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Two More Classes of Games with the Fictitious Play Property

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Abstract Fictitious play is the oldest and most studied learning process for games. Since the already classical result for zero-sum games, convergence of beliefs to the set of Nash equilibria has been established for some important classes of games, including weighted potential games, supermodular games with diminishing returns, and 3×3 supermodular games. Extending these results, we establish convergence for ordinal potential games and quasi-supermodular games with diminishing returns. As a by-product we obtain convergence for $3 \times m$ and 4×4 quasi-supermodular games.

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1 Introduction

The idea of *Fictitious Play* (FP) is over half a century old. It was originally introduced by Brown (1949, 1951) as an algorithm to calculate the value of a zero-sum game. In the realm of learning in games (Fudenberg and Levine, 1998), Fictitious Play serves as the prime example of myopic belief learning, and it has its place in almost any modern textbook on game theory. The new literature on models of bounded rationality and learning in interactive decision contexts has revived the study of Fictitious Play and its various smooth or stochastic offsprings. Apart from purely academic interest, this learning process has recently also proven useful as a simple optimization heuristic (Garcia et al., 2000, Lambert et al., 2002).

In a Fictitious Play process two players are engaged in the repeated play of a bimatrix game. Each player believes that her opponent plays a stationary mixed strategy. After an arbitrary initial move, in each round, she takes the empirical distribution of pure strategies played by the opponent as her belief and responds with a pure strategy that maximizes her expected payoff given this belief, i.e., with a *myopic best response*. We say that an FP process approaches equilibrium, if the sequence of beliefs converges to the set of Nash equilibria of the game. A game is said to have the *Fictitious Play property* (FPP), if every FP process approaches equilibrium in this game.

It is well known that not every game has the FPP. Shapley (1964) demonstrated this with an example of a 3×3 game, where the beliefs converge to a limit cycle. More recent studies of nonconvergent FP processes include Cowan (1992), Jordan (1993), Gaunersdorfer and Hofbauer (1995), Krishna and Sjöström (1998), and Foster and Young (1998). However, most of the research concerned with Fictitious Play tried to identify classes of games with the FPP. Often cited convergence results[1] are those for zero-sum games (Robinson, 1951), 2×2 games (Miyazawa, 1961), supermodular games with a unique equilibrium (Milgrom and Roberts, 1991), supermodular games with diminishing returns (Krishna, 1992), and weighted potential games (Monderer and Shapley, 1996a).[2]

The results of Miyazawa and Krishna are subject to a technical constraint: both use a particular tie-breaking rule. With Brown's original definition of FP, without the assumption of a tie-breaking rule, their results need not hold, at least for degenerate games. This has been demonstrated by Monderer and Sela (1996) with an example of a 2×2 supermodular game with diminishing returns, for which FP need not converge.[3] In their example, it is a degeneracy of the game which permits nonconvergent FP processes. Subsequently, Monderer and Shapley (1996a) showed, that without assuming a tie-breaking rule, one must restrict the analysis to nondegenerate games in order to save Miyazawa's result.[4] Specifically, they proved that every 2×2 game satisfying a particular nondegeneracy assumption has the FPP. This result was later extended to 2×3 games by Sela (2000), and finally to $2\times n$ games by Berger (2003a).

Referring to supermodular games with diminishing returns, however, Monderer and Sela (1996, p. 145) state that they *"do not know whether such*

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[1] See Krishna and Sjöström (1997) for a short survey. Beyond these well known results, convergence has also been shown for symmetric games with an interior ESS (Hofbauer, 1995), certain 'one-against-all' games (Sela, 1999), and special classes of 'cyclic' games (Berger, 2001, 2002).

[2] There are also similar results for other, closely related variants of FP, e.g. Fudenberg and Kreps' (1993) *stochastic Fictitious Play*, see Benaim and Hirsch (1999) and Hofbauer and Sandholm (2002), or the *smooth best response dynamics*, see Hopkins (1999). Our work is not concerned with these variants.

[3] For a helpful visualization of this example see Cressman (2003, p. 84).

[4] See also Metrick and Polak (1996) for an analysis of FP in 2×2 games.

a generic result holds for Krishna's games as well". Hence the question if, without using a tie-breaking rule, a nondegeneracy condition similar to the one of Monderer and Shapley (1996a) can save Krishna's result, remained open.

It has also been conjectured that the assumption of diminishing returns is not really necessary for supermodular games to have the FPP. A small step in this direction was done by Hahn (1999), who, using Krishna's tie-breaking rule, proved that 3×3 supermodular games have the FPP.

Milgrom and Shannon (1994) showed that many of the known results for supermodular games can already be derived under the weaker condition of *quasi-supermodularity.* Among these results is the one of Milgrom and Roberts (1991). Hence FP also converges in quasi-supermodular games with a unique equilibrium. In his paper, Krishna (1992) also raised the question, if quasi-supermodularity could already suffice for his result. However, as he demonstrates, his proof does not extend to this larger class of games. He gives no counterexample either, and leaves this question unanswered.

