Shedding light on El Farol

Damien Challet
*Theoretical Physics, Oxford University, 1 Keble Road, Oxford OX1 3NP, United Kingdom**

M. Marsili
International Center for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy†

Gabriele Ottino
Département de Physique, Université de Fribourg, Pérolles, 1700 Fribourg, Switzerland
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We mathematize El Farol bar problem and transform it into a workable model. In general, the average convergence to optimality at the collective level is trivial and does not even require any intelligence on the side of agents. Secondly, specializing to a particular ensemble of continuous strategies yields a model similar to the Minority Game. Statistical physics of disordered systems allows us to derive a complete understanding of the complex behavior of this model, on the basis of its phase diagram.

Keywords: El Farol, Minority Game, resource level, biased strategies

INTRODUCTION

Statistical mechanics has developed powerful tools to tackle analytically disordered systems with many degrees of freedom. These tools were recently shown to be applicable to systems of inductive heterogeneous agents such as the Minority Game [1–4]. Even if the latter is inspired by El Farol's bar problem [5], the literature on these two models are rather separate. In particular what the MG has brought to the understanding of the El Farol problem as defined by Arthur is not clear. Here we show that all results known about the MG directly apply to the El Farol problem.

In the El Farol's bar problem, $N = 100$ customers have to decide independently whether to go or not to the bar, which has a capacity of $L = 60$ seats, the resource level. Attending when the bar is crowded is not enjoyable. Customers are inductive rational agents. They use simple predictor rules, based on the past attendances, to predict whether the bar will be crowded or not, and behave accordingly.

One important issue is whether the customers, that do not communicate with each other, are able to synchronize their actions so that the attendance A is on average equal to the resource level L. The main result of Arthur is that agents need not be endowed by a sophisticated deductive rationality in order to synchronize. Even inductively rational agents can do. This results is probably responsible for the large interest his model aroused [6]. Here we show that even inductive rationality is not necessary because even zero-intelligence agents, acting as simple automaton, are able to self-organize to the comfort level. The convergence of the average attendance to the comfort level L is trivial under very generic and reasonable conditions. The really non-trivial question is whether agents are able or not to reduce stochastic fluctuations of the attendance A around L. The Minority Game was introduced [1] exactly to address this question, though in a simplified model. In what follows, by focusing on a particular ensemble of strategies for the El Farol bar customers, we derive a model for which we can use all the machinery used in the theory of Minority Games to derive a complete picture of the El Farol bar problem.

The main results are that there is an optimal complexity of the strategies which agents should consider, depending on their number N. More precisely, for a fixed ratio $\ell = L/N$ of seats to agents, coordination is optimal when agents use predictor strategies based on $m \approx \alpha_c \log_2 N$ values of the past attendance. If $m \ll \alpha_c \log_2 N$ crowd effects occur whereas when $m \gg \alpha_c \log_2 N$ the information on past attendance is way too redundant.

We also show the importance for the agents to use consistent predictors. On the one hand, a small systematic bias affects considerably the results. On the other, an El Farol problem with consistent strategies is equivalent to a Minority Game. This implies that a large part of literature on the latter model is also directly relevant to the study of the former without any modification. Inconsistent strategies correspond to biased strategies, or equivalently to a tunable resource level, in Minority Games. Such issues were investigated numerically in Refs. [7–9] with numerical simulations.

A MATHEMATICAL FORMALISM FOR EL FAROL BAR PROBLEM

In Arthur's paper, each customer uses the public knowledge of past m weeks' attendance

$$\mathcal{I}_t = \{A(t-m), A(t-m-1), \cdots, A(t-1)\}$$

in order to determine whether to go the the bar, or to stay at home. She crafts a prediction of the next attendance $A(t)$. If her prediction is larger than the resource level, she stays at home, else she goes to the bar. The learning procedure is inductive: she has a personal set of S fixed strategies based on S different attendance *predictors* $\mathcal{A}_{i,1}, \cdots, \mathcal{A}_{i,S}$. These are functions mapping the information $\mathcal{I}$ about the past m attendances to the integer prediction $\mathcal{A}_{i,s}(\mathcal{I}) \in [0, N]$ of the next attendance $A(t)$.

