CATEGORIZATION GENERATED BY PROTOTYPES – AN AXIOMATIC APPROACH

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Abstract. We present a model of categorization based on prototypes. A prototype is an image or template of an idealized member of the category. Once a set of prototypes is defined, entities are sorted into categories on the basis of the prototypes they are closest to. We provide a characterization of those categorizations that are generated by prototypes.

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1. Introduction

1.1. **Categorization – a general background.** Categorization is sorting things. Some entities are members of a particular category, others are not. For instance, anchovy, bull shark and flying fish belong to the category "fish".

Categorization is one of the most fundamental tasks human beings do. Whenever we use the word 'dog' to refer to two different animals, we actually perform acts of categorization. Categories are sets of entities to which we react in the same or a similar way. For instance, we smile at all their members, or we call all their members by the same name. Classifying the world around us into categories is an efficient way to store and to have quick access to a great deal of information using minimal resources. Indeed, if a creature belongs to the category 'dog', we know that it probably has tail, barks, and if you annoy it, may bite.

The classical perspective of categorization is that items are classified into their proper categories on the basis of features. Every category is characterized by a list of features. The entities belonging to some category are those, and only those, having all the appropriate features. For instance, mammals are creatures that (1) give birth, and (2) suckle their young. Therefore, a cat belongs to the category 'mammal'. On the other hand, sea perches of the Pacific coast give birth to living young, but do not suckle them and they are thus not considered mammals. A famous line that perfectly reflects this theory is:

> What kind of bird are you, if you cannot fly, said the little bird to the duck. What kind of bird are you, if you cannot swim, said the duck and dived (Sergei Prokofiev, Peter and the Wolf).

Ludwig Wittgenstein (1953) disagreed with this perspective. He examined the category of "game" and claimed that there is nothing common to board games, card games, ball games and Olympic games.

The work of Eleanor Rosch (see, Rosch and Lloyd (1978)) challenged the classical theory. She demonstrated that when people label an object, they rely less on abstract definitions than on a comparison with what they regard as the best representative of the category designated by that word. Specifically, she found that

(1) People cannot tell what features they rely on in order to categorize.
(2) People can identify some members of categories that are more typical than others.
(3) People categorize more typical members faster than less typical ones.

Rosch concluded that people do not perform a categorization on the basis of features. Rather, they categorize on the basis of how close something is to some prototypical or ideal member of a category. An anchovy is closer to the fish prototype than a Pacific coast sea perch, but both are closer to it than they are to the mammal prototype. Therefore we refer to them both as fish.

1.2. **Entities defined by attributes.** When one tries to categorize, one should first have a description of the entities to be categorized. This description should include the relevant information about the entities under consideration in order to allow the classifier to perform the act of categorizing. Entities lumped together probably have a similar description, while those belonging to different categories do not resemble each other.

Usually, such a description is composed of a list of attributes. Assume that a finite set of attributes is given. Every entity is defined by the intensity of each attribute. For instance, different people may be described by the different intensity of the attributes 'speaks Swahili', 'likes red wine' and 'dresses elegantly'. Formally, an entity is identified with a finite vector of non-negative numbers. Each coordinate of this vector corresponds to an attribute of the entity. When the number of attributes considered is d, the set of entities is the non-negative orthant of $\mathbb{R}^d$, $\mathbb{R}^d_+$. Throughout this paper we will assume that there are at least two attributes, that is, $d \geq 2$.

1.3. **Categorization as a partition and prototypes.** A categorization is a partition of the set of entities into pairwise disjoint sets. Each set of the partition contains the members of a different category. Thus, if the set of categories considered is $L = \{1, 2, ..., \ell\}$, and the set of entities is $\mathbb{R}^d_+$ then a categorization is a function $\varphi : \mathbb{R}^d_+ \to L$ which assigns a category to every entity. Different agents may categorize differently (i.e., use different assignment rules φ) even when the set of categories considered is the same. These differences can be the result of different experiences, different opinions, etc.

Our main interest in this paper is in a special kind of such assignment rules - those which are determined by a set of prototypes. A prototype is a distinguished member of a category. We say that a categorization is generated by a set of prototypes if the closest prototype to a member of a certain category is the designated prototype of that category. Thus, the assignment rule in this case takes a simple form: every entity is assigned to the category that corresponds to its closest prototype.

1.4. **The main objective.** Whether people categorize on the basis of proximity to prototypes or not is still subject to debate. The main task of this paper is to find conditions that characterize those partitions that are generated by prototypes. Such conditions may be useful in experimentally examining the validity of the prototype model.

It is relatively easy to find partitions of $\mathbb{R}^d_+$ that are not generated by prototypes. However, figuring out whether or not an individual's categorization is based on prototypes, may involve intricate calculations. Finding necessary and sufficient conditions for a categorization to be based on prototypes may significantly facilitate this task.

1.5. **A necessary condition – Hierarchic Consistency.** Suppose that an individual has in mind prototypes of Spanish, French and Italian. Thus, according to her categorization, a member of the 'French' category would be closer to, or better match, the French prototype than the other two. If asked to categorize entities between two, say, Spanish and French, a member of the 'French' category would be closer to the French prototype than to the Spanish one. However, a member of the French category, when only French and Spanish are considered, may be categorized as Italian, if only French and Italian are considered, or when all three are considered.

When the categorization is based on prototypes, those people who are categorized as French, rather than Spanish or Italian, are precisely those who were categorized as French when examined separately versus Spanish or Italian.

In general, when L is a set of prototypes, any subset of L that consists of, say, p prototypes, induces a partition of the set of entities into p

categories. Furthermore, when a categorization is generated by the prototypes in A, where A is a subset of L, the members of the category that correspond to a specific prototype a, when all prototypes in A are considered, are those that correspond to a, when examined against any subset of A. This property is called **Hierarchic Consistency**.
Hierarchic Consistency is a necessary condition for a categorization to be generated by prototypes. This property provides a simple and straightforward experimental way to refute the prototype theory. The model of a prototypical categorization is refuted if, for instance, when asked to categorize an individual as Spanish, French or Italian, she is categorized as French, while when asked to categorize the same individual as French or Italian, she is categorized as Italian.

1.6. **Measuring closeness.** Even when there is a prototypical representative for each category, in order to determine the category attribution, one should specify the meaning of closeness. What does it mean that a bat is more similar to a cow than to Hoopoe? The metric by which closeness is measured determines the partition into different categories.
Consider a set of prototypes. If the Euclidean metric defines the notion of closeness, the categories form convex sets. It means that if two entities are similarly categorized, then any convex combination of them is also classified to the same category. Other well-known metrics, like the one determined by the maximal difference[1] between attributes, induce non-convex categories.
In this paper we assume **Convexity**: All the entities that belong to the same category form a convex set. Based on this assumption we derive a prototypical categorization induced by the Euclidean metric.

1.7. **Extended prototypes.** Entities are defined by d attributes, and the set of entities is $\mathbb{R}_+^d$. The characterization we provide utilizes an additional attribute, the hidden one. The original entities are defined by the original intensities of the d original attributes and by the intensity 0 of the hidden attribute. A prototype in the extended setting is a $d+1$ dimensional vector. Such a prototype is called an *extended prototype.*

[1]If $x = (x_1, ..., x_d)$ and $y = (y_1, ..., y_d)$ are two entities, then the distance determined by the maximal difference is $\max_{1 \leq i \leq d} |x_i - y_i|$.

Our main result axiomatizes the categorizations that are generated by extended prototypes. In other words, we axiomatize a categorization that utilizes d attributes by prototypes defined by the d original attributes and a hidden one.