Monderer and Shapley (1996a, 1996b) define the class of ordinal potential games, which contains the class of weighted potential games.[5] For the latter, they prove convergence of FP to the equilibrium set. The relation between ordinal potential games and weighted potential games is somehow analogous to the relation between quasi-supermodular games and supermodular games, since the former are ordinal concepts while the latter are cardinal concepts. In the light of Milgrom and Shannon's (1994) results, the conjecture that the FPP carries over from weighted potential games to (nondegenerate) ordinal potential games suggests itself. However, it has also been explicitly doubted, e.g. by Monderer and Sela (1997).

The present paper extends the described results of Krishna (1992), Hahn (1999), and Monderer and Shapley (1996a). For supermodular games with diminishing returns, we first clarify the question of Monderer and Sela (1996) insofar, as we prove that without assuming a tie-breaking rule, Krishna's result indeed continues to hold for nondegenerate games. Second, and more importantly, we show that even all nondegenerate *quasi*-supermodular games with diminishing returns have the FPP. Concerning potential games, we prove that also nondegenerate games with an *ordinal* potential have the FPP. As a by-product, we finally obtain convergence of FP in nondegenerate $3\times m$ and 4×4 quasi-supermodular games.

The remainder of this paper is structured as follows. In Section 2 we introduce the notation and terminology we use, and define (quasi-) supermodular games, diminishing returns, nondegeneracy, ordinal potential games, Fictitious Play, and games with the pure Nash equilibrium property. Section 3 contains two important properties of Fictitious Play. In Section 4 we derive the first main result on ordinal potential games. Section 5 is

[5] Weighted potential games include Rosenthal's (1973) *congestion games* and have previously been called *rescaled partnership games* by Hofbauer and Sigmund (1988).

concerned with quasi-supermodular games with diminishing returns, and contains the second main result. In Section 6 we prove convergence for $3\times m$ and 4×4 quasi-supermodular games, and Section 7 concludes.

2 Notation and Definitions

2.1 Bimatrix Games and Best Responses

Let (A, B) be a bimatrix game where player 1, the row player, has pure strategies $i \in N = \{1, 2, \ldots, n\}$, and player 2, the column player, has pure strategies $j \in M = \{1, 2, \ldots, m\}$. A and B are the $n\times m$ payoff matrices for players 1 and 2. Thus, if player 1 chooses $i \in N$ and player 2 chooses $j \in M$, the payoffs to players 1 and 2 are a_{ij} and b_{ij}, respectively. The set of mixed strategies of player 1 is the $n-1$ dimensional probability simplex S_n, and analogously S_m is the set of mixed strategies of player 2. With a little abuse of notation we will not distinguish between a pure strategy i of player 1 and the corresponding mixed strategy representation as the i-th unit vector $\mathbf{e}_i \in S_n$. Analogously we identify player 2's pure strategy j with the j-th unit vector $\mathbf{f}_j \in S_m$. Sometimes we will also speak of the players choosing a row, or column, respectively, of the bimatrix.

The expected payoff for player 1 playing strategy i against player 2's mixed strategy $\mathbf{y} = (y_1, \ldots, y_m)^t \in S_m$ (where the superscript t denotes the transpose of a vector or matrix) is $(A\mathbf{y})_i$. Analogously $(B^t\mathbf{x})_j$ is the expected payoff for player 2 playing strategy j against the mixed strategy $\mathbf{x} = (x_1, \ldots, x_n)^t \in S_n$. If both players use mixed strategies $\mathbf{x}$ and $\mathbf{y}$, respectively, the expected payoffs are $\mathbf{x}\cdot A\mathbf{y}$ to player 1 and $\mathbf{y}\cdot B^t\mathbf{x}$ to player 2, where the dot denotes the scalar product of two vectors. We denote by $BR_2(\mathbf{x})$ player 2's pure strategy best response correspondence, and by $br_2(\mathbf{x})$ her mixed strategy best response correspondence. Analogously, $BR_1(\mathbf{y})$ and $br_1(\mathbf{y})$ are the sets of player 1's pure and mixed best responses, respectively, to $\mathbf{y} \in S_m$. Let $BR(\mathbf{x}, \mathbf{y}) = BR_1(\mathbf{y})\times BR_2(\mathbf{x})$ and $br(\mathbf{x}, \mathbf{y}) = br_1(\mathbf{y})\times br_2(\mathbf{x})$. We say that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a best response to $(\mathbf{x}, \mathbf{y}) \in S_n\times S_m$, if $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in br(\mathbf{x}, \mathbf{y})$. Also, we call $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ a pure best response to $(\mathbf{x}, \mathbf{y})$, if $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in BR(\mathbf{x}, \mathbf{y})$. A strategy profile $(\mathbf{x}^*, \mathbf{y}^*)$ is a Nash equilibrium if and only if $(\mathbf{x}^*, \mathbf{y}^*) \in br(\mathbf{x}^*, \mathbf{y}^*)$. It is called a *pure* Nash equilibrium, if $(\mathbf{x}^*, \mathbf{y}^*) \in BR(\mathbf{x}^*, \mathbf{y}^*)$.