Each attendance predictor $\mathcal{A}_{i,s}$ $s = 1, \cdots, S$ should not be rewarded depending on their precision but rather on the payoff they give to an agent who follows their advice. So if $A(t) = 59$, a prediction of 5 is better that a prediction of 61. More precisely, every predictor s has a score $\mathcal{U}_{i,s}$ associated to it, that evolves according to

$$\mathcal{U}_{i,s}(t+1) = \mathcal{U}_{i,s}(t) + \Theta\{[\mathcal{A}_{i,s}(\mathcal{I}_t) - L][A(t) - L]\},$$

where $\Theta(x)$ is the Heaviside function [$\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x \geq 0$],

$$A(t) = \sum_{i=1}^{N} a_i(t),$$

is the attendance at time t and $a_i(t)$ is the choice of customer i at time t, which is determined by following her best predictor at that time. Mathematically,

$$s_i(t) = \arg\max_{s'} \mathcal{U}_{i,s'}(t)$$

and

$$a_i(t) = \Theta[L - \mathcal{A}_{i,s_i(t)}(\mathcal{I}_t)].$$

This is the setup of the game proposed by Arthur [19]. He did not specify what the predictor space was, but gave only a few examples of predictors, such as the average of the last attendance numbers $\mathcal{A}(\mathcal{I}_t) = 1/m \sum_{t'=1}^{m} A(t - t')$, or a mirror number of the attendance of $t - 3$, i.e., $\mathcal{A}(\mathcal{I}_t) = L - A(t-3)$. Fogel *et al* [10] used auto-regressive functions $\mathcal{A}(\mathcal{I}_t) = \sum_{k=1}^{m} f_k A(t - k)$, where f_k are real numbers.

Specifying the ensemble of predictors from which agents draw strategies is a key issue in the El Farol bar problem.

Predictors and strategies

In general, a predictor is a function from $\mathcal{I} \in [0, N]^m$ to $\mathcal{A} \in [0, N]$. There are $(N+1)^{(N+1)^m}$ different such functions. A *strategy* instead, is a function $a(\mathcal{I})$ from the set of possible informations $\mathcal{I} \in [0, N]^m$ to an action $a \in \{0, 1\}$. Each strategy can be be considered as the result of the prescription of a predictor: $a(\mathcal{I}_t) = \Theta[L - \mathcal{A}(\mathcal{I}_t)]$.

However, simple counting shows that there are $2^{(N+1)^M}$ possible strategies, which is way smaller than the number of predictors for large N. Hence many different predictors $\mathcal{A}(\mathcal{I})$ correspond to the same strategy $a(\mathcal{I}_t)$. For a particular strategy $a(\mathcal{I})$ with $\sum_{\mathcal{I}} a(\mathcal{I}) = \mathcal{K}$, there are

$$\mathcal{N}(a) = (N - L)^{\mathcal{K}} L^{(N+1)^M - \mathcal{K}} \tag{1}$$

predictors $\mathcal{A}$ which are consistent with that strategy.

Eq. (1) implies that not all strategies are equivalent, in principle. In order to illustrate this point, let us consider the case of a strategy resulting from a predictor taken at random. When $N, L \gg 1$, we almost surely pick a strategy which prescribes to go a fraction $L/(N+1)$ of times. Indeed by Eq. (1), almost all strategies have this property.

As a byproduct we see that Arthur's result that the attendance self-organizes to the comfort level L is trivial if agents draw predictors uniformly and at random from the whole predictor space. The attendance self-organizes to the comfort level L for the simple reason that agents following predictor strategies will attend with a probability $L/(N+1)$.

Actually it is desirable to restrict the ensemble of predictors from which agents draw, to those having some consistency and continuity properties. Consistency means that the predictor should be consistent with past observations: for example if $A(t - k)$ fluctuates around some value L a consistent predictor would also have $\mathcal{A} \approx L$. A predictor of the Fogel type $\mathcal{A}(\mathcal{I}) = \sum_{k=1}^{M} f_k A(t-k)$ should be such that $\sum_{k=1}^{M} f_k = 1$, else it would predicts systematically an attendance which is larger than the true one, hence, not be consistent.

A minimal requirement of consistency is that the resulting strategies be unbiased, i.e. that "on average" they prescribe to attend a fraction $L/(N+1)$ of the times. A mathematical formalization of this property entails non-trivial considerations and it will not be pursued here[20]. Rather we shall later introduce explicitly the bias of strategies as an external parameter and study the collective behavior as a function of it.

Strategies derived from a random predictor, are unbiased but fail to have a minimal degree of "continuity". Loosely speaking, continuity of $\mathcal{A}$ means that if the change in the information $\mathcal{I}$ is small, the change in the prediction $\mathcal{A}(\mathcal{I})$, or at least in the prescribed action $\Theta[L - \mathcal{A}(\mathcal{I})]$ should also be small (or rare). For example, the action prescribed by the strategy should change only rarely when the attendance of a past day changes by a small amount. Considering that past attendance may also be subject to observation errors, continuity is a quite desirable robustness property of strategies.

A workable model

In the following we focus on a particular ensemble of strategies, which is obtained by reducing the information space. The intuition is that what is really telling about the value of a past attendance $A(t-k)$ is whether that was below or above the comfort level L, which is a binary information. In other words we consider strategies

$$a_{s,i}^{\mu} = \Theta[L - \mathcal{A}_{s,i}(\mathcal{I})] \quad (2)$$

which depend only on the information

$$\mu(t) = \{\Theta[L - A(t-1)], \dots, \Theta[L - A(t-m)]\}. \quad (3)$$

Clearly strategies derived in this way have a high degree of continuity.

