Learning versus Diversification in Project Choice*

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Abstract

We study the issue of project choice when a risk-averse agent must choose whether to invest in two projects of the same type (*focus*) or of different types (*diversification*). Projects of the same type are subject to common type-specific shocks. Hence focusing is more risky within each period, but enables faster learning across periods. Optimal project choice involves balancing these two considerations. We demonstrate how an agent's choice of whether to focus or diversify is related to (i) the speed of learning, (ii) the type-specific risk and (iii) his risk-aversion and investment horizon. We show that, contrary to intuition, an increase in type-specific risk may lead to a decrease in diversification. Our theory is applicable to occupational choice within households, project choice under group lending, and corporate diversification.

1 Introduction

There are several models of learning-by-doing in economics, in which agents learn the optimal action (or technology) over time by observing the relationship between their actions and the resulting output from period to period. In many of these models, agents are also able to learn from the experience of others *provided that they are engaged in the same kind of activity*. If one agent is a farmer and the other a potter, they are unlikely to learn much regarding their own technologies from each other. Given that learning might be faster when agents choose common activities, it follows that if agents decide on project choice jointly, and if they are risk-neutral or if there is no opportunity for risk-sharing among them, then the optimal policy is for all of them to be engaged in the same activity (within the bounds dictated by competition and returns to scale), at least until learning is complete.

But consider the problem of occupational choice within a poor household. Suppose that two members of the same household, say father and son, with scant access to capital markets, must decide on which trade to specialize in. Since they belong to the same household, they may be concerned about the total household income (or consumption) rather than individual income. Further, because of lack of access to capital markets, they are likely to be risk-averse. If the son enters the same trade as the father, then he will be able to pick up the necessary skills faster as he learns from his father's past experience. However, this also makes the income of the household more vulnerable to the shocks affecting that particular trade. Hence, the optimal choice of activity will depend on the trade-off between learning and diversification benefits.

In this paper, we model this trade-off between learning and insurance motives in the choice of projects. To our knowledge, this is the first attempt at incorporating risk and learning into a cohesive framework. Prior research on learning has considered only the benefits of learning, with agents being assumed to be risk-neutral. Prior research on diversification, on the other hand, has ignored learning. In our model, a risk-averse agent (or a pair of agents) must choose two projects for investment. There are two types of projects, and the agent may choose to *focus*, by investing in two projects of the same type, or to *diversify*, by investing in two projects of two different types. In the beginning, there is some uncertainty about an underlying parameter of the technology for each type of project. Over time, through experience, the agent learns this parameter. There are two kinds of risks in the economy – *type-specific risk*, that results from a shock common to all projects of the same type; and *idiosyncratic risk*, that results from a shock that varies from project to project. Focusing on the same type of project makes the total output more susceptible to the type-specific risk in each period. However, it also enables faster learning-by-doing, since each period's outcomes provide two experimental data points rather than one. This implies that while focus imposes higher risks in the initial stages of the projects, it may be lead to lower risks in subsequent periods (until learning is advanced). Thus, for an agent who is concerned about discounted expected utility over the life of the projects, the optimal project choice will depend on the relative importance of type-specific risks and learning.

We present two models in this paper. In the first model, the standard Bayesian learning process with normal shocks is so slow that focus is always inferior to diversification – in each period, the faster reduction in the variance of beliefs about the underlying parameter that occurs with focus is more than offset by the higher type-specific risk. This holds even if the type-specific shocks are arbitrarily small. The model uses a fairly general set-up to highlight the importance of considering the risks involved in learning-by-doing.

Next, we consider the target-input model that is often used in the literature. In this case, the relationship between learning and type-specific risk is more complex. We study how the optimal choice of strategy (focus or diversification) depends on the parameters of the technologies, i.e., the ex-ante uncertainty and type-specific risk, and on the agent's characteristics, i.e., risk-aversion parameter, intertemporal discount factor and investment horizon. We show that there always exists a threshold level of type-specific risk above which diversification will be optimal for all agents irrespective of the speed of learning. Another striking result that emerges from our analysis is that an *increase in type-specific risk may lead to a decrease in diversification.* This happens because a small increase in type-specific risk has two effects – first, it increases the relative cost of focus (versus diversification) in each period. However, it also slows down learning since the precision of the signals that the agent gets in each period is lower. Slower learning implies that the reduction in variance of beliefs (or the increase in precision of beliefs) is slower under focus. However, it also implies that the reduction in variance is *even slower* under diversification. Hence the difference in risk between focus and diversification during the learning process is wider for a longer period of time, which increases the relative benefits of focus in the subsequent periods.

In addition to occupational choice within households, these results are also applicable to other economic settings such as project choice under group lending and corporate diversification. In group lending, the agents within the group share the risks of default, since one agent's non-payment of dues is treated by the lender as default by the whole group. This risk-sharing arrangement may induce agents to diversify project choice even at the cost of slower learning. Similarly, a corporate manager who is forced to hold a lot of restricted stock[1] for incentive alignment purposes would likely consider the risk to his wealth (which is tied up with the firm), as well as the benefits of learning the technology quickly. Hence, he may find it optimal to diversify across industries. Our results suggest that such diversification is dependent on the importance of learning, the investment horizon of the manager, and the pervasiveness of type-specific risk.

The paper proceeds as follows: in Section 2, we discuss prior research in this area. In section 3, we present our first model of learning-by-doing with risk-averse agents where we show that if learning is relatively slow, then diversification will always be better than focus. In Section 4, we develop the target input model and analyze how the optimal project choice is related to the various parameters of the model. In Section 5, we discuss the relevance of the results and conclude. The proofs of the propositions are given in the appendix.

2 Prior research

Wilson (1975) studies a model of learning in the context of the theory of the firm under uncertainty. Using a target input model (example 1 of the paper), he explores the

[1] Restricted stock is stock that must be held by the employee or manager until a certain date, and can only be sold under certain conditions. Such stock may also not be hedged away by taking short positions in other stock, for example.

optimal degree of sampling for information by a firm that follows the mean-variance criterion. In the context of our model, greater sampling implies investing in more projects of the same type. In contrast, we study the issue of whether to invest in projects of the same type or of different types. Holmstrom (1982, 1999) studies a Bayesian learning problem that is similar to the one in our first model (section 3), albeit with a risk-neutral agent. Foster and Rosenzweig (1995) and Jovanovich and Nyarko (1995, 1996) are other papers that deal with learning-by-doing and the choice of technology.

3 Learning-by-doing with risk-averse agents

We begin with a model that incorporates risk in a standard model of learning-by-doing. There are two types of projects, poultry farming and rice cultivation, denoted by P and R respectively. There are two agents – we may view them as proprietor-entrepreneurs, each of whom must choose the type of project to invest in. Investment in the projects must be made at the beginning (at time t=0). By investing in a project, the agent specializes his/her human capital to the project, which precludes the gaining of skills specific to the other type of project. In other words, neither agent can invest in both projects. In the following analysis, we will compare the case where agents invest individually versus the case of joint investment with pooling of cash flows. We examine the outcome in terms of focus or diversification in each case.