2.2 Quasi-Supermodular Games and Supermodular Games

Definition 1 *(i) A bimatrix game (A, B) is* quasi-supermodular*, if for all $i < i'$ and $j < j'$:*

$$a_{i'j} > a_{ij} \Longrightarrow a_{i'j'} > a_{ij'} \quad \text{and} \quad b_{ij'} > b_{ij} \Longrightarrow b_{i'j'} > b_{i'j}.$$

(ii) A bimatrix game (A, B) is supermodular*, if for all $i < i'$ and $j < j'$:*

$$a_{i'j'} - a_{ij'} > a_{i'j} - a_{ij} \quad \text{and} \quad b_{i'j'} - b_{i'j} > b_{ij'} - b_{ij}.$$

We write QSMG short for *quasi-supermodular game*, and SMG for *supermodular game*. In a QSMG, the payoff difference between two payoffs in a column of A or a line of B can change its sign at most once, and only from -1 to $+1$, if the players move up to a higher column or line, respectively. In a broader context, these games have been studied by Milgrom and Shannon (1994). From Definition 1, quasi-supermodularity is implied by supermodularity. In an SMG, the advantage of switching to a higher strategy increases when the opponent chooses a higher strategy.

Both QSMGs and SMGs have frequently been called games with *strategic complementarities*. We refrain from using this term to avoid confusion. Originally, the term "strategic complementarities" was coined by Bulow et al. (1985) to denote games with increasing best response correspondences. This property is already implied by quasi-supermodularity. SMGs have been introduced (in a much more general framework) by Topkis (1979) and studied by Vives (1990) and Milgrom and Roberts (1990). This class of games has important applications in economics, e.g. in models of oligopolistic competition, R&D competition, macroeconomic coordination, bank runs, network externalities, and many others.

2.3 Games with Diminishing Returns

Another condition we use is *diminishing returns*, sometimes also referred to as *diminishing marginal returns*. As the name suggests, this property means that the payoff advantage of increasing one's strategy is decreasing.

Definition 2 *A bimatrix game* (A, B) *has* diminishing returns *(DR), if for all* $i = 2, \ldots, n-1$ *and* $j \in M$*,*

$$a_{i+1,j} - a_{ij} < a_{ij} - a_{i-1,j},$$

and for all $i \in N$ *and* $j = 2, \ldots, m-1$*,*

$$b_{i,j+1} - b_{ij} < b_{ij} - b_{i,j-1}.$$

Krishna (1992) observed that DR restrict the best response correspondence in the following way.

Lemma 1 *Let* (A, B) *be a game with DR. Then for any* $(\mathbf{x}, \mathbf{y}) \in S_n \times S_m$*, the sets* $BR_1(\mathbf{y})$ *and* $BR_2(\mathbf{x})$ *contain at most two strategies. If one of these sets contains two strategies, they are numbered consecutively.*

2.4 Nondegenerate Games

As mentioned in the introduction, without assuming a tie-breaking rule, one must impose a nondegeneracy assumption in order to keep the FPP, even in the class of 2×2 games. We work with games which are nondegenerate in the following specific sense.

Definition 3 *We call a bimatrix game* (A,B) degenerate*, if for some* $i,i' \in N$*, with* $i \neq i'$*, there exists* $j \in M$ *with* $a_{i'j} = a_{ij}$*, or if for some* $j,j' \in M$*, with* $j \neq j'$*, there exists* $i \in N$ *with* $b_{ij'} = b_{ij}$*. Otherwise, the game is said to be* nondegenerate*.*

We write ND((Q)SM)G short for *nondegenerate ((quasi-)supermodular) game.*

2.5 Ordinal Potential Games

Monderer and Shapley (1996b) define several classes of games with a so-called *potential*. The class of *ordinal potential games* contains the class of *weighted potential games*, for which Monderer and Shapley (1996a) proved the FPP.

Definition 4 *We say that a bimatrix game* (A,B) *is an* ordinal potential game *(OPG), if there exists an ordinal potential function, i.e. a function* $F: N \times M \to \mathbb{R}$*, such that for all* $i,i' \in N$ *and* $j,j' \in M$*,*

$$a_{i'j} > a_{ij} \quad \Leftrightarrow \quad F(i',j) > F(i,j),$$

and

$$b_{ij'} > b_{ij} \quad \Leftrightarrow \quad F(i,j') > F(i,j).$$

They also define *improvement paths* and games with the *finite improvement property*. We extend this definition slightly by defining *improvement steps*.

