Single NTU-value solutions*

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Abstract

We propose a variation of the non-cooperative bargaining model for n-person games in coalitional form, introduced in Hart and Mas-Colell (1996). This strategic game implements, in the limit, a new NTU-value for the class of monotonic games. This value coincides with the Consistent NTU-value (Maschler and Owen,1989) for hyperplane games, and with the Shapley value for TU games (Shapley, 1953). The main characteristic of this proposal is that always select a unique payoff allocation. This value can also be considered as an extension of the Nash bargaining solution (Nash,1950). Variations of this model yield extensions of the Discrete Raiffa solution (Raiffa, 1953), and the Kalai-Smorodinsky solution (Kalai and Smorodinsky, 1975).

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KEYWORDS: Shapley value; NTU-value solutions; Nash Bargaining; Raiffa solution; Kali-Smorodinsky solution.

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1 Introduction

We consider in this paper n-person cooperative games in coalitional form. In this setting partial cooperation is allowed between players. Lotteries over outcomes are also feasible and we assume that players' preferences over outcomes are represented by von Neuman-Morgenstern utility functions. A *n-person coalitional form* is given by a couple (N, V), where N is a finite set of n players and V is a function that assigns a subset $V(S)$ of $\mathbb{R}^S$ to every coalition $S \subset N$. $V(S)$ is interpreted as the set of payoff vectors to the members of S that would be feasible if S where the group of deciding players.

When utility is transferable across players (*TU-games*) the most prominent solution concept is the *Shapley* (1953) *value.* It yields a unique expected payoff allocation for the players in the game. The original Shapley's support for the value was axiomatic. Other relevant axiomatizations of the value are in Myerson (1980), and Hart and Mas-Colell (1989). Bargaining models that yields the value in the TU-case have been also proposed. Among them can be cited Gul (1989), Hart and Moore (1990), Hart and Mas-Colell (1996), Winter (1994), and Pérez-Castrillo and Wettstein (2001).

When utility is not transferable (*NTU-games*), different ways to extend the value have been considered. The most relevant are due to Harsanyi (1963), Shapley (1969)[1], and Maschler and Owen (1992)[2]. These three solutions were constructed in such way that they coincide with the Nash (1950) solution for pure bargaining games (i.e., only the coalition of all players can reach an agreement). Axiomatic support for these solutions have been done by Aumann (1985) for the Shapley NTU-value, by Hart (1985) for the Harsanyi NTU-value, and by de Clippel, Peters and Zank (2002) and by Hart (2003) for

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[1] The Shapley NTU-value is also called as λ-transfer value.

[2] First introduced for hyperplane games in Maschler and Owen (1989), and known also as the Consistent NTU-value.

the Maschler-Owen NTU-value. Only for the Consistent NTU-value has been proposed a n-person bargaining procedure in Hart and Mas-Colell (1996).

All these three extensions of the value have in common the same well known problem: *They do not guarantee uniqueness in the payoffs vector solution.*

Perhaps the multiplicity of the outcomes is not a trouble if we wish to interpret the solutions from a *descriptive* point of view. The same phenomena happens in many others contexts, as in the Walrasian equilibria, or in the Nash equilibria and its refinements for non-cooperative games. But from a *normative* point of view, two different Pareto optimal outcomes put the players in a conflict. Which outcome should be selected? As far as the election is done, some players have been benefited and the rest harmed with respect the alternatives discarded. If we wish to solve this impasse by using a fair lottery among the outcomes selected the expected payoff will usually be inefficient, which is not a good criteria for a cooperative solution.

In the present paper we follow a strategic approach, starting with a modified version of the bargaining procedure of Hart and Mas-Colell (1996), in which the only change is produced in the way to solve the *breakdown* in the negotiations. It will be showed that the equilibria of this bargaining game implements in the limit a new single-valued solution. Restricted to pure bargaining problems, the point selected is the maximization of utility gains from a reference breakdown point, so it is very similar to the Nash solution. When we deal with TU-games it is selected the Shapley value, and in Hyperplane games it is selected the Consistent NTU-value.

The interest of this breakdown technology lies also in the fact that maintaining the same breakdown rule, but changing the negotiation procedure, it can be considered single value extensions of solutions that were defined only for pure bargaining games. We present two examples: The *Discrete Raiffa* solution (Raiffa, 1953), made by de Clippel (2002), and the *Kalai-Smorodinsky* solution (Kalai and Smorodinsky, 1975).

Following this Introduction, Section 2 discusses the non-uniqueness aspect with a two-person problem, and the way to solved this problem by the modification of the breakdown rule is offered. In Section 3 it is defined with more detail the general bargaining procedure, differentiating, in each round, the Negotiating stage from the Breakdown stage. The next Sections consider different ways to define the Negotiation stage: Section 4 considers the Alternating Offers with Risk of Breakdown model, yielding an extension of the Nash solution. Section 5 considers the Alternating Offers with Finete Horizon model, yielding an extension of the Discrete Raiffa solution. Finally, Section 6 considers the Auctioning Counteroffer's Chance model, yielding an extension of the Kalai-Smorodinsky solution.

2 A two-person example

We start with a simple bankruptcy problem to illustrate the problems in which players can be involved when they follow a solution concept that don't satisfy the uniqueness requirement.

The state is 100 units and there are two potential claimants arguing 75 units of debts each one. Assume that these debts are not well documented, so each claimant must put his claim in a Court in order to determine his legitimateness. We normalize by cero the payoff of a claimant which decide to not litigate. If only one claimant litigates, we assume that the magistrate is ready to accept his claim, and then his payoff is 75. In case both litigate, the magistrate must to check carefully the validity of the proofs presented by the claimants. Assume that only four outcomes are considered by law. If the magistrate find that both claims are well-grounded then the state must be shared equally, so the payoffs are (50,50). If only one demand is valid, the claimant accepted receives his debt and the claimant rejected receives cero, so the payoffs are either (75,0) or (0,75). If both are rejected they will receive cero. In addition, the claimants are allowed to agree about what of these four alternatives be the final outcome proposed in the Court. They can bargaining previously any lottery among these four alternatives, and we assume that utilities over lotteries of claimants are von Neumann-Morgenstern type, and risk neutral. Therefore the feasible expected payoffs that both claimants can guarantee by cooperation are done by the convex hull of $A = \{(50, 50), (75, 0), (0, 75), (0, 0)\}$.

Here the set of players is $N = \{1, 2\}$ and V is defined by

$$V(\{i\}) = \{x : x \leq v_i\} \text{ where } v_i = 75,\ i \in N,$$

and

$$V(N) = conv(A) - \mathbb{R}^2_+,$$

("*conv*" denotes "convex hull"). The sets $V(\cdot)$ are also comprehensive (utility is freely disposable). $V(N)$ is represented in Figure 1.[3].

Now applying the Consistent NTU-value to this example[4] the solution selects three possible outcomes: $c^1 = (56.25, 37.5)$, $c^2 = (37.5, 56.25)$, and $c^3 = (50, 50)$. If we wish to follow a strategic approach, we should offer some prespecified bargaining rule that allows to the players find an agreement themselves.

