

Nonspecific Networking

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Abstract

A new model of strategic network formation is developed and analyzed, where an agent's investment in links is nonspecific. The model comprises a large class of games which are both potential and super- or submodular games. We obtain comparative statics results for Nash equilibria with respect to investment costs for supermodular as well as submodular networking games. We also study logit-perturbed best-response dynamics for supermodular games with potentials. We find that the associated set of stochastically stable states forms a sublattice of the lattice of Nash equilibria and derive comparative statics results for the smallest and the largest stochastically stable state. Finally, we provide a broad spectrum of applications from social interaction to industrial organization.

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To network or not to network,
that is the question.

1 Introduction

We develop and analyze a new model of strategic network formation where an agent's effort or investment in links is nonspecific. Models of strategic network formation typically assume that each agent selects his direct links to other agents in which to invest. Yet in practice, a person's networking efforts may not only establish or strengthen desirable links to specific agents, but also create or reinforce links to many if not all other individuals. Beneficial links may come along with detrimental ones. For example, being better connected and more accessible implies potentially more calls from phone banks, more "spam", more encounters with annoying or hostile people.

Here we focus on nonspecific networking, meaning that an agent cannot select a specific subset of feasible links which he wants to establish or strengthen. Rather, each agent chooses an effort level or intensity of networking. In the simplest case, the agent faces a binary choice: to network or not to network. If an agent increases his networking effort, all direct links to other agents are strengthened to various degrees. We assume that benefits accrue only from direct links. The set of agents or players is finite. Each agent has a finite strategy set consisting of the networking levels to choose from. For any pair of agents, their networking levels determine the individual benefits which they obtain from interacting with each other. An agent derives an aggregate benefit from the pairwise interactions with all others. This aggregate benefit is a function of the chosen profile of networking levels. In addition, the agent incurs networking costs, which are a function of the agent's own networking level. The agent's payoff is his aggregate benefit minus his cost. The set of agents together with the individual strategy sets and payoff functions constitute a game in strategic form. Equilibrium means Nash equilibrium.¹

¹The recent literature employs mainly two alternative equilibrium concepts — and combinations thereof. Jackson and Wolinsky (1996) introduced pairwise stability as solution

Despite its apparent simplicity, our hitherto unexplored model of network formation proves very promising for two reasons. First, it covers a broad spectrum of applications. It allows for social networking where some persons are more attractive than others, and some even possess negative attraction. Attraction or repulsion can be mutual or not. Certain individuals can have greater advantages from networking or smaller costs of networking than others and, therefore, may be considered natural networkers. To the extent that benefits are positive, under-investment in links can occur in equilibrium. When one allows for the possibility that benefits from interactions with certain agents are negative, a player prefers not to have links and interactions with such “bad neighbors”. Therefore, agents may refrain from networking even when link formation is costless. But an agent cannot prevent bad neighbors from networking and, consequently, may suffer from their efforts. Thus, there can be over-investment in the sense that less investment would increase aggregate welfare. In Section 6, we shall present an example where under-investment by one group of agents and over-investment by a second group coexist in equilibrium. In addition to social interaction and networking, the model is applicable in economics, in particular in the context of industrial organization. Subsection 4D is devoted to the formation of user networks with various degrees of compatibility across technologies.

The second reason why our model holds much promise are two classes of very interesting games it includes. Specifically, the model comprises a large class of games which are both potential and supermodular (strategic complements) games. Finite potential games and finite supermodular games have in common that a Nash equilibrium in pure strategies exists. The literature on games which share both properties is scarce. Dubey, Haimanko, and Zapechelnjuk (2002) show that games of strategic substitutes or complements with aggregation are “pseudo-potential” games.² As a consequence, they obtain existence of a Nash equilibrium and convergence to Nash equilibrium of certain deterministic best response processes. Brânzei, Mallozzi and Tijs (2003) investigate the relationship between the class of potential games and the class of supermodular games. They essentially focus on two-person zero-sum games (and a special case of Cournot duopoly). Their main result is

concept for strategic models of network formation. Here we follow Bala and Goyal (2000) in adopting Nash equilibrium as solution concept.

²The notion of pseudo-potential games is a generalization of the notion of best-response potential games introduced by Voorneveld (2000).

that two-player zero-sum supermodular games are potential games and conversely that two-player zero-sum potential games can be transformed into supermodular games. In our model, suitable assumptions on the benefits from pairwise interaction give rise to a novel class of games which are both potential and supermodular games. A different set of assumptions generates an equally rich family of networking games which have both a potential and the strategic substitutes property.

In section 5, we obtain comparative statics results for Nash equilibria with respect to networking costs for either class of networking games, those with strategic complements and those with strategic substitutes. If a game is a potential game as well, then perturbed best response dynamics with logit trembles yield the maximizers of the potential as the stochastically stable states, as shown by Blume (1993, 1997), Young (1998), Baron *et al.* (2002b), among others. Logit trembles have the appealing feature that mistake probabilities are state-dependent and the probability of making a specific mistake, that is of playing a specific non-best response, is inversely related to the opportunity cost of making the mistake.³ Furthermore, Mattsson and Weibull (2002) and Baron *et al.* (2002a, 2002b) derive a logit rule as the solution of a maximization problem involving a trade-off between the magnitude of trembles and control costs. The fact that the stochastically stable states are the maximizers of the potential, has several strong implications. First, the set of stochastically stable states is a subset of the set of Nash equilibria. Further, in the case of supermodular games, the set of stochastically stable states forms a sublattice of the lattice of Nash equilibria. Consequently, a symmetric game has at least one symmetric stochastically stable state. Finally, again in the case of supermodular games, one obtains comparative statics results for the smallest and the largest stochastically stable state. The structural properties of the set of stochastically stable states are derived in Section 3. Comparative statics is performed in Section 5.

The investigation of logit perturbed best response dynamics for supermodular games with potentials and the associated set of stochastically stable states is one of the original contributions of the current paper. Dubey, Haimanko, and Zapechelnyuk (2002) do not consider stochastic perturba-

³The most prominent alternative, Bernoulli or uniform trembles, does not have this feature. Both types of trembles often, but not always lead to the same set of stochastically stable states or long-run equilibria.

tions or “noise” and stochastic stability. To our knowledge only two earlier papers, Kandori and Rob (1995) and Kaarboe and Tieman (1999), combine stochastic stability and supermodularity in a general setting.⁴ These two papers focus on a class of global interaction games based on two-player and symmetric strict supermodular games. Players gradually adjust their behavior in taking a summary statistic into account. The adjustment process is perturbed by Bernoulli or uniform trembles or slight generalizations thereof. All authors obtain monotonicity results of best responses over the set of states and show that the limit sets of the unperturbed process correspond one-to-one with the set of (strict) Nash equilibria of the recurrent game. Consequently, the set of stochastically stable states is contained in the set of Nash equilibria. Hence supermodular games exposed to uniform trembles and potential games exposed to logit trembles both induce perturbed dynamics under which the stochastically stable states form a subset of the set of Nash equilibria. Unlike the present paper, the earlier literature does not examine the structure of the set of stochastically stable states and its variation in response to parametric changes.

Nonspecific networking admits a differential impact of an agent’s networking efforts on the strength of links to various other agents. In particular, undirected graphs serve as a descriptive tool throughout the paper to distinguish between pairs of agents which can form links among themselves and those pairs which cannot reach each other. Such a graph represents a “social structure” in the sense of Chwe (2000). At the end of Section 4, we indicate our original yet easily overlooked contribution to the nascent economic literature on the role of social structure.

We set the stage in Section 2, where we develop the general model and discuss some of the most pertinent issues by means of prototype case studies. In Section 3, we introduce logit perturbed best-response dynamics and analyze the corresponding set of stochastically stable states. In Section 4, we present pairwise symmetry conditions which guarantee existence of a potential for the networking game. This section also includes the analysis of several classes of games which satisfy pairwise symmetry and most of which exhibit either strategic substitutes or strategic complements. Section 5 is devoted

⁴Other papers on stochastic stability and supermodularity (or submodularity) exist but they exclusively deal with symmetric aggregative games that are either submodular or supermodular [Alós-Ferrer and Ania (2002), Schipper (2003)].

to comparative statics. In Section 6, we present two classes of games which violate pairwise symmetry. Section 7 contains conclusions and extensions.

2 Model and Key Issues

Our model of nonspecific networking constitutes a game in strategic form. There is a finite player set $I = \{1, \dots, N\}$ where $N > 1$. Every player $i \in I$ has strategy set $S_i = K = \{k_0, k_1, \dots, k_T\}$, with $T \geq 1$ and $0 = k_0 < k_1 < \dots < k_T$. The $T + 1$ individual strategies $0, k_1, \dots, k_T$ are the **networking levels** a player can choose and are the same for all players. Depending on the context, a higher networking level may mean more effort in socializing, more investment in networking skills, more investment in communication and information hardware or software, subscription to better network services. The joint strategy space is $S = S_1 \times \dots \times S_N$, with generic elements s . The space of joint strategies or strategy profiles of all players except i is denoted $S_{-i} = \prod_{j \neq i} S_j$, with generic elements s_{-i} . Players receive **benefits from pairwise interaction** with others:

For any pair $(i, j) \in I \times I, i \neq j$, player i receives a benefit $b_{ij}(s_i, s_j) \in \mathbb{R}$ from interacting with j , if i chooses $s_i \in S_i$ and j chooses $s_j \in S_j$. At this preliminary stage, the benefit function b_{ij} should be viewed as a reduced form that convolutes several effects. Subsequently, special cases of benefit functions will be considered, where the different aspects of network formation become more explicit and transparent. Player $i \in I$ incurs a **cost** $c_i(s_i)$ when choosing $s_i \in S_i$. As a rule, the choice of a higher networking level is more costly: $0 = c_i(0) < c_i(k_1) < \dots < c_i(k_T)$. However, in some applications, k_0, k_1, \dots, k_T may just be labels for different technologies, user networks, natural or artificial languages, etc. which cannot be unambiguously ranked in terms of benefits or costs. The **payoff** $u_i(s)$ for player i depends on the strategy profile (joint strategy) of all players, $s = (s_1, \dots, s_N) \in S$, and consists of i 's total benefit from interacting with other players minus i 's cost:

$$u_i(s) = \sum_{j \neq i} b_{ij}(s_i, s_j) - c_i(s_i). \quad (1)$$

The list $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ constitutes a **game in strategic or normal form** and summarizes our model of nonspecific networking and network formation. The equilibrium concept is Nash equilibrium. When appropriate,

we shall employ the concept of a **potential** P for G pioneered by Monderer and Shapley (1996), i.e. a function $P: S \rightarrow \mathbb{R}$ such that

$$u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) = P(s_i, s_{-i}) - P(s'_i, s_{-i})$$

for all $i \in I$, $s_i, s'_i \in S_i$, $s_{-i} \in S_{-i}$.