1.8. **Main result.** It turns out that **Hierarchic Consistency** is the key property of the categorization partitions that are generated by prototypes. Those categorization partitions induced by Euclidean proximity to prototypes consist of convex categories. Thus, convex categories and **Hierarchic Consistency** are necessary conditions for categorization partitions to be generated by prototypes. However, these conditions alone are not sufficient. Two additional axioms described in Section 3 are added to **Hierarchic Consistency** and **Convexity** to axiomatize the categorizations generated by extended prototypes.

As shown in Subsection 4.2, these axioms do not guarantee that a categorization is generated by non-extended prototypes. Thus, the hidden attribute is necessary. What characterizes categorizations that are generated by non-extended prototypes is still an open question.

1.9. **Categorization and decision.** A decision maker has a finite set of available actions. Any decision problem is characterized by a set of attributes. For instance, for a coach of a basketball team the following attributes of the competing team may be of interest: how tall the point guard is, how many offensive rebounds the center collects per game, etc. The decision maker may categorize all possible decision problems according to the best action: all those problems to which a certain action is a best response are lumped together as one category. This categorization might not be a result of an observable decision process. Rather, the classification of the decision problems into the various categories might be based solely on the observable actions taken.

A natural question is whether the observable decisions are consistent with a categorization made on the basis of how close a decision problem is to a prototypical member of some category. In such a case, there is a prototypical decision problem that stands out as the epitome of all instances characterized by a certain action which is the best response to all of them. Furthermore, any case is examined against these epitomic examples. A decision maker looks for the prototype that matches best the attributes of the decision problem under his or her consideration, and the decision is taken accordingly. Such a process is called prototype-oriented decision making.

1.10. **The structure of the paper.** Section 2 describes the model. In Section 3 we introduce the axioms and the main results. The roles of the various axioms are exemplified in Section 4. Section 5 examines the scope of the axioms **Hierarchic Consistency** and **Convexity**. Section 6 links this discussion with decision theory. Section 7 surveys relevant literature related to behavioral economics as well as that related to computational geometry. The proofs appear in Section 8 and the paper terminates with final comments in Section 9.

2. The model

Any entity is characterized by the intensity of d attributes. Thus, an entity is represented by a vector in $\mathbb{R}^d_+$. An *open partition* of $\mathbb{R}^d_+$ is a collection of non-empty, pairwise disjoint open sets, say, $A_1, A_2, ...$ such that[2] $\text{cl} \cup A_i = \mathbb{R}^d_+$.
Consider a set of categories $L = \{1, 2, \ldots, \ell\}$. An agent is asked to classify the set of entities. When a subset $A \subseteq L$ is considered, the agent divides the set of entities among the categories of A. That is, for any $A \subseteq L$ (with $|A| \geq 2$), there is an open partition of $\mathbb{R}^d_+$, denoted by P_A, that consists of $|A|$ sets, one for each category in A. Such a system of partitions is called a *categorization system* and is a primitive of our model. Formally,

Definition 1. 1. *A* categorization system *is a collection of open partitions* $P_A = \{P_A(i)\}_{i \in A}$ *of* $\mathbb{R}^d_+$, $A \subseteq L$.
2. *When* $x \in P_A(i)$ *we say that* x *is* categorized as i when A is considered.

3. Axioms and axiomatization

3.1. **Axioms.** This subsection describes four axioms of categorization systems. The first two properties seems to be natural from a behavioral point of view. The other two are of a technical nature and can be considered as genericity properties.

[2] cl$B$ denotes the closure of B.

Convexity: For every $A \subseteq L$ and for each $i \in A$, $P_A(i)$ is a convex set.

Convexity states that if two entities y and y' are categorized as i when A is considered, then so is any convex combination of y and y'.

Hierarchic Consistency: For every $A \subseteq L$ and for each $i \in A$, $P_A(i) = \bigcap_{B \subsetneq A, i \in B} P_B(i)$.

Hierarchic Consistency states that the entities categorized as i when A is considered are those entities categorized as i when B is considered for every $B \subsetneq A$.

Non-Redundancy: For every three categories $\{i, j, k\} \in L$, $P_{\{i,j\}}(i) \not\subseteq P_{\{i,k\}}(i)$.

Non-Redundancy states that there is always an entity categorized as i, when the two categories i, j are considered, but categorized as k when i, k are considered.

Variety: For every four distinct categories $\{i, j, k, m\} \in L$,

$$\begin{aligned} &\mathrm{cl}(P_{\{i,j,k\}}(i)) \cap \mathrm{cl}(P_{\{i,j,k\}}(j)) \cap \mathrm{cl}(P_{\{i,j,k\}}(k)) \neq \\ &\mathrm{cl}(P_{\{m,j,k\}}(m)) \cap \mathrm{cl}(P_{\{m,j,k\}}(j)) \cap \mathrm{cl}(P_{\{m,j,k\}}(k)). \end{aligned}$$

The set $\mathrm{cl}(P_{\{i,j,k\}}(i)) \cap \mathrm{cl}(P_{\{i,j,k\}}(j)) \cap \mathrm{cl}(P_{\{i,j,k\}}(k))$ consists of all entities that belong to (the closure of) the three categories $\{i, j, k\}$. **Variety** states that this set and the set of entities which are in the (closure of the) three categories $\{m, j, k\}$ are not the same.

3.2. **Axiomatization – Categorization generated by extended prototypes.** A prototype of a category is an entity that reflects best the properties of a category.

Definition 2. *A categorization system, $P_A = \{P_A(i)\}_{i\in A}$, $A \subseteq L$, is* generated by extended prototypes, *if there are ℓ points $x_1, x_2, \ldots, x_\ell$ in $\mathbb{R}_+^{d+1}$, such that for any $A \subseteq L$ and any $i \in A$, $P_A(i) = \{y \in \mathbb{R}_+^d : d_i(y) < d_j(y)$ for every $j \in A, j \neq i\}$, where $d_i(y) = \|(y,0) - x_i\|^2$ $(i = 1, \ldots, \ell)$ and $(y, 0)$ is the vector in $\mathbb{R}_+^{d+1}$ whose first d coordinates coincide with y and the last coincides with 0.*

When a categorization system is generated by extended prototypes an additional attribute is added to the d existing ones. Every category i has an extended prototype, $x_i \in \mathbb{R}_+^{d+1}$ (i.e., an entity characterized by $d+1$ attributes). Whether or not a point $y \in \mathbb{R}_+^d$ is categorized as i (when A is considered) is determined by the distances, d_i, of $(y, 0)$ to the prototype x_i, $i \in A$. An entity y is categorized as i when A is considered, if the closest extended prototype to $(y, 0)$ is x_i.
The objective of the paper is to axiomatize the categorization systems that are generated by extended prototypes. We use the four axioms described in the previous subsection in order to axiomatize the categorization systems that are generated by extended prototypes. Our main result is:

The Main Theorem

(1) *If $\{P_A\}_{A\subseteq L}$ is a categorization system generated by extended prototypes, then it satisfies **Convexity** and **Hierarchic Consistency**;*

(2) *If a categorization system $\{P_A\}_{A\subseteq L}$ satisfies **Convexity** and **Hierarchic Consistency**, and in addition satisfies **Non-Redundancy** and **Variety**, then it is generated by extended prototypes.*

4. Examples

4.1. **Categorization which is not generated by extended prototypes.** Let the set of entities be $\mathbb{R}_+^2$ and consider a set of three categories $L = \{i, j, k\}$. Assume that the partitions when only pairs of categories are considered are as follows (we write only one of the atoms in each partition; the other one is the complementary open set):

$$P_{\{i,j\}}(i) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_1 < 1\}$$
$$P_{\{i,k\}}(i) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_2 < 1\}$$
$$P_{\{j,k\}}(j) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_2 < 2 - x_1\}.$$

By part (1) of the Main Theorem, if a categorization system is generated by prototypes then it must satisfy **Hierarchic Consistency**. Thus, when all three categories are considered, the entities categorized as, say i, are those that are categorized as i when both $\{i, j\}$ and $\{i, k\}$ are considered. It follows that when L is considered:

$$P_L(i) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_1 < 1,\ x_2 < 1\},$$
$$P_L(j) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_1 > 1,\ x_2 < 2 - x_1\},\ \text{and}$$
$$P_L(k) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_2 > 1,\ x_2 > 2 - x_1\}.$$

However, the resulting P_L is not an open partition of $\mathbb{R}^2_+$, since the closure of the union of these three sets is not $\mathbb{R}^2_+$.