In Hart and Mas-Colell (1996) it is offered a bargaining procedure that supports the Consistent NTU-value[5], that in our two person case goes as follows:

- "Choose a player $i \in N = \{1, 2\}$ by tossing a coin.

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[3] This example is basically an adaptation of the "reverse pure bargaining problem" given in Hart and Mas-Colell (1996), Section 4, pag. 365.

[4] For a detailed description of the Consistent NTU-value the reader is referred to Hart and Mas-Colell (1996), and in particular Section 4, pags. 366-367.

[5] We follow the description given in Mas-Colell (1997).

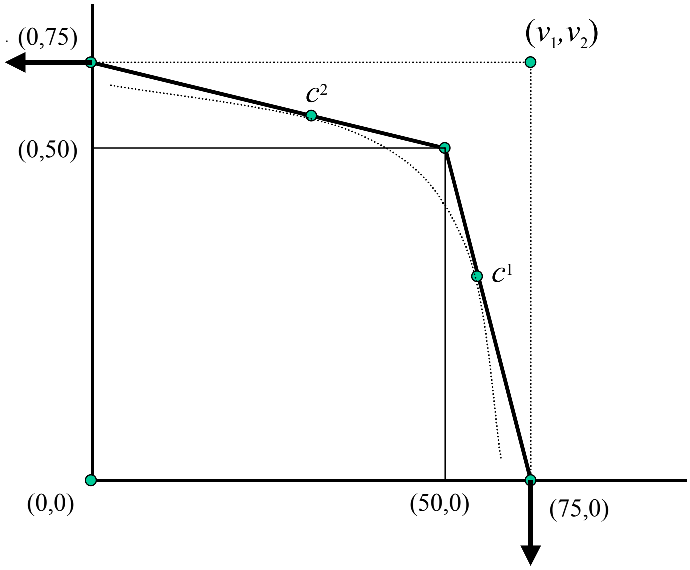

Figure 1:

- Player i proposes a payoff vector $a^i \in V(N)$.
- Player j is asked if agree or dissent.
  - Agree $\Rightarrow$ a^i is implemented.
  - Dissents $\Rightarrow$
    - $\star$ With probability ρ, the game repeats.
    - $\star$ With probability $1 - \rho$, breakdown occurs."

The key aspect in this model is how *breakdown* is defined. Because it is intended that feasible payoffs of subcoalitions play a role in determination of final payoffs, it is given a chance that agreements for subcoalitions can happen in the end. With this intention at hand, breakdown means that the proposer i disappears, and the game repeats with the remainder player. In our example player i leaves player j alone to make the claim in the court. This bargaining procedure have always a stationary perfect equilibria. The game is solved by induction. When a player i claims alone his/her payoff is v_i. The equilibrium equations which determine the proposals in N are:

$$a^i(\rho) \in \partial V(N),\ (i \in N), \tag{1}$$

i.e., the proposals are efficient, and

$$a^i_j(\rho) = \rho a_j(\rho) + (1-\rho)v_j,\ (j \neq i), \tag{2}$$

where $a(\rho)$ is the expected vector payoffs for coalition N, i.e.:

$$a(\rho) = \frac{1}{2}a^1(\rho) + \frac{1}{2}a^2(\rho).$$

That is, player i offers to player j just what will get in case to reject the proposal: $a(\rho)$ in case the game repeats, and v_j in case breakdown happens.

It can be checked that equations (1) and (2) implies that

$$\left(a^1_1(\rho) - v_1\right)\left(a^1_2(\rho) - v_2\right) = \left(a^2_1(\rho) - v_1\right)\left(a^2_2(\rho) - v_2\right),$$

which, in our example, yield two different solutions: $\{a^1(\rho), a^2(\rho)\}$ and $\{e^1(\rho), e^2(\rho)\}$ that, when $\rho \to 1$, converge to c^1 and c^2 respectively (see Figure 2).

First note that this bargaining procedure not allways allows to approximate to all payoffs solutions: In our example the point $c^3 = (50, 50)$ is excluded. Second, we have multiplicity: we can approximate to either c^1 or c^2. If we have no reasons a priori to discriminate between claimants 1 and 2, the only fair way to choose between c^1 and c^2 is by tossing a coin. But therefore the expected payoffs are $(46.875, 46.875)$ that are Pareto dominated by $(50, 50)$. From a *normative* point of view, propose rules of cooperation that yield *asymmetric* outcomes in *symmetric* problems is a bit hardly task. Both aspects, uniqueness and symmetry, should merit to be into account when is intended to design

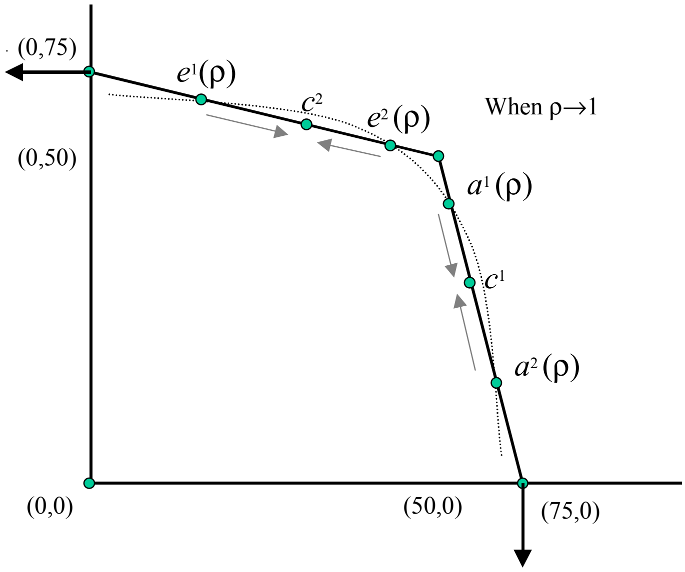

Figure 2:

rules of cooperation. Can we change in some way the bargaining procedure to avoid this multiplicity of solutions?

Given the usual convexity assumption, the Nash bargaining solution (Nash, 1950) always select a unique point if the disagreement point belongs to the interior of the feasible set. Hence, a natural strategy to solve this question could be by changing the breakdown process in such a way that the expectations of the players in case of breakdown be also feasible. There is a natural and simple way for that, and this is what we want to explore here.

We propose the following *breakdown procedure*: Assume as before that $N = \{1, 2\}$ is the set of players negotiating. If breakdown happens:

- "Choose a player $i \in N = \{1, 2\}$ by tossing a coin.
- Player i proposes a payoff vector $u^i \in V(N)$.

- Player j is asked if agree or dissent.

    - Agree $\Rightarrow$ u^i is implemented.

    - Dissents $\Rightarrow$player i leaves the game, receiving a payoff of cero, and player j receives his claim v_j.

With this breakdown rule the equilibrium proposals u^i are characterized by

$$u^i \in \partial V(N), \; (i \in N), \tag{3}$$

and

$$u^i_j = v_j, \; (\, j \neq i). \tag{4}$$

Therefore, it follows that, in case of breakdown, the expected payoffs vector u is

$$u = \frac{1}{2}u^1 + \frac{1}{2}u^2,$$

which, given the convexity assumption on $V(N)$, belongs to the feasible set too.