We shall further use the following terminology and notation related to graphs and networks. Let $W: S \rightarrow \mathbb{R}$ be the aggregate or utilitarian welfare function given by

$$W(s) = \sum_i u_i(s)$$

for $s \in S$. A strategy profile s is called **efficient** if it is a maximizer of W and called **inefficient** otherwise. Let $F = \{J \subseteq I : |J| = 2\}$. A pair $\Gamma = (I, E)$ with $E \subseteq F$ is called an **undirected graph** with vertex set I and edge set E . Then the elements of I are called the vertices or nodes of the graph and the elements of E are called the edges or links of the graph. In case $\{i, j\} \in E$, i.e. in case $\{i, j\}$ is an edge (link) of the graph, we also say that $\{i, j\}$ “belongs to the graph” and that i and j are “neighbors”. Throughout, without further mention, we are restricting ourselves to graphs without isolated nodes. In such a graph, every node has at least one neighbor. Finally, we use the shorthand notation ij for (i, j) .

We are going to explore the implications of two opposite conditions on the benefits from interaction of a pair of players ij , $i \neq j$:

Definition 1 b_{ij} satisfies **decreasing differences** in $(s_i, s_j) \in S_i \times S_j$ if for all pairs (s_i, s_j) and (s'_i, s'_j) , it is the case that $s_i \geq s'_i$ and $s_j \geq s'_j$ implies

$$b_{ij}(s_i, s_j) - b_{ij}(s'_i, s_j) \leq b_{ij}(s_i, s'_j) - b_{ij}(s'_i, s'_j).$$

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$$b_{ij}(s_i, s_j) - b_{ij}(s'_i, s_j) \geq b_{ij}(s_i, s'_j) - b_{ij}(s'_i, s'_j).$$

Decreasing differences is a strategic substitutes condition when a reaction function for i exists and equivalent to $\partial^2 b_{ij} / \partial s_j \partial s_i \leq 0$ when K is an interval and b_{ij} is sufficiently smooth. Increasing differences is a strategic complements condition when a reaction function for i exists and equivalent to $\partial^2 b_{ij} / \partial s_j \partial s_i \geq 0$ when K is an interval and b_{ij} is sufficiently smooth. For details and further references, see Topkis (1998) and Chapter 2 of Vives (1999). As a first result, we obtain

Proposition 1 *Let G be a networking game where pairwise benefits satisfy the following assumption:*

- (A) *There exists an undirected graph (without isolated nodes) (I, E) such that $b_{ij} = 0$ for $\{i, j\} \notin E$ and b_{ij} satisfies increasing differences in $(s_i, s_j) \in S_i \times S_j$ for $\{i, j\} \in E$.*

Then the set of Nash equilibria $S^N \subseteq S$ is nonempty and the partially ordered set S^N is a lattice.

PROOF. $S = \prod_{i \in I} S_i$ is a finite lattice as the cartesian product of finite lattices. For each $i \in I$, $u_i(s_i, s_{-i})$ is trivially supermodular in s_i on S_i for each $s_{-i} \in S_{-i}$. Pick any $i \in I$. For each $j \neq i$, $u_i(s_i, s_j, s_{-ij})$ has increasing differences in (s_i, s_j) on $S_i \times S_j$ because of the functional form (1) and assumption (A) which means that $u_i(s_i, s_{-i})$ has increasing differences in (s_i, s_{-i}) on $S_i \times S_{-i}$. Hence G is a supermodular game. The assertion follows from Zhou's Theorem (1994, p. 299). ■■

The specific cases examined in subsections 2C, 4B and Section 6, the case of downward compatibility in subsection 4D, and the examples given in Section 5 satisfy the assumptions of the proposition. If in addition, the game is symmetric, one obtains existence of a symmetric equilibrium as a corollary.

For the remainder of this section, we are going to examine several special cases of the model, which allows us to bring out some of its most important features and pertinent issues. In particular, we shall address the issue of inefficiency of equilibrium outcomes. As mentioned in the Introduction, there can be both under-investment and over-investment in links. Another interesting aspect is the equilibrium investment pattern. As a rule, a symmetric model gives rise to a symmetric equilibrium. But asymmetric equilibria can exist as well. A further property to be investigated is stochastic stability of equilibrium outcomes. It turns out that stochastic stability does not necessarily introduce a bias in favor of the symmetric equilibrium of the symmetric game.

2A: Case of Homogeneous Player Population

We consider a homogeneous population of $N = 2M$ players with $M \geq 2$. The players form a circular undirected graph $\Gamma = (I, E)$ with $E =$

$\{\{1, 2\}, \{2, 3\}, \dots, \{N-1, N\}, \{N, 1\}\}$. The set of available networking levels is $K = \{h/2 : h = 0, 1, \dots, 10\}$. The pairwise benefit functions are

$$b_{ij}(s_i, s_j) = \sqrt{s_i + s_j} \text{ for } \{i, j\} \in E; b_{ij}(s_i, s_j) = 0 \text{ for } \{i, j\} \notin E.$$

Costs are of the specific linear form $c_i(s_i) = s_i$. The corresponding game $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ is symmetric and has the symmetric equilibrium $s^* = (s_1^*, \dots, s_N^*) = (1/2, \dots, 1/2)$. This equilibrium is inefficient, since maximization of the welfare function W requires that $s_i + s_j = 4$ for $\{i, j\} \in E$. Indeed, everybody would be better off at $s = (2, \dots, 2)$ than at s^* . The reason for the under-investment in equilibrium — which is characteristic for most of the existing network formation models — is that players neglect the positive externality that their investment creates for their neighbors.

G has at least two asymmetric equilibria, $s^\Delta = (0, 1, 0, 1, \dots, 0, 1)$ and $s^\nabla = (1, 0, 1, 0, \dots, 1, 0)$.⁵ All three equilibria are inefficient, with the same value $W = 2N - M$ whereas the maximum value of W is $4N - 2N = 2N$. The game has a potential P given by (3) below. All three equilibria are maximizers of P , and therefore stochastically stable states with respect to best response dynamics with logit perturbations, as studied by Blume (1993, 1997), Young (1998), and Baron et al. (2002a, b), among others. Consequently, under very small random perturbations, asymmetric outcomes are more likely (since they outnumber the symmetric one) than the symmetric equilibrium.⁶ While there may be a preference for concentrating on symmetric equilibria because of analytical convenience, our analysis suggests that the symmetric equilibrium is not necessarily the predominant outcome. To recapitulate, the main features of this particular case with a homogeneous player population are:

- (a) Existence of symmetric and asymmetric equilibria.
- (b) Under-investment in equilibrium.
- (c) Stochastic stability of symmetric and asymmetric equilibria.

Co-existence of symmetric and asymmetric equilibria and under-investment can be expected in many other cases with homogeneous player populations.

⁵For $M = 2$, these are obviously the only other equilibria. For $M > 2$, there exist also equilibria with strings $1/2, 1, 0, 1, 1/2$.

⁶For details, see Section 3.

The stochastic stability result (c) seems to be less robust: In subsection 4D, we refer to a numerical example in Baron *et al.* (2002a, p. 574) where stochastic stability rules out all symmetric equilibria.

2B: Case of Variably Attractive Players

Like in the previous case, we consider a population of $N = 2M, M \geq 2$ players, which form a circular undirected graph $\Gamma = (I, E)$ with $E = \{\{1, 2\}, \{2, 3\}, \dots, \{N-1, N\}, \{N, 1\}\}$. As before, the set of available networking levels is $K = \{h/2 : h = 0, 1, \dots, 10\}$ and the costs functions are $c_i(s_i) = s_i$. But the population is no longer homogeneous with respect to benefits. We assume that the even numbered players $i = 2, 4, \dots, 2M$ are more attractive to their neighbors than the odd numbered players $j = 1, 3, \dots, 2M-1$. Specifically, the pairwise benefit functions are now:

$$\begin{aligned} b_{ij}(s_i, s_j) &= \sqrt{s_i + s_j} \text{ for } \{i, j\} \in E, i \text{ even}; \\ b_{ij}(s_i, s_j) &= 2 \cdot \sqrt{s_i + s_j} \text{ for } \{i, j\} \in E, i \text{ odd}; \\ b_{ij}(s_i, s_j) &= 0 \text{ for } \{i, j\} \notin E. \end{aligned}$$

Hence, *ceteris paribus*, it is more beneficial for an odd numbered player to interact with his two even numbered neighbors than for an even numbered player to interact with her two odd numbered neighbors. As before, for an even numbered player, a best response s_i against any $s_{-i} \in S_{-i}$ satisfies $s_i \leq 1$. But now if j is an odd numbered player, $s_{-j} \in S_{-j}$ where every even numbered i plays $s_i \leq 1$, and $s_j \in S_j$ is a best response against s_{-j} , then $s_j \geq 3$. In turn, if i is an even numbered player, $s_{-i} \in S_{-i}$ where every odd numbered j plays $s_j \geq 3$, and $s_i \in S_i$ is a best response against s_{-i} , then $s_i = 0$. Finally, if j is an odd numbered player, $s_{-j} \in S_{-j}$ where every even numbered i plays $s_i = 0$, and $s_j \in S_j$ is a best response against s_{-j} , then $s_j = 4$. These best response properties of the game imply that $s^{**} = (4, 0, 4, 0, \dots, 4, 0)$ is the unique equilibrium and that the singleton $\{s^{**}\}$ is the only minimal set closed under asynchronous best replies (minimal cuabr set) in the sense of Baron *et al.* (2002a). By Proposition 1 of Baron *et al.* (2002a), s^{**} is the only stochastically stable state with respect to best response dynamics with logit perturbations. The strategy profile s^{**} emerges as the only equilibrium and stochastically stable outcome, because

interaction is very beneficial for odd numbered players. The odd numbered players make a substantial investment in networking, which is much higher than their investment in the corresponding asymmetric equilibrium s^∇ of the homogeneous player case 2A. Now the even numbered players enjoy a guaranteed and larger free ride because of their attractiveness. Still, there is underinvestment in networking: $W(s^{**}) = 8M$ whereas $s = (5, 0, 5, 0, \dots, 5, 0)$ yields $W(s) \approx 8.41M$. However, the associated welfare gain can only benefit all players, if side-payments are possible.