Each project lasts for two periods. The cash flows y_{ijt} from project $j \in \{1, 2\}$ (or agent j) of type $i \in \{P, R\}$ in period t are given by:

$$y_{ijt} = a_i + u_{it} + e_{ijt} \tag{1}$$

where a_i is a measure of the underlying project quality, which does not change over time. u_{it} is a type-specific shock which affects all projects of type i in period t, and e_{ijt} is an idiosyncratic shock that varies from project to project. Agents know that the two shocks are distributed as follows:[2]

$$u_{it} \sim \mathrm{N}(0, \sigma_{\mathrm{u}}^2); \quad e_{ijt} \sim \mathrm{N}(0, \sigma_{\mathrm{e}}^2). \tag{2}$$

The underlying project quality is initially unknown (but is learnt over time). Agents begin with a common prior belief that

$$a_i \sim \mathrm{N}(\mathrm{a}_0, \sigma_0^2) \; \forall \; i. \tag{3}$$

Over time, agents learn the project quality from experience. Since the priors and shocks are distributed normally, the Bayesian process for updating beliefs is well known (De Groot, 1970). Let τ denote the precision of a random variable. Thus,

$$\tau_0 = \frac{1}{\sigma_0^2}; \;\; \tau_u = \frac{1}{\sigma_u^2}; \quad \text{and} \quad \tau_e = \frac{1}{\sigma_e^2}.$$

After n observations of the output, the posterior beliefs of an agent who has invested in project of type i will be as follows:

[2]We are assuming here that the distribution of the type-specific shock is identical across types. This is to emphasize that the difference in risk between focus and diversification is due to the role of correlation in shocks across projects of the same type, and not due to differences in the profitability of the types themselves. The extension to the case where $u_{it} \sim N(\mu_i, \sigma_i^2)$ is straightforward.

$$
\begin{aligned}
a_i &\sim \mathrm{N}(a_{in}, \tau_n^{-1}) & (4)\\
a_{in} &= a_0 \left(\frac{\tau_0}{\tau_n}\right) + \left(\frac{\tau_{u+e}}{\tau_n}\right) \sum_{k=1}^{n} y_{ik} & (5)\\
\tau_n &= \tau_0 + n\tau_{u+e}. & (6)
\end{aligned}
$$

where $\tau_{u+e} = (\sigma_u^2 + \sigma_e^2)^{-1}$ and y_{ik} denotes the output from *all projects of type* i in period k.

In case the agents invest in different types of projects, after t periods, the posterior beliefs of each agent are given by equations (4)-(6) with $n = t$.

We wish to allow for the possibility that agents who have invested in the same type of projects may learn from each other's experience. In order to do this, we assume that each agent is able to observe the output of the other agents. (While we do not incorporate noise in this observation process, that is quite easily done. Since we will show that even perfect observation does not lead to sufficient learning, introducing noise will only reinforce our results.) This implies that there will be two observations for each period in the case where they focus on the same type of project. Each agent is therefore able to update her beliefs (on project quality) twice in each period. In contrast, in the case where the agents invest in different types of projects, each agent updates her beliefs only once per period. This leads to the following lemma.

Lemma 1.1 *After t periods, the posterior beliefs of each agent are given by equations (4)-(6), with $n = t$ in case they diversify and $n = 2t$ in case they focus.*

Before we deal with the case of joint investment, it is useful to consider optimal project choice when there is no risk-sharing between the agents. It is immediately obvious from lemma 1 that each agent's optimal strategy in this case is to mimic the other agent, so that she can learn faster. Thus, equilibrium always involves focus (on either project type P or R). In this case, learning is at the highest possible level.

Next, we consider the case where risk-averse agents pool their income and maximize utility of total income. Agents may pool their income because they belong to the same household. Alternatively, they may belong to the same lending group and risk-sharing within the group may be approximated to income-pooling in a simplified formulation, as in Prescott (1997). More generally, we may think of the same agent as having to invest in two different projects, which could be chosen from the same type or different types. The agent is then concerned with total income rather than income from each project separately. In the following exposition, we will treat the problem as a single-agent problem, with the implicit equivalence to the multiple agent case with income pooling.

To introduce risk-aversion, we assume that agents have the following utility function:[3]

$$U(c_1, c_2) = -\exp(-\gamma c_1) - \exp(-\gamma c_2) \tag{7}$$

where c_1 and c_2 are the consumption amounts in the two periods. γ is the Arrow-

[3]The results of this section do not change if we assume a utility function based on total lifetime income, of the form $U(c_1, c_2) = -e^{-\gamma(c_1+c_2)}$. Such a utility function would be more appropriate if the agent were able to transfer wealth costlessly across periods.

Pratt coefficient of risk aversion. We ignore intertemporal discounting (and assume that the interest rate is zero) to keep the exposition simple, but introducing it will not change the results. Discounting will change the size of the relative benefit to diversification over focus, but not its sign.

The agent chooses c_1 and c_2 to maximize expected utility. The following lemma incorporates optimal consumption choice into (7). Let $Y_1 = y_{11}+y_{21}$ and $Y_2 = y_{12}+y_{22}$ denote the total income from both projects combined in periods 1 and 2, and y_{11} and y_{21}, the first period income from projects 1 and 2. $E_t[\cdot]$ denotes the expectation based on information at the end of period t.

Lemma 1.2 *When $U(c_1, c_2)$ is given by the expression in (7), we have*

$$\begin{aligned} \max_{c_1,c_2} \quad & E_0\left[\mathrm{U}(\mathrm{c}_1, \mathrm{c}_2)\right] \;\; \text{s.t.} \;\; c_1 + c_2 = Y_1 + Y_2 \\ = \quad & -2E_0\Big[\exp\big(-0.5\gamma\{Y_1 + E_1[Y_2|y_{11}, y_{21}] - 0.5\gamma Var_1[Y_2|y_{11}, y_{21}]\}\big)\Big] \qquad (8) \end{aligned}$$

In lemma 1.2 and in the following analysis, we use the well-known result that if a variable z is normally distributed with mean μ and variance σ^2, then,

$$E[\exp(-kz)] = \exp(-k\{\mu - 0.5k\sigma^2\}).$$

Lemma 1.2 simplifies the comparison between focus and diversification. The difference between the two cases arises in the calculation of $E_1[Y_2|y_{11}, y_{21}]$ and $Var_1[Y_2|y_{11}, y_{21}]$. While each of these terms is important, it is useful to pay special attention to the variance term. Note that the variance at the end of the first period is a function of two variables – the variance of beliefs regarding project quality, which is lower under focus due to faster learning, and the correlation in outcomes between the two projects, which is higher under focus. If learning is fast enough, the benefits of increased precision of beliefs may outweigh the cost of greater correlation under focus. We now turn to the calculation of the expected utility under each case.

Lemma 1.3 *The expected utility under diversification, $E_0[U^D]$, and under focus, $E_0[U^F]$, when $U(c_1, c_2)$ is given by the expression in (7), are*

$$E_0[U^D] = -2 * \exp\left(-0.5\gamma\{4a_0 - 0.5\gamma\sigma_{1D}^2 - 0.25\gamma d^2\sigma_{0D}^2\}\right). \quad (9)$$

$$E_0[U^F] = -2 * \exp\left(-0.5\gamma\{4a_0 - 0.5\gamma\sigma_{1F}^2 - 0.25\gamma f^2\sigma_{0F}^2\}\right). \quad (10)$$

Lemma 1.3 states that the expected utility is a function of the expected total income and the variance of income. The (ex ante) expected total income from both periods is $4a_0$ from the law of iterated expectations. The variance of total income is split into two terms, σ_{0D}^2 and σ_{1D}^2, denoting the variance in periods 1 and 2 under diversification (σ_{0F}^2 and σ_{1F}^2 denote the corresponding variances under focus). Under Bayesian learning with normal shocks, the variance of beliefs changes from period to period in a deterministic way. Hence, σ_{1D}^2, which is actually the conditional variance term $Var_1[Y_2|y_{11}, y_{21}]$ in lemma 1.2, is independent of the exact realizations of y_{11} and y_{21}. In equations (9) and (10), d $(= \frac{\tau_2}{\tau_1})$ and f $(= \frac{\tau_4}{\tau_2})$ are the weights on the first period income in the agent's utility function, taking into account its direct effect on utility and its effect on beliefs regarding the second period income. A higher value of d or f implies that utility is more sensitive to first period income, because of the information it contains about the second period income. Hence, a higher value of d or

f increases the variance component of expected utility. We show below that $f > d$, since faster learning implies that the effect of the first period income on the expected second period income is greater under focus.

