berger (2003).

Referring to Krishna's result, however, Monderer and Sela (1996, p. 145) state that they *"do not know whether such a generic result holds for Krishna's games as well"*. Hence the question if, without using a tie-breaking rule, a nondegeneracy condition similar to the one of Monderer and Shapley (1996a) can save Krishna's result, remained open.

Milgrom and Shannon (1994) have shown that many of the known results for games with strategic complementarities can already be derived under the weaker conditions called *ordinal* complementarities in this paper. Krishna

[1] The cited results also hold for the continuous-time version of fictitious play (see below) and for the mathematically equivalent *best response dynamics* of Gilboa and Matsui (1991) and Matsui (1992). Another well known variant of FP is Fudenberg and Kreps' (1993) *stochastic fictitious play*, which has recently been studied thoroughly by Hofbauer and Sandholm (2002).

[2] For a helpful visualization of this example see Cressman (2003, p. 84).

(1992) raised the question if these ordinal conditions could also be sufficient for his result. However, as he demonstrates, his proof does not extend to the larger class of games with ordinal complementarities, and hence he must leave this question unanswered.

Hahn (1999) studies 3×3 games with strategic complementarities and asks if these games have the FPP. Assuming the same tie-breaking rule as Krishna, he shows that this is indeed the case for the *continuous-time* version of FP.

The present paper accomplishes three goals: First, we clarify the question of Monderer and Sela insofar, as we prove that without assuming a tie-breaking rule, Krishna's result indeed continues to hold for nondegenerate games. Second, we show that Krishna's question, if ordinal complementarities would also suffice for this result, can be answered in the affirmative.

The remainder of this paper is structured as follows. In Section 2 we introduce the notation and terminology we use. Section 3 contains some important properties of fictitious play. In Section 4 we discuss the condition of diminishing marginal returns and derive the main result. Section 6 concludes.

2 Notation and Definitions

2.1 Fictitious Play

Let (A, B) be a bimatrix game where player 1, the row player, has pure strategies in the set $N = \{1, 2, \ldots, n\}$, and player 2, the column player, has pure strategies in $M = \{1, 2, \ldots, m\}$. A and B are the $n \times m$ payoff matrices for players 1 and 2. Thus, if player 1 chooses $i \in N$ and player 2 chooses $j \in M$, the payoffs to players 1 and 2 are a_{ij} and b_{ij}, respectively. The set of mixed strategies of player 1 is the $n-1$ dimensional probability simplex S_n, and analogously S_m is the set of mixed strategies of player 2. With a little abuse of notation we will not distinguish between a pure strategy i and the corresponding mixed strategy representation as the i-th unit (column-) vector $\mathbf{e}_i$ in the respective probability simplex. Sometimes we will also speak of the players choosing a row, or column, respectively, of the bimatrix.

The expected payoff for player 1 playing strategy i if player 2 plays the mixed strategy $\mathbf{y} = (y_1, \ldots, y_m)^t \in S_m$ (where the superscript ^{t} denotes the transpose of a vector or matrix) is $(A\mathbf{y})_i$. Analogously $(B^t\mathbf{x})_j$ is the expected payoff for player 2 playing strategy j against the mixed strategy $\mathbf{x} = (x_1, \ldots, x_n)^t \in S_n$. If both players use mixed strategies $\mathbf{x}$ and $\mathbf{y}$, respectively, the expected payoffs are $\mathbf{x} \cdot A\mathbf{y}$ to player 1 and $\mathbf{y} \cdot B^t\mathbf{x}$ to player 2, where the dot denotes the scalar product of two vectors. We denote by $BR_2(\mathbf{x})$ player 2's pure strategy best response correspondence, and by $br_2(\mathbf{x})$ her mixed strategy best response correspondence. Analogously, $BR_1(\mathbf{y})$ and $br_1(\mathbf{y})$ are the sets of player 1's pure and mixed best responses, respectively, to $\mathbf{y} \in S_m$. A pair of mixed strategies $(\mathbf{x}^*, \mathbf{y}^*)$ is a Nash equilibrium if and only if $\mathbf{x}^* \in br_1(\mathbf{y}^*)$ and $\mathbf{y}^* \in br_2(\mathbf{x}^*)$.