2C: Case of Good and Bad Neighbors

Again, we consider a population of $N = 2M$, $M \geq 2$ players and the circular undirected graph $\Gamma = (I, E)$ with $E = \{\{1, 2\}, \{2, 3\}, \dots, \{N-1, N\}, \{N, 1\}\}$. As before, the set of available networking levels is $K = \{h/2 : h = 0, 1, \dots, 10\}$ and the cost functions are $c_i(s_i) = s_i$. As in case 2B, the population is heterogeneous with respect to benefits. But now we assume that the even numbered players $i = 2, 4, \dots, 2M$ are “good neighbors” whereas the odd numbered players $j = 1, 3, \dots, 2M - 1$ are “bad neighbors”. Specifically, the pairwise benefit functions are now:

$$\begin{aligned} b_{ij}(s_i, s_j) &= -4(s_i + s_j) \text{ for } \{i, j\} \in E, i \text{ even;} \\ b_{ij}(s_i, s_j) &= 2(s_i + s_j) \text{ for } \{i, j\} \in E, i \text{ odd;} \\ b_{ij}(s_i, s_j) &= 0 \text{ for } \{i, j\} \notin E. \end{aligned}$$

Under these circumstances, an even numbered player i wants to minimize interaction with her bad neighbors and has the strictly dominant strategy $s_i = 0$ while an odd numbered player j prefers to maximize interaction with his good neighbors and has the strictly dominant strategy $s_j = 5$. Consequently, $s^+ = (5, 0, 5, 0, \dots, 5, 0)$ is the equilibrium point in strictly dominant strategies and $\{s^+\}$ is the unique minimal cuabr set. By Proposition 1 of Baron *et al.* (2002a), s^+ is the only stochastically stable state with respect to best response dynamics with logit perturbations. The extremely stable outcome s^+ represents an instance of over-investment in networking: W is maximized at $s = (0, 0, \dots, 0, 0)$. Yet in equilibrium, the even numbered “good neighbors” refrain from networking, but cannot prevent their odd numbered “bad neighbors” from making heavy investments. Everybody could be made better off, if the even numbered players were able to compensate their neighbors for abstention from networking.

3 Stochastic Stability

Our concept of stochastic stability is based upon best response dynamics with logit perturbations. Throughout, we consider dynamics with asynchronous updating and persistent noise, with discrete time $t = 0, 1, \dots$ and states $s \in S$. Let $q = (q_1, \dots, q_n) \gg 0$ be an n -dimensional probability vector. The recurrent game G is played once in each period. In each period t , one player, say i , is drawn with probability $q_i > 0$ from this population to adjust his strategy and does so according to a perturbed adaptive rule. The non-selected players repeat the strategies they have played in the previous period.

The perturbed adaptive rule is a logit rule. Suppose the current state is $s = (s_j)_{j \in I}$. In principle, the updating player i wants to play a best reply against $s_{-i} = (s_j)_{j \neq i}$. But with some small probability, the player trembles and plays a non-best reply. If the player follows a logit rule, then for all $t_i \in S_i$, the probability that i chooses t_i in state s is given by

$$p_i^{t_i}(s) = \frac{\exp[u_i(t_i, s_{-i})/\epsilon]}{\sum_{k_i} \exp[u_i(k_i, s_{-i})/\epsilon]}, \quad (2)$$

where $\epsilon > 0$ is a noise parameter. For given ϵ , two choices that yield the same payoff to i are equally likely. If one of them yields a higher payoff, it will be chosen with a higher probability. In particular, any best reply to s_{-i} is more likely to be chosen than a non-best reply. As $\epsilon \rightarrow 0$, the probability that a best reply is chosen goes to 1. For given $\epsilon > 0$, one obtains a stationary Markov process on S with transition matrix $M(\epsilon)$. $M(\epsilon)$ has entries $m_{s,s'}(\epsilon) \in S \times S$ with the following properties. If s and s' differ in more than one component, then $m_{s,s'}(\epsilon) = 0$. If s and s' differ only in the i th coordinate and $s' = (t_i, s_{-i})$, then $m_{s,s'}(\epsilon) = q_i \cdot p_i^{t_i}(s)$. If $s = s'$, then $m_{s,s}(\epsilon) = \sum_{j \in I} q_j \cdot p_j^{s_j}(s)$. The process is irreducible and aperiodic, hence it is ergodic and has a unique stationary distribution, represented by a row probability vector $\mu(\epsilon)$. Like in many prior studies of perturbed evolutionary games we want to determine the behavior of the system when $\epsilon \rightarrow 0$, that is when the noise becomes arbitrarily small. If the limit stationary distribution $\mu^* = \lim_{\epsilon \rightarrow 0} \mu(\epsilon)$ exists, we write S^* for its support:

$$S^* \equiv \{s \in S : \mu_s^* > 0\}$$

The profiles in S^* will be referred to as **stochastically stable states**. These are the states in which the system stays most of the time when very little, but still some noise remains. Baron *et al.* (2002a) show that S^* can be partitioned into minimal sets closed under asynchronous best replies. It turns out that the limit stationary distribution exists and the stochastically stable states are the maximizers of the potential, if the underlying game G has a potential.

Proposition 2 *If G has a potential P , then $S^* = \arg \max_{s \in S} P(s)$ and all stochastically stable states have equal probability.*

PROOF. See Blume (1993, 1997), Young (1998), Baron *et al.* (2002a,b) for the key argument. ■ ■

A further immediate implication is given by

Proposition 3 *If G has a potential P , then $\arg \max_{s \in S} P(s)$ is a non-empty set of Nash equilibria.*

PROOF. Since G is a finite game, there exists $s \in S$ which maximizes P . Every maximizer of P is a Nash equilibrium of G . ■ ■

Moreover, for a potential game which satisfies assumption (A) of Proposition 1, the set of stochastically stable states forms a nonempty sublattice of the equilibrium set S^N . For the definition and characterization of supermodular functions, see Topkis (1998).

Proposition 4 *Suppose G is a networking game which has a potential $P: S \rightarrow \mathbb{R}$ and satisfies (A). Then:*

- (α) *The potential P is supermodular on S .*
- (β) *S^* is a nonempty sublattice of S^N .*

Remark. Observe that if in addition, G is a symmetric game, then assertion (β) of the proposition implies that G has at least one symmetric stochastically stable state. The result that the set of stochastically stable state forms a nonempty sublattice of S (rather than merely a lattice), is also of some practical interest. Namely, then one can easily find

a new stochastically stable state knowing that two profiles (equilibria) are stochastically stable: If $s = (s_1, \dots, s_N)$ and $s' = (s'_1, \dots, s'_N)$ are in S^* , then so are $\sup_S\{s, s'\} = (\max\{s_1, s'_1\}, \dots, \max\{s_N, s'_N\})$ and $\inf_S\{s, s'\} = (\min\{s_1, s'_1\}, \dots, \min\{s_N, s'_N\})$. One cannot necessarily proceed this way within the equilibrium set S^N . For the conclusion of Proposition 1 that the set of Nash equilibria S^N is a nonempty lattice can be hardly replaced by the stronger assertion that S^N is a sublattice of the set of strategy profiles S . The reason is that Zhou's Fixed-Point Theorem (1994, p. 297) cannot be generalized to show the set of fixed points of an increasing correspondence from a nonempty complete lattice X into itself to be a sublattice of X ; see Zhou (1994, p. 298) and Example 2.5.1 of Topkis (1998, p. 40). For the specific case of a two-player supermodular game where players' strategy sets are totally ordered, Echenique (2003) establishes that the set of Nash equilibria is a sublattice of the set of strategy profiles. But he observes that a supermodular game with more than two players need not have an equilibrium set that is a sublattice even if players' strategy sets are totally ordered.

PROOF. (α): Pick any $i \in I$. For all $j \neq i$ and for all $s, s' \in S$ such that $s_i \geq s'_i$ and $s_j \geq s'_j$, we have

$$\begin{aligned} & P(s_i, s_j, s_{-ij}) - P(s'_i, s_j, s_{-ij}) \\ &= u_i(s_i, s_j, s_{-ij}) - u_i(s'_i, s_j, s_{-ij}) \\ &\geq u_i(s_i, s'_j, s_{-ij}) - u_i(s'_i, s'_j, s_{-ij}) \\ &= P(s_i, s'_j, s_{-ij}) - P(s'_i, s'_j, s_{-ij}). \end{aligned}$$

The two equalities follow from the definition of a potential P . The inequality follows from (A). This means that P has increasing differences on $S_i \times S_j$ for each $j \neq i$. As this property holds for all $i \in I$, we conclude that P has increasing differences on S . Assertion (α) follows from Corollary 2.6.1 in Topkis (1998).

(β): By Proposition 2, the set of stochastically stable states S^* is the set of maximizers of P . Because S is a finite set, S^* is nonempty. By (α) and Theorem 2.7.1 of Topkis (1998), S^* is a sublattice of S . Moreover, by Proposition 1, S^N is a lattice. Now $S^* \subseteq S^N$ by Proposition 3. Thus we have that $S^* \subseteq S^N \subseteq S$ and S^* is a sublattice of S^N . ■ ■

Remark. The results contained in Propositions 1 and 4 do not depend on the particular form of the payoff functions (1). They hold if the finite game $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ satisfies the following set of conditions:

(X1) For each $i \in I$, S_i is a finite subset of \mathbb{R} .

(X2) For each $i \in I$, $u_i : S_i \times S_{-i} \rightarrow \mathbb{R}$ has increasing differences in (s_i, s_{-i}) on $S_i \times S_{-i}$.

(X3) G is a potential game with potential $P : S \rightarrow \mathbb{R}$.

