

Additivity in cost spanning tree problems*

Gustavo Bergantiños. Research Group in Economic Analysis.

Universidade de Vigo.

Juan J. Vidal-Puga. Departamento de Estatística.

Universidade de Vigo.

May 4, 2004

Abstract

We characterize a rule in cost spanning tree problems using an additivity property and some basic properties. If the set of possible agents has at least three agents, these basic properties are symmetry and separability. If the set of possible agents has two agents, we must add positivity. In both characterizations we can replace separability by population monotonicity.

JEL Codes: C71, D70, D85.

Keywords: cost spanning tree problems, additivity, characterization.

1 Introduction

Many problems involving network formation have been studied in the operations research and the economic literature. In operations research two issues have been extensively explored: the design of efficient algorithms and the computational complexity. The economic literature focuses on aspects such like the cost sharing of the network and the design of mechanisms trying to explain the way in which the network forms.

In this paper we focus on the cost sharing aspect. Our contribution can be considered in the well-known literature of cost allocation. We assume that there are no external forces (for example, the market) which determine the final allocation. Agents can achieve agreements directly among themselves, or indirectly by letting the final decision to a neutral referee. In both cases the important issue is to find a "fair allocation" of the cost.

*Latest version at: <http://webs.uvigo.es/vidalpuga/>. Financial support by the Spanish Ministerio de Ciencia y Tecnología and FEDER through grant BEC2002-04102-C02-01 and Xunta de Galicia through grant PGIDIT03PXIC30002PN is gratefully acknowledged.

In particular we study cost spanning tree problems (*cstp*). Consider that a group of agents, located at different geographical places, want some particular service which can only be provided by a common supplier, called the source. Agents will be served through connections which entail some cost. However, they do not care whether they are connected directly or indirectly to the source. This situation is described by a symmetric matrix C , which denotes the connection costs.

We assume that agents construct a minimal cost spanning tree (*mcst*). The question is how to divide the cost associated with the *mcst* between the agents. Different rules give different answers to this question. One of the most important topics is the axiomatic characterization of rules. The idea is to propose desirable properties and to find out which of them characterize each rule. Properties often help agents to compare different rules and to decide which rule is preferred in a particular situation.

Additivity is a standard property and it has been used in many situations. Additivity says, roughly speaking, that if our problem is the sum of small problems, the solution to our problem must be the sum of the solutions of the small problems.

The justification of additivity, as a fairness property, is not as clear as with other properties (for example, efficiency or symmetry). Nevertheless, from the mathematical point of view it is a very appealing property because if a rule is additive we only need to care about small problems. This property simplifies our work because small problems are usually much easier to solve.

In many problems it is possible to characterize rules with additivity and very "basic" properties. Usually, the rules satisfying additivity are very nice rules. This provides an ex-post justification of additivity as a fairness property. Let us mention some examples.

The Shapley value, the most important value in cooperative games with transferable utility, is characterized by additivity and other properties (efficiency, symmetry, and dummy). If we compare the Shapley value with other prominent values (for example the nucleolus) we realize that these values satisfy all the properties characterizing the Shapley value except additivity. In bankruptcy problems and other related problems, it is possible to characterize the three rules based on the principles of proportionality, equal award, and equal loss with additivity and other properties, like symmetry and continuity. See Bergantiños and Vidal-Puga (2004a) for a summary of these results. In cost sharing problems, Moulin and Shenker (1994) characterize the serial cost sharing rule with additivity and other properties.

The natural formulation of additivity in *cstp* is the following: a rule ψ satisfies additivity if and only if $\psi(C + C') = \psi(C) + \psi(C')$ for all *cstp* C and C' . This property is very demanding and no rule satisfies it. The reason is that we are comparing very different problems, and thus the *mcst* of C , C' , and $C + C'$ can be very different.

Hence, we introduce a property called restricted additivity. The idea is to claim the condition $\psi(C + C') = \psi(C) + \psi(C')$ only when the problems C , C' , and $C + C'$ are "similar". We consider as similar problems those which satisfy two conditions. First, they have a common *mcst*. Second, if we order the arcs of the common *mcst* by non-decreasing cost, we can obtain the same order in C as in C' .

We prove that if a rule satisfies restricted additivity then it is enough to know how this rule works in small problems. Our small problems are those in which the connection costs could be either 0 or x for some fixed x .

We present four "basic" properties of rules: Symmetry, positivity, separability, and population monotonicity. Symmetry says that symmetric agents must pay the same. Two agents are symmetric if their connection costs are the same. Positivity says that every agent must pay at least 0.

Separability appears in Megiddo (1978), Granot and Huberman (1981), and Granot and Maschler (1998) with the name of decomposition. It also appears in Bergantiños and Vidal-Puga (2004b) with the name of separability. Two subsets of agents can connect to the source separately or jointly. If there are no savings when they connect jointly, separability says that agents must pay the same in both circumstances.

Population monotonicity says that if new agents join the society, no former agent can be worse. This is a standard property, which has been used in many different situations.

Our results are the following. If the set of possible agents has at least three members, then there is a unique rule satisfying restricted additivity, symmetry, and separability. If the set of possible agents has exactly two members, then there is a unique rule satisfying positivity, restricted additivity, symmetry, and separability. In both cases we can replace separability by population monotonicity.

The rule we obtain in these characterizations is already present in the literature. This rule was introduced by Feltkamp, Tijs, and Muto (1994) and studied later by Branzei, Moretti, Norde, and Tijs (2003) and Bergantiños and Vidal-Puga (2004b).

Our paper is very related to the paper of Branzei *et al* (2003) because they

also characterize this rule using an additivity property and other properties. Our additivity property implies their additivity property because we claim the condition $\psi(C + C') = \psi(C) + \psi(C')$ in more cases. This is a small improvement with respect to their results. The most important improvement is in relation to the other properties. We claim that our properties are much more appealing. In the last section we comment this aspect in more detail.

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we present our results. In Section 4 we compare our results with Branzei *et al* (2003).

2 Cost spanning tree problems

In this section we introduce cost spanning tree problems. We do it following the notation of Kar (2002) and Dutta and Kar (2002).

Let \mathcal{N} be the set of all possible agents. \mathcal{N} can be finite or infinite ($\mathcal{N} = \{1, 2, \dots\}$). We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1, \dots, n\}$. Our interest lies on networks where each node in N is (directly or indirectly) connected to the source.

Let Π_N be the set of all orders over the finite set N . Given $\pi \in \Pi_N$, let $Pre(i, \pi)$ denote the set of elements of N which come before i in the order given by π , *i. e.* $Pre(i, \pi) = \{j \in N \mid \pi(j) < \pi(i)\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on N represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$.

We denote by \mathcal{C}^N the set of all cost matrices on N . Given $C, C' \in \mathcal{C}^N$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *cost spanning tree problem*, briefly *cstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is the finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

Given a *cstp* (N_0, C) , we denote the *cstp* induced by C in $S \subset N$ as (S_0, C) .

A *network* g over N_0 is a subset of $\{(i, j) \mid i, j \in N_0\}$. The elements of g are called *arcs*. Since $c_{ij} = c_{ji}$ we will work with undirected arcs, *i.e.* $(i, j) = (j, i)$. Given a network g and a pair of nodes i and j , a *path* from i to j in g is a sequence of arcs $\{(i_{h-1}, i_h)\}_{h=1}^l$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, \dots, l\}$, $i = i_0$ and $j = i_l$. We denote by \mathcal{G}^N the set of all networks over N_0 .