Not all strategies derived by a predictor $a_{s,i}^{\mathcal{I}} = \Theta[L - \mathcal{A}_{s,i}(\mathcal{I})]$ are of the form described above. There are only 2^{2^M} strategies of this type, which means that a reduction of the information space also implies a strong reduction of the strategy space [21].

The fact that agents use a strategy space whose size is independent of N makes much sense. In context where agents interact with a crowd, their behavior is insensitive to the exact size N of the population.

Henceforth we assume that each agent is assigned S strategies randomly drawn from this pool. More precisely, for each i, s and μ we draw $a_{s,i}^{\mu}$ independently from the distribution

$$P(a) \equiv \mathrm{Prob}\{a_{s,i}^{\mu} = a\} = \bar{a}\delta(a-1) + (1-\bar{a})\delta(a).$$

As explained above, the induced strategy space of the El Farol problem is such that on average, agents attend the bar with a frequency L/N. In other words, binary strategies of the El Farol bar problem account *a priori* for the convergence of the attendance to L on average. Hence we shall consider below an ensemble of strategies such that the average of $a_{s,i}^{\mu}$ is $\bar{a} \approx L/N$. Actually we shall see that small deviations of $\bar{a}$ from L/N may have a large effect in the limit $N \to \infty$.

A further simplification, which does not change the qualitative nature of the results [11] amounts to consider a linear dynamics of the strategy scores

$$U_{i,s}(t+1) = U_{i,s}(t) - (2a_{i,s}^{\mu(t)} - 1)[A(t) - L]. \quad (4)$$

Note that the strategies that predict the correct choice are rewarded whereas those prescribing a wrong choice are punished.

The understanding of the behavior of this model is made complex by the feedback of the fluctuations of $A(t)$ with the dynamics of information $\mu(t)$. Notice that Eq. (3) is equivalent to assuming that $\mu(t)$ follows the non-linear dynamics

$$\mu(t+1) = |2\mu(t) + \Theta[L - A(t)]|_{2^m} \quad (5)$$

where $|\dots|_P$ is the modulus P operation. The behavior of the Minority Game is largely unaffected if this dynamics is replaced by a random draw of $\mu(t)$ [12, 13]. As we shall see this is not the case in our case. Still it is very helpful to introduce a variation of the model with *random information*, where $\mu(t)$ is just randomly drawn, with uniform probability, from the integers $1, \dots, P = 2^m$ (if the history is random, P can be any integer number). This is because the model with random information can be understood in detail within a theory such as that developed for the MG. This is a quite useful intermediate step toward understanding the behavior of the El Farol bar problem with true information. In addition it is also possible to quantify the effects of the dynamics of true information Eq. (5) along the lines of Ref. [12].

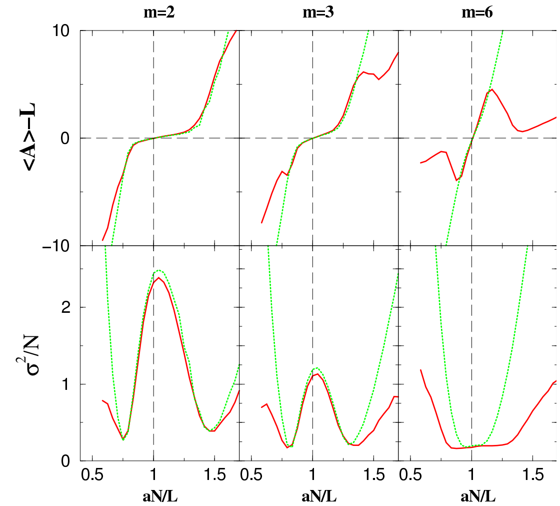

FIG. 1: Behavior of the average attendance (top) and of the fluctuations (bottom) in the El Farol bar problem with $L = 60$ seats, $\bar{a} = 1/2$ and $m = 2, 3$ and 6 from left to right. In both cases, the solid (dotted) line refers to true (random) information.

In order to illustrate the behavior of the model, Fig. 1 shows the results of simulations with $L = 60$, $\bar{a} = 1/2$ and $m = 2, 3, 6$ (from left to right) fixed as a function of N. This shows what happens in a system where the "environment" (L) and the adaptive capabilities of agents ($\bar{a}$ and m) are kept fixed, while the number N of agents increases. The top graph shows the deviation $\langle A \rangle - L$ of the average attendance from the comfort level. As we see when $N\bar{a} \approx L$ the attendance converges to the comfort level $\langle A \rangle \approx L$. However, for small m there is a whole interval around $N\bar{a} = L$ where agents are still able to coordinate efficiently $\langle A \rangle \approx L$. For $m = 2$, the results are qualitatively the same with true and random information. For larger values of m the region where $\langle A \rangle \approx L$

shrinks. In addition, while $\langle A \rangle$ maintains a monotonic behavior with random information, it develops a maximum and minimum for intermediate values of $\bar{a}N/L$ beyond which the behavior with true information markedly departs from that with random information.