Equations (9) and (10) give the expected utility under the diversification and focus strategies. Comparing the two expressions, we note that

$$E_0[U^F] \geq E_0[U^D] \quad \text{if and only if} \quad \sigma_{1F}^2 + 0.5f^2\sigma_{0F}^2 \leq \sigma_{1D}^2 + 0.5d^2\sigma_{0D}^2.$$

From (36) and (41) in the appendix, (in the proof of lemma 1.3),

$$\sigma_{1F}^2 - \sigma_{1D}^2 = \left(\frac{4}{\tau_2} - \frac{2}{\tau_1}\right) + 2\sigma_u^2.$$

From (6), it is easy to show that

$$\frac{4}{\tau_2} > \frac{2}{\tau_1}, \tag{11}$$

Hence, $\sigma_{1F}^2 > \sigma_{1D}^2$. From (6), it also follows that $f > d$. Moreover,

$$\sigma_{0F}^2 = 4\sigma_0^2 + 4\sigma_u^2 + 2\sigma_e^2 \ > \ \sigma_{0D}^2 = 2\sigma_0^2 + 2\sigma_u^2 + 2\sigma_e^2.$$

The above analysis leads to Proposition 1.

Proposition 1 *For the process in (1) and the utility function in (7), the expected utility under focus* $E_0[U^F]$ *is always less than the expected utility under diversification* $E_0[U^D]$.

The reason that proposition 1 holds is that the drop in variance of beliefs caused by faster learning under the focus strategy is not enough to offset the greater type-specific risk involved. This is sharply highlighted by inequality 11.

In the model presented above, learning affects the agent's utility in a relatively passive manner – the reduction in uncertainty regarding the technological parameter caused by learning leads to a higher utility for a risk-averse agent without any direct effect on the actions of the agent in the next period. In the next section, we present a model where learning is more "active," and show that diversification may be inferior to learning in this context.

4 Risk in a target-input model

Another model of learning that has been used in the literature (Wilson, 1975; Foster and Rosenzweig, 1995; Jovanovich and Nyarko, 1996) is the target input model. In this model, output from each project depends on a target level of input, which varies from period to period about a long run mean. Specifically,

$$\begin{aligned} z_{ijt} &= I\left[1 - (y_{ijt} - a_{ijt})^2\right], \text{ where,} \\ y_{ijt} &= a_i + u_{it} + e_{ijt}. \end{aligned} \tag{12}$$

y_{ijt} is the optimal or target input level in period t. Output z_{ijt} is higher when the deviation between actual input a_{ijt} and the target input y_{ijt} is lower. I is a parameter that denotes maximum output in any period. It also affects the sensitivity of output to errors in input choice. Following the literature, we assume that inputs are costless, so that output is equal to profits.

Equation 12 is the analogue of (1), except that it pertains to the target input and not the output. The target input for project j of type i in period t consists of three components: (i) a long run component that is stable over time, denoted by a_i; (ii) u_{it}, which is a type-specific shock that affects all projects of type i in period t; and

(iii) e_{ijt}, which is an idiosyncratic shock that varies from project to project. Agents do not know a_i, but begin with normal priors which are updated over time. We retain the distributional assumptions of the previous section (equations 2 and 3) regarding these shocks. Hence the process of Bayesian updating is given by (4)-(6).[4] As in the previous section, we assume that there are two types of projects, P and R. The agent also has the same utility function as in the previous section, given by (7).

4.1 The agent's problem and its solution

The agent chooses focus or diversification based on the expected utility from each at time t=0. However, the expected utility in this case depends not only on the first period consumption-savings decision, but also on the choice of inputs in the two periods. When the agent is risk-neutral, optimal input choice is quite straightforward, with input levels being chosen to equal expected target input. But when the agent is risk averse, as in this model, the agent's expected utility maximizing choice of input need not coincide with the risk-neutral level. The problem is particularly complex when the projects are of the same type (i.e., under focus), since the project cash flows are correlated in this case. Further, the first period input choice might affect the second period choice since the first period income contains information regarding the distribution of the second period income. Hence, the agent's problem (given focus or diversification) is:

$$\max_{\tilde{a}_{11},\tilde{a}_{21}} E_0 \left[\max_{c_1,\tilde{a}_{12},\tilde{a}_{22}} e^{-\gamma c_1} + E_1 \left[e^{-\gamma c_2} | y_{11}, y_{21} \right] \right] \quad \text{s.t.} \quad z_1 + z_2 = c_1 + c_2 \tag{13}$$

[4]In the previous section, agents could observe y_{ijt} directly. With the target input model, agents can infer y_{ijt} from z_{ijt}.

where $z_1 = z_{11} + z_{21}$ and $z_2 = z_{12} + z_{22}$ are the first and second period incomes, which follow the process in (12), and $\tilde{a}_{ij}$ is the input level for project i in period j. Note that while the target input y_{ij} has a normal distribution, the total income in any period involves the sum of two squared normal variates which are correlated under focus. Hence, calculating the expected utility is not straightforward.

We solve this problem backwards, beginning with the optimal second period input choice conditional on first period income. It turns out that for the utility function considered here, there is no distortion in input choice from the risk-neutral level.[5] This leads to the following lemma.

Lemma 2.1 *The optimal input choice in period t for an agent maximizing expected utility as given in (13) under the target input model is*

$$\tilde{a}_t = E_{t-1}[a|y_{11}, y_{21}, y_{12}, y_{22}, ..., y_{1,t-1}, y_{2,t-1}] \tag{14}$$

In the two-period case, this implies that $\tilde{a}_2 = E_1[a|y_{11}, y_{21}]$ for both projects. Further, given the distributional assumptions here, the expected second period target input is only a function of the variance of beliefs and shocks and not the actual realization of the first period incomes z_{11} and z_{21}. Moreover, given the Bayesian learning process with normal shocks, the variance of beliefs at the beginning of the second period is also deterministic and independent of z_{11} and z_{21}.

[5]This holds more generally for any utility function for which the agent's expected utility depends only on the mean and variance of the distribution of income.

Lemma 2.1 substantially simplifies the consumption-savings problem at the end of the first period, as it implies that the expected second period utility does not depend on the first period income. Next, we solve for the first period consumption function and incorporate that into (13), to get the expected first period utility. Finally, we solve for the optimal first period input choice, which is also found to be equal to the level under risk neutrality.

This analysis leads to the following proposition.

Proposition 2 *The expected utility of the agent under focus and diversification are given by the following expressions:*

$$E_0[U^F] = \frac{-2e^{-2\gamma I}}{F_1 F_2} \tag{15}$$

$$E_0[U^D] = \frac{-2e^{-2\gamma I}}{D_1 D_2} \tag{16}$$

where F_1, F_2, D_1 and D_2 are defined below:

$$F_1 = (1 - \gamma I \sigma_e^2)^{1/2}(1 - \gamma I(\sigma_e^2 + 2\sigma_u^2 + 2\sigma_0^2))^{1/2} \tag{17}$$

$$F_2 = (1 - 2\gamma I \sigma_e^2)^{1/4}(1 - 2\gamma I(\sigma_e^2 + 2\sigma_u^2 + 2\sigma_2^2))^{1/4} \tag{18}$$

$$D_1 = (1 - \gamma I(\sigma_e^2 + \sigma_u^2 + \sigma_0^2)) \tag{19}$$

$$D_2 = (1 - 2\gamma I(\sigma_e^2 + \sigma_u^2 + \sigma_1^2))^{1/2} \tag{20}$$

From proposition 2, it is clear that the comparison between focus and diversification hinges on the terms in the denominators of (15) and (16). Of these, the terms F_1 and D_1 correspond to the expected utility in the first period, while F_2 and D_2 pertain to the second period. Before analyzing these in detail, it is useful to recall that learning affects the focus-diversification trade-off through its effect on the variance of beliefs in the second period. Accordingly, F_2 is a function of σ_2^2, while D_2 is a function of σ_1^2, and recall from (6) that $\sigma_2^2 < \sigma_1^2$.