A *tree* is a network satisfying that for each $i \in N$ there exists a unique path from i to the source. If t is a tree we usually write $t = \{(i^0, i)\}_{i \in N}$ where i^0 represents the first agent in the unique path in t from i to 0. We denote by \mathcal{G}_0^N

the set of trees over N_0 .

Given a $cstp(N_0, C)$ and $g \in \mathcal{G}^N$, we define the *cost* associated to g as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there are no ambiguities, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum cost spanning tree* for (N_0, C) , briefly a *mcst*, is a tree $t \in \mathcal{G}_0^N$ such that $c(t) = \min_{g \in \mathcal{G}^N} c(g)$. It is well-known in the literature about *cstp* that there exists a *mcst*, even though it does not need to be unique. Given a $cstp(N_0, C)$ we denote by $m(N_0, C)$ the cost associated with any *mcst* t in (N_0, C) .

Given a $cstp(N_0, C)$, Prim (1957) provides an algorithm for computing the *mcst*. The idea of this algorithm is quite simple: starting from the source we construct a network by consecutively adding arcs with the lowest cost, without introducing cycles.

Formally, Prim's algorithm is defined as follows. We start with $S^0 = \{0\}$ and $g^0 = \emptyset$.

Stage 1: Take the arc $(0, i)$ such that $c_{0i} = \min_{j \in N} \{c_{0j}\}$. If there are several arcs $(0, i)$ satisfying this condition, select one of them. Now, $S^1 = \{0, i\}$ and $g^1 = \{(0, i)\}$.

Stage $p+1$. Assume that we have defined S^p and g^p . We now define S^{p+1} and g^{p+1} . Take an arc (i^0, i) such that $c_{i^0 i} = \min_{k \in S^p, l \in N_0 \setminus S^p} \{c_{kl}\}$. If there are several arcs (i^0, i) satisfying this condition, select one of them. Now, $S^{p+1} = S^p \cup \{i\}$ and $g^{p+1} = g^p \cup \{(i^0, i)\}$.

This process finishes in n stages. We say that g^n is a tree obtained via Prim's algorithm. Notice that this algorithm leads to a tree, but not necessarily unique.

A *TU* game is a pair (N, w) where $N \subset \mathcal{N}$ and $w : 2^N \rightarrow \mathbb{R}$ satisfies that $w(\emptyset) = 0$. We denote by $Sh(N, w)$ the *Shapley value* (Shapley, 1953) of the *TU* game (N, w) .

Bird (1976) associates each $cstp(N_0, C)$ with a *TU* game (N, v_C) where for each coalition $S \subset N$,

$$v_C(S) = m(S_0, C).$$

Usually, we write v instead of v_C .

Given a $cstp(N_0, C)$, Bergantiños and Vidal-Puga (2004b) say that C is a *canonical matrix* if there exists a *mcst* t in (N_0, C) satisfying the two following conditions:

- (A1) $t = \{(i_{p-1}, i_p)\}_{p=1}^n$ where $i_0 = 0$ (the source).
(A2) Given $i_p, i_q \in N_0$, $p < q$, then $c_{i_p i_q} = \max_{p < r \leq q} \{c_{i_{r-1} i_r}\}$.

Bergantiños and Vidal-Puga (2004b) introduce an algorithm for associating each arbitrary matrix $C \in \mathcal{C}^N$ with a canonical matrix C^* . This algorithm is defined as follows:

Let $t^0 = \{(i^0, i)\}_{i \in N}$ be a *mcst* in (N_0, C) . Take $C^0 = C$, $S^0 = \{0\}$, and $g^0 = \emptyset$.

Stage 1. We define $T_1 = \{i \in N \mid i^0 = 0\}$. Take an arc $(0, i_1)$ such that $i_1 \in T_1$ and $c_{0i_1} = \min_{i \in T_1} \{c_{0i}\}$. If there are several arcs satisfying this condition, take one of them. Now $C^1 = C^0$, $S^1 = \{0, i_1\}$, and $g^1 = \{(0, i_1)\}$.

Assume that we have defined Stage r for all $r \leq p$. We now define Stage $p+1$ when $p+1 \leq n$.

Stage $p+1$. We define $T_{p+1} = \{i \in N \mid i \in N_0 \setminus S^p \text{ and } i^0 \in S^p\}$. Take an arc (i_{p+1}^0, i_{p+1}) such that $c_{i_{p+1}^0 i_{p+1}} = \min_{i \in T_{p+1}} \{c_{i^0 i}\}$. If there are several arcs satisfying this condition, take one of them. We define C^{p+1} such that $c_{kj}^{p+1} = c_{kj}^p$ if $(k, j) \neq (i_p, i_{p+1})$ and $c_{i_p i_{p+1}}^{p+1} = c_{i_{p+1}^0 i_{p+1}}^0$. Moreover, $S^{p+1} = S^p \cup \{i_{p+1}\}$ and $g^{p+1} = g^p \cup \{(i_p, i_{p+1})\}$.

Stage $n+1$. We define C^* such that $c_{i_p i_q}^* = \max_{p < r \leq q} \{c_{i_{r-1} i_r}^n\}$ for all $i_p, i_q \in N_0$, $p < q$. Moreover, $t = g^n$.

One of the most important issues addressed in the literature about *cstp* is how to divide the cost of connecting agents to the source among them.

A (*cost allocation*) rule is a function ψ such that $\psi(N_0, C) \in \mathbb{R}^N$ for each *cstp* (N_0, C) and $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$. As usually, $\psi_i(N_0, C)$ represents the cost assigned to agent i . Notice that we implicitly assume that agents build a *mcst*. As far as we know, all the rules proposed in the literature make this assumption.

There are several rules studied in the literature. In this paper we focus on a rule introduced by Feltkamp, Tijs, and Muto (1994) and called Equal Remaining Obligations (*ERO*). Kruskal (1956) introduced an algorithm for computing the *mcst* of a *cstp*. Feltkamp, Tijs, and Muto (1994) define *ERO* through Kruskal's algorithm. Initially, each agent has an obligation 1 and the network is empty. We now apply Kruskal's algorithm and the obligations of the agents decrease when we add an arc to the network. This obligation is $\frac{1}{n_i}$, where n_i is the number of agents connected directly or indirectly to agent i through the network. At each step of the algorithm, each agent pays the part of the cost of the arc we add given by the difference between his obligation before adding the arc, and his

obligation after adding the arc. See Feltkamp *et al* (1994) for a formal definition.

On the other hand, Bergantiños and Vidal-Puga (2004b) define the rule φ as

$$\varphi(N_0, C) = Sh(N, v_{C^*})$$

where C^* is the canonical matrix associated to C through the algorithm. They prove that, surprisingly, φ coincides with *ERO*.

In our proofs the canonical form plays an important role. Thus, we use the same notation as in Bergantiños and Vidal-Puga (2004b).