It is important to quantify the model's behavior also beyond the properties of $\langle A \rangle$. Indeed, we have seen that convergence to the comfort level is a trivial result in the limit $N \to \infty$. It is a built-in property of the model which arises from the requirement of unbiased strategies ($\bar{a} \approx L/N$).

The non-trivial cooperative behavior of this system, as forcefully remarked by the literature on the Minority Game, lies in how and whether agents manage to decrease fluctuations of the attendance $A(t)$ around the comfort level L. Indeed even if $A(t)$ equals L on average, the distance $|A(t)-L|$ measures the amount of wasted resources, either unexploited $A(t) < L$ or over-exploited $A(t) > L$. Therefore, the quality of the cooperation among agents is measured, at a finer level, by the fluctuations around the resource level, defined as

$$\sigma^2 = \langle (A - L)^2 \rangle \tag{6}$$

where $\langle \ldots \rangle$ is the average on the stationary state. Given that $A(t)$ is the sum of N contributions, it is natural to study the quantity σ^2/N which, as we shall see has a finite limit value in the limit $N \to \infty$.

The behavior of σ^2/N is shown in Fig. 1. Away from the point $\bar{a}N = L$, the increase of σ^2/N is mainly due to the deviation $\langle A \rangle - L$ and as before, it differs in the cases of true and random information. For small m, σ^2/N displays a maximum at $\bar{a}N = L$ which becomes shallower as m increases and it disappears for $m = 6$. This non-trivial behavior suggests that a small bias in the strategies, of either sign, is beneficial as it decreases the fluctuations (see also [8]).

This is evident from Fig. 2, where we compute the fraction $\langle |A - L| \rangle / N$ of unsatisfied agents and compare it with the behavior of agents who attend the bar at random, either with probability $\bar{a}$ or with probability L/N. Here we clearly see that in the region $\bar{a}N \approx L$ adaptive agents behave less efficiently than random agents. This effect is related to the emergence of fluctuations and is stronger for small values of m. Efficiency increases for larger values of m.

We shall devote the rest of this paper to explain the non-trivial behavior displayed by $\langle A \rangle$ and σ^2. The first step will be extending the analytic approach of Refs. [3, 11] to the model with random information, which is essentially equivalent to a Minority Games with biased strategies and tunable resource level. Then we shall analyze the case with true information.

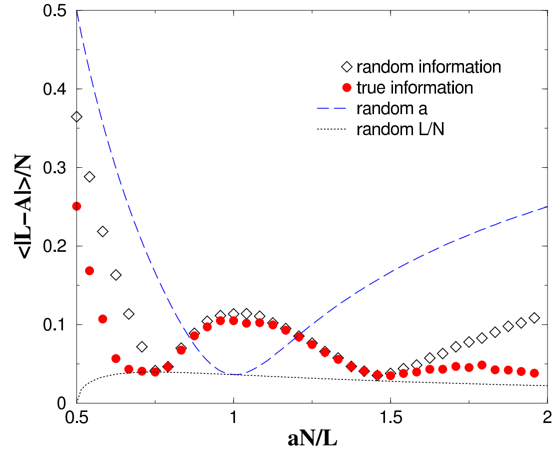

FIG. 2: Fraction of losers in the El Farol bar problem with $L = 60$ seats, $\bar{a} = 1/2$ and $m = 2$ for random ($\diamond$) and true ($\bullet$) information. The results for a population of random agents who attend the bar independently with probability $\bar{a}$ and L/N is also shown for comparison (dashed and dotted lines).

STATISTICAL MECHANICS OF THE EL FAROL BAR PROBLEM WITH RANDOM INFORMATION

Following Ref [11], one deduces that agent i ends up playing strategy s with frequency $f_{i,s}$ that minimizes the quantity

$$H = \sum_{\nu=1}^{P} \rho^{\nu} (\langle A|\nu \rangle - L)^2$$

where $\rho^{\nu} = \text{Prob}\{\mu(t) = \nu\} = 1/P$ and

$$\langle A|\nu \rangle = \sum_{i=1}^{N} \sum_{s_i=1}^{S} f_{s,i} a_{s,i}^{\nu}$$

is the average of $A(t)$ conditional on the event $\mu(t) = \nu$. This result can be obtained in a straightforward manner by taking the average of Eq. (4) in the stationary state, and comparing the resulting equations with the first order conditions of the minimization of H with respect to $f_{s,i}$.