4.2. **The hidden attribute.** Consider the following partitions when pairs of categories are considered ($L = \{i, j, k, m\}$ and the set of entities is $\mathbb{R}^2_+$):

$$P_{\{i,j\}}(i) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ 2x_2 > 4 - x_1\},$$
$$P_{\{i,k\}}(i) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_2 > 1\},$$
$$P_{\{i,m\}}(i) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ 2x_2 > x_1 - 1\},$$
$$P_{\{j,k\}}(j) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ 2x_2 > x_1\},$$
$$P_{\{j,m\}}(j) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ 2x_1 < 5\},\ \text{and}$$
$$P_{\{k,m\}}(k) = \{(x_1, x_2) \in \mathbb{R}^2_+;\ 2x_2 < 5 - x_1\}.$$

When more than two categories are considered the partitions are determined by **Hierarchic Consistency**. One may verify that for any subset A of L the resulting P_A is indeed an open partition of $\mathbb{R}^2_+$. Moreover, this categorization system satisfies the four axioms and is therefore generated by extended prototypes.

However, this categorization system is not generated by non-extended prototypes. That is, if we restrict ourselves to the original two attributes without allowing for a hidden attribute, this categorization system is not generated by prototypes. This fact follows from Corollary 10 in Ash and Bolker (1985).

4.3. **Convexity and Hierarchic Consistency are insufficient.** Consider a categorization system where

$$\begin{aligned} P_{\{i,j\}}(i) &= \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_1 < 1\}, \\ P_{\{i,k\}}(i) &= \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_1 < 1\}, \text{ and} \\ P_{\{j,k\}}(j) &= \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_2 > x_1\}. \end{aligned}$$

Assuming that the categorization system satisfies **Hierarchic Consistency**, the partition when the three categories are considered should be:

$$\begin{aligned} P_L(i) &= \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_1 < 1\}, \\ P_L(j) &= \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_1 > 1,\ x_2 > x_1\}, \text{ and} \\ P_L(k) &= \{(x_1, x_2) \in \mathbb{R}^2_+;\ x_1 > 1,\ x_2 < x_1\}. \end{aligned}$$

It is clear that this categorization system satisfies **Convexity** and **Hierarchic Consistency**. However, **Non-Redundancy** is not satisfied since $P_{\{i,j\}}(i) \subseteq P_{\{i,k\}}(i)$.

This categorization system is not generated by extended prototypes. Suppose on the contrary that (x_i, y_i, z_i), (x_j, y_j, z_j) and (x_k, y_k, z_k) are the extended prototypes of the categories i, j and k, respectively. Then, the lines $(x_i, y_i) - (x_j, y_j)$ and $(x_i, y_i) - (x_k, y_k)$ are perpendicular to the line $x_1 = 1$ (which is the border between $P_L(i)$ and $P_L(j)$ and the border between $P_L(i)$ and $P_L(k)$). However, the line $(x_j, y_j) - (x_k, y_k)$ is perpendicular to the line $x_1 = x_2$, which contradicts the previous conditions.

Thus, **Convexity** and **Hierarchic Consistency** are not sufficient to ensure that a categorization system is generated by extended prototypes.

4.4. **Variety is not satisfied.** The following is an example of a categorization system which does not satisfy **Variety**. The open partitions when pairs of categories are considered are as follows:

$$\begin{aligned}
P_{\{i,j\}}(i) &= \{(x_1,x_2) \in \mathbb{R}^2_+;\ x_1 < 1\},\\
P_{\{i,k\}}(i) &= \{(x_1,x_2) \in \mathbb{R}^2_+;\ x_2 > 1\},\\
P_{\{i,m\}}(i) &= \{(x_1,x_2) \in \mathbb{R}^2_+;\ x_2 > 2x_1 - 1\},\\
P_{\{j,k\}}(j) &= \{(x_1,x_2) \in \mathbb{R}^2_+;\ 2x_2 > 3 - x_1\},\\
P_{\{j,m\}}(j) &= \{(x_1,x_2) \in \mathbb{R}^2_+;\ x_2 > x_1\},\ \text{and}\\
P_{\{k,m\}}(k) &= \{(x_1,x_2) \in \mathbb{R}^2_+;\ x_2 < 2 - x_1\}.
\end{aligned}$$

Defined the entire categorization system in a manner that satisfies **Hierarchic Consistency**. Consider the triplets $\{i,j,k\}$ and $\{m,j,k\}$. The set $\mathrm{cl}(P_{\{i,j,k\}}(i)) \cap \mathrm{cl}(P_{\{i,j,k\}}(j)) \cap \mathrm{cl}(P_{\{i,j,k\}}(k))$ is the intersection point of the three lines $\{x_1 = 1\}$, $\{x_2 = 1\}$ and $\{2x_2 = 3 - x_1\}$, which consists of the point $(1,1)$. Similarly, $\mathrm{cl}(P_{\{m,j,k\}}(m)) \cap \mathrm{cl}(P_{\{m,j,k\}}(j)) \cap \mathrm{cl}(P_{\{m,j,k\}}(k))$ is the intersection point of the lines $\{2x_2 = 3 - x_1\}$, $\{x_2 = x_1\}$ and $\{x_2 = 2 - x_1\}$, which is, again, the point $(1,1)$. Thus, $\mathrm{cl}(P_{\{i,j,k\}}(i)) \cap \mathrm{cl}(P_{\{i,j,k\}}(j)) \cap \mathrm{cl}(P_{\{i,j,k\}}(k)) = \mathrm{cl}(P_{\{m,j,k\}}(m)) \cap \mathrm{cl}(P_{\{m,j,k\}}(j)) \cap \mathrm{cl}(P_{\{m,j,k\}}(k))$, and therefore **Variety** is not satisfied.

5. Discussion

The axioms described above seem to be plausible in a wide array of settings. However, there are some instances where the axioms are not applicable. In this section we try to examine the limitations of the axioms **Convexity** and **Hierarchic Consistency**.

The axioms **Non-Redundancy** and **Variety** are of a technical nature and reduce the generality of our discussion. However, they seem to raise no conceptual problems. We can just say that it would be interesting to find relaxed axioms that together with the other two necessary axioms guarantee a prototypical representation.

5.1. **More than one prototypical example.** There are cases in which categories are defined by resemblance to one of a few ideal representatives. A board of managers examines a few candidates. It may have a few prototypical examples of what constitutes a good CEO and some others that are prototypical examples of a bad CEO. These prototypes may have been collected from the list of thefirm's ex CEOs judged according to their performance.

Candidates are sorted according to their resemblance to one of the prototypical individuals. It may happen that two individuals categorized as potentially good CEOs resemble different prototypes: one resembles a former CEO in her assertiveness and the other resembles another former CEO in his business creativity. A combination of these two may resemble a former bad CEO. In other words, the category of potentially good CEOs is not necessarily convex.