Finally, we define *C-components* following Norde *et al* (2004). This concept will be used in some of the proofs. Given a *cstp* (N_0, C) and $S \subset N$, we say that $i, j \in N, i \neq j$ are *(C, S)-connected* if there exists a path g from i to j satisfying that $g \in \mathcal{G}^S$ and $c_{kl} = 0$ for all $(k, l) \in g$. We say that $S \subset N$ is a *C-components* if two conditions hold. First, for all $i, j \in S, i$ and j are *(C, S)-connected*. Second, S is maximal, *i.e.* if $S \subsetneq T$ there exist $i, j \in T, i \neq j$ such that i and j are not *(C, T)-connected*.

Norde *et al* (2004) prove that the set of *C-components* is a partition of N .

3 The axiomatic characterization

In this section we present an axiomatic characterization of φ using a (restricted) additivity property.

We first present the natural formulation of additivity in *cstp* problems. We argue that this property is very demanding, which produces that no rule satisfies it. Then, we restrict additivity to a subclass of problems in order to find rules satisfying it.

Later, we present some "basic" properties of rules. We present four properties. Symmetry (symmetric agents must pay the same) and positivity (every agent must pay at least 0) do not need any further justification. We also consider separability, introduced by Granot and Huberman (1984), and population monotonicity, which has been used in many different situations.

Our results are the following. If the set of possible agents \mathcal{N} has at least three members, φ is the unique rule satisfying restricted additivity, symmetry, and separability. If the set of possible agents has exactly two members, φ is the unique rule satisfying positivity, restricted additivity, symmetry, and separability.

We say that ψ satisfies *additivity* (*ADD*) if for all *cstp* (N_0, C) and (N_0, C') ,

$$\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C').$$

Unfortunately, there is no rule satisfying *ADD*. Take

$$C^1 = \begin{pmatrix} 0 & 10 & 50 \\ 10 & 0 & 10 \\ 50 & 10 & 0 \end{pmatrix} \text{ and } C^2 = \begin{pmatrix} 0 & 50 & 10 \\ 50 & 0 & 10 \\ 10 & 10 & 0 \end{pmatrix}.$$

Then,

$$C^1 + C^2 = \begin{pmatrix} 0 & 60 & 60 \\ 60 & 0 & 20 \\ 60 & 20 & 0 \end{pmatrix}.$$

Notice that $m(N_0, C^1) = m(N_0, C^2) = 20$ and $m(N_0, C^1 + C^2) = 80$. By definition, $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$ for all *cstp* (N_0, C) . Then, $\psi(N_0, C^1 + C^2) = \psi(N_0, C^1) + \psi(N_0, C^2)$ is impossible.

If we insist on additive rules, we cannot divide the minimum cost among agents. Of course, we do not insist on additivity.

We take a different approach. Namely, we claim additivity when the pair of problems are "similar", and not in any case (as with *ADD*). Moreover, we are also interested to claim additivity in a large class of problems.

In *cstp* there exists an additivity property called *cone-wise positive linearity* (*CPL*), which has been introduced by Branzei *et al* (2003).

We say that ψ satisfies *CPL* if

$$\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$$

for all *cstp* (N_0, C) and (N_0, C') such that given $i, j, k, l \in N_0$, then $c_{ij} \leq c_{kl}$ if and only if $c'_{ij} \leq c'_{kl}$.

Notice that according to this definition two problems are "similar" when, ordering the arcs by their cost we can obtain the same order in C and C' .

Remark 1. Branzei *et al* (2003) define *CPL* in a way a little bit different. Take $x, x' \geq 0$. They say that a rule ψ satisfies *CPL* if

$$\psi(N_0, xC + x'C') = x\psi(N_0, C) + x'\psi(N_0, C')$$

when (N_0, C) and (N_0, C') are "similar"; xC is the cost matrix where the connection cost between i and j is xc_{ij} .

It is not difficult to check that the characterization of Branzei *et al* (2003) also holds with our definition. We present it in a different way only because it is simpler.

We now introduce our additivity property. If we want to claim $\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$, we need, as we argue before, that $m(N_0, C + C') = m(N_0, C) + m(N_0, C')$.

Assume that t is a *mcst* in $(N_0, C + C')$ and $m(N_0, C + C') = m(N_0, C) + m(N_0, C')$. We know that $c(N_0, C, t) \geq m(N_0, C)$ and $c(N_0, C', t) \geq m(N_0, C')$. Then,

$$\begin{aligned} m(N_0, C + C') &= c(N_0, C + C', t) = c(N_0, C, t) + c(N_0, C', t) \\ &\geq m(N_0, C) + m(N_0, C'). \end{aligned}$$

Thus, t is a *mcst* in (N_0, C) and (N_0, C') . Assume that we order the arcs of t by non-decreasing cost. If we obtain the same order in (N_0, C) and (N_0, C') we claim additivity on these problems. This is our idea of "similar" problems.

We now present the definition formally. We say that ψ satisfies *restricted additivity (RA)* if

$$\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$$

for all *cstp* (N_0, C) and (N_0, C') satisfying that there exists a *mcst* $t = \{(i^0, i)\}_{i \in N}$ in (N_0, C) , (N_0, C') , and $(N_0, C + C')$ and an order $\pi = (i_1, \dots, i_n) \in \Pi_N$ such that $c_{i_1^0 i_1} \leq c_{i_2^0 i_2} \leq \dots \leq c_{i_n^0 i_n}$ and $c'_{i_1^0 i_1} \leq c'_{i_2^0 i_2} \leq \dots \leq c'_{i_n^0 i_n}$.

It is not difficult to check that if we define *RA* as $\psi(N_0, xC + x'C')$ where $x, x' \geq 0$ our results do not change.

In the next proposition we prove that there exists a relation between both additivity properties.

Proposition 1. *If ψ satisfies RA, then ψ also satisfies CPL.*

Proof. Even though this result is very intuitive we give a formal proof. Let ψ be a rule satisfying *RA*.

Take (N_0, C) and (N_0, C') as in the definition of *CPL*. Since ψ satisfies *RA* it is enough to prove that there exists a *mcst* $t = \{(i^0, i)\}_{i \in N}$ in (N_0, C) , (N_0, C') , and $(N_0, C + C')$ and an order $\pi = (i_1, \dots, i_n) \in \Pi_N$ such that $c_{i_1^0 i_1} \leq c_{i_2^0 i_2} \leq \dots \leq c_{i_n^0 i_n}$ and $c'_{i_1^0 i_1} \leq c'_{i_2^0 i_2} \leq \dots \leq c'_{i_n^0 i_n}$.

Let $t = \{(i^0, i)\}_{i \in N}$ be a *mcst* obtained following Prim's algorithm when we apply it to (N_0, C) . Without loss of generality we assume that for all $i = 1, \dots, n$ we select in Stage i the arc (i^0, i) . We now prove that t is also a *mcst* in (N_0, C') .

Since arc $(0, 1)$ is selected in the first stage of Prim's algorithm when we apply it to (N_0, C) we know that $c_{01} = \min_{j \in N} \{c_{0j}\}$. Then, $c'_{01} = \min_{j \in N} \{c'_{0j}\}$ and $c_{01} + c'_{01} = \min_{j \in N} \{c_{0j} + c'_{0j}\}$ because C and C' satisfy the conditions of the definition of *CPL*. Thus, arc $(0, 1)$ can be selected in the first stage of Prim's algorithm when we apply it to (N_0, C') and $(N_0, C + C')$.