Definition 1 *For $t \in \mathbb{N}$ the sequence $f(t) := (\mathbf{x}(t), \mathbf{y}(t))$ is a* fictitious play process *(FP process), if*

$$(\mathbf{x}(1), \mathbf{y}(1)) \in S_n \times S_m$$

and for all $t \in \mathbb{N}$,

$$\mathbf{x}(t+1) = \frac{t\mathbf{x}(t) + \mathbf{b}_1(t)}{t+1} \quad \text{and} \quad \mathbf{y}(t+1) = \frac{t\mathbf{y}(t) + \mathbf{b}_2(t)}{t+1}, \tag{1}$$

where $\mathbf{b}_1(t) \in BR_1(\mathbf{y}(t))$ and $\mathbf{b}_2(t) \in BR_2(\mathbf{x}(t))$. The sequence $(\mathbf{b}_1(t), \mathbf{b}_2(t))_{t\in\mathbb{N}}$ is called the sequence of play *along this process. The Euclidian distance between $f(t)$ and $f(t+1)$ in $\mathbb{R}^{n+m}$ is denoted by $|f(t) - f(t+1)|$ and called the* step size *of the process at time t.*

The step size of any FP process goes to zero as $t \to \infty$, and if the sequence of play converges, it must be constant from some stage on, implying that the process converges to the respective pure strategy equilibrium. Even if the sequence of play does not converge, it is easily established that if the FP process does, then the limit must be a Nash equilibrium. As noted above, however, there are games where FP need not converge.

In general an FP process is not defined uniquely by its initial point $(\mathbf{x}(1), \mathbf{y}(1))$. Indeed, if at some point one of the best response sets is multi-valued, there are at least two possible continuations of a process. To handle this multiplicity of solutions, particular *tie-breaking rules* have sometimes been assumed. Krishna (1992) e.g. assumed that both players, whenever indifferent between two or more pure strategies, choose the strategy with the highest number. However, any tie-breaking rule is somehow artificial, and therefore we stick to Brown's (1951) original definition of the FP process, which does not impose any tie-breaking rule.

For any FP process, if at time t player 1 plays strategy i as a best response to his belief $\mathbf{y}(t)$, then the empirical distribution $\mathbf{x}(t+1)$ is a convex combination of $\mathbf{x}(t)$ and $\mathbf{e}_i$. If player 1 switches from playing i at time t to playing i' at time $t+1$, then $\mathbf{y}(t)$ and $\mathbf{y}(t+1)$ must lie (weakly) on different sides of the set of "indifference points" between strategies i and i'. Geometrically, the set of such points $\mathbf{y} \in S_m$ is the (possibly empty) intersection of S_m with an $m-1$-dimensional linear subspace of $\mathbb{R}^m$, the hyperplane $\{\mathbf{y} : (A\mathbf{y})_i = (A\mathbf{y})_{i'}\}$.

Definition 2 *We say that an FP process $(\mathbf{x}(t), \mathbf{y}(t))$* switches (at time t) from (i, j) to (i', j')*, if $i \neq i'$ or $j \neq j'$, and*

$$(\mathbf{b}_1(t), \mathbf{b}_2(t)) = (i, j) \quad \text{and} \quad (\mathbf{b}_1(t+1), \mathbf{b}_2(t+1)) = (i', j').$$

It is often much easier to work with the continuous-time analog of FP instead of the discrete version. The basic idea here is to let the time between successive rounds of the game shrink to zero. In the limit, we obtain a system of differential inclusions, which also goes back to Brown (1951).

Definition 3 *For* $t \in \mathbb{R}$ *the path* $f(t) := (\mathbf{x}(t), \mathbf{y}(t))$ *is a* continuous-time fictitious play process *(CFP process), if*

$$(\mathbf{x}(1), \mathbf{y}(1)) \in S_n \times S_m$$

and for almost all $t \geq 1$*,*

$$\dot{\mathbf{x}}(t) \in \frac{br_1(\mathbf{y}(t)) - \mathbf{x}(t)}{t} \quad \text{and} \quad \dot{\mathbf{y}}(t) \in \frac{br_2(\mathbf{x}(t)) - \mathbf{y}(t)}{t}. \tag{2}$$

This dynamics has been studied by Rosenmüller (1971), Hofbauer (1995), and Harris (1998), see also the surveys of Krishna and Sjöström (1997) and Hofbauer and Sigmund (2003). Hofbauer (1995) proved that solutions, possibly multiple ones, exist, and are essentially piecewise linear. In all known cases, the behavior of CFP is very similar to that of FP. Only a few formal results on the relation between these two versions are available, however, see e.g. Hofbauer and Sorin (2002).