5.2. **Non-consistent prototypes.** Our model assumes that the prototype of a category is not influenced by the set of categories considered. That is, if x_i is the prototype of a category i when the set of categories considered is A ($i \in A$), then it is also the prototype of i when the set of categories considered is B ($B \neq A,\ i \in B$).
Consider the following example: A school has basketball and volleyball teams. A student is included in the basketball team if he is closer to some prototypical basketball player than to a prototype of a volleyball player, and vice versa.
If the same school had to classify the students between basketball and soccer teams, the prototypical players would probably be different.
For instance, the speed of the prototypical basketball player in the first categorization (basketball vs. volleyball) would be higher than in the second (basketball vs. soccer), because speed is important in basketball and in soccer, and less so in volleyball. On the other hand, the height would be a significant attribute of the prototypical basketball player in the second categorization, because height is an advantageous attribute in both basketball and in volleyball, and less so in soccer.
This phenomenon can result in a violation of the **Hierarchic Consistency** axiom. Indeed, it may happen that a student is assigned to the basketball team when only basketball and volleyball are considered, but when all three sports are considered, he is assigned to the volleyball team.

6. Decision Theory and Categorization

6.1. **On prototype-oriented decision making.** Let D be a decision problem characterized by ℓ attributes. For instance, an individual needs to take a decision whether to take an umbrella or not. The decision maker may have two prototypical weather conditions in mind, one in which he should take an umbrella and another in which he should not. The decision is taken according to which of the two prototypes is closer to the actual weather conditions.

Categorization of decision problems may be done according to the actions taken. That is, all problems that share the same best response are lumped together. The decision process should not be consciously prototype-oriented, but it might seem that way.

Categorization of decision problems according to their best response satisfies **Hierarchic Consistency**. Let $A = \{a_1, ..., a_\ell\}$ be the set of all available actions and let D be a decision problem. The decision maker needs to take one of the actions in A in response to D. If among all actions in A, a_1 is the best response to D, it means that a_1 is a better response to D than any other action in A. That is, if for instance, only a_1 and a_2 are available, a_1 is a better response to D than a_2. The problem D is, therefore, categorized as "a decision problem for which the best response is a_1". The problem D is categorized in the same fashion no matter how many other actions in A are available.

In summary, a decision problem D is categorized as "a decision problem for which the best response is a_1" when all actions in A are available, if and only if it is categorized the same way when any subset of actions in A is available. Therefore, **Hierarchic Consistency** is satisfied. If it satisfies the other axioms, it appear to be prototype-oriented decision making.

6.2. **Best-response categorization.** Suppose that a decision problem is defined by a distribution over the state space, Ω. As before, A stands for the action set. Suppose that the utility function u specifies the utility the decision maker derives from taking the action a when the distribution over states is P. That is, $u : \Delta(\Omega) \times A \to \mathbb{R}$, where $\Delta(\Omega)$ is the set of distributions over Ω. In other words, $u(P, a)$ is the utility derived from taking the action a and the states are drawn from Ω according to the distribution P.

In this setup, decision problems are merely distributions over the state space, Ω. Thus, the set of problems is $\Delta(\Omega)$. Consider a categorization of distributions in $\Delta(\Omega)$ according to their best response. P is classified to the category corresponding to a if a is the best response to P (i.e., $u(P, a) > u(P, b)$ for any $b \in A \setminus \{a\}$). If for any $P, P' \in \Omega$ and

$\alpha \in (0, 1)$, u satisfies

$$u(P, a) \geq u(P, b) \text{ and } u(P', a) \geq u(P', b) \text{ imply}$$
$$u(\alpha P + (1-\alpha)P', a) \geq u(\alpha P + (1-\alpha)P', b),$$

then this categorization satisfies **Convexity**. In particular, if u is defined according to the expected utility, then the best-response categorization satisfies **Convexity**.

7. Related Literature

7.1. **Categorization and behavioral economics.** An extensive amount of effort has been put into finding alternatives to the classical expected utility theory. This is due to cumulative evidence that the predictions of this theory are not always consistent with the actual decisions being made by individuals. Some alternative theories are strongly inspired by psychological concepts, and categorization has an important role in many of them.

Fryer and Jackson (2003) use a model of categorization to explain discrimination of minorities. They claim that discrimination against minorities in hiring is a result of a cognitive process whereby majority groups are better sorted on the basis of qualifications than minority groups.

Mullainathan (2003) suggests an alternative for Bayesian updating of probabilities based on the idea of coarse categories. He claims that people tend to consider similar (but not equal) cases as the same (i.e., belonging to the same category). Based on categorical estimations people obtain biased probabilities. Furthermore, upon observing new data, beliefs about the actual state of the world are not updated in a continuous manner, as in the Bayesian case.

Gilboa and Schmeidler (2001) developed a case-based decision theory based on past experience. The action chosen when a new decision problem is encountered, is the one that performed best in past problems which are weighted according to their similarity to the decision problem under consideration.

In view of the case-based decision theory, the prototype-oriented decision making, as described in Subsection 6.1, may be considered as follows. All past experience is encapsulated in some prototypical cases.

The similarity of any decision problem to a prototypical case is polar: either perfectly similar or perfectly dissimilar. When a new case is encountered, the action taken is the best response to the perfectly similar prototype (to the case under consideration).

7.2. **Voronoi diagrams.** Partitions of the space that are generated by a set of center points are well known as Voronoi diagrams or Dirichlet tessellations. This concept is fundamental in computational geometry (see for example Preparata and Shamos (1985)) and has applications in almost every field of science. To appreciate the variety of topics where this idea is used, see Boots et al. (1992).

In its simplest form, a Voronoi diagram is a partition of some Euclidean space into a finite number of sets. Given the set of center points (generators), every point in the space is assigned to its closest generator (in terms of the Euclidean distance). However, there are many different generalizations of this concept. One such generalization considers generators that are not necessarily points. These generators may be subsets of the space such as lines, arcs and circles. Other examples use other metrics than the Euclidean metric.

In this paper we use a generalization of the Voronoi diagram known as the Power diagram (Aurenhammer (1987)) or sectional Dirichlet tessellation (Ash and Bolker (1986)). Every generator x_i has a non-negative "weight" w_i, and the distance between some point y and x_i is $d_i(y) = \|y - x_i\|^2 + w_i$. This distance has a strong connection to Laguerre geometry and therefore the resulting diagram is also referred to as the Laguerre diagram (Imai et al. (1985)).

A relatively small part of the literature is dedicated to the problem of characterizing and recognizing Voronoi diagrams. Ash and Bolker (1985) give a geometrical characterization of those partitions of the plane which are the Voronoi diagram of some set of points. For algorithms that check whether or not a given partition is a Voronoi diagram the reader is referred to Evans and Jones (1987) and to Heath and Kasif (1993).

8. Proof of the main theorem

8.1. **Preliminary lemmas.**

Lemma 1. *Let $\{P_A\}_{A\subseteq L}$ be a categorization system. Then $\{P_A\}_{A\subseteq L}$ satisfies* ***Hierarchic Consistency*** *if and only if for every $A \subseteq L$ ($|A| \geq 2$) and for each $i \in A$, $P_A(i) = \bigcap_{j\in A\setminus\{i\}} P_{\{i,j\}}(i)$.*

Proof: Assume that for every $A \subseteq L$ ($|A| \geq 2$) and for each $i \in A$, $P_A(i) = \bigcap_{j\in A\setminus\{i\}} P_{\{i,j\}}(i)$. We show that the categorization system satisfies **Hierarchic Consistency**. Let $A \subseteq L$ be such that $|A| = 3$. Then for each $i \in A$, $P_A(i) = \bigcap_{j\in A\setminus\{i\}} P_{\{i,j\}}(i) = \bigcap_{B;\ B\subsetneqq A \text{ and } i\in B} P_B(i)$. The proof proceeds by induction. Assume that A consists of more than three elements, and that the assertion holds for any $B \subsetneqq A$. Then, for any $i \in A$, $\bigcap_{B;\ B\subsetneqq A \text{ and } i\in B} P_B(i) = \bigcap_{B;\ B\subsetneqq A \text{ and } i\in B} \bigcap_{j\in B\setminus\{i\}} P_{\{i,j\}}(i) = \bigcap_{j\in A\setminus\{i\}} P_{\{i,j\}}(i) = P_A(i)$, as desired.