In our previous example, if a claimant is compelled to make an ultimatum offer to the other one, he must offer 75 units because this what the other claimant would obtain if the proposer is forced to leave the game in case of rejection. Hence the expectations in case of breakdown are

$$u = \frac{1}{2}(0, 75) + \frac{1}{2}(75, 0) = (37.5, 37.5)\,.$$

The Hart and Mas-Colell bargaining model, with this new breakdown rule, applied to our example reduces to the well known “random order proposer with risk of breakdown event equal to u” (See Binmore, Rubinstein and Wolinsky, 1986). The equilibrium conditions for this two-person case are

$$a^i\,(\rho) \in \partial V(N), \;\; (i \in N),$$

and

$$a_j^i(\rho) = \rho a_j(\rho) + (1-\rho)u_j,\ (j \neq i).$$

It is easy to check that with this conditions it holds that

$$\begin{aligned}\left(a_1^1(\rho) - u_1\right)\left(a_2^1(\rho) - u_2\right) &= \left(a_1^2(\rho) - u_1\right)\left(a_2^2(\rho) - u_2\right) = \\ &\quad (2-\rho)\rho\left(a_1(\rho) - u_1\right)\left(a_2(\rho) - u_2\right).\end{aligned}$$

Therefore, we fall into the classical implementation of the Nash Bargaining solution from the disagreement point u. That is, when $\rho \to 1$, we have that $a^i(\rho) \to a(\rho)$ for $i = 1, 2$, and $a(\rho) \to (50, 50)$, which, in our example, is the point that maximizes the product of the utility gains from the reference point u (see Figure 3).

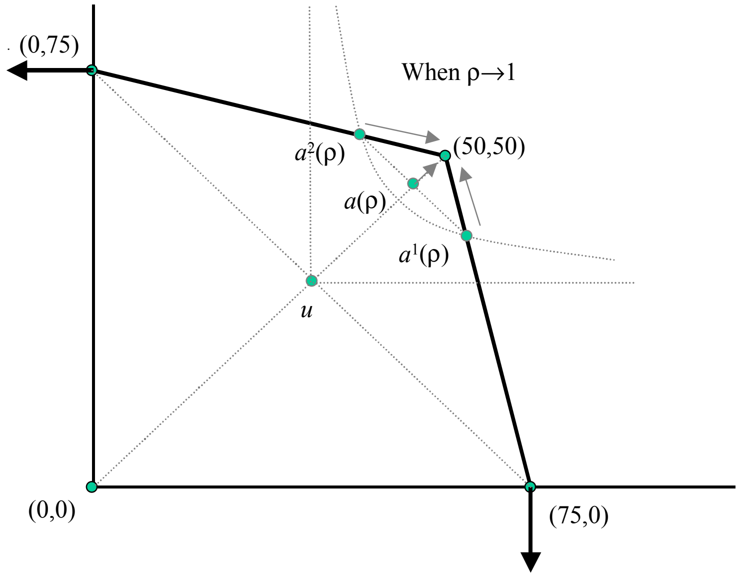

Figure 3:

Although in this example our procedure yields, in the limit, a point that belongs to the set of Consistent NTU-value allocations, i.e. the point $c^3 = (50, 50)$, this is not true in general (see Remark 3, in Section 4 above).

In next two Sections we extend this procedure for more than two players.

3 Breakdown

Let $N = \{1, ..., n\}$ be a finite set of players. A *coalition* is a subset of N. If $x, y \in \mathbb{R}^N$, we write $x \geq y$ if $x_i \geq y_i$ for all $i \in N$, and $x > y$ if $x_i > y_i$ for all $i \in N$. If $x \in \mathbb{R}^N$ and $\emptyset \neq S \subset N$, we write x_S as the restriction of x to S, i.e., $x_S = (x_i)_{i \in S} \in \mathbb{R}^S$. Let $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N \mid x \geq 0\}$ and $\mathbb{R}^N_{++} := \{x \in \mathbb{R}^N \mid x > 0\}$. Let $A \subset \mathbb{R}^N$, A is called *comprehensive* if $A - \mathbb{R}^N_+ \subset A$. The boundary of A is denoted by ∂A. We say that the boundary is *non level* if for all $x \in \partial A$ it holds that $\{x\} - \mathbb{R}^N_+ \cap \partial A = \{x\}$.

A *non-transferable utility game* (NTU-game for short), is a pair (N, V), where N is a finite set of players, and V is a map assigning to each coalition S, $\emptyset \neq S \subset N$, a subset $V(S) \subset \mathbb{R}^S$ of *attainable payoff vectors* such that

(A.1) $V(S)$ is non-empty, closed, convex and comprehensive.

(A.2) $\partial V(S)$ is non level.

(A.3) $0_S \in V(S)$ and $V_0(S) := V(S) \cap \mathbb{R}^S_+$ is bounded.

(A.4) Monotonicity: $V_0(S) \times \{0_{T \setminus S}\} \subset V_0(T)$ whenever $S \subset T$.

The last assumption A.4 is just the extension on NTU-games of the classical Monotonicity assumption for TU-games. The family of all NTU-games on N satisfying A.1-A.4 will be denoted by G^N.

For each $i \in N$, let $v_i := \max\{x : x \in V(i)\}$. Denote by $v = (v_i)_{i \in N} \in \mathbb{R}^N$. Some particular classes of NTU-games have been played a relevant role in the analysis: The *transferable utility* games (TU-games), when there is a real-valued function $v(\cdot)$ such that $V(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\}$ for all $S \subset N$. The *Hyperplane* games (H-games), when $\partial V(S)$ is a hyperplane for all $S \subset N$. And *Pure Bargaining* games (PB-games), when $v = 0 \in V(N)$, and $v_S = 0_S \in \partial V(S)$, for all $S \neq N$.

A *payoff configuration* is a family $\mathbf{p} = \left(p^S\right)_{S \subset N}$ where $p^S \in \mathbb{R}^S$ for all $S \subset N$. For notational simplicity, we use $S\backslash i$ and $S \cup i$ instead of $S\backslash\{i\}$ and $S \cup \{i\}$ respectively.

We describe here the general rules of the multilateral bargaining procedure in which the solutions will be based.

• "In each *round* there is a set $S \subset N$ of *active* players. When we deal with single coalitions, $S = \{i\}$, $i \in N$, every player obtains its own value v_i.

• In the first round, the active set is $S = N$.

• Every round, with active player set S, ($|S| \geq 2$), is divided in two *Stages*: The *Negotiation* stage and the *Breakdown* stage.

• *Negotiation stage.* We don't want to be precise at this point. Here players make offers that are accept or reject by the other players following some specified rules. The only important at this step is that we can exit from this stage only in two ways: Either, at some moment one player makes an offer that is accepted, then the game finish with an *agreement* and this offer is implemented; either, after some rejection, the stage stops without agreement and we go to the *Breakdown stage.*

• *Breakdown stage.* Choose a player i at random from S using a uniform distribution. Then i plays the following *ultimatum game,* $U(i, S)$:

– Player i proposes a payoff vector $u^{i,S} \in V(S)$.

– Other players in $S\backslash i$ are asked if they agree or dissent.

- All agree $\Rightarrow$ $u^{i,S}$ is implemented.

- Any player dissents $\Rightarrow$ player i leaves the game, receiving a payoff of cero, and go to a new round with active players set $S\backslash i$."