2.2 Complementarities, Nondegeneracy, and Diminishing Marginal Returns

Definition 4 *(i) A bimatrix game* (A, B) *has* ordinal complementarities*, if for all* $i < i'$ *and* $j < j'$*:*

$$a_{i'j} > a_{ij} \Longrightarrow a_{i'j'} > a_{ij'} \quad \text{and} \quad b_{ij'} > b_{ij} \Longrightarrow b_{i'j'} > b_{i'j}.$$

(ii) A bimatrix game (A, B) *has* strategic complementarities*, if for all* $i < i'$ *and* $j < j'$*:*

$$a_{i'j'} - a_{ij'} > a_{i'j} - a_{ij} \quad \text{and} \quad b_{i'j'} - b_{i'j} > b_{ij'} - b_{ij}.$$

We write OC short for *ordinal complementarities*, and GOC for *game with ordinal complementarities*. Similarly, SC and GSC stands for *(game with) strategic complementarities*.

In a GOC, payoffs satisfy a single-crossing property: the difference between consecutive payoffs in a column of A or a line of B can change its sign at most once, and only from -1 to $+1$. In a broader context, these games have been studied by Milgrom and Shannon (1994). From Definition 4, ordinal complementarities are implied by strategic complementarities. In a game with strategic complementarities, the advantage of switching to a higher strategy increases when the opponent chooses a higher strategy.

Originally, the term "strategic complementarities" was coined by Bulow et al. (1985) to denote games with increasing best response correspondences. This is actually a weaker property, which is already implied by ordinal complementarities. GSC, sometimes also called *supermodular games*, have been introduced (in a much more general framework) by Topkis (1979) and studied by Vives (1990) and Milgrom and Roberts (1990). This class of games has important applications in economics, e.g. in models of oligopolistic competition, R&D competition, macroeconomic coordination, bank runs, network externalities, etc.

As mentioned in the introduction, without assuming a tie-breaking rule, one must impose a nondegeneracy assumption in order to keep the FPP, even in the class of 2×2 games. We work with games which are nondegenerate in the following specific sense.

Definition 5 *We call a bimatrix game* (A, B) degenerate*, if some column of* A *or some line of* B *contains two identical payoffs. Otherwise, the game is said to be* nondegenerate*.*

We write NDGOC (NDGSC) short for *nondegenerate game with ordinal (strategic) complementarities*.

Another condition we use is *diminishing marginal returns*. As the name suggests, this property means that the payoff advantage of increasing one's strategy is decreasing.

Definition 6 *A bimatrix game* (A, B) *has* diminishing marginal returns *(DMR), if for all* $i \in \{2, 3, \ldots, n-1\}$ *and for all* $j \in \{2, 3, \ldots, m-1\}$,

$$a_{i+1,j} - a_{ij} < a_{ij} - a_{i-1,j} \quad \text{and} \quad b_{i,j+1} - b_{ij} < b_{ij} - b_{i,j-1}.$$

3 Some Properties of FP in Nondegenerate Games

Whenever a switch occurs along an FP process, at least one of the players changes her strategy. The next lemma shows that the "new" strategy of this player must be a *better* response than her "old" strategy against the "old" strategy of the opponent. This was called the *Improvement Principle* by Monderer and Sela (1997), see also Sela (2000).

Lemma 1 *If an FP process for the nondegenerate bimatrix game* (A, B) *switches from* (i, j) *to* (i', j')*, then*

$$i \neq i' \Longrightarrow a_{i'j} > a_{ij} \quad \text{and} \quad j \neq j' \Longrightarrow b_{ij'} > b_{ij}.$$

Proof Assume $i \neq i'$. Since $\mathbf{y}(t+1)$ is a convex combination of $\mathbf{y}(t)$ and $\mathbf{e}_j$, we can write $\mathbf{e}_j = c\mathbf{y}(t+1) + (1-c)\mathbf{y}(t)$ for some $c > 1$. Left-multiplying with the matrix A yields $A\mathbf{e}_j = cA\mathbf{y}(t+1) + (1-c)A\mathbf{y}(t)$. Subtracting the i-th line of this vector equation from the i'-th line gives us

$$\begin{aligned}&a_{i'j} - a_{ij} = \\ &= c[(A\mathbf{y}(t+1))_{i'} - (A\mathbf{y}(t+1))_i] + (1-c)[(A\mathbf{y}(t))_{i'} - (A\mathbf{y}(t))_i].\end{aligned}$$

The right-hand side of this is nonnegative, since $i \in BR_1(\mathbf{y}(t))$, $i' \in BR_1(\mathbf{y}(t+1))$, and $c > 1$. The payoff difference on the left-hand side can not be zero if $i \neq i'$ and the game is nondegenerate, hence $a_{i'j} > a_{ij}$. By the same reasoning we get $b_{ij'} > b_{ij}$ if $j \neq j'$. □

Assume for the moment, that along some FP process, the sequence of play switches from (i, j) to (i', j'), where $i \neq i'$ *and* $j \neq j'$, i.e. both players change strategies at the same time. Then Lemma 1 assures us that $a_{i'j} > a_{ij}$ (and an analogous ordering for the other player's payoffs). However, we can not say anything about the ordering of $a_{ij'}$ and $a_{i'j'}$. This ordering is of course determined, if we observe a switch from (i, j') to (i', j'), or vice versa, at some other time in the FP process. However, the next result shows that sometimes it can also be determined without such additional information.