4.2 The focus-diversification choice

In section 3, we found that even though $\sigma_2^2 < \sigma_1^2$, the fact that $2\sigma_2^2 > \sigma_1^2$ implied that the benefits of faster learning were outweighed by the costs of lack of insurance under focus. With the target input model, however, the variance terms enter the expected utility function in a more complex manner, with a slightly different trade-off between risk and learning. Therefore, it is possible for focus to be optimal under certain conditions. The following proposition states this existence result formally, and is proved in the appendix.

Proposition 3 *There exists a set of parameters γ, I, σ_0^2, σ_u^2 and σ_e^2 such that the expected utility under focus (given in (15)) is greater than that under diversification (given in (16)).*

Proposition 3 contrasts with proposition 1 which ruled out the possibility that focus might be optimal. The difference between the two is caused by the different functional forms of the lifetime expected utility, which, in turn, is due to the differences in technology. Since our purpose here is to examine the trade-off between learning and risk in a general context, we do not wish to emphasize specific technological differences between the two models. Rather, we study the role of factors such as the prior variance of beliefs, the type-specific and idiosyncratic risks, and the agent's risk-aversion in the choice between focus and diversification.

We begin by noting that since U^F and U^D are negative, $U^F > U^D$ if $F_1F_2 > D_1D_2$. Therefore we will examine what happens to the ratio $\frac{F_1F_2}{D_1D_2}$ for some illustrative values of the risk and learning parameters. These examples are useful in developing the intuition regarding how the speed of learning affects the focus-diversification choice.

Example 1

Consider the case when $\sigma_u^2 = \sigma_e^2 = 0$. This implies that the agent is able to observe the technology parameter a without any noise. Hence, at the end of the first period, the precision of beliefs regarding a is infinite. Another way to see this is to rewrite (6) as follows:

$$\sigma_1^2 = \frac{\sigma_0^2(\sigma_u^2 + \sigma_e^2)}{\sigma_0^2 + \sigma_u^2 + \sigma_e^2} \text{ and } \sigma_2^2 = \frac{\sigma_0^2(\sigma_u^2 + \sigma_e^2)}{2\sigma_0^2 + \sigma_u^2 + \sigma_e^2} \tag{21}$$

Thus, if $\sigma_u^2 = \sigma_e^2 = 0$, it follows that $\sigma_1^2 = \sigma_2^2 = 0$. Hence,

$$\frac{F_1F_2}{D_1D_2} = \frac{(1 - 2\gamma I\sigma_0^2)^{\frac{1}{2}}(1)^{\frac{1}{4}}}{(1 - \gamma I\sigma_0^2)(1)^{\frac{1}{2}}} < 1$$

.

Thus, $U^F - U^D < 0$, that is, focus is worse than diversification in this case. This happens because learning is *too fast*. Since learning is complete within one period irrespective of whether the agent chooses focus or diversification, it is optimal for the agent to diversify.

Example 2

Let $\sigma_e^2 = 0$ and let σ_u^2 be large relative to σ_0^2. From (21), this implies that $\sigma_1^2 \approx \sigma_2^2 \approx \sigma_0^2$. In this case,

$$\frac{F_1 F_2}{D_1 D_2} = \frac{(1-2\gamma I\sigma_u^2)^{\frac{1}{2}}}{(1-\gamma I\sigma_u^2)} \cdot \frac{(1-4\gamma I\sigma_u^2)^{\frac{1}{4}}}{(1-2\gamma I\sigma_u^2)^{\frac{1}{2}}} < 1$$

In contrast to example 1, learning is *too slow* in this case, so that there is hardly any change in the variance of beliefs after one period. Hence the agent is better off ignoring learning altogether and investing in different types of projects.

The above two examples highlight the fact that for focus to be optimal, the speed of learning should be such that the difference between the variance of beliefs in the second period under the two strategies should be significant. When learning is very fast or very slow, σ_1^2 is too close to σ_2^2, and the trade-off between focus and diversification is dominated by the first period terms F_1 and D_1, and clearly, $F_1 < D_1$. Next we will consider a case where the learning speed is intermediate, and examine the conditions under which focus is optimal.

Example 3

Let $\sigma_e^2 = 0$ and let $\sigma_u^2 = \sigma_0^2$. Then, from (21),

$$\sigma_1^2 = \frac{\sigma_0^2\sigma_0^2}{\sigma_0^2 + \sigma_0^2} = \frac{\sigma_0^2}{2} \text{ and } \sigma_2^2 = \frac{\sigma_0^2\sigma_0^2}{2\sigma_0^2 + \sigma_0^2} = \frac{\sigma_0^2}{3}$$

.

$$\frac{F_1F_2}{D_1D_2} \approx \frac{(1-4\gamma I\sigma_0^2)^{\frac{1}{2}}}{(1-2\gamma I\sigma_0^2)}.\frac{[1-2\gamma I(2\sigma_0^2+2\sigma_2^2)]^{\frac{1}{4}}}{[1-2\gamma I(\sigma_0^2+\sigma_1^2)]^{\frac{1}{2}}} = \left[\frac{(1-4\gamma I\sigma_0^2)^{\frac{1}{2}}}{(1-2\gamma I\sigma_0^2)}\right]\left[\frac{(1-\frac{16}{3}\gamma I\sigma_0^2)^{\frac{1}{4}}}{(1-3\gamma I\sigma_0^2)^{\frac{1}{2}}}\right]$$

Since the ratio in the first pair of brackets is < 1, a sufficient condition for diversification to be optimal is:

$$(1-\frac{16}{3}\gamma I\sigma_0^2)^{\frac{1}{4}} > (1-3\gamma I\sigma_0^2)^{\frac{1}{2}} \Rightarrow 1-\frac{16}{3}\gamma I\sigma_0^2 > (1-3\gamma I\sigma_0^2)^2$$

Solving the quadratic equation, we get the condition to be $\gamma I\sigma_0^2 > \frac{2}{27}$. If this condition is violated, it is possible that focus is better than diversification. For example, if $\gamma I\sigma_0^2 = \frac{1}{27}$, it can be verified that $F_1F_2 > D_1D_2$, which implies that $U^F > U^D$. This example also highlights the role of the aggregate risk level (σ_0^2 or σ_u^2 in this case) on the trade-off. Focus is optimal only if the speed of learning is neither too high nor too low, and the risk level is sufficiently low.

Finally, comparing example 1 to example 3, we see that an increase in type specific risk may actually lead to focus becoming more dominant relative to diversification. The following proposition formalizes this result.

Proposition 4 *There exists a set of parameters (γ,I,σ_0^2, σ_u^2, σ_e^2) such that an increase in type-specific risk within that set leads to an increase in U^F relative to U^D.*

It is also easy to see from examples 1 to 3 that the positive effect of σ_u^2 on $U^F - U^D$ is limited - as σ_u^2 increases beyond a certain point, the learning effect slows while the aggregate risk level continues to increase, leading to a decline in $U^F - U^D$. These effects are highlighted in figure 1, in which $U^F - U^D$ is graphed against variance of prior beliefs (σ_0^2) and type-specific risk (σ_u^2). The values of the other parameters used in this numerical simulation are - $\sigma_e^2 = 1; \gamma = 1,$ and $I = 0.025$. The figure illustrates the concave relationship between σ_0^2 and $U^F - U^D$ for a fixed value of σ_u^2 and between σ_u^2 and $U^F - U^D$ for fixed σ_0^2. While the latter relationship is largely negative, there does exist a region of σ_0^2 over which σ_u^2 has a positive effect on $U^F - U^D$. (This is more sharply highlighted in figure 3, which we discuss below.)