The function H measures the predictability in the system, i.e. the amount of useful information about the fluctuations of the attendance which is left in the signal $\mu(t)$. Indeed if e.g. $\langle A|\nu \rangle \neq L$, the signal $\mu(t)$ carries information which is useful to predict whether one should attend or not to the bar when $\mu(t) = \nu$. The fact that the stationary state corresponds to minimal H means that agents exploit to their best the system's predictability.

In terms of statistical mechanics, H can be considered as a Hamiltonian and its minima can be studied with standard methods. As long as $H > 0$, the stationary state is unique, and the replica trick [14] gives exact results [3, 4, 11]. For the sake of simplicity, we will focus on the $S = 2$ case (see Ref. [15] for a generalization to $S > 2$). The details of the calculus are of no special interest, as they mostly replicate previously published calculations [3].

We shall consider the thermodynamic limit $N \to \infty$ with

$$\ell \equiv \frac{L}{N} \quad \text{and} \quad \alpha = \frac{P}{N}$$

fixed [16]. Furthermore, in order to study the effect of a deviation of $\bar{a}$ around ℓ we introduce the convenient parameter γ with the equation:

$$\bar{a} = \ell + \sqrt{\frac{\ell(1-\ell)}{P}}\gamma \tag{7}$$

and we shall consider γ finite in the limit $N \to \infty$. As we shall see this is a non-trivial limit. For example, the case where $\bar{a} - \ell \approx O(1)$ is finite, is trivial because then each agent uses just one strategy, that which prescribes to go more (less) often if $\bar{a} < \ell$ ($\bar{a} > \ell$) and there is no dynamics at all. Eq. (7) implies that we consider small deviation of $\bar{a} = \ell + O(1/\sqrt{N})$ from the correct value $\ell = L/N$. Even such a small deviation, which vanishes as $N \to \infty$, has a finite effect on the global behavior as we shall see. At any rate, other limits can be obtained analyzing the cases where either ℓ, α or γ vanish or diverge, in the analysis that follows.

After some routine calculations, we find that the predictability H/N is given by

$$H = N\frac{\sqrt{\ell(1-\ell)}}{2}\,\frac{1 + Q(\zeta) + 2\gamma^2/\alpha}{[1 + \chi(\zeta)]^2} \tag{8}$$

and the fluctuations, as long as the stationary state is unique (see later), are equal to

$$\sigma^2 = H + N\frac{\sqrt{\ell(1-\ell)}}{2}[1 - Q(\zeta)] + N\Sigma. \tag{9}$$

In these two equations, Q and χ are given by

$$Q(\zeta) = 1 - \sqrt{\frac{2}{\pi}}\frac{e^{-\zeta^2/2}}{\zeta} - \left(1 - \frac{1}{\zeta^2}\right)\text{erf}\left(\frac{\zeta}{\sqrt{2}}\right)$$
$$\chi(\zeta) = \left[\frac{\alpha}{\text{erf}\,(\zeta/\sqrt{2})} - 1\right]^{-1} \tag{10}$$

whereas the parameter ζ is uniquely determined by the transcendental equation

$$\frac{\alpha}{\zeta^2} - Q(\zeta) - 1 - 2\frac{\gamma^2}{\alpha} = 0 \tag{11}$$

as a function of α and γ. Finally Σ in Eq. (9) is a term which arises from collective fluctuations and its calculation requires, in principle, a detailed theory of the stochastic dynamics of the model (see Ref. [11]) which we shall not pursue here, but whose importance is discussed below.

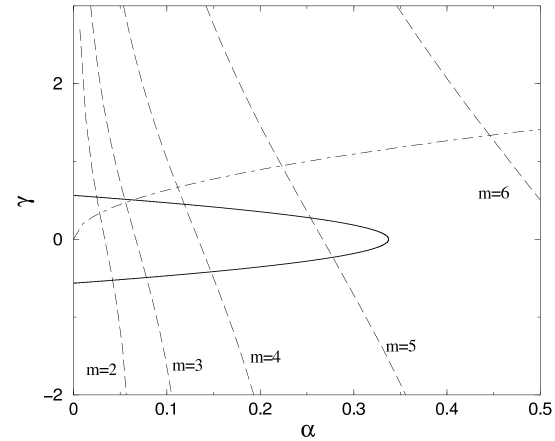

FIG. 3: Phase diagram of the El Farol bar problem. The dashed lines correspond to the trajectories of systems with $L = 60$, $\bar{a} = 1/2$ and $m = 2, \ldots, 6$ as the number of agents increases (from bottom to top). The dot-dashed line corresponds to a typical trajectory of a system with fixed L, N and $\bar{a} > L/N$ as the agents' memory changes.