Lemma 2 *Let $f(t) := (\mathbf{x}(t), \mathbf{y}(t))$ be an FP process for the nondegenerate bimatrix game (A, B). Let $i \neq i'$ and $j \neq j'$ and assume there is a strictly increasing, infinite sequence of times t_k, and an $\epsilon > 0$, such that for all k, the process switches from (i, j) to (i', j') at time t_k, but does not switch again until some time $t_k + T_k$ with $|f(t_k + T_k) - f(t_k + 1)| \geq \epsilon$. Then*

$$a_{i'j'} > a_{ij'} \quad \text{and} \quad b_{i'j'} > b_{i'j}.$$

Proof Since under the assumptions made, $\mathbf{y}(t_k+T_k)$ is a convex combination of $\mathbf{y}(t_k + 1)$ and $\mathbf{e}_{j'}$, and has a distance of at least ϵ from $\mathbf{y}(t_k + 1)$ for all k, we can find a constant $0 < c < 1$ and values $c_k \in [c, 1]$ such that for all k, $\mathbf{y}(t_k + T_k) = c_k\mathbf{e}_{j'} + (1 - c_k)\mathbf{y}(t_k + 1)$. Left-multiplying with the matrix A yields $A\mathbf{y}(t_k + T_k) = c_kA\mathbf{e}_{j'} + (1 - c_k)A\mathbf{y}(t_k + 1)$. Subtracting the i-th line of this vector equation from the i'-th line gives us

$$\begin{aligned}&(A\mathbf{y}(t_k + T_k))_{i'} - (A\mathbf{y}(t_k + T_k))_i = \\ &= c_k[a_{i'j'} - a_{ij'}] + (1 - c_k)[(A\mathbf{y}(t_k + 1))_{i'} - (A\mathbf{y}(t_k + 1))_i].\end{aligned} \tag{3}$$

The step size of the FP process vanishes as $k \to \infty$, so the distance of $\mathbf{y}(t_k)$ and $\mathbf{y}(t_k + 1)$ shrinks to zero. But this means that the advantage of i' over i against $\mathbf{y}(t_k + 1)$ shrinks to zero as well, i.e. the second term on the right-hand side of (3) vanishes. As opposed to this, the first term on the right-hand side can not vanish, since the c_k's are bounded from below by c. The left-hand side of the above equation is nonnegative, since i' is always a best response to $\mathbf{y}(t_k + 2)$. Hence the first term on the right-hand side is also nonnegative, i.e. $c_k[a_{i'j'} - a_{ij'}] \geq 0$. Nondegeneracy then implies $a_{i'j'} > a_{ij'}$. A similar argument shows that $b_{i'j'} > b_{i'j}$. □

Next we define a relation on the space of pure strategy pairs, which can be used to describe the possible sequences of play of an FP process in a nondegenerate game.

Definition 7 *Let (A, B) be a nondegenerate $n \times m$ game. On the set $N \times M$ of pure strategy pairs, we define the following binary relation.*

$$(i, j) \to (i', j') \quad \Leftrightarrow \quad \left\{ \begin{array}{l} (i, j) \neq (i', j') \\ i \neq i' \Longrightarrow a_{i'j} > a_{ij} \\ j \neq j' \Longrightarrow b_{ij'} > b_{ij} \end{array} \right\}$$

If $(i, j) \to (i', j')$, we say that this is a *profitable deviation step* (PD step). If $i = i'$ or $j = j'$, we call it a *unilateral profitable deviation step* (UPD step), otherwise we call it a *diagonal* PD step. The length of a PD step $(i, j) \to (i', j')$ is defined by $\max(|i' - i|, |j' - j|)$. We say that the game is *PD-acyclic (UPD-acyclic)*, if there is no PD (UPD) cycle, i.e. if any sequence of PD (UPD) steps is finite.[3]

Note that if $(i, j) \to (i', j')$ is a diagonal PD step, then we necessarily have the UPD steps $(i, j) \to (i', j)$ and $(i, j) \to (i, j')$.

With Definition 7, from Lemma 1 we can see that if an FP process switches from (i, j) to (i', j'), then these two pairs are connected by a PD step: $(i, j) \to (i', j')$. We can even infer more.