Next, we examine the effect of idiosyncratic risk. As earlier, it is instructive to look at specific examples.

Example 4

Let $\sigma_u^2 = 0$ and σ_e^2 be large relative to σ_0^2. Then, $\sigma_1^2 \approx \sigma_2^2 \approx \sigma_0^2$ from (21). This implies that

$$\frac{F_1 F_2}{D_1 D_2} \approx \frac{(1-\gamma I \sigma_e^2)^{\frac{1}{2}}(1-\gamma I \sigma_e^2)^{\frac{1}{2}}}{(1-\gamma I \sigma_e^2)} . \frac{(1-2\gamma I \sigma_e^2)^{\frac{1}{4}}(1-2\gamma I \sigma_e^2)^{\frac{1}{4}}}{(1-2\gamma I \sigma_e^2)^{\frac{1}{2}}} = 1$$

This shows that when the risk is mainly idiosyncratic and learning is slow, focus and diversification are not very different. It can be verified that if $\sigma_e^2 = 2\sigma_0^2$, diversification is marginally better than focus, while if $\sigma_e^2 = 3\sigma_0^2$, focus is marginally better than diversification if $\gamma I \sigma_0^2$ is $< \frac{12}{117}$. This is in contrast to example 2 where learning was slow, but the risk was mainly type-specific, leading to focus being the inferior strategy.

Example 5

Let $\sigma_u^2 = 0$ and σ_0^2 be large relative to σ_e^2. In this case, $\sigma_1^2 \approx 2\sigma_2^2 \approx \sigma_e^2$. Therefore,

$$\frac{F_1F_2}{D_1D_2} = \left[\frac{(1-\gamma I\sigma_e^2)^{\frac{1}{2}}(1-2\gamma I\sigma_0^2)^{\frac{1}{2}}}{(1-\gamma I\sigma_0^2)}\right] \cdot \left[\frac{(1-2\gamma I\sigma_e^2)^{\frac{1}{4}}(1-4\gamma I\sigma_e^2)^{\frac{1}{4}}}{(1-4\gamma I\sigma_e^2)^{\frac{1}{2}}}\right]$$

While the term in the first pair of square brackets is < 1, the term in the second is greater than 1. Thus, it is possible for focus to be better than diversification. It can be verified, for example, that if $\sigma_e^2 = \frac{1}{5}\sigma_0^2$, then focus is the superior strategy for all valid values of other parameters.

As with prior variance and type specific risk, idiosyncratic risk affects the focus-diversification choice in two ways, through its direct impact on risk and through its impact on the speed of learning. This leads to the possibility that the effect of σ_e^2 on $U^F - U^D$ is positive at low levels of σ_e^2 and negative at high levels. Figure 2 presents a surface plot of $U^F - U^D$ against σ_0^2 and σ_e^2. The figure illustrates the effects discussed above. It is, of course, less surprising that the effect of idiosyncratic risk on focus may be positive, than that type-specific risk may have a positive effect.

Figure 3 presents a surface plot of $U^F - U^D$ against σ_u^2 and σ_e^2. It demonstrates that the effect of each kind of risk on focus is positive when the risk level is low. It is useful to consider what happens to $U^F - U^D$ when $\sigma_u^2 + \sigma_e^2$ is constant and the balance shifts from one kind of risk to the other. In the figure, this may be done, for example, by cutting the surface by a vertical plane that passes through the 45° line connecting the points A=$(\sigma_u^2 = 0, \sigma_e^2 = 2)$ and B=$(\sigma_u^2 = 2, \sigma_e^2 = 0)$. It is clear from the figure that $U^F - U^D$ unambiguously decreases when we move from point A to point B along the surface. From (21), it is clear that changing σ_u^2 or σ_e^2 while keeping $\sigma_u^2 + \sigma_e^2$ constant leaves σ_1^2 and σ_2^2 unchanged. Thus, the learning speed is unaffected, while the nature of risk changes as σ_e^2 decreases and σ_u^2 increases. This leads to focus being less preferred relative to diversification.

To summarize, focus is optimal when the type-specific risk is low relative to the prior variance of beliefs, but not so low that learning is substantially complete within one period. For a given speed of learning, a shift in risk from idiosyncratic to type-specific risk leads to focus becoming less attractive relative to diversification. Figure 4 presents these results in the form of graphs of $U^F - U^D$ against each of the three kinds of risk. Finally, it may be noted that the dependence of the focus-diversification choice on the learning speed also implies that the agent's investment horizon should neither be too long nor too short for focus to be optimal.

4.3 Capital market imperfections

So far, we have assumed that the agent is able to costlessly borrow and lend, so that consumption can be freely transferred from one period to the other. It could be argued that this is implausible in a context such a rural household in a developing country that has scarce access to bank loans for consumption purposes. In this case, focus becomes even more risky since it trades off greater risk in the initial periods in return for lower risk in the later periods. It is therefore necessary to examine whether it is still possible for focus to be better than diversification under these conditions. To do so, we assume in this section that the agent's consumption equals income in each period.

It is relatively easy to show that the optimal input choice is still unaffected by the inability to transfer consumption across periods. Hence, following an argument similar to that presented in the proof of lemma 2.1 and proposition 2, it can be shown that the agent's two-period utility in this case is

$$E_0[U^F] = -2e^{-2\gamma I}\left(\frac{1}{F_1'} + \frac{1}{F_2^2}\right) \tag{22}$$

$$E_0[U^F] = -2e^{-2\gamma I}\left(\frac{1}{D_1'} + \frac{1}{D_2^2}\right) \tag{23}$$

$$\tag{24}$$

where F_2 and D_2 are defined in proposition 2, and F_1' and D_1' are defined below:

$$F_1' = (1 - 2\gamma I\sigma_e^2)^{1/2}(1 - 2\gamma I(\sigma_e^2 + 2\sigma_u^2 + 2\sigma_0^2))^{1/2} \tag{25}$$

$$D_1' = (1 - 2\gamma I(\sigma_e^2 + \sigma_u^2 + \sigma_0^2)). \tag{26}$$

$$\tag{27}$$

It is possible to apply the logic of propositions 3 and 4 to show that focus may be better than diversification, and that type-specific risk may have a positive effect on focus even in this case. The proofs are omitted to conserve space. However, in figure 5, we present the analogue to figure 3, with no borrowing or lending. The figures are seen to be quite similar, except that $U^F - U^D$ is uniformly lower in figure 5 due to the constrained nature of the optimization.

5 Conclusion

In this paper, we study the issue of project choice when a risk-averse agent must choose whether to invest in two projects of the same type (*focus*) or of different types (*diversification*). Investing in projects of the same type is more risky within each period, but enables faster learning across periods. Optimal project choice involves balancing these two considerations. We first show that in a standard model of learning, it is possible for learning to be so slow that diversification is always better than learning. This is even with a very small amount of type-specific risk. Next, we consider the target input model and show how an agent's choice of whether to focus or diversify is related to (i) the speed of learning, (ii) the type-specific risk and (iii) risk-aversion and investment horizon. There always exists a threshold level of type-specific risk, above which diversification is optimal for all agents.

The trade-off between learning and insurance motives is likely to occur in several economic settings. For example, occupational choice within households in less developed countries which are subject to large weather shocks is likely to take into account the potential benefits of diversifying across trades. Similarly, when members of the same group choose projects under group lending, they may trade off risk-sharing against learning. Members of the same group share the risks of default, since one agent's non-payment of dues is treated as default by the whole group. This risk-sharing arrangement may induce agents to diversify project choice even at the cost of slower learning. A corporate manager who holds a lot of firm-specific wealth may also find it optimal to diversify across industries, even when this imposes a cost on the risk-neutral shareholders in the form of slower accumulation of skills and knowledge by the management. Our results suggest that such diversification is dependent on the importance of learning, the investment horizon of the manager, and the pervasiveness of type-specific risk.