When $\gamma = 0$, these equations are identical to those which describe the MG behavior. We briefly recall the resulting picture: When α is large, the system is in an information rich phase with positive predictability $H > 0$. The predictability H decreases as α decreases. This can be understood by observing that, at fixed $P = 2^M$, a decrease in α means that the number of agents increases, and hence their ability to exploit the information. At a critical value $\alpha_c = 0.3374\ldots$ the predictability vanishes and a phase transition to a symmetric phase with $H = 0$ takes place. The fact that H measures an asymmetry in the signal means that the phase transition is related to symmetry breaking [12]. The phase transition is signaled by the divergence of the spin susceptibility χ, which is infinite in the whole symmetric phase. The stationary state is unique, independent of initial conditions, for $\alpha > \alpha_c$ and these facts conspire [11] in such a way that $\Sigma \cong 0$ for $\alpha > \alpha_c$. On the contrary, for $\alpha < \alpha_c$, the stationary state is not unique but it depends on the initial conditions. Then $\Sigma > 0$ can be computed within a very accurate self-consistent approximation [11]. In particular this shows that $\Sigma \propto 1/\alpha$ for small α.

Therefore, the El Farol bar problem, with the strategy ensemble studied here, and the MG have the same behavior of fluctuations when $\gamma = 0$, that is, when $N\bar{a} = L$. When $\gamma \neq 0$ the picture changes in the following way: First we observe that all quantities depend on γ^2, then it is enough to consider only the case $\gamma > 0$ since all conclusions extend directly to the case $\gamma < 0$. When γ is small, we still have a phase transition at the point

$\alpha_c(\gamma) = \text{erf}(\zeta_c/\sqrt{2})$ where $\chi \to \infty$. The parameter ζ_c is the solution of

$$\frac{\text{erf}\,(\zeta/\sqrt{2})}{\zeta^2} - Q(\zeta) - 1 - \frac{2\gamma^2}{\text{erf}\,(\zeta/\sqrt{2})} = 0 \qquad (12)$$

Figure 3 plots the phase diagram of the game. The critical line separates the asymmetric phase ($H > 0$) from the symmetric phase ($H = 0$). It crosses $\gamma = 0$ at $\alpha_c(0) = 0.3374\ldots$, the critical point of the standard MG [3, 4]; when γ increases α_c decreases and $\alpha_c = 0$ for $\gamma \geq 1/\sqrt{\pi}$.

The meaning of the phase diagram is clear: Indeed $H = 0$ implies $\langle A \rangle = L$. The symmetric phase is the region of parameters where the average attendance converges to the comfort level. This region is also characterized by large collective fluctuations σ^2 and by a dependence on initial conditions; in particular, the fluctuations decrease if the difference of strategy *a priori* valuation increases as discussed in the MG literature [4, 11, 17, 18]. On the contrary, there is no equality between $\langle A \rangle$ and L in the asymmetric phase if $\gamma > 0$.

These results explains the complex behavior reported in Fig. 1. Indeed as N varies with L, $\bar{a}$ and m fixed, the system follows the trajectories shown in Fig. 3. For small values of m these cross the symmetric phase in the region $\bar{a}N \approx L$. Fig. 4 indeed shows that H computed along these trajectories fully agrees with the theoretical results (for random information). The symmetric phase is characterized by large fluctuations, mainly due to dynamic fluctuations (the term Σ). This explains the non-monotonic behavior of σ^2 in Fig. 1 for small values of m. When m increases, the trajectory in Fig. 3 moves toward larger values of α and, for $m > m_c$, it remains all in the asymmetric phase. Then $\Sigma = 0$, which means that σ^2 displays a single minimum at $\bar{a}N = L$ (i.e. $\gamma = 0$).

In spite of the fact that the theory is derived in a particular $N \to \infty$ limit, our results show that it reproduces accurately results for moderately small values of N. At the same time, it clearly predicts how the collective behavior depends on the parameters N, $L, \bar{a}$ and m. Any "experiment" where one of these parameters is changed, corresponds to a precise trajectory in the (α, γ) phase diagram and a corresponding collective behavior. For example, for fixed m and $\ell = L/N$, anomalous fluctuations will arise in an interval of size $1/\sqrt{P} = 2^{-m/2}$ as $\bar{a}$ changes around ℓ along a vertical trajectory in Fig. 3. The memory size controls fluctuations. Indeed generally, as P increases, keeping all other parameters fixed, the systems moves away from the symmetric phase (see dot-dashed line in Fig. 3).

It is precisely at the boundary of the two phases that coordination is most efficient. This means that there is an intermediate memory length which is optimal for the collective behavior.