It is worth noting that we have making the implicit assumption that the utilities are previously normalized in such a way that when every player leaves the game the payoff that he/she obtains is cero. Therefore, Monotonicity allows the chance to reach benefit agreements.

The next Lemma gives the payoffs associated to every ultimatum game defined in $(N, V) \in G^N$.

Lemma 1 *Let* $U(i, S)$ *be an ultimatum game for player* i *in* S, $|S| \geq 2$. *Assume that* $p^{S\backslash i} \in V_0(S\backslash i)$ *is the unique payoff vector that players in* $S\backslash i$ *expect to obtain in the round with active set* $S\backslash i$. *Then* $u^{i,S}$, *defined as the vector in* $\mathbb{R}^S$ *such that* $u^{i,S} \in \partial V(S)$ *and* $u_j^{i,S} = p_j^{S\backslash i}$ *for all* $j \in S\backslash i$, *is the unique payoff vector associated to any subgame perfect equilibria (SPE) of* $U(i, S)$.

Proof. First note that under the Monotonicity assumption, if $p^{S\backslash i} \in V_0(S\backslash i)$ it holds that $u_i^{i,S} \geq 0$. By construction, $u^{i,S}$ is unique and $u^{i,S} \in V_0(S)$. The players j in $S\backslash i$ only will accept offers such that $a_j^{i,S} \geq p_j^{S\backslash i}$. Therefore, if $u_i^{i,S} > 0$, the best player i can do is to offer $a^{i,S} = u^{i,S}$, that will be accepted for all $j \in S\backslash i$. If $u_i^{i,S} = 0$, player i is indifferent between either offering $u^{i,S}$ that will be accepted, and offering a different proposal $a^{i,S} \neq u^{i,S}$ such that $a_i^{i,S} > 0$. In this latter case, some player $j \neq i$ necessarily must have $a_j^{i,S} < p_j^{S\backslash i}$, because A.1 and A.2 imply that $\partial V(S)$ coincides with the Pareto frontier of $V(S)$, therefore $a^{i,S}$ will be rejected by player j. In both cases i will obtain 0 and the rest of players $j \in S\backslash i$ obtain $p_j^{S\backslash i}$, which again coincides with $u^{i,S}$[6]. ■

We define the *breakdown point* u^S by

$$u^S := \sum_{i \in S} \frac{1}{|S|} u^{i,S} \tag{5}$$

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[6] In this indiffernt case, also mixed strategies yield the same outcome.

Note that, because all $u^{i,S} \in \partial V_0(S)$, the convexity of $V(S)$ implies that $u^S \in V_0(S)$. If the rules of the negotiation stage are such that for subcoalitions $S \backslash i$ $(i \in S)$ yield unique expected payoffs $p^{S \backslash i} \in V_0(S \backslash i)$, we have that u^S are the *expectations* for players in S when breakdown happens.

Given this breakdown rule, some results can be advanced for particular games.

Let us assume that the rules of the negotiation stage are defined in such a way that the equilibrium payoffs p^S are always greater or equal than the breakdown payoffs u^S. In this case we have the next straightforward Theorem.

Theorem 2 *Let* $\mathbf{p} = \left(p^S\right)_{S \subset N}$ *be the equilibrium payoffs configuration associated to the multilaterateral bargaining, such that* $p^S \geq u^S$, *for all* $S \subset N$, *Then* $\mathbf{p}$ *coincides with the* Consistent *NTU-value for Hyperplane games, and with the* Shapley *value for TU-games.*

Proof. In Hyperplane games $u^S \in \partial V(S)$, hence $p^S = u^S$, for all $\emptyset \neq S \subset N$. This means that

$$p_i^S = \frac{1}{|S|} u_i^{i,S} + \sum_{j \in S \backslash i} \frac{1}{|S|} p_i^{S \backslash j}, \ (i \in S). \tag{6}$$

The formula above determines recursively the *Consistent* NTU-value of Maschler and Owen (1989) for H-games. In TU-games, $u_i^{i,S} = v(S) - v(S \backslash i)$, therefore formula (6) yields the *Shapley* (1953) value for TU-games. ■

Remark 1. If we wish a true implementation (that is, not in expectations) we only need to consider either players break ties in favor of quick termination of the game, or the player responsible for the breakdown stage (player who *has made* the last offer rejected) must to pay to the Referee a positive amount of money. We will make the same assumptions for the negotiation stages of the three models considered below.

In the next sections we consider three different bargaining procedures in the negotiation stage.

4 Alternating offers with risk of breakdown

Perhaps one of the most popular models of bargaining is the alternating offers with risk of breakdown (first introduced in Binmore, Rubinstein and Wolinsky, 1986). In that context, breakdown enforces the status quo payoffs. The remarkable result obtained is that, when the probability of breakdown goes to cero, the equilibrium payoffs converge towards the Nash bargaining solution.

This alternating offers procedure was also the Hart and Mas-Colell's model chosen for the Negotiation stage. Formally:

- *Negotiation stage.* "Assume that $S \subset N$ $(|S \geq 2|)$ is the active players set and let ρ be a fixed parameter, $0 \leq \rho < 1$.

Choose a player i at random from S using a uniform distribution.

Player i proposes a payoff vector $a^{i,S} \in V(S)$.

Other players are asked if they agree or dissent.

  - All agree $\Rightarrow$ $a^{i,S}$ is implemented.
  - Any player dissents $\Rightarrow$
    - $\star$ With probability ρ , repeat the process.
    - $\star$ With probability $1 - \rho$, go to Breakdown stage."

It is worth noting that this way to define the Negotiation stage is completely consistent with the Breakdown stage. Each time is chosen a proposer randomly: If the offer is rejected, with probability ρ next time we repeat the process, and with probability $1 - \rho$ *next time will be the last time* in which a random proposer will be chosen.

The negotiation stage have potentially infinitely many periods, and with more than two active players it is well known that many subgame perfect equilibria strategies appear. Hence, as usual, we restrict to consider only *stationary* strategies. The characterization of the stationary subgame perfect equilibrium (SSPE) strategies of the Negotiation stages are given by the next Lemma.

Lemma 3 *Let a round with active player set $S \subset N$ ($|S| \geq 2$), the proposals corresponding to an SSPE in the negotiation stage are always accepted, and they are characterized by:*

$$\begin{aligned} a^{i,S}(\rho) &\in \partial V(S) \text{ for all } i \in S, && (n.1) \\ a_j^{i,S}(\rho) &= \rho a_j^S(\rho) + (1-\rho) u_j^S(\rho) \text{ for all } i, j \in S \ (i \neq j), && (n.2) \end{aligned}$$

where $a^S(\rho) = \frac{1}{|S|}\sum_{i \in S} a^{i,S}(\rho)$, and $u^S(\rho)$ is the breakdown point. Moreover, $a_i^{i,S}(\rho) \geq u_i^S(\rho) \geq 0$, for all $i \in S$, and $a^S(\rho) \geq u^S(\rho)$.