Definition 5 *For a bimatrix game* (A,B)*, define the following binary relation on* $N \times M$*:*
$(i,j) \to (i',j') \Leftrightarrow (i = i'$ *and* $b_{ij'} > b_{ij})$ *or* $(j = j'$ *and* $a_{i'j} > a_{ij})$*.*
(i) If $(i,j) \to (i',j')$*, we say that this is an* improvement step*. We denote by* $|i'-i| + |j'-j|$ *the* length *of the improvement step.*
(ii) An improvement path *is a (finite or infinite) sequence of improvement steps* $(i_1,j_1) \to (i_2,j_2) \to (i_3,j_3) \to \cdots$ *in* $N \times M$*.*
(iii) An improvement path $(i_1,j_1) \to \cdots \to (i_k,j_k)$ *is called an* improvement cycle*, if* $(i_k,j_k) = (i_1,j_1)$*.*
(iv) A bimatrix game is said to have the finite improvement property *(FIP), if every improvement path is finite, i.e., if there are no improvement cycles.*

It is clear that every NDG with an ordinal potential has the FIP. Monderer and Shapley (1996b) show that also the opposite direction holds.

Lemma 2 *A nondegenerate bimatrix game has the FIP if and only if it is an ordinal potential game.*

We write NDOPG short for *nondegenerate ordinal potential game.*

2.6 Fictitious Play

Definition 6 *For a bimatrix game (A, B), the sequence*

$$(\mathbf{p}(t))_{t\in\mathbb{N}} := (\mathbf{x}(t), \mathbf{y}(t))_{t\in\mathbb{N}} ,$$

called the sequence of beliefs*, is a* discrete-time Fictitious Play process *(DFP process), if* $\mathbf{p}(1) \in S_n \times S_m$*, and for all* $t \in \mathbb{N}$*,*

$$\mathbf{p}(t+1) \in \frac{1}{t+1}[t\mathbf{p}(t) + BR(\mathbf{p}(t))]. \tag{1}$$

The pure best response $(t+1)\mathbf{p}(t+1) - t\mathbf{p}(t)$ played at time t is denoted by $\mathbf{B}(t)$. The sequence $(\mathbf{B}(t))_{t\in\mathbb{N}}$ is called the sequence of actual play *of this process. The Euclidian distance $||\mathbf{p}(t) - \mathbf{p}(t+1)||$ between $\mathbf{p}(t)$ and $\mathbf{p}(t+1)$ in $\mathbb{R}^{n+m}$ is called the* step size *of the process at time t.*

The step size of any DFP converges to zero as time goes to infinity. Continuous-time Fictitious Play could roughly be described as the limiting "zero step size" version of DFP.

Definition 7 *For a bimatrix game (A, B), the path*

$$(\mathbf{p}(t))_{t\in\mathbb{R},t\geq 1} := (\mathbf{x}(t), \mathbf{y}(t))_{t\in\mathbb{R},t\geq 1} ,$$

called the belief-path*, is a* continuous-time Fictitious Play process *(CFP process), if* $\mathbf{p}(1) \in S_n \times S_m$*, and for almost all* $t \geq 1$*,*

$$\dot{\mathbf{p}}(t) \in \frac{1}{t}[BR(\mathbf{p}(t)) - \mathbf{p}(t)]. \tag{2}$$

The pure best response $\mathbf{p}(t) + t\dot{\mathbf{p}}(t)$ played at time t is denoted by $\mathbf{B}(t)$. The path $t \mapsto \mathbf{B}(t)$ for $t \geq 1$ is called the path of actual play *of this process.*

If actual play converges, it must be constant from some stage on, implying that the process converges to the respective pure Nash equilibrium. Even if actual play does not converge, it is easily established that if the FP process does, then the limit must be a Nash equilibrium. As noted above, however, there are games where FP need not converge.

In general an FP process is not defined uniquely by its initial value $\mathbf{p}(1)$. Indeed, if at some point in time the best response set is multivalued, there may be several possible continuations of a process. To handle this multiplicity of solutions, particular *tie-breaking rules* have sometimes been assumed. Krishna (1992) e.g. assumed that both players, whenever indifferent between two or more pure strategies, choose the strategy with the highest number. However, any tie-breaking rule is somehow artificial, and should therefore be avoided, as long as there is no (external) reason why some pure best responses should not be chosen in a Fictitious Play.