If we repeat this argument until Stage n we deduce that t can be obtained through Prim's algorithm when we apply it to (N_0, C') and $(N_0, C + C')$. Then, t is a *mcst* in (N_0, C) and $(N_0, C + C')$.

Take $\pi = (i_1, \dots, i_n) \in \Pi_N$ such that $c_{i_1^0 i_1} \leq c_{i_2^0 i_2} \leq \dots \leq c_{i_n^0 i_n}$. Since (N_0, C') satisfies the conditions of the definition of *CPL* we conclude that $c'_{i_1^0 i_1} \leq c'_{i_2^0 i_2} \leq \dots \leq c'_{i_n^0 i_n}$. Then, (N_0, C) and (N_0, C') satisfy the conditions of the definition of *RA* and hence $\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$. ■

As we argue at the beginning of this section we can not claim $\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C')$ in all cases. From the proof of Proposition 1 we deduce that *RA* claims this condition in more cases than *CPL*. Then, *RA* is closer to *ADD* than *CPL*.

We now introduce the "basic" properties we use in our characterization results.

Given a *cstp* (N_0, C) , we say that $i, j \in N$, $i \neq j$ are *symmetric* if for all $k \in N_0 \setminus \{i, j\}$, $c_{ik} = c_{jk}$.

We say that ψ satisfies *symmetry* (*SYM*) if for all *cstp* (N_0, C) and all pair of symmetric agents $i, j \in N$,

$$\psi_i(N_0, C) = \psi_j(N_0, C).$$

We say that ψ satisfies *positivity* (*POS*) if for all *cstp* (N_0, C) and all $i \in N$,

$$\psi_i(N_0, C) \geq 0.$$

The meaning of *SYM* and *POS* is clear.

We say that ψ satisfies *separability* (*SEP*) if for all *cstp* (N_0, C) and $S \subset N$ satisfying $m(N_0, C) = m(S_0, C) + m((N \setminus S)_0, C)$,

$$\psi_i(N_0, C) = \begin{cases} \psi_i(S_0, C) & \text{if } i \in S \\ \psi_i((N \setminus S)_0, C) & \text{if } i \in N \setminus S. \end{cases}$$

Two subset of agents, S and $N \setminus S$, can connect to the source separately or can connect jointly. If there are no savings when they connect jointly, separability says that agents must pay the same in both circumstances.

This property appears in Megiddo (1978), Granot and Huberman (1981), and Granot and Maschler (1998). They use the name decomposition, instead of separability, and study its relation to the core and the nucleolus of (N, v_C) . For instance, Granot and Huberman (1981) prove that the core and the nucleolus satisfy *SEP*. Bergantiños and Vidal-Puga (2004b) use the name separability.

We say that ψ satisfies *population monotonicity (PM)* if for all *cstp* (N_0, C) , all $S \subset N$, and all $i \in S$,

$$\psi_i(N_0, C) \leq \psi_i(S_0, C).$$

PM says that if new agents join a society no agent of the initial society can be worse. This is a well-known property, which has been used in many different problems.

We now present two propositions. In Proposition 2 we prove that φ satisfies *SYM*, *SEP*, and *RA*. In Proposition 3 we prove that there exists a unique rule satisfying these properties when \mathcal{N} has at least three members.

Proposition 2. φ satisfies *SYM*, *SEP*, and *RA*.

Proof. Bergantiños and Vidal-Puga (2004b) prove that φ satisfies *SEP*.

We now prove that φ satisfies *SYM*. Let (N_0, C) be a *cstp* and let $i, j \in N$ be two symmetric agents. We prove that $\varphi_i(N_0, C) = \varphi_j(N_0, C)$.

Given a *cstp* (N_0, C) we denote by (S_0, C^{+T}) the *cstp* obtained from (N_0, C) assuming that agents of S have to be connected and agents of T are already connected. This means that $c_{ij}^{+T} = c_{ij}$ for all $i, j \in S$ and $c_{i0}^{+T} = \min_{j \in T_0} c_{ij}$ for all $i \in S$.

Now, we associate with each *cstp* (N_0, C) a *TU* game (N, v_C^+) where for each $S \subset N$,

$$v_C^+(S) = m\left(S_0, C^{+(N \setminus S)}\right).$$

Bergantiños and Vidal-Puga (2004b) prove that $\varphi(N_0, C) = Sh(N, v_C^+)$.

Take $S \subset N \setminus \{i, j\}$. Since i and j are symmetric agents, $c_{ik}^{+(N \setminus (S \cup \{i\}))} = c_{jk}^{+(N \setminus (S \cup \{j\}))}$ for all $k \in S_0$ and $c_{lk}^{+(N \setminus (S \cup \{i\}))} = c_{lk}^{+(N \setminus (S \cup \{j\}))}$ for all $l, k \in S_0$.

Assume that $t = \{(k^0, k)\}_{k \in S \cup \{i\}}$ is a *mcst* in $((S \cup \{i\})_0, C^{+(N \setminus (S \cup \{i\}))})$. We define $S_i = \{k \in S \mid k^0 = i\}$ and let t' be the network obtained when we replace agent i by agent j in t , *i.e.*

$$t' = \{(j, k)\}_{k \in S_i} \cup \{(i^0, j)\} \cup \{(k^0, k)\}_{k \in S \setminus S_i}.$$

It is not difficult to see that t' is a *mcs*t in $((S \cup \{j\})_0, C^{+(N \setminus (S \cup \{j\}))})$. Then,

$$\begin{aligned} v_C^+(S \cup \{i\}) &= m\left((S \cup \{i\})_0, C^{+(N \setminus (S \cup \{i\}))}\right) \\ &= m\left((S \cup \{j\})_0, C^{+(N \setminus (S \cup \{j\}))}\right) \\ &= v_C^+(S \cup \{j\}). \end{aligned}$$

Thus, agents i and j are symmetric in the *TU* game (N, v_C^+) . Since the Shapley value satisfies symmetry,

$$\varphi_i(N_0, C) = Sh_i(N, v_C^+) = Sh_j(N, v_C^+) = \varphi_j(N_0, C).$$

We now prove that φ satisfies *RA*. Let (N_0, C) and (N_0, C') be two *cstp* and let $t = \{(i^0, i)\}_{i \in N}$ be a *mcs*t in (N_0, C) , (N_0, C') , and $(N_0, C + C')$ satisfying that there exists an order $\pi = (i_1, \dots, i_n) \in \Pi_N$ such that $c_{i_1^0 i_1} \leq c_{i_2^0 i_2} \leq \dots \leq c_{i_n^0 i_n}$ and $c'_{i_1^0 i_1} \leq c'_{i_2^0 i_2} \leq \dots \leq c'_{i_n^0 i_n}$.

Assume that we compute the canonical form associated with the problems (N_0, C) , (N_0, C') , and $(N_0, C + C')$ through the algorithm. It is not difficult to prove that there exists $(j_1, \dots, j_n) \in \Pi_N$ satisfying that:

- $t^* = \{(j_{k-1}, j_k)\}_{k=1}^n$ is a *mcs*t in the canonical matrices associated to (N_0, C) , (N_0, C') , and $(N_0, C + C')$. For convenience we take $j_0 = 0$.
- For all $k = 1, \dots, n$, $(c + c')_{j_{k-1} j_k}^* = c_{j_{k-1} j_k}^* + c'_{j_{k-1} j_k}^*$.