The proposition says that i makes offers such that he will obtain his maximum payoff compatible by giving to the rest of players what they would expect to obtain if the offer were rejected

Proof. The proof of this Lemma is just a straightforward adaptation of the arguments given in Hart and Mas-Colell (1996, Proposition 1). The only differences are that, in case of breakdown, player j will obtain now $u_j^S(\rho)$ instead of $a_j^{S \setminus i}(\rho)$, because the different way in which it is defined. Starting with $|S| = 2$, it is easy to see that Monotonicity implies that $u^S(\rho) \geq 0$, and the following strategy will guarantee to i a payoff of at least $a_i^S(\rho) \geq u_i^S(\rho)$: accept only if offered at least $u_i^S(\rho)$ and, when proposing, propose $u^S(\rho) \in V(S)$. This implies that $a_i^{i,S}(\rho) \geq u_i^S(\rho)$, and $a_i^{j,S}(\rho) \geq u_i^S(\rho)$. Therefore it holds that $a^{i,S}(\rho) \geq u^S(\rho)$, and $a^S(\rho) \geq u^S(\rho)$. The rest of the proof is left to the reader. ■

We define now the payoff configuration that will be supported in the limit by the equilibria of the bargaining game. The construction is made recursively, starting with single coalitions $\{i\}$, for all $i \in N$, up to the grand coalition N.

Definition 1 *The payoff configuration* $\boldsymbol{\alpha}= \left(\alpha^S\right)_{S\subset N}$ *is defined by:*

(α.i) $\alpha^i = v_i$ *for all* $i \in N$.

(α.ii) For all $S \subset N$, *with* $|S| \geq 2$, *let* $u^S := \frac{1}{|S|}\sum_{i\in S} u^{i,S}$, *where, for all* $i \in S$, $u^{i,S}$ *are the vectors in* $\mathbb{R}^S$ *such that* $u^{i,S} \in \partial V(S)$ *and* $u_j^{i,S} = \alpha_j^{S\backslash i}$ *for all* $j \in S\backslash i$.

(α.iii) Let α^S *given by*

$$\alpha^S := \arg\max \prod_{\substack{x\geq u^S \\ x\in V(S)}} \left(x_i - u_i^S\right),$$

for all $S \subset N$, *with* $|S| \geq 2$.

Note that under the regularity conditions imposed to (N, V), each $u^{i,S}$ is unique, and if $\alpha^{S\backslash i} \in V_0(S\backslash i)$, by Monotonicity, we have that $u_i^{i,S} \geq 0$, and then $u^{i,S} \in V_0(S)$. Moreover, by Convexity, $u^S \in V_0(S)$. The point α^S is the Nash bargaining solution for the PB-problem $(u^S, V_0(S))$, where $V_0(S)$ is the feasible set, and u^S play the role of the disagreement point. We have that α^S is unique, $\alpha^S \geq u^S$ and $\alpha^S \in \partial V_0(S)$.

Remark 2. It should be clear that Monotonicity is not needed in order to define the payoff configuration $\boldsymbol{\alpha}$. Monotonicity is needed to guarantee that $u_i^{i,S} \geq 0$ because $\alpha^{S\backslash i} \in V_0(S\backslash i)$, that will be of importance only when the ultimatum game $U(i, S)$ be played in the breakdown stage. Convexity guarantee the uniqueness of α^S.

We are in condition to establish the main result of this Section.

Theorem 4 *Suppose that* (N, V) *is an NTU-game satisfying the regularity conditions A.1-A.4, with the additional Smoothness assumption: For each* $S \subset N$, *at each point of* $\partial V(N)$ *there is a single outward normal direction. Then for each* $0 \leq \rho < 1$ *there is an SSP equilibrium. Moreover, as* $\rho \rightarrow 1$ *every SSP equilibrium payoff configuration* $\mathbf{a}(\boldsymbol{\rho})$ *converges to* $\boldsymbol{\alpha}$.

Proof. This is almost a corollary of previous results in the Nash bargaining implementation literature[7]. First note that, by Monotonicity, Lemmas (1) and (3) altogether yield that $u^S(\rho) \in V_0(S)$, $a^{i,S}(\rho) \in \partial V_0(S)$, for all $i \in S$, and $a^S(\rho) \in V_0(S)$ for all $S \subset N$. Therefore, let $(M, ..., M) \in \mathbb{R}_+^N$ be an upper bound of $V_0(N)$, then $\left| a_j^{i,S}(\rho) - a_j^S(\rho) \right| \leq M(1-\rho)$ for all $i, j \in S$, and all $S \subset N$. Therefore, when $\rho \to 1$, $a^{i,S}(\rho) \to a^S(\rho)$ for all $i \in S$. Moreover, it can be checked that

$$\prod_{j \in S} \left(a_j^{i,S}(\rho) - u_j^S(\rho) \right) = \rho^{(|S|-1)} \left(|S| - (|S|-1)\rho \right) \prod_{j \in S} \left(a_j^S(\rho) - u_j^S(\rho) \right), \quad (i \in S). \qquad (7)$$

Hence, the equilibrium offers $a^{i,S}(\rho)$ are in the same contour set of $\prod_{j \in S} \left(x_j - u_j^S(\rho) \right)$. The only difference with the implementation of the Nash bargaining solution α^S of the PB-problem $(u^S, V_0(S))$, is that the disagreement point $u^S(\rho)$ is not fixed. But following an straightforward induction argument, when $\rho \to 1$, we have that $u^S(\rho) \to u^S$, and then $a^{i,S}(\rho) \to \alpha^S$, for all $S \subset N$. Note here that for $|S \geq 3|$ the set of equilibrium offers $a^{i,S}(\rho)$ are not necessarily unique; and without smoothness on $\partial V(S)$, the convergence to α^S may fail[8]. ■

Remark 3. It is clear that in PB-games, α^N do not coincide with the Nash solution, because the maximization of the product of utility gains is taken from the breakdown point, instead the disagreement point as in the Nash solution (see Figure 4).

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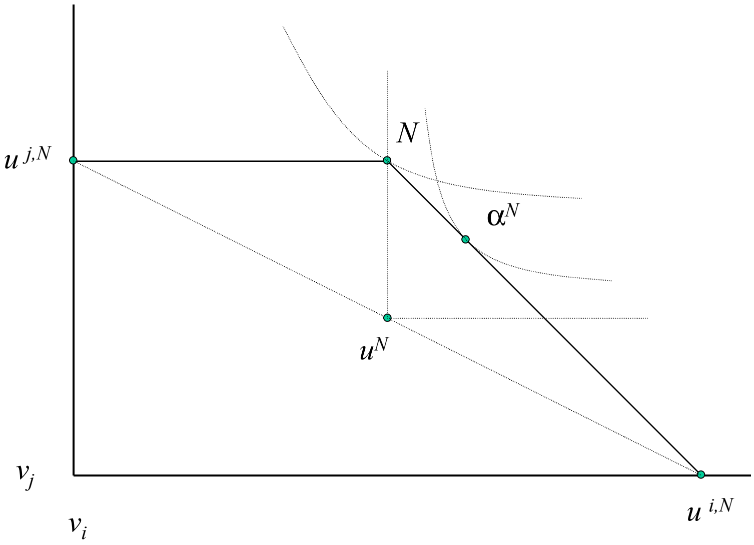
[7] See for example Hart and Mas-Colell (1996), Theorem 3; and Thomson and Lensberg (1989).