In prior work, learning and the associated risks have usually been dealt with separately. The analysis on the learning side has mainly dealt with issues such as when to switch to a new technology based on the relative profitability, and the analysis on the risk side has concentrated on risk-shifting behavior (which occurs when risk-averse managers do not invest in some positive NPV projects). This paper combines both the costs and benefits of learning in one model. By incorporating learning and risk into a cohesive framework, this research contributes to the literature on optimal project choice in environments where such concerns matter.

Appendix

Proof of lemma 1.2

First consider the problem of the agent at the end of the first period, when the first period income Y_1 is known, but the second period income is unknown. The maximand of the agent's expected utility maximization problem at this point is

$$\begin{aligned} \mathrm{E}_1[\mathrm{U}] &= -\exp(-\gamma c_1) - E\left[\exp(-\gamma\{Y_2 + Y_1 - c_1\}|y_{11}, y_{21}\right] \\ &= -\exp(-\gamma c_1) - \exp(-\gamma\{Y_1 - c_1\}) * E\left[\exp(-\gamma Y_2)|y_{11}, y_{21}\right]. \end{aligned} \tag{28}$$

It is easily seen from (1) and (4) that the distribution of Y_2 conditional on $\{y_{11}, y_{21}\}$ is normal. Further, for a random variable z that is normally distributed, the following result can also be easily derived:

$$E\left[-\exp(-\gamma z)\right] = -\exp\left(-\gamma\{E[z] - 0.5\gamma \mathrm{Var}[z]\}\right) \tag{29}$$

Applying these results to 28, we have

$$\mathrm{E}_1[\mathrm{U}] = -\exp(-\gamma c_1) - \exp(-\gamma\{Y_1 - c_1 + E[Y_2|y_{11}, y_{21}] - 0.5\gamma \mathrm{Var}[Y_2|y_{11}, y_{21}]\}). \tag{30}$$

The first order condition for maximizing $\mathrm{E}_1[\mathrm{U}]$ is seen to be

$$\gamma\exp(-\gamma c_1) - \gamma\exp(-\gamma\{Y_1 - c_1 + E[Y_2|y_{11}, y_{21}] - 0.5\gamma \mathrm{Var}[Y_2|y_{11}, y_{21}]\}) = 0,$$

which simplifies to

$$c_1 = 0.5 * \left(Y_1 + E[Y_2|y_{11}, y_{21}] - 0.5\gamma \mathrm{Var}[Y_2|y_{11}, y_{21}]\right). \tag{31}$$

Substituting for c_1 in (30) gives lemma 1.2.

Proof of Lemma 1.3

We deal with diversification first and focus next.

Diversification

When the two projects are of different types, then their outputs in each period are independent. Therefore, assuming without loss of generality that project 1 is of type P and project 2 of type R,

$$y_{12} + y_{22} = a_P + u_{P2} + e_{12} + a_R + u_{R2} + e_{22} \tag{32}$$

From lemma 1.1 and equations (4)-(6),

$$\begin{aligned} E_1[y_{12} + y_{22}|y_{11}, y_{21}] &= E_1[a_P|y_{11}] + E[a_R|y_{21}] \\ &= 2a_0\left(\frac{\tau_0}{\tau_1}\right) + (y_{11} + y_{21})\left(\frac{\tau_{u+e}}{\tau_1}\right). \end{aligned} \tag{33}$$

Hence,

$$y_{11} + y_{21} + E_1[y_{12} + y_{22}|y_{11}, y_{21}] = 2a_0\left(\frac{\tau_0}{\tau_1}\right) + (y_{11} + y_{21})\left(\frac{\tau_1 + \tau_{u+e}}{\tau_1}\right). \tag{34}$$

Let $d = \frac{\tau_1+\tau_{u+e}}{\tau_1} = \frac{\tau_0+2\tau_{u+e}}{\tau_0+\tau_{u+e}} = \frac{\tau_2}{\tau_1}$. Then, we have

$$y_{11} + y_{21} + E_1[y_{12} + y_{22}|y_{11}, y_{21}] = 2a_0(2-d) + d(y_{11} + y_{21}). \tag{35}$$

Since the types are independent and $Var_1[a_P|y_{11}] = Var_1[a_R|y_{21}] = \tau_1^{-1}$,

$$\begin{aligned} \sigma_{1D}^2 = Var_1[y_{12} + y_{22}|y_{11}, y_{21}] &= Var_1[y_{12}|y_{11}] + Var_1[y_{22}|y_{21}] \\ &= 2\tau_1^{-1} + 2\sigma_u^2 + 2\sigma_e^2 = 2\tau_1^{-1} + 2\tau_{u+e}^{-1} \end{aligned} \tag{36}$$

Substituting (35) and (36) in the maximand in (8),

$$E_0[U^D] = -2 * \exp\left(-0.5\gamma\{2a_0(2-d) - 0.5\gamma\sigma_{1D}^2\}\right) * E_0\left[\exp\left(-0.5\gamma d(y_{11} + y_{21})\right)\right] \tag{37}$$

Since y_{11} and y_{21} are normally distributed and independent, it follows that

$$\begin{aligned} E_0\left[\exp\left(-0.5\gamma(y_{11}+y_{21})d\right)\right] &= \exp\left(-0.5\gamma d\{E_0[y_{11}+y_{21}] - 0.25\gamma d Var_0[y_{11}+y_{21}]\}\right) \\ &= \exp\left(-0.5\gamma d\{2a_0 - 0.25\gamma d(2\sigma_0^2 + 2\sigma_u^2 + 2\sigma_e^2)\}\right) \quad (38) \end{aligned}$$

Let $\sigma_{0D}^2 = Var_0[y_{11}+y_{21}] = 2(\sigma_0^2 + 2\sigma_u^2 + 2\sigma_e^2)$. From (37) and (38), we have (9) of lemma 1.3.

<u>*Focus*</u>

In this case, the output from the two projects in each period are correlated. The agent starts with the same prior regarding the project quality of either project. After the first period, she updates her beliefs twice based on the output from each project and begins the next period with the same updated prior for both projects. Therefore, suppressing type subscripts, we have

$$y_{12} + y_{22} = a + u + e_1 + a + u + e_2 = 2a + 2u + e_1 + e_2 \quad (39)$$

Hence, applying lemma 1.1 to equations (4)-(6), we have

$$\begin{aligned} E_1[y_{12}+y_{22}|y_{11},y_{21}] &= 2 * E_1[a|y_{11},y_{21}] \\ &= 2a_0\left(\frac{\tau_0}{\tau_2}\right) + 2(y_{11}+y_{21})\left(\frac{\tau_{u+e}}{\tau_2}\right), \text{ and} \quad (40) \end{aligned}$$

$$\begin{aligned} \sigma_{1F}^2 = Var_1[y_{12}+y_{22}|y_{11},y_{21}] &= 4 * Var_1[a|y_{11},y_{21}] + 4\sigma_u^2 + 2\sigma_e^2 \\ &= 4 * \tau_2^{-1} + 4\sigma_u^2 + 2\sigma_e^2. \quad (41) \end{aligned}$$

Comparing (33) with (40) and (36) with (41), the impact of faster learning under focus is seen in the greater precision of beliefs, $\tau_2 > \tau_1$. However, the sign of $\sigma_{1F}^2 - \sigma_{1D}^2$ is not immediately obvious, and the net effect benefit of focus over diversification is, as yet, ambiguous.