Applying Proposition 3 (a) of Bergantiños and Vidal-Puga (2004b, page 20) it is not difficult to conclude that for all $S \subset N$,

$$v_{(C+C')^*}(S) = v_{C^*}(S) + v_{C'^*}(S).$$

Since the Shapley value is additive in the characteristic function,

$$\begin{aligned} \varphi(N_0, C + C') &= Sh(N, v_{(C+C')^*}) = Sh(N, v_{C^*} + v_{C'^*}) \\ &= Sh(N, v_{C^*}) + Sh(N, v_{C'^*}) = \varphi(N_0, C) + \varphi(N_0, C'). \end{aligned}$$

This concludes the proof. ■

Assume that \mathcal{N} has at least three agents. Then we have the following result:

Proposition 3. There is a unique rule satisfying *SYM*, *SEP*, and *RA*.

Proof. Norde *et al* (2004) prove that if (N_0, C) is a *cstp*, then there exists a family $\{C^p\}_{p=1}^a$ of cost matrices satisfying three conditions:

1. $C = \sum_{p=1}^a C^p$.
2. For each p there exist $x^p \in \mathbb{R}$ and a network g^p such that $c_{ij}^p = x^p$ if $(i, j) \in g^p$ and $c_{ij}^p = 0$ otherwise.
3. Take $p \in \{1, \dots, a\}$ and $\{i, j, k, l\} \subset N$. Then, $c_{ij}^p \leq c_{kl}^p$ if and only if $c_{ij} \leq c_{kl}$.

We give the intuition of this result for the cost matrix C where

$$C = \begin{pmatrix} 0 & 5 & 9 \\ 5 & 0 & 3 \\ 9 & 3 & 0 \end{pmatrix}.$$

Then $C = C^1 + C^2 + C^3$ where

$$C^1 = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}, \quad C^2 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad \text{and } C^3 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

$$x^1 = 3, \quad x^2 = 2, \quad \text{and } x^3 = 4.$$

$$g^1 = \{(0, 1), (0, 2), (1, 2)\}, \quad g^2 = \{(0, 1), (0, 2)\}, \quad \text{and } g^3 = \{(0, 2)\}.$$

Let ψ be a rule satisfying *SYM*, *SEP* and *RA*.

$$\text{Claim 1. } \psi(N_0, C) = \sum_{p=1}^a \psi(N_0, C^p).$$

By Proposition 1 we know that ψ satisfies *CPL*. Now Claim 1 is a consequence of Conditions 1 and 3.

As a consequence of Claim 1 and Condition 2, it is enough to prove that ψ is unique on the subclass of *cstp* (N_0, C) satisfying that there exist $x \in \mathbb{R}$ and a network g such that $c_{ij} = x$ if $(i, j) \in g$ and $c_{ij} = 0$ otherwise.

We denote by $\{N_1, N_2, \dots, N_q\}$ the partition of N in *C-components*.

$$\text{Claim 2. For each } r = 1, \dots, q \text{ and } i \in N_r, \psi_i(N_0, C) = \psi_i((N_r)_0, C).$$

$$\text{Since } \psi \text{ satisfies } \textit{SEP} \text{ it is enough to prove that } m(N_0, C) = \sum_{r=1}^q m((N_r)_0, C).$$

It is trivial to see that for each *C-component* N_r ,

$$m((N_r)_0, C) = \begin{cases} x & \text{if } c_{0i} = x \text{ for all } i \in N_r \\ 0 & \text{if there exists } i \in N_r \text{ such that } c_{0i} = 0. \end{cases}$$

We assume, without loss of generality, that $m((N_r)_0, C) = 0$ for all $r = 1, \dots, q'$ and $m((N_r)_0, C) = x$ for all $r = q' + 1, \dots, q$.

We now apply Prim's algorithm to (N_0, C) . Since $c_{0j} = 0$ for all $j \in \bigcup_{r=1}^{q'} N_r$, in Stage 1 we can select an arc $(0, i)$ such that $i \in \bigcup_{r=1}^{q'} N_r$.

In any Stage k with $k = 2, \dots, \left| \bigcup_{r=1}^{q'} N_r \right|$ we can select an arc (i^0, i) where $i \in \bigcup_{r=1}^{q'} N_r$ and $c_{i^0 i} = 0$.

Because of the second condition in the definition of C -components, it is not difficult to see that if $i \in N_r$, $j \in N_{r'}$, and $r \neq r'$, then $c_{ij} = x$.

This means that in Stage $\left| \bigcup_{r=1}^{q'} N_r \right| + 1$ we can select an arc (i^0, i) such that $i \in N_{q'+1}$, $i^0 = 0$, and $c_{i^0 i} = x$.

In any Stage k with $k = \left| \bigcup_{r=1}^{q'} N_r \right| + 2, \dots, \left| \bigcup_{r=1}^{q'+1} N_r \right|$ we can select an arc (i^0, i) such that $i \in N_{q'+1}$ and $c_{i^0 i} = 0$.

Using similar arguments as those used before we can conclude that for all $s = q', \dots, q - 1$,

- In Stage $\left| \bigcup_{r=1}^s N_r \right| + 1$ we can select an arc (i^0, i) such that $i \in N_{s+1}$, $i^0 = 0$, and $c_{i^0 i} = x$.
- In any Stage k with $k = \left| \bigcup_{r=1}^s N_r \right| + 2, \dots, \left| \bigcup_{r=1}^{s+1} N_r \right|$ we can select an arc (i^0, i) such that $i \in N_{s+1}$ and $c_{i^0 i} = 0$.

Then,

$$m(N_0, C) = \sum_{r=q'+1}^q x = \sum_{r=1}^q m((N_r)_0, C).$$

This completes the proof of Claim 2.

By Claim 2, we can assume that (N_0, C) has a unique C -component.

Let $t = \{(i^0, i)\}$ be a $mcst$ in (N_0, C) . Then, there exists $\alpha \in N$ such that $\alpha^0 = 0$ and $c_{i^0 i} = 0$ for all $i \in N \setminus \{\alpha\}$. We define C^1 and C^2 where

$$c_{ij}^1 = \begin{cases} c_{ij} & \text{if } 0 \in \{i, j\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad c_{ij}^2 = \begin{cases} 0 & \text{if } 0 \in \{i, j\} \\ c_{ij} & \text{otherwise.} \end{cases}$$

Now it is not difficult to conclude that t is also a $mcst$ in (N_0, C^1) and (N_0, C^2) . Moreover, $c_{i^0 i}^1 = c_{i^0 i}^2 = 0$ for all $i \in N \setminus \{\alpha\}$, $c_{0\alpha}^1 \geq 0 = c_{0\alpha}^2$, and $C = C^1 + C^2$. By RA we conclude that

$$\psi(N_0, C) = \psi(N_0, C^1) + \psi(N_0, C^2).$$

Since $c_{0i}^2 = 0$ for all $i \in N$, $m(\{i\}_0, C^2) = 0$ for all $i \in N$ and $m(N_0, C) = 0$.
By *SEP*,

$$\psi_i(N_0, C^2) = \psi_i(\{i\}_0, C^2) = 0.$$

Then, $\psi(N_0, C) = \psi(N_0, C^1)$.