One can draw stronger conclusions if (X2) is replaced by the stronger condition that for each $i \in I$, $u_i : S \rightarrow \mathbb{R}$ is supermodular on S :

Proposition 5 *Assume (X1), (X3), and supermodularity of all $u_i, i \in I$. Then the set of states $s \in S$ which are both efficient and stochastically stable is a sublattice of S .*

PROOF. Because the payoff function u_i is supermodular on S for each $i \in I$, the utilitarian welfare function W is supermodular on S as the finite sum of supermodular functions by Lemma 2.6.1 in Topkis (1998). It follows that $S^\infty = \arg \max_{s \in S} W(s)$ is a sublattice of S by Theorem 2.7.1 from Topkis (1998). Now the set of states which are both efficient and stochastically stable is defined by $S^\infty \cap S^*$. Because S^* is a sublattice of S by the generalized version of Proposition 4, $S^\infty \cap S^*$ is the sublattice of S as the intersection of sublattices of S by Lemma 2.2.2 from Topkis (1998). ■■

Remark. Note that Proposition 5 does not assert that $S^\infty \cap S^*$ is nonempty. Consider the well known class of coordination games: $I = \{1, 2\}$, $S_1 = S_2 = \{0, 1\}$, and $u_1(0, 1) = u_2(1, 0) = b$, $u_1(1, 1) = u_2(1, 1) = d$, $u_1(0, 0) = u_2(0, 0) = a$, $u_1(1, 0) = u_2(0, 1) = c$, where $a > c$, $d > b$, $a < d$ and $(a - c) > (d - b)$. Here $P(s) = (a - c)(1 - \max\{s_1, s_2\}) + (d - b)s_1s_2$, and we see that $S^* = \{(0, 0)\}$, $S^\infty = \{(1, 1)\}$.

4 Pairwise Symmetry

In this section, we explore the implications of sufficiently strong restrictions on the pairwise benefit functions which render the networking game G a potential game. Suppose that indeed, G has a potential P . Then the most

important implications are given by Propositions 2 and 3: Stochastic stability selects a subset of Nash equilibria of a potential game. Under further assumptions, the set of stochastically stable states forms a lattice. But which restrictions on benefit functions would yield a potential game? To this end, let us consider for any pair of distinct players ij , the two-player game G_{ij} with player set $I_{ij} = \{i, j\}$, strategy sets $S_i = S_j = K$, and payoffs $b_{ij}(s_i, s_j)$ for i and $b_{ji}(s_j, s_i)$ for j when they play the joint strategy $(s_i, s_j) \in K \times K$. Suppose β_{ij} is a potential for G_{ij} . We say that β_{ij} is **symmetric**, if $\beta_{ij}(s_i, s_j) = \beta_{ij}(s_j, s_i)$ for all $(s_i, s_j) \in K \times K$. Existence of a symmetric potential for all pairwise interactions is sufficient for the existence of a potential of the entire networking game:

Proposition 6 *If for each distinct pair ij , β_{ij} is a symmetric potential of G_{ij} , then the function P given by*

$$P(s) = \frac{1}{2} \sum_i \sum_j \beta_{ij}(s_i, s_j) - \sum_i c_i(s_i) \quad (3)$$

for $s \in S$, is a potential of G .

PROOF. Analogous to proof of Proposition 1 in Baron *et al.* (2002b). ■■

Next we impose directly certain restrictions on the pairwise benefit functions and discuss how they relate to the existence of symmetric potentials. For any distinct pair of players ij , we consider the following three conditions:

- (I) *Identical Benefits:* $b_{ij}(s_i, s_j) = b_{ji}(s_j, s_i)$ for all $(s_i, s_j) \in K \times K$.
- (II) *Symmetric Benefits:* $b_{ij}(s_i, s_j) = b_{ji}(s_i, s_j)$ for all $(s_i, s_j) \in K \times K$.
- (III) *Interchangeable Actions:* $b_{ij}(s_i, s_j) = b_{ij}(s_j, s_i)$, $b_{ji}(s_i, s_j) = b_{ji}(s_j, s_i)$ for all $(s_i, s_j) \in K \times K$.

Condition (I) is tantamount to b_{ij} being a (not necessarily symmetric) potential of G_{ij} and b_{ji} being a (not necessarily symmetric) potential of G_{ji} . Condition (II) implies existence of a symmetric potential of G_{ij} in case $T = 1$, but not otherwise. Conditions (I) and (II) combined are equivalent to $b_{ij} = b_{ji}$ being a symmetric potential of G_{ij} . Any two of the three conditions imply the third one. As an immediate consequence of the foregoing propositions, we obtain

Corollary 1 *If any two of the conditions (I)–(III) hold, then the game G has a potential P of the form (3) with $\beta_{ij} = b_{ij}$ and the maximizers of P are Nash equilibria and the stochastically stable states.*

While obviously restrictive, existence of a symmetric potential for G_{ij} still leaves a lot of flexibility in terms of functional form and interpretation. To illustrate the scope of applications, let us specialize and assume that

$$b_{ij}(s_i, s_j) = \pi_{ij}(s_i, s_j) \cdot v_{ij}, \quad (4)$$

where $\pi_{ij} \geq 0$ is the probability or intensity of i interacting with j and v_{ij} is i 's benefit, appreciation or valuation of an interaction with j . We can normalize so that $\pi_{ij} \leq 1$ and π_{ij} can be interpreted as a probability, but for computational convenience will not always do so. If $0 \leq \pi_{ij} \leq 1$ and π_{ij} is interpreted as a probability, then player i receives benefit v_{ij} with probability π_{ij} , zero benefit with probability $1 - \pi_{ij}$, and expected benefit b_{ij} . It is possible that players are linked without any effort or investment, that is $\pi_{ij}(0, 0) > 0$. It is also possible that the strength or probability of certain links proves irresponsive to effort or investment, that is π_{ij} is constant. (I)–(III) have the following counter-parts:

- (i) *Identity:* $\pi_{ij}(s_i, s_j) = \pi_{ji}(s_j, s_i)$ for all $(s_i, s_j) \in K \times K$.
- (ii) *Symmetry:* $\pi_{ij}(s_i, s_j) = \pi_{ji}(s_i, s_j)$ for all $(s_i, s_j) \in K \times K$.
- (iii) *Interchangeability:* $\pi_{ij}(s_i, s_j) = \pi_{ij}(s_j, s_i)$, $\pi_{ji}(s_i, s_j) = \pi_{ji}(s_j, s_i)$ for all $(s_i, s_j) \in K \times K$.

We also consider a symmetry condition for the valuations v_{ij} ,

- (iv) *Mutual Affinity:* $v_{ij} = v_{ji}$.

Mutual affinity can result, e.g., from similarity (kindred spirits) or from complementarity (attraction of opposites). There can be mutual lack of interest, $v_{ij} = v_{ji} = 0$, and mutual dislike or disadvantage, $v_{ij} = v_{ji} < 0$. Any two of the conditions (i)–(iii) imply the third one. Conditions (i)–(iv) imply (I)–(III).

For the remainder of the section, we specialize further and investigate several interesting cases.

4A: Homogeneity and Regularity

Case 2A can be generalized as follows. First, there exists a positive integer $r \leq N - 1$ such that the population I forms an r -regular graph $\Gamma = (I, E)$, i.e. an undirected graph where every player $i \in I$ has r neighbors. In an abstract sense, this reflects spatial homogeneity of the player population. Next, we assume a corresponding homogeneity of pairwise benefits. Namely, there exists a C^2 -function $B : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $B(0) = 0, B' > 0, B'' < 0$ such that for any pair ij and networking levels $s_i, s_j \in K$: $\pi_{ij}(s_i, s_j) = B(s_i + s_j)$ if $\{i, j\} \in E$ and $\pi_{ij} = 0$ if $\{i, j\} \notin E$. Finally, we assume uniform mutual affinity across players: There exists $v > 0$ such that $v_{ij} = v_{ji} = v$ for any distinct pair ij . Then (i)–(iv) are satisfied. Furthermore, if costs are of the linear form $c_i(s_i) = C \cdot s_i$, then for a wide range of specifications of this particular model, the three main results (a)–(c) of case 2A obtain.

4B: Mutual Sympathy or Antipathy

Sympathy or antipathy among people need not be mutual, but often they are and here we assume that they are. We consider pairwise benefits of the form (4) and concentrate on the simplest special case $\pi_{ij}(s_i, s_j) = B(s_i + s_j) = s_i + s_j$. In addition, we postulate the mutual affinity condition (iv) and linear networking cost functions, $c_i(s_i) = C_i \cdot s_i$, with $C_i > 0$ for all $i \in I$. Then conditions (i)–(iv) hold. Finally, we assume that players make binary choices, to network, $s_i = 1$, or not to network, $s_i = 0$. Accordingly, $K = \{0, 1\}$.

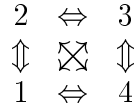
To analyze the specific game G , define $W_i \equiv \sum_{j \neq i} v_{ij}$ for $i \in I$. Each player i has weakly dominant strategies. Namely, the player's best responses are 1 if $W_i - C_i > 0$; 0 if $W_i - C_i < 0$; 0 and 1 if $W_i = C_i$.

A player's decision creates the surplus $(2W_i - C_i)s_i$. Hence a player's best response is inefficient in two instances, if $W_i = C_i$ and the player chooses $s_i = 0$ and if $W_i < C_i < 2W_i$. Therefore, inefficiencies always constitute under-investments. The aggregate functions P and W assume correspondingly simple forms:

$$P(s) = \sum_i (W_i - C_i)s_i; \quad W(s) = \sum_i (2W_i - C_i)s_i.$$

In particular, all equilibria are stochastically stable. Depending on model parameters and tie-breaking, equilibria may be efficient or inefficient.

Now mutual affinity allows for mutual lack of interest, $v_{ij} = v_{ji} = 0$ and mutual dislike, disadvantage, animosity, antipathy, enmity, or hostility, $v_{ij} = v_{ji} < 0$. Let us consider an example of four players where each has two friends and one enemy. Specifically, $N = 4$ and $C_i = 1.5$ for all i . We assume $v_{ij} = v_{ji} = 1$ for $ij = 12, 23, 34, 41$ and $v_{ij} = v_{ji} = -1$ for $ij = 13, 24$. Mutual affinity can also be indicated by means of the following diagram



where a link \Leftrightarrow signifies friendship, in which case $v_{ij} = 1$ for the two players, and a link \boxtimes signifies negative affinity, in which case $v_{ij} = -1$ for the two players. In this game, $W_i = 1$ for all players. $s^0 = (0, 0, 0, 0)$ is the unique equilibrium and stochastically stable outcome whereas $s^1 = (1, 1, 1, 1)$ is the efficient outcome. Obviously, affinities and adversities can give rise to a host of interesting social spill-overs, where a player is affected by affinities between other players. We confine ourselves to one more instructive example.