Both CFP and DFP have been introduced by Brown (1949). Which version to prefer is mainly a matter of taste. In recent studies of the FP process, CFP is often preferred to DFP, because it is usually much easier to work with. For example, Robinson's (1951) proof for DFP is several pages long, while the proof of the same result[6] for CFP, based on a Ljapunov function, requires only a few lines. In this paper we therefore stick to the continuous-time version of FP as well.[7]

CFP is closely related to the *best response dynamics*,

$$\mathbf{p}(1) \in S_n \times S_m, \quad \dot{\mathbf{p}}(t) \in br(\mathbf{p}(t)) - \mathbf{p}(t) \quad \text{for almost all } t \geq 1, \qquad (3)$$

a learning process for population games, introduced by Gilboa and Matsui (1991) and Matsui (1992). These continuous-time processes have also been studied by Rosenmüller (1971), Monderer et al. (1997), and Harris (1998), they are surveyed by Krishna and Sjöström (1997) and Hofbauer and Sigmund (2003). Hofbauer (1995) proved that solutions, possibly multiple ones, exist, and are essentially piecewise linear. In all known cases, the behavior of DFP is very similar to that of CFP. Only a few formal results on the relation between these two versions are available, however, see e.g. Hofbauer and Sorin (2002).

In the definition of CFP, (2) is a *differential inclusion* (see Aubin and Cellina, 1984). The definition of CFP does not guarantee existence of solutions from all initial values, because the pure best response correspondence $BR(\mathbf{p})$ admits nonconvex values in general. For instance, starting with initial beliefs arbitrarily close to the completely mixed equilibrium in a Matching Pennies game, we get unique solutions of CFP (which can be shown to spiral inwards to the equilibrium), but starting directly in the equilibrium, the CFP path cannot be continued with players using only pure strategies. To get existence of solutions for all initial values, we would have to allow for mixed best responses in the definition of CFP, as this is done for the best response dynamics. As Hofbauer (1995) argues, this is meaningful, and even essential, for the latter. However, it would be against the spirit of CFP as a learning process, since players cannot observe their opponent's mixed strategies.

The analysis of Hofbauer (1995) shows that piecewise linear solutions of CFP exist through any initial value, and can be continued for all positive times, in every game with the *pure Nash equilibrium property*. To explain this property, let $N' \subset N$ and $M' \subset M$ be nonempty subsets of the players' pure strategy sets. We call the restriction of the original game to these pure strategy subsets a *bimatrix-subgame*.[8] The following definition is from Takahashi and Yamamori (2002).

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[6] See Cowan (1992), or Hofbauer (1995). A sketch of this proof already appears in Brown (1949).

[7] A partial result for DFP is worked out in Berger (2003b).

[8] Takahashi and Yamamori (2002) simply call it a *subgame*. To avoid confusion with the notion of a subgame in the context of extensive form games, we refrain from using this term.

Definition 8 *A game has the* pure Nash equilibrium property *(PNEP), if every bimatrix-subgame has a pure Nash equilibrium.*

Existence of piecewise linear solutions through all initial values in games with the PNEP follows, because for every $\mathbf{p}(1) \in S_n \times S_m$, actual play of the process can be continued with a pure Nash equilibrium of the bimatrix-subgame with $N' \times M' = BR(\mathbf{p}(t))$.

Ordinal potential games, quasi-supermodular games, and dominance-solvable games have pure Nash equilibria. Since for games in these classes, any bimatrix-subgame is again a game of the same class, these games have the PNEP. Hence, for such games, CFP paths exist through any initial value. These paths follow straight lines in the simplex product as long as actual play is constant. Whenever one of the players switches to another pure best response, the path changes its direction discontinuously. This can only happen when beliefs cross the so-called *indifference hyperplanes*, the sets of mixed strategy profiles where one of the players is indifferent between several best responses. A formal definition of switching follows.

Definition 9 *We say that a CFP process* $\mathbf{p}(t)$ switches *from* (i, j) *to* (i', j') *at time* t_1*, if* $(i', j') \neq (i, j)$ *and there exists* $\epsilon > 0$ *with* $\mathbf{B}(t) = (i, j)$ *for* $t \in [t_1 - \epsilon, t_1[$ *and* $\mathbf{B}(t) = (i', j')$ *for* $t \in]t_1, t_1 + \epsilon]$.

3 Two Improvement Principles for Fictitious Play

Whenever switching occurs along an FP process, at least one of the players changes her strategy. The next lemma shows that the 'new' strategy of this player must be a weakly *better* response than her 'old' strategy against the 'old' strategy of the opponent. This was called the *Improvement Principle* by Monderer and Sela (1997), see also Sela (2000). To keep the analysis self-contained, we repeat the proof here.