As for the inverse direction, assume that $\{P_A\}_{A\subseteq L}$ satisfies **Hierarchic Consistency**. We show that $P_A(i) = \bigcap_{j\in A\setminus\{i\}} P_{\{i,j\}}(i)$ for any $A \subseteq L$ ($|A| \geq 2$) and $i \in A$. The proof is by induction on $|A|$. The statement is obviously true for $|A| = 2$. Assume that it is true for all $A \subseteq L$ with $|A| \leq k$, and let $\hat{A}$ be a subset of L containing $k + 1$ elements. Fix some $i \in \hat{A}$. Then, $\bigcap_{j\in \hat{A}\setminus\{i\}} P_{\{i,j\}}(i) = \bigcap_{B;\ B\subsetneqq \hat{A} \text{ and } i\in B} \bigcap_{j\in B\setminus\{i\}} P_{\{i,j\}}(i)$. By the induction hypothesis, the right-hand side is equal to $\bigcap_{B;\ B\subsetneqq \hat{A} \text{ and } i\in B} P_B(i)$. Therefore,
$\bigcap_{j\in \hat{A}\setminus\{i\}} P_{\{i,j\}}(i) = \bigcap_{B;\ B\subsetneqq \hat{A} \text{ and } i\in B} P_B(i)$, as required. ■

Lemma 1 asserts that **Hierarchic Consistency** is equivalent to the following seemingly weaker condition: For every $A \subseteq L$ and for each $i \in A$, $P_A(i)$ is the set of entities categorized as i when all the pairs i, j ($j \in A \setminus \{i\}$) are considered.

A *hyperplane*, H, in $\mathbb{R}^d$ is a set of the type $\{y \in \mathbb{R}^d;\ \langle v, y\rangle = c\}$, where v is a non-zero vector in $\mathbb{R}^d$, $\langle v, y\rangle$ is the inner product of v and y, and c is a constant. We denote $H^+ = \{y \in \mathbb{R}^d;\ \langle v, y\rangle > c\}$ and $H^- = \{y \in \mathbb{R}^d;\ \langle v, y\rangle < c\}$.
We say that a hyperplane $H_{i,j}$ *separates* $P_{\{i,j\}}(i)$ *from* $P_{\{i,j\}}(j)$ if $P_{\{i,j\}}(i) = H^-_{i,j} \cap \mathbb{R}^d_+$ and $P_{\{i,j\}}(j) = H^+_{i,j} \cap \mathbb{R}^d_+$.

Lemma 2. *Let $\{P_A\}_{A\subseteq L}$ be a categorization system which satisfies* ***Convexity*** *and let i, j be two categories in L. Then, there exists a hyperplane $H_{i,j}$ that separates $P_{\{i,j\}}(i)$ from $P_{\{i,j\}}(j)$.*

Proof: Follows from the fact that $P_{\{i,j\}}(i)$ and $P_{\{i,j\}}(j)$ are open, non-empty disjoint convex sets such that the closure of their union is the entire $\mathbb{R}^d_+$. ■

The intersection of the hyperplane $H_{i,j}$ and $\mathbb{R}^d_+$ consists of all those entities that are in the boundary of both $P_{\{i,j\}}(i)$ and $P_{\{i,j\}}(j)$. These are the entities that are categorized as both i and j, when the pair of categories i, j is considered.

We say that two hyperplanes H_1 and H_2 are parallel if there are two constants c_1 and c_2 and one vector v such that $H_i = \{y \in \mathbb{R}^d;\ \langle v, y\rangle = c_i\},\ i = 1, 2$.
In case H_1 and H_2 are not parallel, there are two independent vectors v_1 and v_2 such that $H_i = \{y \in \mathbb{R}^d;\ \langle v_i, y\rangle = c_i\},\ i = 1, 2$. Thus, $H_1 \cap H_2 = \{y \in \mathbb{R}^d;\ \langle v_i, y\rangle = c_i,\ i = 1, 2\}$. This means that when H_1 and H_2 are not parallel, $H_1 \cap H_2$ is an affine subspace of dimension $d - 2$.

Lemma 3. ***Non-Redundancy*** *implies that for any three distinct categories i, j, k the hyperplanes $H_{i,j}$ and $H_{i,k}$ are not parallel.*

Proof: If $H_{i,j}$ and $H_{i,k}$ are parallel for some three categories i, j, k then there is a vector v and constants c_1, c_2 such that $H_{i,j} = \{y \in \mathbb{R}^d;\ \langle v, y\rangle = c_1\}$ and $H_{i,k} = \{y \in \mathbb{R}^d;\ \langle v, y\rangle = c_2\}$. Assume without loss of generality that $c_1 \geq c_2$. By Lemma 2, $P_{\{i,j\}}(i) = H^+_{i,j} \cap \mathbb{R}^d_+ = \{y \in \mathbb{R}^d;\ \langle v, y\rangle > c_1\} \cap \mathbb{R}^d_+ \subseteq \{y \in \mathbb{R}^d;\ \langle v, y\rangle > c_2\} \cap \mathbb{R}^d_+ = H^+_{i,k} \cap \mathbb{R}^d_+ = P_{\{i,k\}}(i)$ and this contradicts **Non-Redundancy**. ■

Lemma 4. ***Non-Redundancy*** *and* ***Hierarchic Consistency*** *imply that for any three distinct categories i, j, k, $P_{\{j,k\}}(k) \cap P_{\{i,k\}}(i) \neq \emptyset$, and $P_{\{j,k\}}(k) \cap P_{\{i,k\}}(i) \subseteq P_{\{i,j\}}(i)$.*

Proof: We start by showing that $P_{\{j,k\}}(k) \cap P_{\{i,k\}}(i) \neq \emptyset$. Indeed, if $P_{\{j,k\}}(k) \cap P_{\{i,k\}}(i) = \emptyset$, then $P_{\{j,k\}}(k) \subseteq P_{\{i,k\}}(k)$ which contradicts **Non-Redundancy**.
Next, if $P_{\{j,k\}}(k) \cap P_{\{i,k\}}(i) \nsubseteq P_{\{i,j\}}(i)$, then $B = P_{\{j,k\}}(k) \cap P_{\{i,k\}}(i) \cap P_{\{i,j\}}(j) \neq \emptyset$. B is an open set.

By **Hierarchic Consistency**, $P_{\{i,j,k\}}(i) = P_{\{i,k\}}(i) \cap P_{\{i,j\}}(i) \subseteq P_{\{i,j\}}(i)$ and $B \subseteq P_{\{i,j\}}(j)$. Thus, $B \cap P_{\{i,j,k\}}(i) = \emptyset$. For similar reasons, $B \cap P_{\{i,j,k\}}(j) = B \cap P_{\{i,j,k\}}(k) = \emptyset$. Therefore, $B \cap [P_{\{i,j,k\}}(j) \cup P_{\{i,j,k\}}(j) \cup P_{\{i,j,k\}}(j)] = \emptyset$. However, since $P_{\{i,j,k\}}(i) \cup P_{ijk}(j) \cup P_{ijk}(k)$ is the union of an open partition, the intersection of this union with any open set is not empty. This is a contradiction and the lemma is proven. ■

The next lemma expresses the main geometric property of categorizations which satisfy **Hierarchic Consistency**. This property is later used to prove the main result.