“The enemy of my enemy is my friend” usually means that if j is i ’s enemy and k is j ’s enemy, then i and k might form an alliance against j . Yet in the present situation, i may benefit from hostility between j and k in a different way: If i and j are enemies, $v_{ij} < 0$, then i prefers that j is not networking. This is certainly the case if $v_{jk} < 0$ for all k , that is if j has only enemies. For instance, let $N = 3$, $v_{12} = v_{21} < 0$, and $v_{13} = 0$. Then 1 prefers that 2 is not networking. This is guaranteed if $W_2 = v_{23} + v_{21} < C_2$. Since $v_{21} < 0$ the latter holds if 2 and 3 are enemies, $v_{23} < 0$, or not too close friends, $0 \leq v_{23} < |v_{21}| + C_2$.

4C: Networkers and Networking

Both in traditional and in electronic interactions, some agents are much more active in networking than others and might be called “networkers”. A networker is more eager than others to form and utilize networks because of (actual or perceived) benefit or cost advantages. The homogeneous case 4A represents a model without distinguished networkers. This model can be modified in several ways to differentiate between networkers and other agents. Three major modelling options are the following:

- (*) There exist numbers $v_i > 0$, $i \in I$, which are not all equal and such that $v_{ij} = v_i$ for any distinct pair ij .
- (**) There exist numbers $v_i > 0$, $i \in I$, which are not all equal and such that $v_{ij} = v_i v_j$ for any distinct pair ij .
- (***) There exist numbers $C_i \geq 0$, which are not all equal and such that $c_i(s_i) = C_i \cdot s_i$ for $i \in I$, $s_i \in S_i$.

Option (*) means that i has a greater incentive to network than j if $v_i > v_j$, since i 's benefit from any interaction is higher than j 's. This option violates the mutual affinity condition (iv). Consequently, G need not be a potential game. For the sake of staying within the current framework, we will not pursue this option any further. Notice, however, that (*) yields a weighted potential for G , a generalization dealt with in Baron *et al.* (2002b).

Option (**) means that if $v_i > v_j$ and $n \in I \setminus \{i, j\}$, then $v_{in} > v_{jn}$ and $v_{ni} > v_{nj}$. Not only has i a higher benefit from any interaction and, therefore, a greater incentive to network than j . It is also the case that any third player n would have a higher benefit from interacting with i than from interacting with j . In some cases this implies that being surrounded by networkers, i.e. players with high v_i , may induce a player to make a large networking effort. Hence, under certain circumstances, the presence of networkers fosters networking by networkers and others. To illustrate this possibility, let $N = 2M$, $M \geq 2$, and $\Gamma = (I, E)$ be the circular undirected graph with $E = \{\{1, 2\}, \{2, 3\}, \dots, \{N-1, N\}, \{N, 1\}\}$. Moreover, let $K = \{h/2 : h = 0, 1, \dots, 10\}$ and $B(x) = \sqrt{x}$ for $x \in \mathbb{R}_+$. Finally, let $c_i(s_i) = s_i$. If $v_i = 1$ for all $i \in I$, then the homogeneous case 2A obtains. Now suppose $v_i = 2$ for i odd and $v_j = 1$ for j even. Then the odd numbered players may be called networkers. Now it happens that $v_{ij} = 2$ for all $\{i, j\} \in E$. Hence in this particular instance, the introduction of networkers has the same effect as doubling everybody's benefit. The modified game has the symmetric equilibrium $t^* = (2, \dots, 2)$ which is inefficient and stochastically stable. Like in case 2A, stochastically stable asymmetric equilibria also exist, such as $(1, 3, \dots, 1, 3)$, $(0, 4, \dots, 0, 4)$, etc. Comparison of t^* with the symmetric equilibrium s^* in case 2A (and of other corresponding pairs of equilibria) demonstrates that the presence of networkers can induce others

to increase their networking efforts.

Option (***) means that some individuals have a cost advantage which may induce them to invest more in networks and in turn may cause reduced networking efforts by others. Let us depart again from the benchmark case 2A. Now consider the modification where still $v_i = 1$ for all i , but $c_i(s_i) = (1/2) \cdot s_i$ for i odd and $c_j(s_j) = s_j$ for j even. Then the odd numbered players have a cost advantage and are the networkers. After doubling the payoffs of odd numbered players, this specific model is identical to case 2B. In equilibrium, the odd numbered players make a substantial investment in networking whereas the even numbered players make none. This striking result, that only networkers invest, is primarily due to the fact that networkers and non-networkers alternate on the circular graph. To obtain the opposite, let us consider the following

Example. $N = 4$, $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$, $K = \{0, 1, \dots, T\}$, $B(x) = \sqrt{x}$ for $x \in \mathbb{R}_+$. Put $v_1 = v_2 = 1$, $v_3 = v_4 = R$ where R is an odd integer such that $1 < R$ and $R^2(R + 1)^2 \leq 4T$. Put $c_i(s_i) = s_i$ for $i = 1, 2, 3$; $s_i \in S_i$ and $c_4(s_4) = 2s_4$ for $s_4 \in S_4$. Then players 3 and 4 are the networkers in terms of benefits, but player 4 is somewhat handicapped by a cost disadvantage. Now suppose that $s_2 = s_4 = 0$. Then the best responses s_1 and s_3 by 1 and 3, respectively, are given by $\sqrt{s_1} = (R + 1)/2$ and $\sqrt{s_3} = R(R + 1)/2$. Player 2 would not invest more on his own than his neighbors' best responses. Hence $s_2 = 0$ is optimal against the given values of s_1 and s_3 . It remains to consider player 4. If 4 keeps $s_4 = 0$, his payoff is $u_4(0, s_{-4}) = R\sqrt{s_1} + R^2\sqrt{s_3}$. If 4 chooses $s_4 = 1$, his payoff becomes $u_4(1, s_{-4}) = R\sqrt{s_1 + 1} + R^2\sqrt{s_3 + 1} - 2 < R \cdot (\sqrt{s_1} + 1/(2 \cdot \sqrt{s_1})) + R^2 \cdot (\sqrt{s_3} + 1/(2 \cdot \sqrt{s_3})) - 2$. Hence the marginal benefit is $R/(R + 1) + R^2/(R(R + 1)) < 2$ whereas the marginal cost is 2. Since B is strictly concave, it follows that $s_4 = 0$ is 4's best response against s_1 and s_3 given above. Therefore, $s = ((R + 1)^2/4, 0, R^2(R + 1)^2/4, 0)$ is an equilibrium where the unhandicapped networker and one of the non-networkers invest. When R is large, the bulk of the investment is shouldered by the networker. $\square\square$

4D: User Networks

Here the player set I is interpreted as a finite population of users or adopters. Each player has to adopt exactly one technology or network good from the list K . K may consist of computer systems, word processors, internet providers, etc. The adopters of the same good constitute a user network. Here we consider the case where a user's benefit stems from pairwise interactions with his neighbors. Accordingly, the population I forms an undirected graph (I, E) . We restrict ourselves to the case of uniform mutual affinity across players, $v_{ij} = v_{ji} = 1$ for all ij . We examine three distinct cases of compatibility of different technologies: partial but imperfect compatibility; perfect incompatibility; and downward compatibility.

PARTIAL BUT IMPERFECT COMPATIBILITY. We allow for partial, but not perfect compatibility of different technologies: $\pi_{ij}(s_i, s_j) > \pi_{ij}(s_i, s'_j)$ if $\{i, j\} \in E$ and $s_j = s_i$ and $s'_j \neq s_i$. It is plausible to assume that the symmetry condition (ii) holds. It is unlikely that (i) or (iii) hold. But then, in general, the game G with payoff function (1) is not a potential game. G does become a potential game under the following more restrictive assumption: There exist numbers β_k and γ_k for $k \in K$ such that $\beta_k > \gamma_k \geq 0$ for $k \neq \ell$ and

$$\begin{aligned}\pi_{ij}(s_i, s_j) &= \beta_k \text{ if } s_i = s_j = k; \{i, j\} \in E; \\ \pi_{ij}(s_i, s_j) &= \gamma_\ell \text{ if } s_i \neq s_j = \ell; \{i, j\} \in E; \\ \pi_{ij}(s_i, s_j) &= 0 \text{ if } \{i, j\} \notin E.\end{aligned}$$

$\gamma_\ell > 0$ means **partial but imperfect compatibility**. In case $E = F$, the game G is symmetric. Baron *et al.* (2002a, p. 574) provide a numerical specification of this case where G has N symmetric equilibria and a single asymmetric one. For certain parameter constellations, the asymmetric equilibrium is the unique stochastically stable state — which shows that stochastic stability can have bite as equilibrium selection criterion and may work against the symmetric outcomes.

PERFECT INCOMPATIBILITY. Two prominent classes of spatial games, both analyzed in detail in Baron *et al.* (2002b, pp. 555-557), permit a novel interpretation as user network formation games. The first class consists of

coordination games. There exist numbers $0 < e_0 < e_1 < \dots < e_T$ such that $b_{ij}(s_i, s_j) = e_k$ if $\{i, j\} \in E$ and $s_i = s_j = k$ and $b_{ij}(s_i, s_j) = 0$ if $\{i, j\} \notin E$ or $s_i \neq s_j$. In terms of user networks, this constitutes the case of **perfect incompatibility**. Conditions (I)–(III) are met. Let us proceed with the particular case where all technologies cost the same to each user, that is the cost functions c_i are constant. Then G has $T + 1$ symmetric equilibria $s^k = (k, \dots, k)$, $k \in K$ which are Pareto-ranked. s^T is the Pareto- or payoff- dominant equilibrium and the unique stochastically stable state. These conclusions persist as long as the cost functions vary sufficiently little. As an alternative, let us consider another particular case where again the cost functions c_i are constant or of sufficiently small variation, but now there is an extra usage cost for each pairwise interaction: There exist numbers $z_0 < z_1 < \dots < z_T$ such that

$$\begin{aligned} b_{ij}(s_i, s_j) &= e_k - z_k \text{ if } \{i, j\} \in E \text{ and } s_i = s_j = k; \\ b_{ij}(s_i, s_j) &= -z_k \text{ if } \{i, j\} \in E \text{ and } k = s_i \neq s_j; \\ b_{ij}(s_i, s_j) &= 0 \text{ if } \{i, j\} \notin E. \end{aligned}$$

The extra usage cost can result from the fact that many technologies are bundled and come along with various (in)conveniences of usage. The extra usage cost can also represent the variable component of a two-part pricing scheme, a frequent feature of service fee schedules. The resulting games G_{ij} satisfy (I) but not (II) and (III), yet still have symmetric potentials. If $e_0 - z_0 < e_1 - z_1 < \dots < e_T - z_T$, and $e_0 - 2z_0 > e_1 - 2z_1 > \dots > e_T - 2z_T$, then again s^0, \dots, s^T are symmetric equilibria of G and s^T is Pareto-dominant. However, now s^0 is the stochastically stable equilibrium. This demonstrates that stochastic stability may, but need not select the Pareto-dominant outcome. Rather risk dominance plays a crucial role.