Lemma 3 *If an FP process for the bimatrix game* (A, B) *switches from* (i, j) *to* (i', j')*, then*

$$a_{i'j} \geq a_{ij} \quad \text{and} \quad b_{ij'} \geq b_{ij}.$$

Proof Let t_1 be the time where the process switches from (i, j) to (i', j'). Then both players are indifferent between their respective best responses, $\{(i, j), (i', j')\} \subset BR(\mathbf{p}(t_1))$. By Definition 9 there exists $\epsilon > 0$ with $\mathbf{B}(t) = (i, j)$ for $t \in [t_0, t_1[$, where $t_0 = t_1 - \epsilon$. Hence $\mathbf{p}(t_1)$ is a convex combination of $\mathbf{p}(t_0)$ and (i, j), and we can write $(\mathbf{e}_i, \mathbf{f}_j) = c\mathbf{p}(t_1) + (1 - c)\mathbf{p}(t_0)$ for some $c \geq 1$. Left-multiplying the second component vector with the payoff matrix A yields $A\mathbf{f}_j = cA\mathbf{y}(t_1) + (1 - c)A\mathbf{y}(t_0)$. Subtracting the i-th line of this vector equation from the i'-th line gives

$$\begin{aligned} &a_{i'j} - a_{ij} = \\ &= c[(A\mathbf{y}(t_1))_{i'} - (A\mathbf{y}(t_1))_i] + (1 - c)[(A\mathbf{y}(t_0))_{i'} - (A\mathbf{y}(t_0))_i]. \end{aligned}$$

The first term on the right-hand side of this equation is zero, and the second term is nonnegative, since $i \in BR_1(\mathbf{y}(t_0))$ and $c \geq 1$. Hence $a_{i'j} \geq a_{ij}$. By the same reasoning we get $b_{ij'} \geq b_{ij}$. □

The following complement of Lemma 3 is an essential requirement of our proofs. We call it the *Second Improvement Priciple*.

Lemma 4 *If a CFP process for the bimatrix game (A, B) switches from (i, j) to (i', j'), then*

$$a_{i'j'} \geq a_{ij'} \quad \textit{and} \quad b_{i'j'} \geq b_{i'j}.$$

Proof Again let t_1 be the time where the process switches from (i, j) to (i', j'), then $\{(i, j), (i', j')\} \subset BR(\mathbf{p}(t_1))$. By Definition 9 there exists $\epsilon > 0$ with $\mathbf{B}(t) = (i', j')$ for $t \in]t_1, t_2]$, where $t_2 = t_1 + \epsilon$. Hence $\mathbf{p}(t_2)$ is a convex combination of $\mathbf{p}(t_1)$ and (i', j'), and we can write $(\mathbf{e}_{i'}, \mathbf{f}_{j'}) = c\mathbf{p}(t_2) + (1 - c)\mathbf{p}(t_1)$ for some $c \geq 1$. Left-multiplying the second component vector with the payoff matrix A yields $A\mathbf{f}_{j'} = cA\mathbf{y}(t_2) + (1 - c)A\mathbf{y}(t_1)$. Subtracting the i-th line of this vector equation from the i'-th line gives

$$\begin{aligned} &a_{i'j'} - a_{ij'} = \\ &= c[(A\mathbf{y}(t_2))_{i'} - (A\mathbf{y}(t_2))_i] + (1 - c)[(A\mathbf{y}(t_1))_{i'} - (A\mathbf{y}(t_1))_i]. \end{aligned}$$

The second term on the right-hand side of this equation is zero, and the first term is nonnegative, since $i' \in BR_1(\mathbf{y}(t_2))$. Hence $a_{i'j'} \geq a_{ij'}$. By the same reasoning we get $b_{i'j'} \geq b_{i'j}$. □

If only one of the players switches her strategy at time t_1, i.e. if $i = i'$ or $j = j'$, then in both of these two Lemmas, one of the inequalities is trivially true, while the other inequalities are then identical, and simply state that $(i, j) \to (i', j')$. Only if both players switch simultaneously, Lemma 4 comes into use. If, moreover, the game is nondegenerate, then the strict inequalities hold. In this case we know that $(i, j) \to (i', j) \to (i', j')$, and also $(i, j) \to (i, j') \to (i', j')$.

4 Fictitious Play in Ordinal Potential Games

By the last remark, if an FP process switches from (i, j) to (i', j') in a nondegenerate game, then these two pure strategy pairs are connected by an improvement path, even if both players switch simultaneously.

Lemma 5 *If an FP process for the NDG (A, B) switches from (i, j) to (i', j'), then there is an improvement path from (i, j) to (i', j').*

This is essentially all we need for the first main result.

Theorem 1 *Let (A, B) be an NDOPG. Then it has the FPP.*

Proof Assume there is an NDOPG without the FPP. In a nonconvergent FP process there are infinitely many switches. Since there are only finitely many pure strategy pairs, however, at least two such pairs are played infinitely often. Hence there must be a sequence of switches leading from one of these pairs to the other and back again. By Lemma 5, this means that there is an improvement cycle. By Lemma 2 then, the game cannot have an ordinal potential, which contradicts the assumption. □

It should be noted, that for a nonconvergent FP process in an NDG, actual play need not follow a single improvement cycle over and over. In case of nonconvergence, an improvement cycle must exist, but if there is a second such cycle intersecting the first one, actual play can in principle jump back and forth between these two improvement cycles in an irregular and aperiodic fashion, while the belief-path approaches a chaotic attractor.[9]

It is also worth mentioning that for the best response dynamics (3), there might be piecewise linear paths pointing to a *completely mixed* Nash equilibrium of a bimatrix-subgame. For such paths, however, the two Improvement Principles can no longer be applied to show that nonconvergence implies the existence of an improvement cycle. Hence the proof does not carry over to this dynamics, and we cannot rule out the existence of nonconvergent paths. However, if they do exist, they are unstable, due to the PNEP.