From (40),

$$y_{11} + y_{21} + E_1[y_{12} + y_{22}|y_{11}, y_{21}] = 2a_0\left(\frac{\tau_0}{\tau_2}\right) + (y_{11} + y_{21})\left(\frac{\tau_2 + 2\tau_{u+e}}{\tau_2}\right).$$

Let $f = \frac{\tau_2+2\tau_{u+e}}{\tau_2} = \frac{\tau_0+4\tau_{u+e}}{\tau_0+2\tau_{u+e}}$. Then, we have

$$y_{11} + y_{21} + E_1[y_{12} + y_{22}|y_{11}, y_{21}] = 2a_0(2 - f) + f(y_{11} + y_{21}). \quad (42)$$

Substituting (41) and (42) in the maximand in (8), we have

$$E_0[U^F] = -2 * \exp\left(-0.5\gamma\{2a_0(2 - f) - 0.5\gamma\sigma_{1F}^2\}\right) * E_0\left[\exp\left(-0.5\gamma f(y_{11} + y_{21})\right)\right] \quad (43)$$

Since $(y_{11} + y_{21})$ is normally distributed, it follows that

$$\begin{aligned} E_0\left[\exp\left(-0.5\gamma f(y_{11} + y_{21})\right)\right] &= \exp\left(-0.5\gamma f\{E_0[y_{11} + y_{21}] - 0.25\gamma f Var_0[y_{11} + y_{21}]\}\right) \\ &= \exp\left(-0.5\gamma f\{2a_0 - 0.25\gamma f(4\sigma_0^2 + 4\sigma_u^2 + 2\sigma_e^2)\}\right) \quad (44) \end{aligned}$$

Let $\sigma_{0F}^2 = Var_0[y_{11} + y_{21}] = 4\sigma_0^2 + 4\sigma_u^2 + 2\sigma_e^2$. From (43) and (44), we have (10) of lemma 1.3.

Hence the proof.

<u>Proof of Lemma 2.1</u>

The case of focus is more intricate than that of diversification, since the project cash flows are correlated under focus. Hence, we prove the result for focus, drawing parallels for diversification where appropriate.

Let $\tilde{a}_2$ denote input choice in second period for each project and $\overline{a} = E_1[a|y_{11}, y_{22}]$. We prove the lemma by proving the following two results.

Result 2.1.1 *The expected second period utility under focus, denoted by U_2^F, conditional on the total first period income $Y_1 = y_{11} + y_{21}$, is given by*

$$U_2^F = E_1^F[e^{-\gamma Z_2}|Y_1] = \frac{e^{-2\gamma I} e^{\frac{(k_1^F - k_2^F)}{(2(\sigma_u^2+\sigma_2^2))}(1-k_3^F)}}{(1-2\gamma I\sigma_e^2)^{\frac{1}{2}}(1-2\gamma I(\sigma_e^2+2\sigma_u^2+2\sigma_2^2))^{\frac{1}{2}}} \tag{45}$$

where $k_1^F = \left(\frac{\bar{a}-k_3^F\tilde{a}_2}{1-k_3^F}\right)^2$, $k_2^F = \frac{\bar{a}^2-k_3^F\tilde{a}_2^2}{1-k_3^F}$, and $k_3^F = \frac{4\gamma I(\sigma_2^2+\sigma_u^2)}{1-2\gamma I\sigma_e^2}$.

Result 2.1.2 *The value of $\tilde{a}_2$ that maximizes (45) above is $\tilde{a}_2 = \overline{a} = E_1[a|y_{11}, y_{22}]$.*

Proof of Result 2.1.1

Under focus, since both projects are of the same type, the target inputs in the second period for the two projects are given by $y_{12} = a + u_2 + e_{12}$ and $y_{22} = a + u_2 + e_{22}$. The expected target for both projects is therefore the same, given by $\overline{a}$. Let $a + u_2 = \theta$ and $x = a + u_2 - \tilde{a}_2 = \theta - \tilde{a}_2$. Note that $\theta \sim N(\overline{a}, \sigma_2^2 + \sigma_u^2)$. Income in the second period from the first and second projects are $z_{21} = I[1-(y_{21}-\tilde{a}_2)^2]$ and $z_{22} = I[1-(y_{22}-\tilde{a}_2)^2]$, respectively. This implies that $z_{21} = I[1-(a+u_2+e_1-\tilde{a}_2)^2] = I[1-(x+e_1)^2]$ and $z_{22} = I[1-(a+u_2+e_2-\tilde{a}_2)^2] = I[1-(x+e_2)^2]$. Therefore, $z_{21}+z_{22} = I[2-(x+e_1)^2-(x+e_2)^2]$. Thus, $e^{-\gamma z_2} = e^{-\gamma(z_{21}+z_{22})} = e^{-\gamma I[2-(x+e_1)^2-(x+e_2)^2]} = e^{-2\gamma I}.e^{\gamma I(x+e_1)^2}.e^{\gamma I(x+e_2)^2}$. Substituting this into the expression for U_2^F, we get

$$U_2^F = e^{-2\gamma I} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\gamma I(x+e_2)^2} d\Phi(\theta) d\Phi(e_2) \int_{-\infty}^{\infty} e^{\gamma I(x+e_1)^2} d\Phi(e_1), \tag{46}$$

where $\Phi(.)$ denotes the normal distribution function corresponding to that particular variable. To evaluate the innermost integral, we follow the method of completion of squares. Temporarily suppressing the subscript 1 in e_1,

$$\int_{-\infty}^{\infty} e^{\gamma I(e+x)^2} d\Phi(e) = \int_{-\infty}^{\infty} \frac{exp\{\gamma I(e+x)^2 - \frac{e^2}{2\sigma_e^2}\}}{\sigma_e\sqrt{2\pi}} de = \int_{-\infty}^{\infty} \frac{exp - \{\frac{e^2 - 2\gamma I\sigma_e^2(e+x)^2}{2\sigma_e^2}\}}{\sigma_e\sqrt{2\pi}} de. \tag{47}$$

The numerator of the expression within braces in the integrand above is

$$\begin{aligned} e^2 - 2\gamma I\sigma_e^2(e^2 + x^2 + 2xe) &= (1 - 2\gamma I\sigma_e^2)e^2 - 2\gamma I\sigma_e^2 x^2 - 4(\gamma I\sigma_e^2 x)e \\ &= \left[e^2 - \left(\frac{2\gamma I\sigma_e^2}{1 - 2\gamma I\sigma_e^2}\right)x^2 - \left(\frac{4\gamma I\sigma_e^2 x}{1 - 2\gamma I\sigma_e^2}\right)e\right](1 - 2\gamma I\sigma_e^2) \\ = \left[\left(e - \frac{2\gamma I\sigma_e^2 x}{1 - 2\gamma I\sigma_e^2}\right)^2 - \left(\frac{4\gamma^2 I^2\sigma_e^4}{(1 - 2\gamma I\sigma_e^2)^2}\right)x^2 \right. & \left. - \left(\frac{2\gamma I\sigma_e^2}{1 - 2\gamma I\sigma_e^2}\right)x^2\right](1 - 2\gamma I\sigma_e^2). \end{aligned}$$

Taking the terms that are independent of e outside the integrand, the integral in (47) becomes

$$exp\left\{\frac{x^2}{2\sigma_e^2}\left(\frac{4\gamma^2 I^2\sigma_e^4}{1 - 2\gamma I\sigma_e^2} + 2\gamma I\sigma_e^2\right)\right\}\int_{-\infty}^{\infty} \frac{exp\left\{-\left(e - \frac{2\gamma I\sigma_e^2 x}{1-2\gamma I\sigma_e^2}\right)^2 \frac{(1-2\gamma I\sigma_e^2)}{2\sigma_e^2}\right\}}{\sigma_e\sqrt{2\pi}} de$$

$$= exp\left\{\frac{2\gamma I\sigma_e^2 x^2}{2\sigma_e^2(1 - 2\gamma I\sigma_e^2)}\right\}\int_{-\infty}^{\infty} \frac{exp\left\{-\left(e - \frac{2\gamma I\sigma_e^2 x}{1-2\gamma I\sigma_e^2}\right)^2 \frac{(1-2\gamma I\sigma_e^2)}{2\sigma_e^2}\right\}}{\sigma_e\sqrt{2\pi}} de$$