[8] See the discussion of the Theorem 3, in Hart and Mas-Colell (1996).



Moreover, α^N depends of the breakdown point u^N which depends of points $u^{i,N}$, hence α^N don't satisfies the Independence of Irrelevant Alternatives axiom. An early work in which the breakdown point is used to compute the maximization of the utility gains product can be found in Roth (1977).

Remark 4. Asymmetric solutions can be defined easily. We only need to assume that players are chosen to be the proposer with different probabilities. Let $w \in \mathbb{R}^N_{++}$ be a vector of weights such that players are chosen in proportion to these weights.

Definition 2 *The payoff configuration* $\boldsymbol{\alpha}[w] = \left(\alpha[w]^S\right)_{S\subset N}$ *is defined by:*
(α.i) $\alpha[w]^i = v_i$ *for all* $i \in N$.
(α.ii) For all $S \subset N$, *with* $|S| \geq 2$, *let* $u[w]^S := \sum_{i\in S} \frac{w_i}{\sum_{j\in S} w_j} u[w]^{i,S}$, *where, for all* $i \in S$, $u[w]^{i,S}$ *are the vectors in* $\mathbb{R}^S$ *such that* $u[w]^{i,S} \in \partial V(S)$ *and* $u[w]^{i,S}_j = \alpha[w]^{S\backslash i}_j$ *for all* $j \in S\backslash i$.

(α.iii) Let $\alpha[w]^S$ given by

$$\alpha[w]^S := \arg\max_{\substack{x \geq u^S(w) \\ x \in V(S)}} \prod \left(x_i - u[w]_i^S\right)^{w_i},$$

for all $S \subset N$, with $|S| \geq 2$.

Now, for any $0 \leq \rho < 1$, the conditions that characterizes the equilibrium offers are given by

$$\begin{aligned} a[w]^{i,S}(\rho) &\in \partial V(S) \text{ for all } i \in S, && \text{(wn.1)} \\ a[w]_j^{i,S}(\rho) &= \rho a[w]_j^S + (1-\rho) u[w]_j^S(\rho) \text{ for all } i, j \in S \ (i \neq j), && \text{(wn.2)} \end{aligned}$$

where $a[w]^S(\rho) = \sum_{i \in S} \frac{w_i}{\sum_{j \in S} w_j} a^{i,S}[w](\rho)$. Then, when $\rho \to 1$, we have that $a[w]^{i,S}(\rho) \to \alpha[w]^S$, for all $S \subset N$.

In particular, when (N, V) is a H-game, it holds that $\alpha[w]^S = u[w]^S$ for all $S \subset N$, and therefore, $\alpha[w]^S$ coincides with the *weighted Shapley* (1953) *value* for TU-games (see Kalai and Samet, 1985; and Hart and Mas-Colell, 1989), and with the *weighted Consistent NTU-value* for H-games (See Maschler and Owen, 1989; and Calvo, García and Zarzuelo, 2001).

5 Alternating offers with finite horizon

Replace, in the negotiation stage, the probability of breakdown ρ by a finite time horizon T in which offers can be made. We have this alternative procedure.

- *Negotiation stage.* "Assume that $S \subset N$ ($|S \geq 2|$) is the active players set and let T be a positive integer, $0 < T < \infty$.

Every time t, $t = 1, 2, ..., T$, choose a player i at random from S using a uniform distribution.

Player i proposes a payoff vector $a^{i,S}(t, T) \in V(S)$.

Other players are asked if they agree or dissent.

- All agree $\Rightarrow a^{i,S}(t, T)$ is implemented.
- Any player dissents $\Rightarrow$
  - $\star$ If $t < T$, repeat the process at time $t + 1$.
  - $\star$ If $t = T$, go to breakdown stage."

See again that the breakdown stage can be interpreted as *the last time* $(T + 1)$ in which a random proposer will be chosen. This mechanism applied to PB-games yields, when $T \to \infty$, the *Discrete Raiffa solution*, appeared first in Luce and Raiffa (1957), §6.7. It was conceived as a iterated process of pick up a dictator at random, with equal probabilities, building a sequence of gradual agreements, finding in the limit an efficient point in the boundary. It appear also in Moulin (1984) with the name of *finitely repeated random dictator*. See also Sjöström (1991), Mas-Colell (1997), and Gomes, Hart and Mas-Colell (1999). In the general case of NTU-games, this bargaining model appeared in de Clippel (2002), implementing a payoff allocation called as the *Procedural value*. For completeness, we show only the main results. The proofs are in the de Clippel's paper.

The negotiation stage have at most T periods, hence it can be solved by backwards induction, by using Subgame Perfect Equilibrium (SPE) strategies. Their characterization are given by the next Lemma.

Lemma 5 *Let a round with active player set* $S \subset N$ $(|S| \geq 2)$. *The proposals, for every time* $t \leq T$, *corresponding to an SPE in the Negotiation stage are always accepted, and* *they are characterized by:*

$$a^{i,S}(t,T) \in \partial V(S) \text{ for all } i \in S, \tag{r.1}$$

$$a_j^{i,S}(t,T) = a_j^S(t+1,T) \text{ for all } i,j \in S \ (i \neq j), \tag{r.2}$$

where $a^S(t,T) = \frac{1}{|S|}\sum_{i\in S} a^{i,S}(t,T)$, *for all* $t \leq T$, *and* $a^S(T+1,T) = u^S(T)$ *is the breakdown point defined by* $u^S(T) := \frac{1}{|S|}\sum_{i\in S} u^{i,S}(T)$, *where, for all* $i \in S$, $u^{i,S}(T)$ *are the vectors in* $\mathbb{R}^S$ *such that* $u^{i,S}(T) \in \partial V(S)$ *and* $u_j^{i,S}(T) = a_j^{S\setminus i}(1,T)$ *for all* $j \in S\setminus i$. *Moreover,* $a_i^{i,S}(t,T) \geq a_i^S(t+1,T) \geq 0$, *for all* $i \in S$, *and* $a^S(t,T) \geq a^S(t+1,T)$.

As illustration, in Figure 5 it is shown the equilibrium offers in case $N = \{1,2\}$.

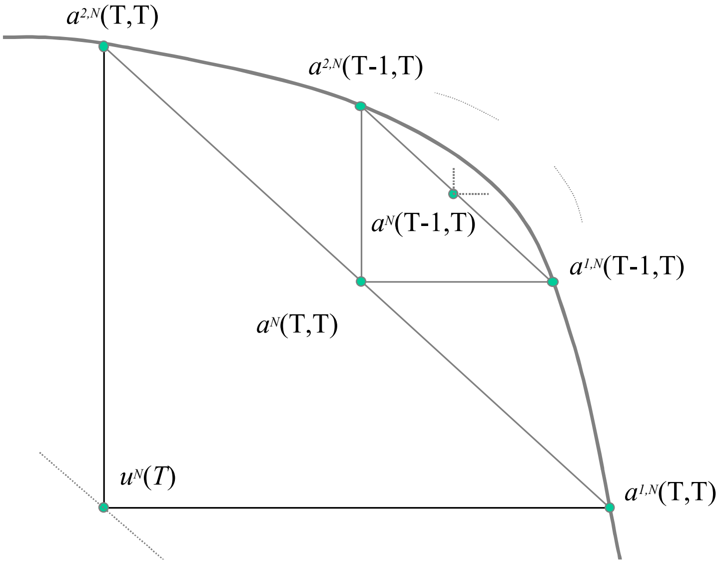


Figure 4:

It is worth noting that the equilibrium offers $\left(a^{i,S}(t,T)\right)_{i\in S}$ are always unique, for every t and S, unlike the equilibrium offers $\left(a^{i,S}(\rho)\right)_{i\in S}$ in the risk of breakdown model.