DOWNWARD COMPATIBILITY. The second class of spatial games is based upon **minimum effort coordination games**, with $k_h = h+1$ and $b_{ij}(s_i, s_j) = \min\{s_i, s_j\}$ if $\{i, j\} \in E$ and $b_{ij}(s_i, s_j) = 0$ if $\{i, j\} \notin E$. In terms of user networks, they form a case of **downward compatibility**. For instance, downward compatibility occurs when the slowest modem determines the speed of transmission or when the oldest and least sophisticated software version determines which applications can be shared. This model satisfies (I)–(III). Put $\lambda \equiv |E|/N$. Now suppose that $c_i(k) = c \cdot k$ for all $i \in I, k \in K$. Then the associated potential (3) is maximized at s^0 if $c > \lambda$ and is maximized at s^T if $c < \lambda$. Hence most of the time, the players coordinate on a specific

technology, but the select technology proves sensitive to who interacts with whom. The ratio λ can be viewed as a connectivity index of the pre-existing physical infrastructure or social fabric underlying E . Higher connectivity tends to sustain the more advanced or upscale technology while a cost increase tends to favor the basic technology. For instance, let $c = 1.5$ and $N > 2$. If $E = F$, then $\lambda = N - 1 > c$ and $k = k_T$ prevails. If the graph is circular, then $\lambda = 1 < c$ and $k = k_0$ prevails. Of course, if $c > N - 1$, then $k = k_0$ prevails regardless of E . Alternatively, let us consider the particular case where $c = 0$ so that $c < \lambda$ would hold, but there is an extra usage cost $z_k = z \cdot k$ for each of a player's pairwise interactions by means of technology $k \in K$. If $z \in (1/2, 1)$, then s^0 turns out to be the unique stochastically stable state regardless of E . Let us add that all technologies k or rather all equilibria $s^k, k \in K$, are equally likely if $c = \lambda$ and $z = 0$. This happens, for example, when $N = 4, E = F, c = 1.5, z = 0$.

Notice that E constitutes a “social structure” in the sense of Chwe (2000). Chwe investigates which social structures are conducive to coordination in a “local information game”. In contrast to Chwe's, our model falls under the rubric of “local interaction games”. Our concern is not whether people coordinate, but on which action (technology) they coordinate. Still, like Chwe we find that social structure matters: Higher connectivity favors coordination on the more advanced technology.

5 Comparative Statics in Networking Costs

Intuitively, one would expect that networking activities intensify if networking costs decline. This conjecture proves at least partially true in the presence of strategic substitutes in pairwise interactions. To be precise, consider the following three additional assumptions:

- (B) There exists an undirected graph (without isolated nodes) (I, E) such that $b_{ij} = 0$ for $\{i, j\} \notin E$ and b_{ij} satisfies decreasing differences in $(s_i, s_j) \in S_i \times S_j$ for $\{i, j\} \in E$.
- (C) There exist $C_1 \geq 0, \dots, C_N \geq 0$ such that $c_i(s_i) = C_i \cdot s_i$ for $i \in I, s_i \in S_i$.
- (D) For $i \in I$, there exists a unique best response against each $s_{-i} \in S_{-i}$.

Condition (B) constitutes the antithesis of condition (A). It is satisfied by the model specifications in 2A, 2B, 4A, and the example in 4C. Both (A) and (B) hold for the linear model of 2C, 4B, and Section 6. Only condition (A) holds in the case of downward compatibility in subsection 4D, and in the examples given later in this section. Neither condition holds when the user networks of subsection 4D exhibit partial but imperfect compatibility or perfect incompatibility.

Proposition 7 *Let G be a networking game satisfying (B)-(D) and let G' be a second networking game that differs from G only in the marginal networking costs, which are $C'_1 > 0, \dots, C'_N > 0$ in G' . Further, let $s \in S$ be an equilibrium of G and $s' \in S$ be an equilibrium of G' . Suppose $C'_i \leq C_i$ for all i and $s' \neq s$. Then $s'_i > s_i$ for some i .*

PROOF. Let $G, G', C_1, \dots, C_N, C'_1, \dots, C'_N, s, s'$ be as hypothesized. Since $s \neq s'$, there is $i \in I$ such that $s_i \neq s'_i$. Consider this player i and suppose the conclusion is false, that is $s'_j \leq s_j$ for all $j \in I$. We have:

$$\begin{aligned}
0 &< \sum_j b_{ij}(s_i, s_j) - C_i \cdot s_i - \left(\sum_j b_{ij}(s'_i, s_j) - C_i \cdot s'_i \right) \\
&= \sum_j \left(b_{ij}(s_i, s_j) - b_{ij}(s'_i, s_j) \right) - C_i \cdot s_i + C_i \cdot s'_i \\
&\leq \sum_j \left(b_{ij}(s_i, s'_j) - b_{ij}(s'_i, s'_j) \right) - C_i \cdot s_i + C_i \cdot s'_i \\
&\leq \sum_j \left(b_{ij}(s_i, s'_j) - b_{ij}(s'_i, s'_j) \right) - C'_i \cdot s_i + C'_i \cdot s'_i \\
&= \sum_j b_{ij}(s_i, s'_j) - C'_i \cdot s_i - \left(\sum_j b_{ij}(s'_i, s'_j) - C'_i \cdot s'_i \right) < 0,
\end{aligned}$$

a contradiction. The first inequality follows from optimality of s_i at s_{-i} , $s_i \neq s'_i$, and (D). The second inequality follows from (B). The third inequality is a consequence of $C'_i \leq C_i$. The last inequality follows from optimality of s'_i at s'_{-i} , $s_i \neq s'_i$, and (D). Hence, to the contrary, the conclusion has to be true. ■■

The assumption (D) of unique best responses can be disposed of if one postulates strict cost reductions instead:

Proposition 8 *Let G be a networking game that satisfies (B) and (C) and let G' be a second networking game that differs from G only in the marginal networking costs, which are $C'_1 > 0, \dots, C'_N > 0$ in G' . Further, let $s \in S$ be an equilibrium of G and $s' \in S$ be an equilibrium of G' . Suppose $C'_i < C_i$ for all i and $s' \neq s$. Then $s'_i > s_i$ for some i .*

PROOF. Let $G, G', C_1, \dots, C_N, C'_1, \dots, C'_N, s, s'$ be as hypothesized. Suppose the conclusion is false, that is $s'_i \leq s_i$ for all $i \in I$. Now take any $i \in I$. By assumption, s_i is a best response of i against s_{-i} in G . Since $s'_j \leq s_j$ for all $j \neq i$ and (B) and (C) hold, the largest best response \widehat{s}_i of i against s'_{-i} in G satisfies $\widehat{s}_i \geq s_i$. Since $C'_i < C_i$, (B) and (C) hold, and G and G' differ only in marginal networking costs, one obtains $\widetilde{s}_i \geq \widehat{s}_i$ for any best response \widetilde{s}_i of i against s'_{-i} in G' and any best response \widehat{s}_i of i against s'_{-i} in G . It follows that $s'_i \geq s_i$ because s'_i is a best response of i against s'_{-i} in G' . But $s'_i \geq s_i$ and $s'_i \leq s_i$ imply $s'_i = s_i$. Since i was arbitrary, $s' = s$, which contradicts the hypothesis of the proposition. Hence, to the contrary, the conclusion has to be true. ■ ■

Notice that the conclusion of Propositions 7 and 8 cannot be substantially strengthened for two reasons. For one, G and G' may have the same equilibria, even if $C'_i < C_i$ for all i . This follows from the discreteness of the model. Secondly, let G be the game of case 2A which satisfies (B)–(D) with $C_i = 1$ for all i . Let G' be a game that differs from G only with respect to marginal networking costs. Specifically, set $C'_i = 1/2$ for i odd and $C'_j = C' < 1$ for j even. If C' is sufficiently close to 1, then the argument given under option (***) in subsection 4C still applies: $s^{**} = (4, 0, \dots, 4, 0)$ is an equilibrium of G' while $s^* = (2, 2, \dots, 2, 2)$ is an equilibrium of G . Obviously $s^{**} \neq s^*$. But some players have lowered their efforts in s^{**} relative to s^* .

Without a strategic substitutes assumption, a cost decline is consistent with a universal reduction of networking activities. Next we provide a numerical example with this property.