Lemma 5. *Let $\{P_A\}_{A \subseteq L}$ be a categorization system which satisfies **Convexity**, **Hierarchic Consistency** and **Non-Redundancy**. For any three distinct categories i, j and k*

$$H_{i,j} \cap H_{i,k} = H_{i,j} \cap H_{j,k} = H_{j,k} \cap H_{i,k}.$$

Proof: It is sufficient to prove that $H_{i,j} \cap H_{i,k} \subseteq H_{i,j} \cap H_{j,k}$. Obviously, $H_{i,j} \cap H_{i,k} \subseteq H_{i,j}$. Thus, it remains to show that $H_{i,j} \cap H_{i,k} \subseteq H_{j,k}$.
Denote $S = P_{\{i,k\}}(i) \cap P_{\{i,j\}}(j)$, $T = P_{\{i,j\}}(i) \cap P_{\{i,k\}}(k)$. By Lemma 4, S and T are both non-empty sets. Moreover, $S \subseteq P_{\{j,k\}}(j)$ and $T \subseteq P_{\{j,k\}}(k)$. It follows that $H_{j,k}$ separates S from T.
Finally, we also have $\text{cl}(S) \cap \text{cl}(T) = \text{cl}(P_{\{i,j\}}(j) \cap P_{\{i,k\}}(i)) \cap \text{cl}(P_{\{i,j\}}(i) \cap P_{\{i,k\}}(k)) = \text{cl}(P_{\{i,j\}}(i)) \cap \text{cl}(P_{\{i,j\}}(j)) \cap \text{cl}(P_{\{i,k\}}(i)) \cap \text{cl}(P_{\{i,k\}}(k)) = H_{i,j} \cap H_{i,k} \cap \mathbb{R}^d_+$.
Lemma 3 implies that $H_{i,j} \cap H_{i,k}$ is an affine space of dimension $d-2$. Since, $H_{j,k}$ is a hyperplane, it follows that $H_{j,k}$ must contain $H_{i,j} \cap H_{i,k}$ which proves the lemma. ■

Notation 1. *The hyperplane $H_{i,j}$ is defined by the vector $s_{i,j}$ and the constant $c_{i,j}$. That is, $H_{i,j} = \{y \in \mathbb{R}^d;\ \langle s_{i,j}, y\rangle = c_{i,j}\}$. Without loss of generality we may assume that $P_{\{i,j\}}(j) \subseteq H^+_{i,j}$. We denote $s_{i,j} = -s_{j,i}$ and $c_{i,j} = -c_{j,i}$.*

Corollary 1. *For every three distinct categories i, j, k, $s_{i,j}, s_{i,k}$ and $s_{k,j}$ are linearly dependent.*

Proof: Let $D_{i,j,k} = H_{i,j} \cap H_{i,k} \cap H_{j,k}$. By Lemma 5, $D_{i,j,k} = H_{i,j} \cap H_{i,k}$. By Lemma 3, $H_{i,j} \cap H_{i,k}$ is an affine subspace of dimension $d-2$. Thus, $s_{i,j}, s_{i,k}$ and $s_{k,j}$ are linearly dependent. ■

Corollary 2. *For any three distinct categories i, j, k there exists $y \in \mathbb{R}^d_+$ such that $\langle y, s_{i,j} \rangle > c_{i,j}$ and $\langle y, s_{j,k} \rangle > c_{j,k}$. Moreover, for every such y, $\langle y, s_{i,k} \rangle > c_{i,k}$.*

Proof: By Lemma 4, the set $P_{\{i,j\}}(j) \cap P_{\{j,k\}}(k)$ is not empty, so there exists some $y \in \mathbb{R}^d_+$ which satisfies the above inequalities. Every such y is by Lemma 4 in $P_{\{i,k\}}(k)$, so $\langle y, s_{i,k} \rangle > c_{i,k}$. ■

Notation 2. *Let t and s be two vectors in $\mathbb{R}^d$. Denote the ray that starts at t and continues in the direction of s by $R(t, s)$. Formally,*

$$R(t, s) = \{t + as;\ a > 0\}.$$

Lemma 6. *Let i, j and k be three distinct categories and t_i and t_j be two points in $\mathbb{R}^d$, such that $t_j - t_i = \gamma s_{ij}$, where $\gamma > 0$. Then, the rays $R(t_i, s_{ik})$ and $R(t_j, s_{jk})$ intersect.*

Proof: By Corollary 1, $s_{i,j}, s_{i,k}$ and $s_{k,j}$ are linearly dependent. Lemma 3 implies that no two are linearly dependent. Thus, there are two non-zero constants α and β such that $s_{i,k} = \alpha s_{i,j} + \beta s_{j,k}$. Recall that $D_{i,j,k} = H_{i,j} \cap H_{i,k} \cap H_{j,k}$ and let $z \in D_{i,j,k}$. Then, $c_{i,k} = \langle z, s_{i,k} \rangle = \langle z, \alpha s_{i,j} + \beta s_{j,k} \rangle = \alpha c_{i,j} + \beta c_{j,k}$.
We prove that both α and β are positive. We prove first that it cannot be the case that both are negative. If, on the contrary, both are negative, then consider $y \in \mathbb{R}^d_+$ such that $\langle y, s_{i,j} \rangle > c_{i,j}$ and $\langle y, s_{j,k} \rangle > c_{j,k}$ (such y exists by Corollary 2). Then, $\langle y, s_{i,k} \rangle = \langle y, \alpha s_{i,j} + \beta s_{j,k} \rangle = \alpha \langle y, s_{i,j} \rangle + \beta \langle y, s_{j,k} \rangle < \alpha c_{i,j} + \beta c_{j,k} = c_{i,k}$. However, by Corollary 2, $\langle y, s_{i,k} \rangle > c_{i,k}$, which is a contraction. This proves that both α and β cannot be negative.
It remains to show that it cannot be the case that either α or β is negative. Assume, on the contrary, that α is negative and β is positive. Consider $y \in \mathbb{R}^d_+$ such that $\langle y, s_{j,k} \rangle > c_{j,k}$ and $\langle y, s_{k,i} \rangle > c_{k,i}$ (again, such y exists by Corollary 2). Then, $\langle y, -\alpha s_{i,j} \rangle = \langle y, \beta s_{j,k} - s_{i,k} \rangle = \langle y, \beta s_{j,k} + s_{k,i} \rangle > \beta c_{j,k} + c_{k,i} = -\alpha c_{i,j}$. Thus, $\langle y, s_{i,j} \rangle > c_{i,j}$. However, by Corollary 2, $\langle y, s_{i,j} \rangle < c_{i,j}$, which is a contradiction. Similarly, it is impossible that α is positive and β is negative.
We conclude that $s_{i,k} = \alpha s_{i,j} + \beta s_{j,k}$, where both α and β are positive. Thus, $s_{i,k} = \frac{\alpha}{\gamma}(t_j - t_i) + \beta s_{j,k}$ and therefore, $t_i + \frac{\gamma}{\alpha} s_{i,k} = t_j + \frac{\gamma\beta}{\alpha} s_{j,k}$. Since the left side is in $R(t_i, s_{ik})$ and the right side is in $R(t_j, s_{jk})$, the

rays $R(t_i, s_{ik})$ and $R(t_j, s_{jk})$ intersect at this point, and the proof is complete. ∎

8.2. **Proof of part (1) of the Main Theorem.** We prove that if $\{P_A\}_{A\subseteq L}$ is generated by extended prototypes then it satisfies **Convexity** and **Hierarchic Consistency**.

Let $\{x_1, ..., x_\ell\} \subseteq \mathbb{R}^{d+1}$ be the set of extended prototypes. First, by the definition of a categorization system generated by extended prototypes, we have for every $A \subseteq L$ and for each $i \in A$:

$P_A(i) = \{y \in \mathbb{R}^d :\ d_i(y) < d_j(y) \text{ for every } j \in A, j \neq i\} =$

$\cap_{j\in A, j\neq i}\{y \in \mathbb{R}^d :\ d_i(y) < d_j(y)\} = \cap_{j\in A,j\neq i} P_{\{i,j\}}(i)$.