Also only subgame perfect strategies are needed, instead of the stronger requirement of stationary strategies used in the first model.

Definition 3 *The payoff configuration* $\mathbf{r} = \left(r^S\right)_{S \subset N}$ *is defined by:*
(r.i) $r^i = v_i$ *for all* $i \in N$.
(r.ii) For all $S \subset N$, *with* $|S| \geq 2$, *let* $u^S := \frac{1}{|S|}\sum_{i \in S} u^{i,S}$, *where, for all* $i \in S$, $u^{i,S}$ *are the vectors in* $\mathbb{R}^S$ *such that* $u^{i,S} \in \partial V(S)$ *and* $u_j^{i,S} = r_j^{S \backslash i}$ *for all* $j \in S \backslash i$.
(r.iii) Let $r^S(T) := a^S(1,T)$ *and*

$$r^S := \lim_{T \to \infty} r^S(T),$$

for all $S \subset N$, *with* $|S| \geq 2$.

Under the regularity assumptions in G^N, the payoff configuration $\mathbf{r}$ is well defined and $r^S \in \partial V(S)$, for all $S \subset N$. Also Monotonicity is not needed in the construction of $\mathbf{r}$, only applies for the resolution of the game.

Theorem 6 *Suppose that* (N, V) *is an NTU-game satisfying the regularity conditions A.1-A.4. Then for each* $0 \leq T < \infty$ *there is an SP equilibrium. Moreover, as* $T \to \infty$ *the SP equilibrium payoff configuration* $\mathbf{r}(T) = \left(r^S(T)\right)_{S \subset N}$ *converges to* $\mathbf{r}$.

Remark 5. Asymmetries can be considered in the same way as in the previous model, yielding again weighted Shapley values in TU-games and weighted Consistent values in H-games.

Remark 6. A combination of the two previous models can be done also. That is, the Negotiation stage game depends of two parameters: The probability of breakdown $1 - \rho$, and the finite horizon T. This approach was first considered in Gomes, Hart and Mas-Colell (1999), studying a finite horizon version of the bargaining model introduced in Hart and Mas-Colell (1996). All their results can also be translated here in a natural way. We only define the Negotiation stage and the characterization of the equilibrium proposals.

- *Negotiation stage.* "Assume that $S \subset N$ ($|S \geq 2|$) is the active players set and let T be a positive integer, $0 < T < \infty$, and ρ be a fixed parameter, $0 \leq \rho < 1$.

Every time t, $t = 1, 2, ..., T$, choose a player i at random from S using a uniform distribution.

Player i proposes a payoff vector $a^{i,S}(t, \rho, T) \in V(S)$.

Other players are asked if they agree or dissent.

- All agree $\Rightarrow a^{i,S}(t, \rho, T)$ is implemented.
- Any player dissents $\Rightarrow$
  - $\star$ If $t < T$, $\diamond$ With probability ρ, repeat the process at time $t + 1$.
    - $\diamond$ With probability $1 - \rho$, go to Breakdown stage.
  - $\star$ If $t = T$, go to Breakdown stage."

Note that this is a finite horizon game, hence it is solved by backwards induction and the equilibrium proposals are given by next Lemma.

Lemma 7 *Let a round with active player set $S \subset N$ ($|S| \geq 2$). The proposals, for every time $t \leq T$, corresponding to an SPE in the Negotiation stage are always accepted, and they are characterized by:*

$$a^{i,S}(t, \rho, T) \in \partial V(S) \;\; \textit{for all } i \in S, \tag{c.1}$$

and for all $i, j \in S$ ($i \neq j$):

$$\begin{aligned} a_j^{i,S}(t, \rho, T) &= \rho a_j^S(t+1, \rho, T) + (1-\rho)\, u_j^S(\rho, T), \textit{ for } t < T, & \text{(c.2)} \\ a_j^{i,S}(t, \rho, T) &= u_j^S(\rho, T), \;\; \textit{for } t = T, & \text{(c.3)} \end{aligned}$$

where $a^S(t,\rho,T) = \frac{1}{|S|}\sum_{i\in S} a^{i,S}(t,\rho,T)$, *for all* $t \leq T$, *and* $u^S(\rho,T)$ *is the breakdown point defined by* $u^S(\rho,T) := \frac{1}{|S|}\sum_{i\in S} u^{i,S}(\rho,T)$, *where, for all* $i \in S$, $u^{i,S}(\rho,T)$ *are the vectors in* $\mathbb{R}^S$ *such that* $u^{i,S}(\rho,T) \in \partial V(S)$ *and* $u_j^{i,S}(\rho,T) = a_j^{S\backslash i}(1,\rho,T)$ *for all* $j \in S\backslash i$. *Moreover,* $a_i^{i,S}(t,\rho,T) \geq a_i^S(t+1,\rho,T) \geq 0$, *for all* $i \in S$, *and* $a^S(t,\rho,T) \geq a^S(t+1,\rho,T)$, *for all* $t \leq T$.

6 Auctioning counteroffer's chance

Perhaps the second most cited solution for PB-problems is the so called Kalai-Smorodinsky (1975) solution. In this solution the gains of cooperation are shared in proportion with the *utopia* payoffs, that is, the maximal utility that each player can gets compatible with the disagreement payoffs for the rest of players. In this Section we offer an extension of this solution on NTU-games.

The first implementation of the KS solution was given in Moulin (1984), with the name of "*auctioning of fractions of a dictatorship*". One important difference with the previous negotiation models considered above is that the Moulin's game yield an *exact* implementation of the KS solution, in contrast with the *approximate* implementation of the Nash and Raiffa solutions. The Negotiation stage that we present now is inspired by the Moulin's game. Informally, first players auction to determine who makes a *fair* offer. The auction is over probabilities, and wins the player who announces the highest probability p^*. If this offer is rejected, a player is selected to make a counteroffer, where the player that rejects the offer is selected with probability p^*. If this counteroffer is also rejected, then go to the Breakdown stage[9].

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[9] At first glance the model sounds different from the Moulin's game, but it is mathematically equivalent. The changes of this version are done only to be more in the spirit of "acceptance/rejection" offers

- *Negotiation stage.* "Assume that $S \subset N$ ($|S \geq 2|$) is the active players set.

Round 1. Each player i in S makes a bid $p_i \in \mathbb{R}$, being $0 \leq p_i \leq 1$. Next, the players are renumbered in decreasing order of their bids, i.e. $p_1 \geq p_2 \geq ... \geq p_s$. Players with tied bids are ordered randomly among themselves.

Round 2. Player 1 makes a feasible offer $a^{1,S} \in V(S)$, and the rest of the players in S are asked sequentially, starting first with player s, second with $s-1$, and so on till player 2, whether they accept or reject the offer. If all players accept, $a^{1,S}$ is implemented. If any player rejects the offer, the game goes to round three.