Example. Let $N = 2M, M \geq 2$, and $\Gamma = (I, E)$ be the circular undirected graph with $E = \{\{1, 2\}, \{2, 3\}, \dots, \{N-1, N\}, \{N, 1\}\}$. Put $b_{ij}(s_i, s_j) = 0$ for $\{i, j\} \notin E$ and $b_{ij}(s_i, s_j) = \frac{1}{2} \ln(1 + s_i) \cdot \ln(1 + s_j)$ for $\{i, j\} \in E$. Then the pairwise interactions exhibit weak strategic complements rather

than strategic substitutes. Let $e = \exp(1)$ be the Euler number and set $K = \{0, e^{1/4} - 1, e - 1\}$. With $C_i = e^{-1}$ for all i , we obtain a game G which has two symmetric equilibria, $s^0 = (0, \dots, 0)$ and $s^\bullet = (e - 1, \dots, e - 1)$. Setting $C'_i = e^{-1/4}/4 < C_i$ for all i defines a game G' which has three symmetric equilibria, s^0 , s^\bullet , and $s^{\bullet\bullet} = (e^{1/4} - 1, \dots, e^{1/4} - 1)$. Thus, the example has actually several interesting features. First, there exists the equilibrium s^0 , an instance of mutual obstruction where nobody has an incentive to network if nobody else is networking. Next there exists the equilibrium s^\bullet where everybody exerts maximum networking effort. Further, a cost reduction leads to the emergence of a third equilibrium, $s^{\bullet\bullet}$ where everyone makes a positive but less than maximal effort. Regarding our original point, the conclusion of Propositions 7 and 8 obviously need not hold if the strategic substitutes assumption of the form (B) is violated. Finally, it turns out that with a slight logit perturbation, the dynamics would stay most of the time in the equilibrium s^\bullet , which is the unique stochastically state of the evolutionary model based on G or G' . $\square\square$

The example satisfies assumptions (A) and (C). In addition, the games G and G' are symmetric. As a consequence of Proposition 1, G and G' have smallest and largest equilibria which are symmetric. s^0 is the smallest equilibrium and s^\bullet is the largest equilibrium in both games. Thus, the smallest and the largest equilibrium prove immune to a cost reduction. This observation is consistent with the claim that in response to a cost decrease, the smallest and the largest equilibrium will never decrease. Formally, we obtain a weak monotonicity result by applying an earlier result of Milgrom and Roberts (1990):

Proposition 9 *Consider a family of networking games G^τ satisfying (A) and (C) which differ in the marginal cost parameters $\tau = (C_1, \dots, C_N) \in \mathbb{R}_{++}^N$. Then the smallest and the largest equilibrium of G^τ are non-increasing functions of τ .*

PROOF. Endow the parameter space \mathbb{R}_{++}^N with the reverse of its canonical partial order, that is for $\tau, \tau' \in \mathbb{R}_{++}^N$, $\tau \leq \tau'$ if and only if $\tau_i \geq \tau'_i$ for all i . Then the payoff functions given by (1) satisfy condition (A5) of Milgrom and Roberts (1990). (A) and (C) imply that each game G^τ is supermodular. Hence by Theorem 6 of Milgrom and Roberts, the smallest and the largest

equilibrium of G^τ are non-decreasing in τ with respect to the reverse canonical partial order of \mathbb{R}_{++}^N . Therefore, the assertion holds with respect to the canonical partial order of \mathbb{R}_{++}^N . ■ ■

By Proposition 4, if in addition to satisfying (A) and (C), a networking game is a potential game, then the set of stochastically stable states forms a nonempty sublattice of the set of equilibrium points. As a consequence of this added structure, there exist a smallest and a largest stochastically stable state. Interestingly enough, the comparative statics à la Milgrom and Roberts for supermodular games extend to the smallest and largest stochastically stable state. We choose a more abstract formulation in this instance than before. Let Θ be a nonempty subset of some Euclidean space \mathbb{R}^n , $n \in \mathbb{N}$, with generic elements θ, θ' , and ϑ .

Proposition 10 *Suppose that $G^\theta = (I, (S_i)_{i \in I}, (u_i^\theta)_{i \in I}), \theta \in \Theta$, is a collection of games satisfying conditions (X1)-(X3), with respective potentials $P^\theta, \theta \in \Theta$. Further suppose that for each $i \in I$ and each $s_{-i} \in S_{-i}$, the payoff function $u_i^\theta(s_i, s_{-i})$ has increasing differences in (s_i, θ) on $S_i \times \Theta$. Then the largest (smallest) stochastically stable state for each game G^θ is weakly increasing in θ on Θ .*

PROOF. Pick any $s, s' \in S$ with $s \geq s'$ and any $\theta, \vartheta \in \Theta$ with $\theta \geq \vartheta$. Define $s(0), s(1), \dots, s(N) \in S$ as follows: $s(0) = s$, $s_i(k) = s'_i$ for $i, k \in I$, $i \leq k$, and $s_i(k) = s_i$ for $i, k \in I$, $i > k$. By construction, $s(k) \geq s(k+1)$ for $k = 0, 1, 2, \dots, N-1$. Because G^θ and G^ϑ are potential games and the payoff function of each player i has increasing differences on $S_i \times \Theta$, it is the case that $s \geq s'$ and $\theta \geq \vartheta$ implies

$$\begin{aligned} P^\theta(s) - P^\theta(s') &= \sum_{i=1}^N \left(P^\theta(s(i-1)) - P^\theta(s(i)) \right) \\ &= \sum_{i=1}^N \left(u_i^\theta(s(i-1)) - u_i^\theta(s(i)) \right) \\ &\geq \sum_{i=1}^N \left(u_i^\vartheta(s(i-1)) - u_i^\vartheta(s(i)) \right) \\ &= \sum_{i=1}^N \left(P^\vartheta(s(i-1)) - P^\vartheta(s(i)) \right) \\ &= P^\vartheta(s) - P^\vartheta(s'). \end{aligned}$$

This means that $P^\theta(s)$ has increasing differences in (s, θ) on $S \times \Theta$. For each $\theta \in \Theta$, $P^\theta(s)$ is supermodular in s on S by assertion (α) of Proposition 4. Then the correspondence $S^* : \Theta \rightrightarrows S$, $\theta \mapsto \arg \max_{s \in S} P^\theta(s)$ is increasing⁷ in $\theta \in \Theta$ by Theorem 2.8.1 of Topkis (1998).

Now consider $\theta, \theta' \in \Theta$ with $\theta \geq \theta'$ and pick any $s \in S^*(\theta)$ and $s' \in S^*(\theta')$. Because $S^*(\theta) \geq^p S^*(\theta')$, $\sup_S \{s, s'\} \in S^*(\theta)$ and $\inf_S \{s, s'\} \in S^*(\theta')$. Since $S^*(\theta)$ and $S^*(\theta')$ are finite sublattices of S , $\sup_S S^*(\theta)$ and $\sup_S S^*(\theta')$ are the largest elements of $S^*(\theta)$ and $S^*(\theta')$ respectively. Then $s' \leq \sup_S \{s, s'\} \leq \sup_S S^*(\theta)$ and so $\sup_S S^*(\theta)$ is an upper bound for $S(\theta')$. But $\sup_S S^*(\theta')$ is the least upper bound for $S^*(\theta')$, so $\sup_S S^*(\theta') \leq \sup_S S^*(\theta)$ as asserted. By next comparing $\inf_S S^*(\theta)$ and $\inf_S S^*(\theta')$, we reach a similar conclusion for the smallest elements of $S^*(\theta)$ and $S^*(\theta')$, respectively. The proof is complete. ■ ■

For example, suppose that for some integer $m > 1$, $\Theta = \{1, 2, \dots, m\}$. Moreover, $S_i = \Theta$ for each $i \in I$ and $u_i^\theta(s) = \min\{\theta, s_1, \dots, s_N\}$ for all $i \in I, \theta \in \Theta, s \in S$. Then the game has the potential $P^\theta(s) = \min\{\theta, s_1, \dots, s_N\}$. For any $\theta \in \Theta$, the smallest stochastically stable state is (θ, \dots, θ) and the largest stochastically stable state is (m, \dots, m) .

A second example is given by the case of downward compatibility in subsection 4D. Suppose $c \in \mathbb{R}_{++}$ and $c_i(k) = c \cdot k$ for all $i \in I, k \in K$. Define $\theta = -c$, so that the payoff for a player $i \in I$ is $u_i^\theta(s) = \sum_{j: \{i, j\} \in E} \min\{s_i, s_j\} + \theta s_i$ for a given parameter $\theta \in \Theta \equiv \mathbb{R}_-$. Each payoff function $u_i^\theta(s_i, s_{-i})$ has increasing differences in (s_i, θ) on $S_i \times \Theta$ for all $i \in I$. Moreover, each game G^θ has a potential P^θ . Hence the hypothesis of the proposition is satisfied. Consequently, the conclusion of the proposition holds. Indeed, we found that the smallest stochastically stable state is s^0 for $\theta \leq -\lambda$ and it is s^T for $\theta > -\lambda$. The largest stochastically stable state is s^0 for $\theta < -\lambda$ and it is s^T for $\theta \geq -\lambda$.

Note that the potential does not necessarily depend on θ even when each payoff function does. For instance, suppose that $S_i = \{1, \dots, m\}$ for each

⁷For all $\theta, \theta' \in \Theta$, $\theta \geq \theta'$ implies $S^*(\theta) \geq^p S^*(\theta')$ where \geq^p is the strong set order. Precisely, $S^*(\theta) \geq^p S^*(\theta')$ means that for each $s \in S^*(\theta)$ and $s' \in S^*(\theta')$, $\sup_S \{s, s'\} \in S^*(\theta)$ and $\inf_S \{s, s'\} \in S^*(\theta')$.

$i \in I$, $\Theta \subseteq \mathbb{R}_+$ and the payoff function is defined by $u_i^\theta(s) = \min_{i \in I} \{s_i\} + \theta$ for all $i \in I, \theta \in \Theta, s \in S$. Then, for all $\theta \in \Theta$, $P^\theta(s) = \min_{i \in I} \{s_i\}$. Proposition 10 still applies: the set of stochastically stable states is the singleton set $\{(m, m, \dots, m)\}$ for each $\theta \in \Theta$.

Further note that the pairwise benefit function $b_{ij}(s_i, s_j) = \frac{1}{2} \ln(1 + s_i) \cdot \ln(1 + s_j)$ used in the first example of this section is just one from a rich family of functions of the multiplicative separable form $f(s_i)f(s_j)$ and generalizations thereof, e.g., $f(s_i)f(s_j) + g(s_i)g(s_j)$ with $f, g \geq 0, f', g' \geq 0, f'', g'' \leq 0$, etc., which all present instances of increasing differences.

6 Pairwise Asymmetry

Instances of variably attractive players have been touched upon in case 2B and, with positive and negative attraction, in case 2C. They exhibit pairwise asymmetries and, therefore, have been excluded from Section 4. Next we revisit those two and other similar scenarios.