Thus, it is enough to prove that ψ is unique in a *cstp* (N_0, C) where $c_{ij} = 0$ if $0 \notin \{i, j\}$ and $c_{0i} \in \{0, x\}$ for all $i \in N$.

Three cases are possible:

1. $c_{0i} = x$ for all $i \in N$. This means that all agents in N are symmetric. By *SYM* we conclude that

$$\psi_i(N_0, C) = \frac{x}{n} \text{ for all } i \in N.$$

2. $c_{0i} = 0$ for all $i \in N$. All agents are symmetric. Then, $\psi_i(N_0, C) = 0$ for all $i \in N$.

3. There exist $j, k \in N$ such that $c_{0j} = 0$ and $c_{0k} = x$. We define $N^1 = \{i \in N \mid c_{0i} = x\} \cup \{j\}$ and $N^2 = \{i \in N \mid c_{0i} = 0\} \setminus \{j\}$. Then, $m(N_0, C) = m(N_0^1, C) + m(N_0^2, C)$.

By *SEP*, $\psi_i(N_0, C) = \psi_i(N_0^1, C)$ if $i \in N^1$ and $\psi_i(N_0, C) = \psi_i(N_0^2, C)$ if $i \in N^2$.

By Case 2, $\psi_i(N_0^2, C) = 0$ if $i \in N^2$.

Thus, it is enough to prove that ψ is unique in a *cstp* (N_0, C) where there exists $k \in N$ such that $c_{0i} = x \in \mathbb{R}$ if $i \neq k$ and $c_{ij} = 0$ otherwise.

For each $i \in N \setminus \{k\}$ let C^i be such that $c_{0i}^i = x$ and $c_{ji}^i = 0$ otherwise. Notice that $C = \sum_{i \in N \setminus \{k\}} C^i$. Moreover,

$$t = \{(j-1, j)\}_{j=2}^n \cup \{(0, k)\}$$

is a *mcst* in (N_0, C^i) for all $i \in N \setminus \{k\}$. By *RA*,

$$\psi(N_0, C) = \sum_{i \in N \setminus \{k\}} \psi(N_0, C^i).$$

Thus, it is enough to prove that for all $i \in N \setminus \{k\}$, ψ is unique in each problem (N_0, C^i) .

Take $i \in N \setminus \{k\}$. It is trivial to see that

$$m(N_0, C^i) = m(\{k, i\}_0, C^i) + \sum_{j \in N \setminus \{k, i\}} m(\{j\}_0, C^i).$$

By *SEP*, $\psi_k(N_0, C^i) = \psi_k(\{k, i\}_0, C^i)$, $\psi_i(N_0, C^i) = \psi_i(\{k, i\}_0, C^i)$, and for all $j \in N \setminus \{k, i\}$, $\psi_j(N_0, C^i) = \psi_j(\{j\}_0, C^i) = 0$.

It only remains to prove that ψ is unique in *cstp* $(\{i, k\}, C)$ where $c_{0k} = c_{ik} = 0$ and $c_{0i} = x$.

Since $m(\{i, k\}_0, C) = 0$ we can assume that $\psi(\{i, k\}_0, C) = (y_{ik}, -y_{ik})$. We now prove that $y_{ik} = 0$.

Let (N_0, C') be such that $N = \{i, j, k\}$, $c'_{0i} = x$, and $c'_{hl} = 0$ otherwise. We can find such a (N_0, C') because \mathcal{N} has at least three members.

Then, $m(N_0, C') = m(\{j\}_0, C') + m(\{i, k\}_0, C')$. By *SEP*,

$$\psi_k(N_0, C') = \psi_k(\{i, k\}_0, C') = -y_{ik}.$$

Moreover, $m(N_0, C') = m(\{i, j\}_0, C') + m(\{k\}_0, C')$. By *SEP*, $\psi_k(N_0, C') = \psi_k(\{k\}_0, C') = 0$. Then, $y_{ik} = 0$.

This concludes the proof. ■

The next theorem is a trivial consequence of Propositions 2 and 3.

Theorem 1. If \mathcal{N} has at least three members, then φ is the unique rule satisfying *SYM*, *SEP*, and *RA*.

We now prove that the properties used in Theorem 1 are independent.

- Given $N \subset \mathcal{N}$ we denote by π^N the order induced in N by the index of the agents, *i.e.* given $i, j \in N$, $\pi^N(i) < \pi^N(j)$ if and only if $i < j$. For each *cstp* (N_0, C) and $i \in N$ we define

$$O_i(N_0, C) = v_{C^*}(Pre(i, \pi^N) \cup \{i\}) - v_{C^*}(Pre(i, \pi^N)).$$

It is not difficult to prove that O satisfies *RA* and *SEP*.

Nevertheless, O does not satisfy *SYM*. Consider the *cstp* where $N = \{1, 2\}$ and

$$C = \begin{pmatrix} 0 & 10 & 10 \\ 10 & 0 & 2 \\ 10 & 2 & 0 \end{pmatrix}.$$

Agents 1 and 2 are symmetric but $\psi(N_0, C) = (10, 2)$.

- Consider the egalitarian rule (*E*) defined as $E_i(N_0, C) = \frac{m(N_0, C)}{n}$ for all $i \in N$. It is not difficult to see that *E* satisfies *RA* and *SYM*.

Nevertheless, E does not satisfy SEP . Consider the $cstp$ where $N = \{1, 2\}$ and

$$C = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 0 & 6 \\ 4 & 6 & 0 \end{pmatrix}.$$

Then, $m(\{1\}_0, C) = 2$, $m(\{2\}, C) = 4$, and $m(N_0, C) = 6$. But $E_1(\{1\}_0, C) = 3$ and $E_1(\{2\}_0, C) = 3$.

- Bird (1976) introduced a rule in a particular subclass of $cstp$. We denote this rule as B . Bergantiños and Vidal-Puga (2004b) extend B to the class of all $cstp$ and they prove that it satisfies SEP . It is not difficult to see that B also satisfies SYM .

Nevertheless, B does not satisfy RA . Consider the $cstp$ where $N = \{1, 2\}$ and the cost matrices are:

$$C = \begin{pmatrix} 0 & 10 & 10 \\ 10 & 0 & 2 \\ 10 & 2 & 0 \end{pmatrix}, C' = \begin{pmatrix} 0 & 10 & 11 \\ 10 & 0 & 2 \\ 11 & 2 & 0 \end{pmatrix}$$

$$C + C' = \begin{pmatrix} 0 & 20 & 21 \\ 20 & 0 & 4 \\ 21 & 4 & 0 \end{pmatrix}.$$

$t = \{(0, 1), (1, 2)\}$ is a $mcst$ in (N_0, C) , (N_0, C') , and $(N_0, C + C')$. But $B(N_0, C) = (6, 6)$, $B(N_0, C') = (10, 2)$, and $B(N_0, C + C') = (20, 4)$.

The next corollary gives another characterization of φ .

Corollary 1. φ is the unique rule satisfying SYM , PM , and RA .

Proof. Bergantiños and Vidal-Puga (2004b) prove that φ satisfies PM and that PM implies SEP (if a rule satisfies PM , it also satisfies SEP).