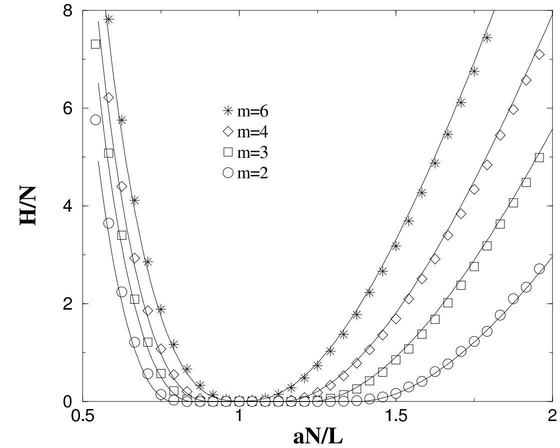

FIG. 4: H/N in the El Farol bar problem with true information and $L = 60$, $\bar{a} = 1/2$ and $m = 2, 3, 4$ and 6 as a function of N. Theoretical results (full lines) fully agree with numerical simulations (symbols).

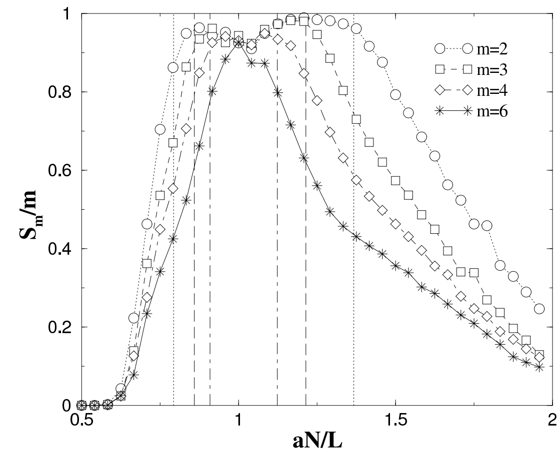

FIG. 5: Entropy S_m/m of the El Farol bar problem with true information and $L = 60$, $\bar{a} = 1/2$ and $m = 2, 3, 4$ and 6 as a function of N. Vertical lines delimit the symmetric phase of the corresponding model with random information.

TRUE INFORMATION

The behavior of the model with true information deviates from that with random information because, under the dynamics Eq. (5), the space of informations is not sampled uniformly. More precisely, if ρ^μ is the probability of information μ in the stationary state, we can make this statement quantitative introducing the entropy

$$S_m = -\sum_{\mu=1}^{2^m} \rho^\mu \log_2 \rho^\mu.$$

When the space of information is sampled uniformly $\rho^\mu = 1/2^m$ we find $S_m = m$, whereas $S_m = 0$ when only one value of μ is sampled recursively. Fig. 5 shows that $S_m \approx m$ only occurs in the region where $H \simeq 0$ in the corresponding model with random information, i.e. in the symmetric phase. This is consistent, because if $H = 0$ then the process of Eq. (5) is a simple diffusion on the so-called De Bruijn graph. We refer the interested reader to Ref. [12] for a detailed account of this process. Here it is sufficient to observe that if $\langle A|\mu\rangle = L$ for all μ, then[22]

$$\text{Prob}\{\mu(t+1) = |2\mu(t)+1|_{2^m}\} = \frac{1}{2}.$$

If this is the case, the stationary state probability $\rho^\mu = 1/2^m$ is uniform [12]. When $H > 0$, for a particular value of μ, we expect that $L - A(t)$ will take more frequently one sign or the other. Hence Eq. (5) will induce a biased diffusion process on $\mu(t)$. In particular, for $\overline{a}N < L$, the attendance will be more often below the comfort level than otherwise. This means that 1's will occur more often that 0's in Eq. (3) for $\mu(t)$. It is easy to check that a systematic bias of this type, produces a distribution ρ^μ which is concentrated on $\mu = \{111\ldots\}$ (using the binary representation). Likewise ρ^μ is peaked on $\mu = \{000\ldots\}$ when $aN > L$ [23].

Hence outside the symmetric phase, when $H > 0$, the process $\mu(t)$ acquires a bias, which reduces the "effective number of information patterns" to a number 2^{S_m}. In order to understand how this changes the collective behavior of players, imagine the extreme case $S_m = 0$ where for some reason, the state $\mu(t) = \{000\ldots\} \equiv 0$ occurs for a large number of periods. Then agents will learn how to respond optimally to this state $\mu = 0$.

There are $N_{++} = \bar{a}^2 N$ agents with $a^0_{i,1} = a^0_{i,2} = 1$. They will go anyway. There are $N_{--} = (1-\bar{a})^2 N$ agents with $a^0_{i,1} = a^0_{i,2} = 0$ who will not go. The remaining $2\bar{a}(1-\bar{a})N$ can decide. Agents can learn to converge to $A(t) = L$ provided that $N_{++} \le L \le N - N_{--}$, i.e. if

$$\frac{L}{1-(1-\bar{a})^2} \le N \le \frac{L}{\bar{a}^2}.$$

Furthermore, if $N > L/\bar{a}$, in particular if it is close to the upper limit $L/\bar{a}^2$, the information $\mu = 0$ will arise very frequently from the dynamics Eq. (5). The same argument runs for $N < L/\bar{a}$ and it shows agents can coordinate quite efficiently when N is close to $L/[1(1-\bar{a})^2]$, because then the information $\mu = \{111\ldots\}$ will almost always occur.