6A: Variably Attractive Players Reconsidered

In case 2B we have seen that the less attractive players may make all the investments, if some players are more attractive than others. The unique equilibrium was stochastically stable and there was under-investment. In case 2C, a mix of players with positive and negative attraction caused over-investment in the unique and stochastically stable equilibrium. In both cases, pairwise benefits were of the form (4) where $\pi_{ij} = B(s_i + s_j)$ for some function $B : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, specifically $B(x) = \sqrt{x}$ and $B(x) = x$. What distinguishes both cases from instances of pairwise symmetry is the existence of numbers V_1, \dots, V_N such that

$$(v) \quad v_{ij} = V_j.$$

If the V_i differ, then (iv) is violated and, as a rule, the pairwise interaction games G_{ij} do not have symmetric potentials. Consequently, Propositions 6, 2, and 3 need not apply. In the sequel, we focus on a linear model which allows a systematic inquiry. This linear model is essentially identical with

the one developed and analyzed in subsection 4B, with the crucial exception of condition (v):

Linear Model. We assume an undirected graph (I, E) such that $b_{ij}(s_i, s_j) = (s_i + s_j) \cdot V_j$ if $\{i, j\} \in E$ and $b_{ij}(s_i, s_j) = 0$ if $\{i, j\} \notin E$. Like in 4B, we assume binary choices, $K = \{0, 1\}$, and linear costs, $c_i(s_i) = C_i \cdot s_i$ with $C_i > 0$ for all i .

Now let N_i be the set of player i 's neighbors and $Z_i = |N_i|$ be the number of his neighbors. Since we always assume that nobody is isolated, $Z_i \geq 1$. Further define $W_i \equiv \sum_{j \in N_i} V_j$. Then $s_i = 1$ is a best response for i iff $W_i \geq C_i$ and $s_i = 0$ is a best response for i iff $W_i \leq C_i$. Moreover, G has the potential $P(s) = \sum_i (W_i - C_i) s_i$. The social welfare function W assumes the particular form $W(s) = \sum_i (W_i + Z_i V_i - C_i) s_i$. It follows that all equilibria are in weakly dominant strategies and stochastically stable. In general, the maximizers of P and W will not coincide. In fact, we know from 2B and 2C that with variably attractive players, there can be under- or over-investment. Let us add two more observations.

First, "bad neighbors" may not only harm "good neighbors", but can also harm each other through their networking efforts. For example, let $N = 4$, $E = F$, $V_1 = V_2 = -1$, $V_3 = V_4 = 1$, $0 < C_i < 1$ for all i . Then the unique equilibrium is $s = (1, 1, 0, 0)$ with utilities $u_1(s) = -C_1$, $u_2(s) = -C_2$ and $u_3(s) = u_4(s) = -2$. Everybody would be better off at $s^0 = (0, 0, 0, 0)$. But given any choices by 3 and 4, players 1 and 2 find themselves in a Prisoner's Dilemma. Incidentally, the efficient outcome would be $t = (0, 0, 1, 1)$ with $W(t) = 4 - (C_3 + C_4)$. Hence, the equilibrium s which is in strictly dominant strategies and stochastically stable, exhibits over-investment by 1 and 2 and under-investment by 3 and 4.

Second, the game G has a potential, even though the games G_{ij} do not have symmetric potentials, thus violating the premise of Proposition 6. By the way, a game G_{ij} does possess a potential β_{ij} given by $\beta_{ij}(0, 0) = 0$, $\beta_{ij}(0, 1) = V_i$, $\beta_{ij}(1, 0) = V_j$, $\beta_{ij}(1, 1) = V_i + V_j$. However, β_{ij} is asymmetric unless $V_i = V_j$.

6B: Adversity

In certain pairwise interactions, one party gains when the other loses and vice versa. One can think of chess matches, instances of gambling, or mutual

industrial espionage. This means that for such a pair of players ij , the game G_{ij} is a zero-sum game: $b_{ij}(s_i, s_j) = -b_{ji}(s_j, s_i)$ for any pair of networking levels $(s_i, s_j) \in K \times K$. If one assumes the functional form (4) and equal intensities of interaction, that is (i), then such an adversarial interaction amounts to $v_{ij} = -v_{ji}$. It turns out that if G_{ij} is zero-sum, then existence of a potential of G_{ij} and supermodularity of G_{ij} are equivalent.

Proposition 11 *Suppose the game G_{ij} is zero-sum. Then the following properties are equivalent:*

(α) G_{ij} has a potential.

(β) There exist functions $f_{ij} : K \rightarrow \mathbb{R}$ and $g_{ij} : K \rightarrow \mathbb{R}$ such that $b_{ij}(s_i, s_j) = f_{ij}(s_i) - g_{ij}(s_j)$, $b_{ji}(s_j, s_i) = g_{ij}(s_j) - f_{ij}(s_i)$ for all $(s_i, s_j) \in K \times K$.

(γ) G_{ij} is supermodular.

PROOF. By Theorem 1 of Brânzei *et al.* (2003), (α) and (β) are equivalent. The separation property (β) implies increasing differences (in fact constant differences) and, since $K \subseteq \mathbb{R}$, supermodularity. Hence (β) implies (γ). By Theorem 4 of Brânzei *et al.* (2003), (γ) implies (α). ■■

If a zero-sum game G_{ij} has a potential, then the function $\beta_{ij}(s_i, s_j) = f_{ij}(s_i) + g_{ij}(s_j)$, with f_{ij} and g_{ij} as in (β), is a potential. The potential is asymmetric unless f_{ij} and g_{ij} are identical up to an additive constant. Hence, in general, Proposition 6 will not apply. Nevertheless, if each basic game G_{ij} satisfies the separation property (β), then G has a potential given by

$$P(s) = \sum_i \sum_{j \neq i} f_{ij}(s_i) - \sum_i c_i(s_i)$$

for $s \in S$. Moreover, then each player i has a nonempty set D_i of weakly dominant strategies and $S^* = S^N = D_1 \times \dots \times D_N$. Essentially the same conclusions hold if each basic game G_{ij} is constant-sum and a potential game.

7 Conclusions and Extensions

Nonspecific networking means that an individual's networking effort establishes or strengthens links to a multitude of people. The individual cannot

single out specific persons with whom she is going to form links. In the simplest case, the individual has a binary choice, to network or not to network. This particular case covers already a variety of interesting scenarios and phenomena. It encompasses scenarios with differential benefits across pairs of individuals, mutual versus non-mutual (positive or negative) affinities, leading for instance to second-order externalities such as the impact of an enemy of an enemy or to the co-existence of under-investment and over-investment in networking as exemplified in Section 6. Often, however, networking efforts are gradual and our model accommodates this possibility as well. Beyond expanding the descriptive scope of the model, the availability of several levels of networking effort makes the question of Section 5 how networking efforts respond to a change in networking costs much more interesting. One conceivable generalization of our analysis, including the comparative statics, would assume multi-dimensional effort choices, like choosing software-hardware combinations.

Supermodularity and increasing differences, utilized in some of our comparative statics, are cardinal properties. As Milgrom and Shannon (1994) point out, comparative statics questions are inherently ordinal questions, and the conditions on objective functions and constraints necessary for comparative statics conclusions should possibly be ordinal. Indeed, Milgrom and Shannon (1994) find such ordinal conditions for monotone comparative statics. They introduce and study quasi-supermodular functions and functions with the single crossing property. These functions generalize supermodular functions and functions with increasing differences and preserve the monotonicity conclusion for parametric optimization problems. A list of a wide variety of problems in economics and in noncooperative games presented by Milgrom and Shannon (1994) makes a convincing case for the value added of their ordinal extension of complementarity conditions. In view of these results, one might ask whether Proposition 4 can be extended further by invoking such ordinal conditions. Precisely, if we assume that each b_{ij} satisfies the single crossing property on $S_i \times S_j$, are we then able to show that the potential P is quasi-supermodular on S ? Unfortunately, one cannot draw such a conclusion. The reason is that the generality of the single crossing property has its drawbacks: Namely, in the proof of Proposition 4 we make use of Corollary 2.6.1 in Topkis (1998) which states that for a function defined on a finite product of totally ordered sets, increasing differences implies

supermodularity. This crucial auxiliary result no longer holds when the single crossing property is substituted for the increasing difference property. Shannon (1995, p. 220) demonstrates that the single crossing property in each pair of variables does not imply quasi-supermodularity in all variables.

Proposition 10 establishes a weak monotonicity result on the set of stochastically stable states. It states that the largest (smallest) stochastically stable state at a lower parameter value is smaller than the largest (smallest) stochastically stable state at a higher parameter value. But this result does not assert that a given stochastically stable state at a lower parameter value is smaller than any other stochastically stable state at a higher parameter value. In a recent paper, Echenique and Sabarwal (2003, p. 309) give a condition on a pair of parameters $\theta, \theta' \in \Theta, \theta \leq \theta'$, which implies $\sup S^N(\theta) \leq \inf S^N(\theta')$ for the two equilibrium sets $S^N(\theta)$ and $S^N(\theta')$. Since $S^*(\theta) \subseteq S^N(\theta)$ and $S^*(\theta') \subseteq S^N(\theta')$, their condition also implies $\sup S^*(\theta) \leq \inf S^*(\theta')$.

A further alternative could make the set of available efforts a (one- or multi-dimensional) interval or convex set and assume sufficient differentiability of the cost and benefit functions. As Brueckner (2003) demonstrates in the context of specific networking, one arrives at some conclusions very elegantly, if such a continuous model is highly symmetric, but does not get very far otherwise. Most of our subcases and examples can be easily embedded into a larger continuous model. But again, while this might produce some eloquence and quickness of derivations in some cases, it would only render the analysis more complicated in others. An added complication stems from the fact that the concept of stochastic stability developed in the literature so far (based on logit or other perturbations) and employed in the present paper relies on a finite state space.

The idea that the strength or probability of a link might depend on the efforts of both agents involved, is also central to the model of Brueckner (2003).⁸ Similarly, Haller and Sarangi (2003) consider the possibility that the reliability of a link depends on the efforts of both agents. Since we

⁸After learning about our work in progress, Sudipta Sarangi pointed out to us Brueckner's paper and a further common trait of the two papers: Brueckner presents two asymmetric examples, one with an agent who creates higher benefits than others and a second example with an agent who is more accessible than others.

allow for negative affinity or attraction, some agents might not only abstain from networking but might take counter-measures against the networking attempts of others and be willing to incur costs in order to weaken or sever links. This eventuality suggests a further extension of the formal model.

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