Now Corollary 1 is a trivial consequence of Theorem 1. ■

It is not difficult to prove that Corollary 1 is also a tight characterization result.

The next theorem is the analogous to Theorem 1 when \mathcal{N} has two agents.

Theorem 2. If \mathcal{N} has two agents, φ is the unique rule satisfying POS , SYM , SEP , and RA .

Proof. Bergantiños and Vidal-Puga (2004b) prove that φ satisfies *POS* and *SEP*. By Proposition 2 we know that φ satisfies *SYM* and *RA*.

We now prove the uniqueness. Let ψ be a rule satisfying *POS*, *SYM*, *SEP*, and *RA*.

If $N = \{i\}$, by definition $\psi_i(N_0, C) = m(N_0, C) = c_{0i}$.

Assume that $N = \{i, j\}$. Using arguments similar to those used in the proof of Proposition 3, we can conclude that it is enough to prove that ψ is unique in *cstp* $(\{i, j\}_0, C)$ where $c_{0i} = c_{ij} = 0$ and $c_{0j} = x$.

Since $m(N_0, C) = 0$, $\psi_i(N_0, C) = -\psi_j(N_0, C)$. By *POS*, $\psi_i(N_0, C) = \psi_j(N_0, C) = 0$. ■

We end this section by proving that the properties used in Theorem 2 are independent.

It is not difficult to see that when \mathcal{N} has two agents, O satisfies *SEP*, *RA*, and *POS* but fails *SYM*; E satisfies *SYM*, *RA*, and *POS* but fails *SEP*; and B satisfies *SEP*, *SYM*, and *POS* but fails *RA*.

We now prove that *POS* is independent of the other properties. Assume, without loss of generality, that $N = \{i, j\}$ and $c_{0i} \leq c_{0j}$. We consider the rule A defined as

$$\begin{aligned} A_i(N_0, C) &= \varphi_i(N_0, C) - \max\{0, c_{0j} - \max\{c_{0i}, c_{ij}\}\} \text{ and} \\ A_j(N_0, C) &= \varphi_j(N_0, C) + \max\{0, c_{0j} - \max\{c_{0i}, c_{ij}\}\}. \end{aligned}$$

A satisfies *SYM*. Assume that i and j are symmetric. Then, $c_{0i} = c_{0j}$ and hence

$$A_i(N_0, C) = \varphi_i(N_0, C) = \varphi_j(N_0, C) = A_j(N_0, C).$$

A satisfies *SEP*. Assume that $m(N_0, C) = m(\{i\}_0, C) + m(\{j\}_0, C)$. Then, $c_{ij} \geq c_{0j}$ and hence $A(N_0, C) = \varphi(N_0, C)$. Since φ satisfies *SEP* we conclude that A satisfies *SEP*.

A satisfies *RA*. Let (N_0, C') and t be as in the definition of *RA*. We only prove that $A_i(N_0, C + C') = A_i(N_0, C) + A_i(N_0, C')$ (the proof for agent j is similar). Three cases are possible:

1. $t = \{(0, i), (i, j)\}$. This means that $c_{ij} \leq c_{0j}$, $c'_{0i} \leq c'_{0j}$, and $c'_{ij} \leq c'_{0j}$.

Thus

$$\begin{aligned} A_i(N_0, C) &= \varphi_i(N_0, C) - c_{0j} + \max\{c_{0i}, c_{ij}\}, \\ A_i(N_0, C') &= \varphi_i(N_0, C') - c'_{0j} + \max\{c'_{0i}, c'_{ij}\}, \text{ and} \\ A_i(N_0, C + C') &= \varphi_i(N_0, C + C') - (c_{0j} + c'_{0j}) + \max\{c_{0i} + c'_{0i}, c_{ij} + c'_{ij}\}. \end{aligned}$$

We know that φ satisfies *RA* and $c_{0i} \leq c_{ij}$ if and only if $c'_{0i} \leq c'_{ij}$. Now, it is easy to conclude that $A_i(N_0, C + C') = A_i(N_0, C) + A_i(N_0, C')$.

2. $t = \{(i, j), (0, j)\}$. Since t is a *mcst* in (N_0, C) and (N_0, C') , we conclude that $c_{0i} \geq c_{0j}$, $c'_{0i} \geq c'_{0j}$, $c_{ij} \leq c_{0i}$, and $c'_{ij} \leq c'_{0i}$. Now it is similar to Case 1, simply by writing i instead of j and j instead of i .
3. $t = \{(0, i), (0, j)\}$. Then, $A_i(N_0, C) = c_{0i}$, $A_i(N_0, C') = c'_{0i}$, and $A_i(N_0, C + C') = c_{0i} + c'_{0i}$.

Nevertheless, A does not satisfy *POS*. Let (N_0, C) be the *cstp* where $c_{0j} = 1$, and $c_{0i} = c_{ij} = 0$. Then, $A(N_0, C) = (-1, 1)$.

4 Concluding remarks

In this section we compare our characterization of φ with the characterization given in Branzei *et al* (2003). In both papers an additivity property plays an important role.

Branzei *et al* (2003) characterize φ as the only rule satisfying *efficiency* (*EF*), *equal treatment* (*ET*), *upper bound contributions* (*UBC*), and *CPL*.

A rule ψ satisfies *EF* if $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$ for all *cstp* (N_0, C) .

Branzei *et al* (2003) do not include *EF* in the definition of a rule. If we do not include it, we will obtain that φ is the only rule satisfying *EF*, *SYM*, *SEP*, and *RA*. Moreover, the independence of *EF* from the other properties is shown by the rule $\psi_i(N_0, C) = 0$ for all *cstp* (N_0, C) and all $i \in N$.

The *C-components* are crucial in the definition of *ET* and *UBC*.

A rule ψ satisfies *ET* if given a *cstp* (N_0, C) and a *C-component* S , $\psi_i(N_0, C) = \psi_j(N_0, C)$ for all $i, j \in S$.

A rule ψ satisfies *UBC* if given a *cstp* (N_0, C) and a *C-component* S , $\sum_{i \in S} \psi_i(N_0, C) \leq \min_{i \in S} \{c_{0i}\}$.

Branzei *et al* (2003) do not give any reason why a fair rule should satisfy *ET* or *UBC*.

We do not know what the principle behind *ET* is. Assume that (N_0, C) is a *cstp* where $N = \{1, 2, 3\}$ and

$$C = \begin{pmatrix} 0 & 12 & 20 & 40 \\ 12 & 0 & 0 & 35 \\ 20 & 0 & 0 & 0 \\ 40 & 35 & 0 & 0 \end{pmatrix}.$$

In this problem N is the unique C -component. Then, ET says that all agents must pay the same. We do not understand why a fair rule must assign the same cost to each agent.

Nevertheless, we see a principle behind UBC .

We say that ψ satisfies *core selection* (CS) if for all $cstp(N_0, C)$ and all $S \subset N$,

$$\sum_{i \in S} \psi_i(N_0, C) \leq m(S_0, C).$$

This property says that no group of agents can be better constructing their own network instead of paying what the rule ψ proposes to them. CS is a property widely used in $cstp$.

Notice that if S is a C -component, then $m(S_0, C) = \min_{i \in S} \{c_{0i}\}$. Then, UBC can be reinterpreted as a property of CS restricted to C -components.