Thus, by Lemma 1 the categorization system satisfies **Hierarchic Consistency**.

Second, let i, j be two different categories in L. The set of entities which are equidistant from these two prototypes, is the set $H_{i,j} = \{y \in \mathbb{R}^d_+;\ d_i(y) = d_j(y)\}$. An elementary calculation shows that this set can be rewritten as $\{y \in \mathbb{R}^d_+;\ \langle y, x'_j - x'_i\rangle = \frac{1}{2}(w_j^2 - w_i^2 + \|x'_j\|^2 - \|x'_i\|^2)\}$, where $x_i = (x'_i, w_i),\ i = 1, ..., \ell$. Therefore, $H_{i,j}$ is the hyperplane perpendicular to $x'_j - x'_i$ which passes through the point $\frac{\|x'_j\|^2-\|x'_i\|^2+w_j^2-w_i^2}{2\|x'_j-x'_i\|^2}(x'_j - x'_i)$. It follows that $P_{\{i,j\}}(i)$, which is the set of entities closer to x_i than to x_j, is an open half space. In particular, it is convex. $P_A(i)$ is the intersection of the sets $P_{\{i,j\}}(i)$ $(j \in A \setminus \{i\})$ and therefore is a convex set, which proves **Convexity**. ∎

8.3. **Proof of part (2) of the Main Theorem.** The proof of this part is divided into two propositions.

Proposition 1. *If a categorization system $\{P_A\}_{A\subseteq L}$ satisfies **Convexity**, **Hierarchic Consistency**, **Non-Redundancy** and **Variety**, then there are ℓ points $x'_1, ..., x'_\ell \in \mathbb{R}^d_+$ such that,*

(1) *$x'_i - x'_j$ is perpendicular to the hyperplane $H_{i,j}$, for every $i, j \in L$; and*
(2) *$\langle x'_j - x'_i, s_{ij}\rangle \geq 0$ for every $i, j \in L$. That is, the direction from x'_i to x'_j is the same as the direction from $P_{\{i,j\}}(i)$ to $P_{\{i,j\}}(j)$ (we call such points 'well oriented').*

Proof: The proof is constructive. We first select ℓ points $r_1, ..., r_\ell$, sequentially. We show that after $r_1, ..., r_{k-1}$ have been selected, it is possible to find r_k such that $r_k - r_j$ is perpendicular to $H_{j,k}$ for every $j = 1, ..., k-1$.

Let r_1 be an arbitrary point in $\mathbb{R}^d_+$. Define $r_2 = r_1 + s_{12}$. Since s_{12} is perpendicular to $H_{1,2}$, so is $r_2 - r_1$. Lemma 6 ensures that the rays $R(r_1, s_{13})$ and $R(r_2, s_{23})$ intersect. The third point, r_3, is placed at the intersection of these rays.
Now comes the key argument of the proof. Similarly to r_3, we place r_4 at the intersection point of the rays $R(r_2, s_{24})$ and $R(r_3, s_{34})$, whose existence is guaranteed by Lemma 6. In particular, $r_4 - r_2$ is perpendicular to $H_{2,4}$ and therefore to $D_{1,2,4}$ and $r_4 - r_3$ is perpendicular to $H_{3,4}$ and therefore to $D_{1,3,4}$. We show now that $r_1 - r_4$ is perpendicular to $H_{1,4}$, and moreover, that r_1 and r_4 are well oriented.
The hyperplane $H_{1,4}$ contains both $D_{1,2,4}$ and $D_{1,3,4}$. **Variety** implies that $D_{1,2,4}$ and $D_{1,3,4}$ are not equal. Furthermore, the dimensions of $D_{1,2,4}$ and $D_{1,3,4}$ are $d-2$. Therefore, in order to show that $r_1 - r_4$ is perpendicular to $H_{1,4}$ it is sufficient to show that $r_1 - r_4$ is perpendicular to any vector of the type $y - y'$, where $y, y' \in D_{1,2,4} \cup D_{1,3,4}$. Let $y, y' \in D_{1,2,4}$. By construction, since $y, y' \in H_{1,2}$, $\langle r_1 - r_2, y - y' \rangle = 0$. Similarly, $\langle r_2 - r_4, y - y' \rangle = 0$. Summing up these equations, we obtain that $\langle r_1 - r_4, y - y' \rangle = 0$. For similar reasons, if $y, y' \in D_{1,3,4}$, then $\langle r_1 - r_4, y - y' \rangle = 0$.
It remains to show that if $y \in D_{1,2,4}$ and $y' \in D_{1,3,4}$, $\langle r_1 - r_4, y - y' \rangle = 0$. Let $z \in D_{1,2,3}$ and $w \in D_{2,3,4}$. Since both z and y are in $H_{1,2}$,

$$\langle z - y, r_1 - r_2 \rangle = 0. \tag{1}$$

Similarly,

$$\langle z - y', r_1 - r_3 \rangle = 0, \tag{2}$$

$$\langle w - y, r_2 - r_4 \rangle = 0, \tag{3}$$

and

$$\langle w - y', r_3 - r_4 \rangle = 0 \tag{4}$$

By summing up equations (1) and (3) and subtracting equations (2) and (4) one obtains,

$$\langle y - y', r_1 - r_4 \rangle = \langle w - z, r_2 - r_3 \rangle. \tag{5}$$

Since w and z are in $H_{2,3}$, $w - z$ is perpendicular to $r_2 - r_3$. Thus, (5) implies that $r_1 - r_4$ is perpendicular to $H_{1,4}$.
It remains to show that every pair of $r_1, ..., r_4$ is well oriented. By construction, every pair of r_1, r_2, r_3 is well oriented. Furthermore, by the choice of r_4, both $\langle r_2 - r_4, s_{24} \rangle$ and $\langle r_3 - r_4, s_{34} \rangle$ are positive. We therefore only need to show that r_1 and r_4 are well oriented.
Consider the triplet r_1, r_2, r_4. By construction, r_4 is on the line perpendicular to $H_{2,4}$ which passes through r_2. By what we showed earlier, r_4 is also on the line perpendicular to $H_{1,4}$ which passes through r_1. It means that r_4 is the intersection point of these two lines. However, by Lemma 6, the rays $R(r_1, s_{14})$ and $R(r_2, s_{24})$ intersect, so r_4 is the intersection point of these rays. It follows that r_1 and r_4 are well oriented.
The procedure is then continued. After $r_1, ..., r_{k-1}$ have been fixed, we place r_k at the intersection of the rays $R(r_{k-2}, s_{k-2,k})$ and $R(r_{k-1}, s_{k-1,k})$. For every $j < k-2$, we use the same argument as before (this time with the categories $j, k-2, k-1, k$), to show that $r_k - r_j$ is perpendicular to $H_{j,k}$ and that r_j, r_k are well oriented.
Note that if $r_1, ..., r_\ell$ satisfy (1) and (2) of the proposition, then so do $r_1 + v, ..., r_\ell + v$ for any $v \in \mathbb{R}^d$. Thus, one can find a vector v such that $x'_1 = r_1 + v, ..., x'_\ell = r_\ell + v$ are all in $\mathbb{R}^d_+$. This proves the proposition.