5 Fictitious Play in Quasi-Supermodular Games with Diminishing Returns

If an FP process switches from (i, j) to (i', j'), then at the switching point $\mathbf{p}(t_1)$, player 1 is indifferent between i and i', and player 2 is indifferent between j and j'. From Lemma 1, an immediate consequence of this is, that in games with DR, FP can only switch to neighboring strategies.

Lemma 6 *Let (A, B) be a game with DR. If an FP process switches from (i, j) to (i', j'), then $|i' - i| \leq 1$ and $|j' - j| \leq 1$.*

As a consequence, in an NDG with DR, actual play of an FP process generates an improvement path where every improvement step has length 1. Lemma 5 then implies the following.

Lemma 7 *Let (A, B) be an NDG with DR. If an FP process does not converge, then there exists an improvement cycle consisting of improvement steps of length 1.*

However, the next result states that in NDGs, quasi-supermodularity prevents the existence of such improvement cycles.[10]

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[9] Cowan (1992) constructed a 3×3 game in which FP shows chaotic behavior.

[10] Note that we do not assert that these games have the FIP. Indeed, an NDQSMG need not have an ordinal potential, as shown below, at the end of Section 6.

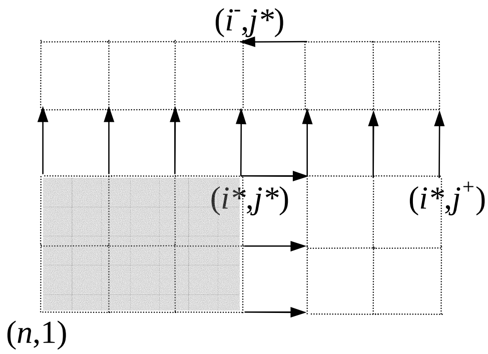

Fig. 1 The construction in the proof of Lemma 8. No sequence of length-1-improvement steps can return to the starting point (i^*, j^*).

Lemma 8 *In an NDQSMG, every improvement path consisting of length-1-steps is finite.*

Proof Let (A, B) be an NDQSMG. Assume the game admits an improvement cycle consisting of length-1-steps. Take any pair (i^*, j^*) in this cycle, where the next step is to $(i^*, j^* + 1)$ — it is easy to see that such a pair always exists. By quasi-supermodularity, we have $(i', j^*) \to (i', j^* + 1)$ for all $i' \geq i^*$, see Figure 1. Since the cycle eventually returns to column j^*, there must be a line i^- with $(i^-, j^* + 1) \to (i^-, j^*)$. We know that $i^- < i^*$ then. Since the improvement cycle leads from $(i^*, j^* + 1)$ to $(i^-, j^* + 1)$, it contains a step $(i^*, j^+) \to (i^* - 1, j^+)$ with $j^+ > j^*$. Then quasi-supermodularity implies $(i^*, j') \to (i^* - 1, j')$ for all $j' < j^+$, including all $j' \leq j^*$. But this implies that no improvement step of length 1 can enter the region of pairs (i, j) with $i \geq i^*$ and $j \leq j^*$ (coloured grey in Figure 1). Hence no sequence of such improvement steps can lead back from $(i^*, j^* + 1)$ to (i^*, j^*), contradicting the initial assumption. □

As a corollary of Lemmas 7 and 8, we obtain our second main result.[11]

Theorem 2 *Every NDQSMG with DR has the FPP.*

6 Fictitious Play in $3 \times m$ and 4×4 Quasi-Supermodular Games

It has been conjectured (e.g. Krishna and Sjöström, 1997) that in (non-degenerate) SMGs, the condition of DR is not really necessary for FP to converge.[12] Hahn (1999) has shown that this is true at least for 3×3 games, if one uses the same tie-breaking rule as Krishna (1992). While we cannot

[11] It can be shown that this result also holds for the best response dynamics.

[12] In the light of Hofbauer and Sandholm's (2002) recent results it may be necessary, however, to weaken this conjecture to only *almost* all initial conditions.

prove the conjecture, we can extend Hahn's result to $3\times m$ and 4×4 NDQSMGs. Basically, these games have the FPP because they have the FIP, and hence an ordinal potential.

Theorem 3 *Every* $3\times m$ *and* 4×4 *NDQSMG has the FPP.*

We prove only the $3\times m$ case. The proof of the 4×4 case is analogous. Like the proof below, it basically consists in going through all possible order relations between payoffs consistent with quasi-supermodularity, and checking the nonexistence of improvement cycles for each case. This is straightforward but tedious, and is therefore omitted here.