Lemma 3 *Let (A, B) be a nondegenerate game. If the game is PD-acyclic, it has the FPP.*

Proof If a nonconvergent FP process exists, then there are infinitely many switches along the process. Since there are only finitely many pure strategy pairs, however, at least two such pairs are played infinitely often. Hence there must be a sequence of switches leading from one of these pairs to the other and back again. By the remark preceeding Lemma 3, this means that there is a PD cycle. □

It is clear by Definition 7, that a PD-acyclic game is UPD-acyclic. The converse need not hold, however. To see this, consider the 2×2 pure coordination game with payoffs of 1 in the diagonal and 0 elsewhere. Clearly, this game is UPD-acyclic. However, both $(1, 2) \to (2, 1)$ and $(2, 1) \to (1, 2)$ are diagonal PD steps in this game, and hence $(1, 2) \to (2, 1) \to (1, 2)$ is a PD cycle.

It should also be noted that PD-acyclicity is sufficient, but not necessary for the FPP. The well known game of Matching Pennies, e.g., has a unique PD cycle, but nevertheless every FP process converges to the unique mixed

[3] UPD-acyclic games are called *1-acyclic* and PD-acyclic games *2-acyclic* by Monderer and Sela (1997). Monderer and Shapley (1996b) refer to UPD-acyclic games as games with the *finite improvement property.*

equilibrium in this game. On the other hand, not every nonconvergent FP process has a sequence of play following a PD cycle. In case of nonconvergence, a PD cycle must exist, but if there is a second PD cycle intersecting the first one, the sequence of play can in principle jump back and forth between these two PD cycles in an irregular and aperiodic fashion, while the beliefs of the FP process approach a chaotic attractor.[4]

4 Fictitious Play in NDGOCs with DMR

We start with Krishna's (1992) observation that DMR restrict the size of the set of pure best responses to a mixed strategy.

Lemma 4 *Let (A, B) be a game with DMR. Then for any $(\mathbf{x}, \mathbf{y}) \in S_n \times S_m$, the sets $BR_1(\mathbf{y})$ and $BR_2(\mathbf{x})$ contain at most two strategies. If such a set contains two strategies, they are numbered consecutively.*

An immediate consequence of this is that in games with DMR, FP can only switch to neighboring strategies from some stage on.

Lemma 5 *Let (A, B) be a game with DMR. For any FP process there exists a time T such that if a switch from (i, j) to (i', j') occurs after time T, then $|i' - i| \leq 1$ and $|j' - j| \leq 1$.*

Proof Assume not. Then w.l.o.g. there are strategies i, i' and j, j' with $i+1 < i'$, such that the process switches from (i, j) to (i', j') infinitely often. Let the times of these switches be t_k. Then $i \in BR_1(\mathbf{y}(t_k))$ and $i' \in BR_1(\mathbf{y}(t_k+1))$. Since the stepsize of the process vanishes, $\mathbf{y}(t_k)$ and $\mathbf{y}(t_k+1)$ have the same set of limit points. If $\hat{\mathbf{y}}$ is such a limit point, then $\hat{\mathbf{y}} \in S_m$ by compactness, and $BR_1(\hat{\mathbf{y}})$ contains i and i' by upper-semicontinuity of the best response correspondence. But this contradicts Lemma 4. □

Another consequence is that in nondegenerate games with DMR, if the sequence of play of an FP process follows some PD cycle infinitely often, then there is also a UPD cycle. The reason for this is that if the PD cycle involves a diagonal PD step $(i, j) \to (i', j')$, followed by some PD step $(i', j') \to (i'', j'')$, then either the diagonal PD step can be "bypassed" by a sequence of two UPD steps, or the step $(i, j) \to (i'', j'')$ is itself a UPD step, or $(i, j) = (i'', j'')$. This is the content of the proof of the following lemma.

Lemma 6 *Let (A, B) be a nondegenerate game with DMR. If an FP process does not converge, then there exists a UPD cycle.*

Proof Let $f(t) = (\mathbf{x}(t), \mathbf{y}(t))$ be a nonconvergent FP process. Then there exists a PD cycle which is followed infinitely often by this process. By Lemma

[4] Cowan (1992) constructed a 3×3 game in which FP shows chaotic behavior.