Round 3. Let player k be the first player who rejects the offer. Choose a player i at random from S, where the probabilities to be chosen are: p_1 for player k, and $\frac{1-p_1}{s-1}$ for every $i \in S \backslash k$. The chosen player, say i, makes a feasible counteroffer $b^{i,S} \in V(S)$. If all agree, $b^{i,S}$ is implemented. If any player dissent, go to the Breakdown stage."

Note that in this way, the only difference between the Round 3 and the Breakdown stage is given by the probability distribution. In the Breakdown stage we use the uniform distribution, whereas in Round 3 the chance to be selected, for the player who rejects the fair offer, is determined by the winner player in the auction round. The fairness of the first offer follows from the fact that the player that wins the auction is who is more ready to give a chance to make a counteroffer to player who dissents.

We define first the payoff configuration that will be the extension of the KS solution.

Definition 4 *The payoff configuration* $\mathbf{k} = \left(k^S\right)_{S \subset N}$ *is defined by:*
(k.i) $k^i = v_i$ *for all* $i \in N$.
For all $S \subset N$, *with* $|S| \geq 2$:
(k.ii) Let $u^S := \frac{1}{|S|} \sum_{i \in S} u^{i,S}$, *where, for all* $i \in S$, $u^{i,S}$ *are the vectors in* $\mathbb{R}^S$ *such that*

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technology applied in the previous Sections.

$u^{i,S} \in \partial V(S)$ and $u_j^{i,S} = k_j^{S\backslash i}$ for all $j \in S\backslash i$.

(k.iii) Let $b^{i,S} \in \partial V(S)$ such that $b_j^{i,S} = u_j^S$, for all $j \in S\backslash i$.

(k.iv) Let $k^S \in \partial V(S)$ and $p^* \in \mathbb{R}_+$ such that

$$k_i^S = u_i^S + p^* \left(b_i^{i,S} - u_i^S \right),$$

for all $i \in S$.

For games in G^N the vectors k^S are unique, and, again, Monotonicity is not needed for its definition. By non-levelness, if $u^S \in \operatorname{int} V(S)$ it holds that $0 < p^* < 1$. Moreover, $k^S \geq u^S$.

Theorem 8 *Suppose that (N, V) is an NTU-game satisfying the regularity conditions A.1-A.4. Therefore, $\mathbf{k} = \left(k^S\right)_{S \subset N}$ is the payoff configuration associated to any SPE equilibrium of the bargaining game.*

Proof. Assume by induction that the payoffs associated to any $T \subsetneq S$ are already given by $\left(k^T\right)_{T \subsetneq S}$. First note that by offering u^S and rejecting any offer different from u^S, in Rounds 2 and 3, every player i can enforce the breakdown payoff u_i^S. Hence the expected payoffs in the Negotiation stage, e^S, must be greater or equal than u^S. Then, if $u^S \in \partial V(S)$ it holds that $e^S = u^S = k^S$.

Assume that $u^S \in \operatorname{int} V(S)$. In Round 3, if any player i is chosen to make a counteroffer, will offer $b^{i,S}$ satisfying $(k.iii)$ and this offer will be accepted by the others. In Round 2, the fair offer $c(p_1)$ of player 1 (winner) will be accepted for all players i, $i > 1$, if and only if

$$\begin{aligned} c_i(p_1) &\geq p_1 b_i^{i,S} + \sum_{j \in S\backslash i} \frac{(1-p_1)}{(s-1)} b_i^{j,S} = p_1 b_i^{i,S} + (1-p_1)\, u_i^S \\ &= u_i^S + p_1 \left(b_i^{i,S} - u_i^S \right). \end{aligned}$$

Let $c(p_1) \in \partial V(S)$ such that $c_i(p_1) = u_i^S + p_1\left(b_i^{i,S} - u_i^S\right)$, for all $i \in S\backslash 1$. Therefore, $c_1(p_1)$ is a concave and strictly decreasing function for $p_1 \in [0,1]$, with $c_1(0) = b_1^{1,S}$. Moreover, for p^* satisfying $(k.iv)$, it holds that $c_1(p^*) = k_1^S$, and for all $p_1 > p^*$ we have that $c_1(p_1) < k_1^S$. Then it is immediate to check that, in any SP equilibrium, at Round 1 all the bids are $p_i \leq p^*$, and at least two players make a bid equal to p^*, and in Round 2, the winner player makes a fair offer $a^{1,S} = c^1(p^*) = k^S$ that is accepted for the rest of players. ■

Note also that in the game reduced to Round 1, there is only one Strong Nash equilibria (Aumann, 1959) in which all players make a bid equal to p^*; and this strong equilibria is of "max min" type. To see this last, note that if a player i makes a bid $p_i' < p^*$, the worst it can happen is that the rest of players tie with the same bid p_i', and i is not the winner. In this case i receives $c_i(p_1') < k_i^S$. If i bids $p_i'' > p^*$, the worst it can happen is that i be the winner, and again he will obtain less than k_i^S.

Remark 7. In PB-games, the point k^N do not coincides with the KS solution point, that is defined by $KS^N \in \partial V(N)$ such that

$$KS_i^S = v_i + p^*\left(v_i^{i,N} - v_i\right) = p^* v_i^{i,N},$$

for all $i \in N$, where $p^* \in \mathbb{R}_+$, and all $v^{i,N}$ satisfy that $v^{i,N} \in \partial V(N)$ and $v_j^{i,N} = v_j = 0$, for all $j \in N\backslash i$ (recall that in PB-games $v = 0$). See the difference in Figure 6 for the case $N = \{1,2\}$.

An early work in which the breakdown point, instead the disagreement point, is used to compute the utility gains was in Salonen (1987). There, the breakdown point is called the minimal compromise point.

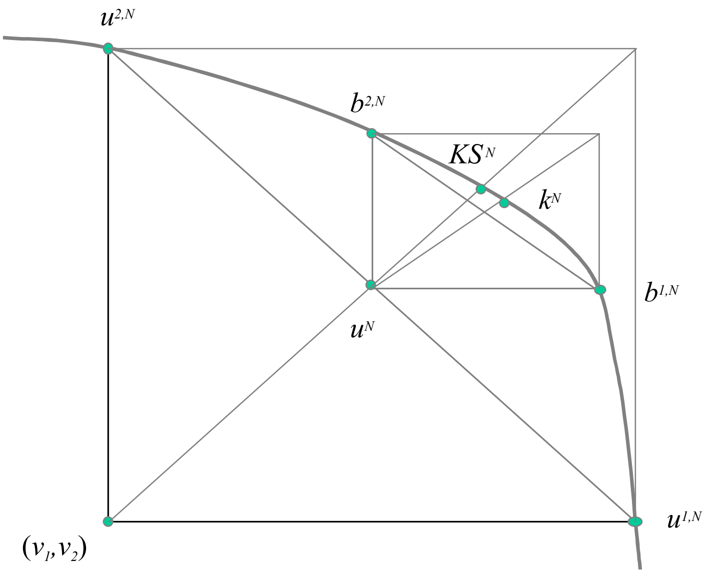

Figure 5:

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