■

Proposition 2. *Let $\{x'_1, ..., x'_\ell\} \subseteq \mathbb{R}^d_+$ be points that satisfy Proposition 1. There are non-negative numbers $w_1, ..., w_l$, such that $P_{\{i,j\}}(i) = \{y \in \mathbb{R}^d_+;\ d_i(y) < d_j(y)\}$ for every $i, j \in L$, where $d_i(y) = \|(y, 0) - (x'_i, w_i)\|^2$.*

Proof: As mentioned before, for extended prototypes $x_i = (x'_i, w_i)$ and $x_j = (x'_j, w_j)$ the set of points in $\mathbb{R}^d_+$ which are equidistant from x_i and x_j, is the hyperplane

$$T_{i,j} = \{y \in \mathbb{R}^d_+;\ d_i(y) = d_j(y)\} =$$
$$\{y \in \mathbb{R}^d_+;\ \langle y, x'_j - x'_i\rangle = \frac{1}{2}(w_j^2 - w_i^2 + \|x'_j\|^2 - \|x'_i\|^2)\}.$$

Note that $T_{i,j}$ is unchanged when the same constant is added to both w_i and w_j. Furthermore, $T_{i,j}$ is perpendicular to $x'_j - x'_i$. When w_j grows to infinity $T_{i,j}$ moves in one direction, while when w_i grows to infinity $T_{i,j}$ moves in the other direction. It follows that every hyperplane perpendicular to $x'_j - x'_i$ can be written with appropriate w_i and w_j. Moreover, the smaller of these two numbers can be as large as needed.

We sequentially choose the numbers $w_1, ..., w_\ell$ and show that they satisfy the proposition. Let $x'_1, x'_2, ..., x'_\ell$ be the prototypes found in Proposition 1. Choose w_1 and w_2 such that $T_{1,2} = H_{1,2}$ (recall that $H_{1,2}$ is the hyperplane separating $P_{\{1,2\}}(1)$ and $P_{\{1,2\}}(2)$). Since x'_1, x'_2 are well oriented, the set of points closer to x'_1 is exactly $P_{\{1,2\}}(1)$ and the set of points closer to x'_2 is $P_{\{1,2\}}(2)$.
Next, choose w_3 such that $T_{2,3} = H_{2,3}$. We need to show that w_3 is consistent with w_1. That is, $T_{1,3} = H_{1,3}$. $T_{1,3}$ is perpendicular to $x'_1 - x'_3$ by definition, while $H_{1,3}$ has the same property by Proposition 1. Moreover, for every $y \in T_{1,2} \cap T_{2,3}$ we have $d_1(y) = d_2(y)$ and $d_2(y) = d_3(y)$. Thus, $d_1(y) = d_3(y)$. It means that $T_{1,3}$ is the unique hyperplane perpendicular to $x'_1 - x'_3$ which contains $T_{1,2} \cap T_{2,3} = H_{1,2} \cap H_{2,3}$. Lemma 5 states that $H_{1,3}$ also contains $H_{1,2} \cap H_{2,3}$. Therefore, $T_{1,3}$ coincides with $H_{1,3}$.
Assume that $w_1, ..., w_k$ have already been found. We choose w_{k+1} such that $T_{k,k+1} = H_{k,k+1}$. For every $j = 1, 2, ..., k-1$ we have $T_{j,k} = H_{j,k}$ and $T_{k,k+1} = H_{k,k+1}$, so similarly to the argument in the case of the first three categories, $T_{j,k+1} = H_{j,k+1}$.
Finally, to make all $w_1, ..., w_\ell$ positive one may add a large constant to all. This completes the proof. ∎

Conclusion of the main result's proof: We have shown that if a categorization system satisfies **Convexity**, **Hierarchic Consistency**, **Non-Redundancy** and **Variety** then it is possible to find points $x_1 = (x'_1, w_1), ..., x_\ell = (x'_\ell, w_\ell)$ in $\mathbb{R}^{d+1}_+$ such that for every two categories $i, j \in L$, $P_{\{i,j\}}(i) = \{y \in \mathbb{R}^d_+;\ \|(y,0) - x_i\|^2 < \|(y,0) - x_j\|^2\}$. By

Lemma 1, for any $A \subseteq L$ and any $i \in A$,

$$\begin{aligned} P_A(i) &= \bigcap_{j \in A \setminus \{i\}} P_{\{i,j\}}(i) = \\ &= \bigcap_{j \in A \setminus \{i\}} \{y \in \mathbb{R}^d_+;\ \|(y,0) - x_i\|^2 < \|(y,0) - x_j\|^2\} = \\ &= \{y \in \mathbb{R}^d_+;\ \|(y,0) - x_i\|^2 < \|(y,0) - x_j\|^2 \text{ for every } j \in A, j \neq i\}. \end{aligned}$$

Thus, the categorization system is generated by extended prototypes.

■

9. Final Comments

9.1. **Categorizations generated by (non-extended) prototypes.** The $d+1$ attribute added to each prototype is rather artificial. By non-extended prototype we mean an entity in $\mathbb{R}^d_+$ and not in $\mathbb{R}^{d+1}_+$. We say that a categorization system is generated by (non-extended) prototypes if it is generated by extended prototypes, and in addition, the extended prototypes can be chosen such that the last coordinate in all of them is 0. Obviously, this is a subset of the categorization systems that are generated by extended prototypes. An interesting question is what characterizes those categorization systems that are generated by (non-extended) prototypes.

9.2. **The axioms of Non-Redundancy and Variety.** As shown in Subsection 4.3, **Convexity** and **Hierarchic Consistency** alone do not guarantee that a categorization system is generated by extended prototypes. Thus, more conditions must be added in order to get the desired characterization.

Regarding the **Non-Redundancy** axiom, we stated it in the above way for the sake of simplicity. It is possible to replace this axiom with a slightly weaker and less transparent one which is also necessary. Here we prefer elegance over generality.

Regarding **Variety**, we suspect that the theorem is true without it. To show that this is indeed the case, one needs to prove that the construction in Proposition 1 can still be made, even without **Variety** being assumed.

9.3. **Restricted domains.** In this discussion we restricted our attention to the positive orthant. That is, an entity is defined by its intensity in any attribute, and the intensity is not bounded. One may consider other domains, of which the most natural is the unit cube: $Q = \{(y_1, ..., y_d);\ 0 \leq y_i \leq 1$ for every $i = 1, ..., d\}$. In this case the intensity of any attribute is bounded between 0 and 1.
The main results (with minor changes in the proofs) hold also when the domain of entities is restricted to Q. One only needs to replace $\mathbb{R}^d_+$ with Q everywhere.

9.4. **Other metrics.** Our main result assumes **Convexity**. That is, the categories are convex sets. Based on this assumption we derived that some categorization systems are generated by prototypes, using the Euclidean metric.
With the same set of prototypes, a different metric would induce different categories. Typically, the categories would not be convex sets. It would be interesting to find what conditions axiomatize categorization systems that are generated by prototypes (or extended prototypes), using other metrics than the Euclidean.

9.5. **Prototypical sets.** We axiomatized categorization systems generated by prototypes. Every category is represented by one prototype. However, as mentioned in Subsection 5.1, there may be cases where categories are defined by a closeness relation to one of a few typical representatives of a category. For instance, the French category may be defined by a proximity to either Charles de Gaulle, Brigitte Bardot or Gerard Depardieu. In such a case the category is generated by a set of prototypes rather than by one.
Formally,

Definition 3. *An open partition $P = (P(i))_{i \in L}$ is* generated by finite sets of prototypes*, if there are ℓ finite sets $B_1, B_2, \ldots, B_\ell$ in $\mathbb{R}^d_+$, such that for any $i \in L$, $P(i) = \{y \in \mathbb{R}^d_+ :\ f_i(y) < f_j(y)$ for every $j \in L, j \neq i\}$, where $f_i(y) = \min_{z \in B_i} \|y - z\|^2$ $(i = 1, \ldots, \ell)$.*

It is clear that an open partition that is generated by finite sets of prototypes consists of categories which are finite unions of convex sets. Further investigation of this subject is beyond the scope of this paper.

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