5, it involves only PD steps of length 1. Assume this cycle is not a UPD cycle, then it contains a diagonal PD step $(i,j) \to (i',j')$. We know that then $(i,j) \to (i',j)$ and $(i,j) \to (i,j')$. If $(i',j) \to (i',j')$ or $(i,j') \to (i',j')$, the diagonal PD step can be replaced by two UPD steps, as indicated in Figure 1 (a). Assume this is not the case. Let the next step in the cycle be (i'',j''). If $(i'',j'') = (i',j)$ or $(i'',j'') = (i,j')$, then the sequence $(i,j) \to (i',j') \to (i'',j'')$ can be replaced by the respective single UPD step, see Figure 1 (b). Assume that also this is not the case. If $(i'',j'') = (i,j)$, as in Figure 1 (c), then the two-cycle $(i,j) \to (i',j') \to (i,j)$ can be disposed of, unless it constitutes the *complete* PD cycle. In this latter case, however, both players use only two strategies in the long run, and we know that the FP process must converge then.

The only cases left are the cases where $i'' \notin \{i,i'\}$ or $j'' \notin \{j,j'\}$. Suppose the former is the case, as in the example in Figure 1 (d). Then there is a subsequence of times t_k and a sequence of "waiting times" T_k such that the switches from $(i,j,)$ to (i',j') occur at times t_k and the subsequent switches to (i'',j'') occur at times $t_k + T_k$, as defined in Lemma 2. Note that this implies $i \in BR_1(\mathbf{y}(t_k))$, $i' \in BR_1(\mathbf{y}(t_k{+}1))$, and $i'' \in BR_1(\mathbf{y}(t_k{+}T_k))$ for all k. Since we have assumed that neither $(i',j) \to (i',j')$ nor $(i,j') \to (i',j')$, by Lemma 2 there is no $\epsilon > 0$ such that the distance between $f(t_k + 1)$ and $f(t_k + T_k)$ is bounded from below by ϵ for all k. In other words, this distance goes to zero as $k \to \infty$. Since also the distance between $f(t_k)$ and $f(t_k + 1)$ (the step size at time t_k) goes to zero, we can conclude that $\mathbf{y}(t_k)$, $\mathbf{y}(t_k{+}1)$, and $\mathbf{y}(t_k{+}T_k)$ have the same set of limit points. Let $\hat{\mathbf{y}}$ be a limit point of these three sequences as $k \to \infty$. By compactness, $\hat{\mathbf{y}} \in S_m$. By upper-semicontinuity of the best response correspondence, $BR_1(\hat{\mathbf{y}})$ contains i, i', and i''. However, this contradicts Lemma 4. □

Together with Lemma 5, Lemma 6 implies that in an NDGOC with DMR, nonconvergence of FP implies the existence of a UPD cycle consisting entirely of UPD steps of length 1. However, the next result states that in nondegenerate games, ordinal complementarities prevent the existence of such UPD cycles.

Lemma 7 *In any NDGOC, every sequence of UPD steps of length 1 is finite.*

Proof Let (A,B) be an NDGOC. Assume the game admits a UPD cycle consisting of length 1 steps. Take any pair (i^*,j^*) in this UPD cycle, where the next step is to (i^*,j^*+1).[5] By OC, we have $(i',j^*) \to (i',j^*+1)$ for all $i' \geq i^*$, see Figure 2. Since the cycle eventually returns to column j^*, there must be a line i^- with $(i^-,j^*+1) \to (i^-,j^*)$. We know that $i^- < i^*$ then. Since the UPD cycle leads from (i^*,j^*+1) to (i^-,j^*+1), it contains a step $(i^*,j^+) \to (i^*-1,j^+)$ with $j^+ > j^*$. Then OC imply $(i^*,j') \to (i^*-1,j')$ for all $j' < j^+$, including all $j' \leq j^*$. But this implies that no UPD step

[5] It is easy to check that such a pair always exists.