It is known that φ satisfies CS . Since CS implies UBC the result of Branzei *et al* (2003) can be reformulated saying that φ is the only rule satisfying EF , ET , CS , and CPL .

We now comment the main differences between our proof of the characterization of φ and the proof of Branzei *et al* (2003).

The proof of the existence (φ satisfies the properties) is completely different. Our proof is made using the canonical matrix and other results stated in Bergantiños and Vidal-Puga (2004b). The proof of Branzei *et al* (2003) is made through Kruskal's algorithm.

The uniqueness has two parts in both proofs. First part. In both papers it is proved that $\psi(N_0, C) = \sum_{p=1}^a \psi(N_0, C^p)$. If a rule satisfies CPL , this part is an immediate consequence of the results obtained in Norde *et al* (2004). In this paper we only need to prove that RA implies CPL (Proposition 1).

Second part. In both papers it is proved that ψ is unique in each problem (N_0, C^p) . In the proof of Branzei *et al* (2003) it is an immediate consequence of the properties of ET and UBC . In our paper we need a more complicated argument.

We end this section comparing the properties used in our characterization result with the properties used in the characterization result given in Branzei *et al* (2003).

We already mentioned that we have included EF in the definition of a rule. As we have proved in Proposition 1, RA implies CPL .

Given a $cstp(N_0, C)$, let $\{N_1, N_2, \dots, N_p\}$ be the partition of N in C -components. This partition will be used in the next rules we define.

SYM and *ET* are unrelated.

- The egalitarian rule satisfies *SYM* but not *ET*.
- We define ψ^1 as follows. If $\{N_1, N_2, \dots, N_p\} = \{\{1\}, \dots, \{n\}\}$ we take $\psi_1^1(N_0, C) = m(N_0, C)$ and $\psi_i^1(N_0, C) = 0$ if $i \neq 1$. If $\{N_1, N_2, \dots, N_p\} \neq \{\{1\}, \dots, \{n\}\}$ we take $\psi^1(N_0, C) = \varphi(N_0, C)$.

It is not difficult to see that ψ^1 satisfies *ET* but fails *SYM*.

SEP and *UBC* are unrelated.

- We define ψ^2 as follows. If $\{N_1, N_2, \dots, N_p\} = \{\{1\}, \dots, \{n\}\}$ we take $\psi_i^2(N_0, C) = \frac{c_{0i}}{\sum_{j \in N} c_{0j}} m(N_0, C)$ for all $i \in N$. If $\{N_1, N_2, \dots, N_p\} \neq \{\{1\}, \dots, \{n\}\}$ we take $\psi^2(N_0, C) = \varphi(N_0, C)$.

Since φ satisfies *UBC*, it is not difficult to see that ψ^2 also satisfies *UBC*.

Nevertheless ψ^2 fails *SEP*. We consider (N_0, C) where

$$C = \begin{pmatrix} 0 & 9 & 9 & 18 \\ 9 & 0 & 3 & 20 \\ 9 & 3 & 0 & 20 \\ 18 & 20 & 20 & 0 \end{pmatrix}$$

Making some computations we obtain that the partition in *C-components* is $\{\{1\}, \{2\}, \{3\}\}$, $m(N_0, C) = 30$, $m(\{1, 2\}_0, C) = 12$, $m(\{3\}_0, C) = 18$, and $\psi^2(N_0, C) = (7.5, 7.5, 15)$. Then, ψ^2 does not satisfy *SEP* because $\psi_3^2(N_0, C) = 15 \neq 18 = \psi_3^2(\{3\}_0, C)$.

- Given $x \in \mathbb{R}^N$ with $1, 2 \in N$, we define $f^N(x) \in \mathbb{R}^N$ as the vector $f_1^N(x) = x_1 + x_2$, $f_1^N(x) = 0$ and $f_i^N(x) = x_i$ for all $i \in N \setminus \{1, 2\}$. Moreover, we define $N^* \subset N$ as the set of agents $i \in N$ such that there exists a *mcst* $t \in \mathcal{G}_0^N$ satisfying that the arc $(0, i)$ belongs to t . Namely,

$$N^* = \{i \in N \mid \exists \text{mcst } t \in \mathcal{G}_0^N \text{ such that } (0, i) \in t\}.$$

We define

$$\psi^3(N_0, C) = \begin{cases} f^N(\varphi(N_0, C)) & \text{if } 1, 2 \in N, c_{01} < c_{12} < c_{02} \text{ and } 1 \in N^* \\ \varphi(N_0, C) & \text{otherwise.} \end{cases}$$

It is not difficult to prove that ψ^3 satisfies *SEP*.

Nevertheless, ψ^3 fails *UBC*. We consider (N_0, C) where $N = \{1, 2\}$, $c_{01} = 1$, $c_{02} = 4$, and $c_{12} = 2$. There are two *C-components* ($\{1\}$ and $\{2\}$) and $\psi_1^3(N_0, C) = 3 > 1 = c_{01}$.

5 References

Bergantiños, G. and J.J. Vidal-Puga, 2004a, Additive rules in bankruptcy problems and other related problems, *Mathematical Social Sciences* 47, 87-101.

Bergantiños, G. and J.J. Vidal-Puga, 2004b, Defining rules in cost spanning tree problems through the canonical form, *EconPapers*, RePEc:wpa:wuwpga:0402004, <http://econpapers.hhs.se/paper/wuwpga/0402004.htm>

Bird, C.G., 1976, On cost allocation for a spanning tree: A game theoretic approach, *Networks* 6, 335-350.

Branzei R., S. Moretti, H. Norde and S. Tijs, 2003, The P-value for cost sharing in minimum cost spanning tree situations, Mimeo, Tilburg University.

Dutta B. and A. Kar, 2002, Cost monotonicity, consistency and minimum cost spanning tree games, *Games and Economic Behavior*, Forthcoming.

Feltkamp V., S. Tijs and S. Muto, 1994, On the irreducible core and the equal remaining obligation rule of minimum cost extension problems, Mimeo.

Granot D. and G. Huberman, 1981, Minimum cost spanning tree games, *Mathematical Programming* 21, 1-18.

Granot D. and M. Maschler, 1998, Spanning network games, *International Journal of Game Theory* 27, 467-500.

Kar A., 2002, Axiomatization of the Shapley value on minimum cost spanning tree games, *Games and Economic Behavior* 38, 265-277.

Kruskal J., 1956, On the shortest spanning subtree of a graph and the traveling salesman problem, *Proceedings of the American Mathematical Society* 7, 48-50.

Megiddo N., 1978, Computational complexity and the game theory approach to cost allocation for a tree, *Mathematics of Operations Research* 3, 189-196.

Moulin H. and S. Shenker, 1994, Average cost pricing versus serial cost sharing: an axiomatic comparison, *Journal of Economic Theory* 64, 178-201.

Norde H., S. Moretti and S. Tijs, 2004, Minimum cost spanning tree games and population monotonic allocation schemes, *European Journal of Operational Research* 154, 84-97.

Prim R.C., 1957, Shortest connection networks and some generalizations, *Bell Systems Technology Journal* 36, 1389-1401.

Shapley L.S., 1953, A value for n-person games, in: H.W. Kuhn and A.W. Tucker, eds., *Contributions to the Theory of Games II* (Princeton, NJ) 307-317.