This complex interaction between the dynamics of $A(t)$ and $\mu(t)$ explains the non-monotonic behavior of $\langle A\rangle - L$ in Fig. 1.

CONCLUSIONS

We have presented a complete theory of the El Farol bar problem. The key issue lies in the definition of the strategy space. First we have shown that, for the most general ensemble of strategies, convergence of the attendance to the comfort level is a trivial consequence of the law of large numbers. It does not even require inductive rationality. Even zero-intelligent agents are able to reach it. This is likely to be true for any reasonable predictor based strategy, in particular for unbiased ones.

We further focus attention on a particular ensemble of strategies, with a desirable continuity property. This leads us to study models very similar to the Minority game. We first introduce the random information version of the game, for which statistical physics provides a complete theoretical understanding and analyze the consequences of the dynamics of true information.

It turns out that as the parameter of the El Farol bar problem change, the system performs a trajectory on a phase diagram characterized by a symmetric phase where $\langle A\rangle = L$ and a phase where $\langle A\rangle \neq L$. Deep in the symmetric phase, anomalous fluctuations similar to crowd effects, develop making the coordination of agents even worse than that of random agents in some cases. It is precisely close to the phase boundary that agents manage to coordinate most efficiently. This, however, requires a small bias of either sign, in the strategies of agents.

These findings not only confirm that the El Farol bar problem is indeed a quite interesting complex system. But they also show that a coherent understanding of its behavior is possible, using tools of statistical physics.

* Electronic address: challet@thphys.ox.ac.uk
† Electronic address: marsili@ictp.trieste.it

[1] D. Challet and Y.-C. Zhang, Physica A **246**, 407 (1997) preprint adap-org/9708006
[2] D. Challet, the Minority Game's web page http://www.unifr.ch/econophysics/minority
[3] D. Challet, M. Marsili and R. Zecchina, Phys. Rev. Lett. **84**, 1824 (2000).
[4] J. A. F. Heimel, A. C. C. Coolen, Phys. Rev. E **63**, 056121 (2001), e-print cond-mat/0012045.
[5] Arthur W. B., Am. Econ. Assoc. Papers and Proc **84**, 406, 1994.
[6] J. L. Casti, Complexity **1**, 7 (1996)
[7] N.F. Johnson *et al.*, Physica A **269**, 493 (1999)
[8] K. F. Yip, P. M. Hui, T. S. Lo, N. F. Johnson, Physica A **321**, 318 (2003)
[9] A. de Cara, P. Guinea, unpublished (2003)
[10] D. B. Fogel, K. Chellapilla, P. J. Angeline, IEEE Trans. Ev. Comp. **3**, 142 (1999)
[11] M. Marsili, D. Challet, Phys. Rev. E **64**, 056138 (2001).
[12] D. Challet and M. Marsili, Phys. Rev. E **60**, R6271 (1999), preprint cond-mat/9904071
[13] A. Cavagna, Phys. Rev. E **59**, R3783 (1999)
[14] V. Dotsenko, *An introduction to the theory of spin glasses and neural networks* ; World Scientific Publishing (1995).
[15] M. Marsili, D. Challet and R. Zecchina, Physica A **280**, 522 (2000), preprint cond-mat/9908480

[16] Savit R., Manuca R., and Riolo R., Phys. Rev. Lett., **82**, 2203 (1999).
[17] J. P. Garrahan, E. Moro, D. Sherrington, Phys. Rev. E 62, R9 (2000)
[18] M. Marsili and D. Challet, Adv. Complex Systems **3**-I (2001), preprint cond-mat/0004376
[19] In practice one should consider a non integer L, else the system can freeze artificially at $A = L$.
[20] It is not even clear that such a formalization is possible *a priori*. Notice that "on average" in the discussion above refers to a probability distribution on $\mathcal{I}$ which is that generated by the game's dynamics itself.
[21] In particular, a randomly drawn $\mathcal{A}$ almost surely leads to a strategy which has not this form.
[22] Notice that $A(t)$ is the sum of N terms ± 1, which is asymptotically symmetric around the mean.
[23] As a side remark, if $\text{Prob}\{\mu(t+1) = |2\mu(t)+1|_{2^m}\} = p$, $\rho^\mu = p^{n(\mu)}(1-p)^{m-n(\mu)}$ where $n(\mu)$ is the number of 1 in the binary representation of μ, and $-S(p)/m = p\log p + (1-p)\log(1-p)$