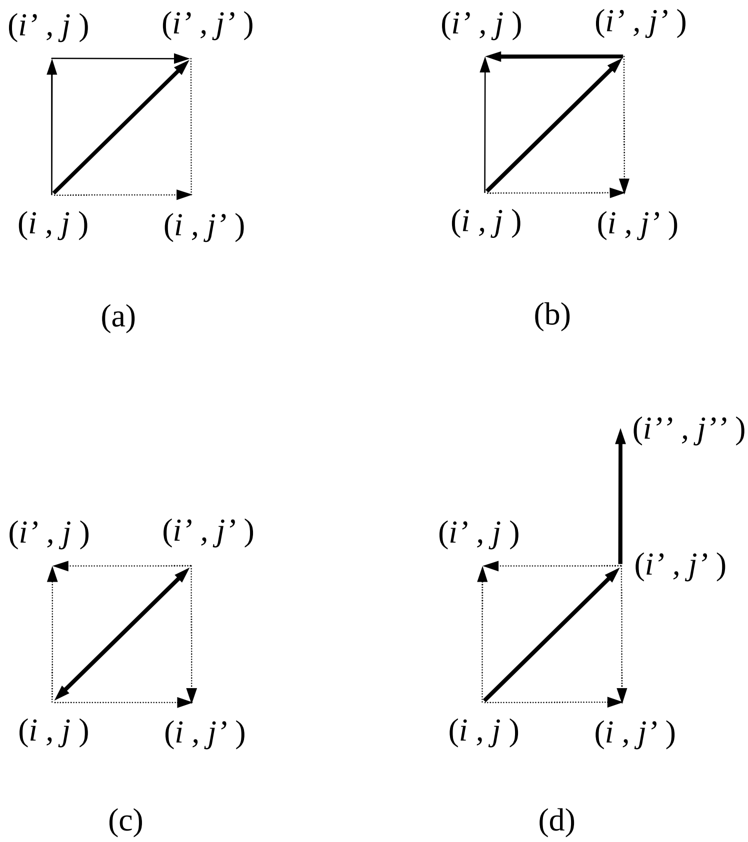

Fig. 1 (a): The diagonal PD step (bold arrow) can be “bypassed” by two UPD steps (thin arrows). (b): The vertical UPD step is a “shortcut” for the sequence of the two PD steps indicated by bold arrows. (c): The two diagonal PD steps lead back to (i, j). (d): Such a sequence of PD steps is impossible in games with DMR.

of length 1 can enter the region of pairs (i, j) with $i \geq i^*$ and $j \leq j^*$ (coloured grey in Figure 2). Hence no sequence of such steps can lead back from $(i^*, j^* + 1)$ to (i^*, j^*), contradicting the initial assumption. □

As a corollary of Lemmas 5, 6 and 7, we obtain our main result.

Theorem 1 *Every nondegenerate game with ordinal complementarities and diminishing marginal returns has the fictitious play property.*

Note that because of DMR, there are no equilibria where one of the players uses more than two pure strategies. Hence convergence is either to a pure strategy equilibrium or to an equilibrium in a 2×2 subgame. This subgame must be UPD-acyclic, and hence a coordination game. It can be shown that convergence to the mixed equilibrium is only possible

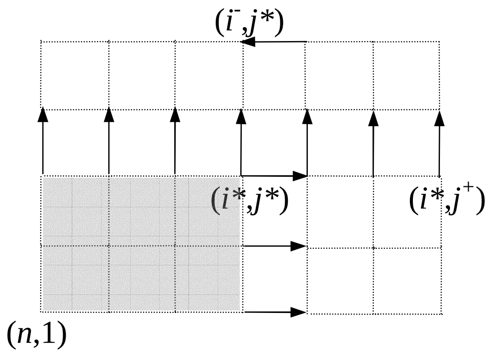

Fig. 2 The construction in the proof of Lemma 7. No sequence of length 1 UPD steps can return to the starting point (i^*, j^*).

then, if the subgame is non-generic in the sense that the mixed equilibrium is symmetric. Hence, if we rule out also this kind of degeneracy, we can establish convergence to pure strategy equilibria in NDGOCs with DMR.

There is also another application of Lemma 6. This lemma says that every nondegenerate, UPD-acyclic game with DMR has the FPP. As Monderer and Shapley (1996b) show, UPD-acyclic games are exactly those which have a generalized ordinal potential. Hence we obtain the following.

Theorem 2 *Every generalized ordinal potential game with diminishing marginal returns has the fictitious play property.*

Note that although GOCs and ordinal potential games share some important properties, the classes of games to which these two theorems refer, are different. Krishna (1992) has demonstrated this by combining an example of an ordinal potential game without OC with Sela's (1992) example of a GOC with DMR which has no ordinal potential.

5 Discussion

In this paper, we have put existing work on learning in games with strategic complementarities under scrutiny. We have established that, analogously to the critical tie-breaking assumption of Miyazawa (1961), the one of Krishna (1992) can be dropped if we restrict the analysis to nondegenerate games. While this might be regarded as a less important technicality, we also extended Krishna's result in the spirit of Milgrom and Shannon (1994), by proving that ordinal instead of strategic complementarities suffice. The critical property of fictitious play, which we use to establish this theorem, also allows us to obtain the result that generalized ordinal potential games with diminishing marginal returns have the FPP. The old conjecture if all (non-degenerate) games with strategic complementarities have the FPP, however, remains unsolved.