Proof Let (A, B) be a $3\times m$ NDQSMG, and assume w.l.o.g. that there are no dominated strategies. Then $(1, 1)$ and $(3, m)$ are equilibria, and $(3, 1) \to (2, 1) \to (1, 1)$ and $(1, m) \to (2, m) \to (3, m)$. Also, $(1, k) \to (1, k-1)$ for all $k = 2, \ldots, m$, and $(3, k) \to (3, k+1)$ for all $k = 1, \ldots, m-1$. Assume there is an improvement cycle. Now suppose there is a step from $(3, j'')$ to $(1, j'')$ in the cycle. Then by quasi-supermodularity, $(3, j) \to (1, j)$ for $j = 1, \ldots, j''$. In the next step, the cycle must move 'left' or 'down' in Figure 2. Since the cycle cannot reach the equilibrium $(1, 1)$, there must eventually be a step down from, say, $(1, j')$. This step cannot be to line 3, since $(3, j') \to (1, j')$. Hence it is $(1, j') \to (2, j')$, as in Case a) of Figure 2. Then by quasi-supermodularity, $(1, j'') \to (2, j'')$. This implies $(3, j'') \to (2, j'')$. Hence $(3, j) \to (2, j)$ for $j = 1, \ldots, j''$. But this means that if we continue the improvement path *backwards* from $(3, j'')$, we move to the left and can never leave line 3, ending up in $(3, 1)$. This is a contradiction, since $(3, 1)$ cannot be part of an improvement cycle.

If, on the other hand, there is a step $(2, j'') \to (1, j'')$ in the cycle, then we also have $(2, j') \to (1, j')$, and the improvement step leaving line 1 must be $(1, j') \to (3, j')$, see Case b) of Figure 2. This implies $(2, j') \to (3, j')$. Hence $(1, j) \to (3, j)$ and $(2, j) \to (3, j)$ for $j = j', \ldots, m$. But then the improvement path moves to the right from $(3, j')$ and can never leave line 3, ending up in the equilibrium $(3, m)$. This is again a contradiction, since $(3, m)$ cannot be part of an improvement cycle. This means the game has the FIP, and hence an ordinal potential. By Theorem 1 then, it has the FPP. □

This method of proof does not extend to NDQSMGs with 'more' strategies, however. The following is an example of a 4×5 NDQSMG with the improvement cycle $(1, 4) \to (1, 3) \to (2, 3) \to (2, 1) \to (2, 5) \to (3, 5) \to (3, 2) \to (4, 2) \to (4, 4) \to (1, 4)$.

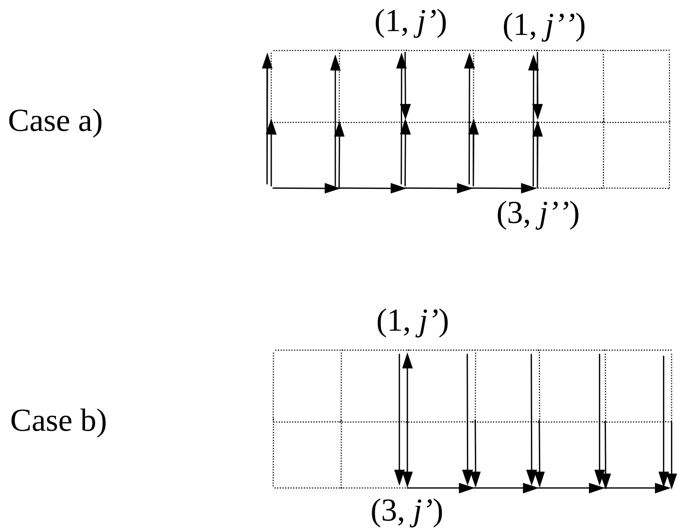

Fig. 2 The construction in the proof of Theorem 3.

$$(A,B) = \begin{bmatrix} 3,4 & 3,3 & 2,2 & 2,1 & 0,0 \\ 2,2 & 2,4 & 3,1 & 3,0 & 1,3 \\ 1,2 & 0,4 & 0,1 & 0,0 & 2,3 \\ 0,0 & 1,1 & 1,2 & 1,3 & 3,4 \end{bmatrix}.$$

7 Discussion

In this paper, we have combined the Improvement Principle of Monderer and Sela (1997) with what we called the Second Improvement Principle to show that actual play of a CFP process essentially follows an improvement path. It is then an easy consequence that in NDOPGs, since these games have the FIP, every FP process converges. The observation that improvement steps in an FP process in games with DR have length 1, allowed us to prove the FPP for NDQSMGs with DR. Finally, $3 \times m$ and 4×4 NDQSMGs could be shown to have the FIP, and hence the FPP. These three main theorems extend the respective results of Monderer and Shapley (1996a), Krishna (1992), and Hahn (1999).

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