References

1. Aubin, J. and Cellina, A. (1984). *Differential inclusions.* Berlin, Springer.
2. Berger, U. (2003). Fictitious play in $2 \times n$ games. Mimeo, Vienna University of Economics.
3. Brown, G. W. (1951). Iterative solution of games by fictitious play. *In* "Activity Analysis of Production and Allocation" (T.C. Koopmans, Ed.), pp. 347-376, New York, Wiley.
4. Bulow, J., Geanakoplos, J., and Klemperer, P. (1985). Multimarket oligopoly: strategic substitutes and complements. *Journal of Political Economy* **93**, 488-511.
5. Cowan, S. (1992). *Dynamical systems arising from game theory.* Dissertation, UC Berkeley.
6. Cressman, R. (2003). *Evolutionary dynamics and extensive form games.* The MIT Press, Cambridge.
7. Foster, D. P. and Young, H. P. (1998). On the nonconvergence of fictitious play in coordination games. *Games and Economic Behavior* **25**, 79-96.
8. Garcia, A., Reaume, D., and Smith, R. L. (2000). Fictitious play for finding system optimal routings in dynamic traffic networks. *Transportation Research B* **34**, 146-157.
9. Gilboa, I. and Matsui, A. (1991). Social stability and equilibrium. *Econometrica* **59**, 859-867.
10. Hahn, S. (1999). The convergence of fictitious play in 3×3 games with strategic complementarities. *Economics Letters* **64**, 57-60.
11. Harris, C. (1998). On the rate of convergence of continuous-time fictitious play. *Games and Economic Behavior* **22**, 238-259.
12. Hofbauer, J. (1995). Stability for the best response dynamics. Mimeo, University of Vienna.
13. Hofbauer, J. and Sandholm, W. (2002). On the global convergence of stochastic fictitious play. *Econometrica* **70**, 2265-2294.
14. Hofbauer, J. and Sigmund, K. (2003). Evolutionary game dynamics. *Bulletin of the American Mathematical Society* **40**, 479-519.
15. Hofbauer, J. and Sorin, S. (2002). Best response dynamics for continuous zero-sum games. Mimeo, Ecole Polytechnique.
16. Krishna, V. (1992). Learning in games with strategic complementarities. Mimeo, Harvard University.
17. Krishna, V. and Sjöström, T. (1997). Learning in games: fictitious play dynamics. In: Hart, S. and Mas-Colell, A. (eds.). Cooperation: game-theoretic approaches. Springer Verlag: Berlin, Heidelberg.
18. Lambert, T. J., Epelman, M. A., and Smith, R. L. (2002). A fictitious play approach to large-scale optimization. Technical Report, University of Michigan.
19. Matsui, A. (1992). Best response dynamics and socially stable strategies. *Journal of Economic Theory* **57**, 343-362.

20. Milgrom, P. and Roberts, J. (1990). Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica* **58**, 1255-1277.
21. Milgrom, P. and Roberts, J. (1991). Adaptive and sophisticated learning in normal form games. *Games and Economic Behavior* **3**, 82-100.
22. Milgrom, P. and Shannon, C. (1994). Monotone comparative statics. *Econometrica* **62**, 157-180.
23. Miyazawa, K. (1961). On the convergence of the learning process in a 2×2 non-zero-sum two-person game. Econometric Research Program, Princeton University, Research Memorandum No. 33.
24. Monderer, D. and Sela, A. (1996). A 2×2 game without the fictitious play property. *Games and Economic Behavior* **14**, 144-148.
25. Monderer, D. and Sela, A. (1997). Fictitious play and no-cycling conditions. Mimeo, The Technion.
26. Monderer, D. and Shapley, L. S. (1996a). Fictitious play property for games with identical interests. *Journal of Economic Theory* **68**, 258-265.
27. Monderer, D. and Shapley, L. S. (1996b). Potential games. *Games and Economic Behavior* **14**, 124-143.
28. Robinson, J. (1951). An iterative method of solving a game. *Annals of Mathematics* **54**, 296-301.
29. Rosenmüller, J. (1971). Über Periodizitätseigenschaften spieltheoretischer Lernprozesse. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **17**, 259-308.
30. Sela, A. (1992). *Learning processes in games.* M.Sc. Thesis, The Technion, Haifa.
31. Sela, A. (2000). Fictitious play in 2×3 games. *Games and Economic Behavior* **31**, 152-162.
32. Shapley, L. S. (1964). Some topics in two-person games. *Annals of Mathematical Studies* **5**, 1-28.
33. Topkis, D. M. (1979). Equilibrium points in nonzero-sum n-person submodular games. *SIAM Journal of Control and Optimization* **17**, 773-787.
34. Vives, X. (1990). Nash equilibrium with strategic complementarities. *Journal of Mathematical Economics* **19**, 305-321.