Let $\left(e - \frac{2\gamma I\sigma_e^2 x}{1-2\gamma I\sigma_e^2}\right)\frac{(\sqrt{1-2\gamma I\sigma_e^2})}{\sigma_e} = w$. Thus, $\frac{de}{\sigma_e} = \frac{dw}{(\sqrt{1-2\gamma I\sigma_e^2})}$. Substituting for $\frac{de}{\sigma_e}$, we get

$$\frac{exp\{\frac{\gamma I x^2}{(1-2\gamma I\sigma_e^2)}\}}{\sqrt{1 - 2\gamma I\sigma_e^2}}\int_{-\infty}^{\infty} \frac{e^{-w^2/2}}{\sqrt{2\pi}} dw = \frac{e^{\gamma I x^2/(1-2\gamma I\sigma_e^2)}}{\sqrt{1 - 2\gamma I\sigma_e^2}}. \tag{48}$$

Substituting (48) in (46),

$$\begin{aligned} U_2^F &= \frac{e^{-2\gamma I}}{\sqrt{1 - 2\gamma I\sigma_e^2}}\int_{-\infty}^{\infty} exp\left\{\frac{\gamma I x^2}{(1 - 2\gamma I\sigma_e^2)}\right\} d\Phi(\theta)\int_{-\infty}^{\infty} exp\{\gamma I(x + e_2)^2\} d\Phi(e_2) \\ &= \frac{e^{-2\gamma I}}{(1 - 2\gamma I\sigma_e^2)}\int_{-\infty}^{\infty} exp\left\{\frac{2\gamma I}{(1 - 2\gamma I\sigma_e^2)}x^2\right\} d\Phi(\theta) \end{aligned} \tag{49}$$

Recall that $x = \theta - \tilde{a}_2$ and $\theta \sim N(\bar{a}, \sigma_2^2 + \sigma_u^2)$. Let $\sigma^2 = \sigma_2^2 + \sigma_u^2$ and $m = \frac{2\gamma I}{(1-2\gamma I\sigma_e^2)}$. Then, (49) becomes

$$U_2^F = \frac{e^{-2\gamma I}}{1-2\gamma I\sigma_e^2}\int_{-\infty}^{\infty}\frac{e^{m(\theta-\tilde{a}_2)^2}.e^{-\frac{(\theta-\bar{a})^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}d\theta = \frac{e^{-2\gamma I}}{1-2\gamma I\sigma_e^2}\int_{-\infty}^{\infty}\frac{e^J}{\sigma\sqrt{2\pi}}d\theta \tag{50}$$

where $J = m(\theta - \tilde{a}_2)^2 - \frac{(\theta-\bar{a})^2}{2\sigma^2}$. Next, completing the squares in the exponent J, we get

$$\begin{aligned}
J &= \left(\frac{m(\theta-\tilde{a}_2)^2 2\sigma^2 - (\theta-\bar{a})^2}{2\sigma^2}\right)\\
&= \frac{1}{2\sigma^2}\left[\theta^2(2m\sigma^2-1) + \theta(2\bar{a} - 4m\sigma^2\tilde{a}_2) + (2m\sigma^2\tilde{a}_2^2 - \bar{a}^2)\right]\\
&= \frac{(2m\sigma^2-1)}{2\sigma^2}\left[\theta^2 + 2\theta\left(\frac{\bar{a}-2m\sigma^2\tilde{a}_2}{2m\sigma^2-1}\right) + \left(\frac{2m\sigma^2\tilde{a}_2^2-\bar{a}^2}{2m\sigma^2-1}\right)\right]\\
&= \frac{(2m\sigma^2-1)}{2\sigma^2}\left[\left(\theta - \frac{2m\sigma^2\tilde{a}_2-\bar{a}}{2m\sigma^2-1}\right)^2 - \left(\frac{2m\sigma^2\tilde{a}_2-\bar{a}}{2m\sigma^2-1}\right)^2 + \left(\frac{2m\sigma^2\tilde{a}_2^2-\bar{a}^2}{2m\sigma^2-1}\right)\right].
\end{aligned}$$

Note that $k_3^F = 2m\sigma^2$, and therefore, $k_1^F = \left(\frac{2m\sigma^2\tilde{a}_2-\bar{a}}{2m\sigma^2-1}\right)^2$, $k_2^F = \left(\frac{2m\sigma^2\tilde{a}_2^2-\bar{a}^2}{2m\sigma^2-1}\right)$, where k_1^F, k_2^F, and k_3^F are defined in the statement of result 2.1.1. Hence,

$$\begin{aligned}
J &= \frac{(2m\sigma^2-1)}{2\sigma^2}\left[(\theta - \sqrt{k_1^F})^2 - k_1^F + k_2^F\right]\\
&= \frac{1-2m\sigma^2}{2\sigma^2}\left[-(\theta - \sqrt{k_1^F})^2 + k_1^F - k_2^F\right]
\end{aligned}$$

Substituting back in the integrand, and taking the terms independent of θ outside,

$$\int_{-\infty}^{\infty}\frac{e^J}{\sigma\sqrt{2\pi}}d\theta = e^{\frac{(1-2m\sigma^2)(k_1^F-k_2^F)}{2\sigma^2}}\int_{-\infty}^{\infty}\frac{e^{-\frac{(\theta-\sqrt{k_1^F})^2(1-2m\sigma^2)}{2\sigma^2}}}{\sigma\sqrt{2\pi}}d\theta.$$

Let $\frac{(\theta-\sqrt{k_1^F})\sqrt{1-2m\sigma^2}}{\sigma} = v$. Then $\frac{d\theta}{\sigma} = \frac{dv}{\sqrt{1-2m\sigma^2}}$. Therefore,

$$\int_{-\infty}^{\infty}\frac{e^J}{\sigma\sqrt{2\pi}}d\theta = \frac{e^{\frac{(1-2m\sigma^2)(k_1^F-k_2^F)}{2\sigma^2}}}{\sqrt{1-2m\sigma^2}}\int_{-\infty}^{\infty}\frac{e^{-\frac{v^2}{2}}}{\sigma\sqrt{2\pi}}dv = \frac{e^{\frac{(1-2m\sigma^2)(k_1^F-k_2^F)}{2\sigma^2}}}{\sqrt{1-2m\sigma^2}}. \tag{51}$$

Substituting (51) in (50),

$$
\begin{aligned}
U_2^F &= \left(\frac{e^{-2\gamma I}}{1-2\gamma I\sigma_e^2}\right)\frac{exp\{\left(1-\frac{4\gamma I(\sigma_u^2+\sigma_2^2)}{1-2\gamma I\sigma_e^2}\right)\frac{(k_1^F-k_2^F)}{2\sigma^2}\}}{\left[1-\frac{4\gamma I(\sigma_u^2+\sigma_2^2)}{1-2\gamma I\sigma_e^2}\right]^{\frac{1}{2}}} \\
&= \frac{e^{-2\gamma I}e^{\frac{(k_1^F-k_2^F)}{2\sigma^2}(1-k_3^F)}}{(1-2\gamma I\sigma_e^2)^{\frac{1}{2}}(1-2\gamma I(\sigma_e^2+2\sigma_u^2+2\sigma_2^2))^{\frac{1}{2}}} \qquad (52)
\end{aligned}
$$

This ends the proof of result 2.1.1.

Proof of Result 2.1.2

Differentiating U_2^F derived in lemma 2.1 with respect to $\tilde{a}_2$ and setting the derivative to zero, we get $\tilde{a}_2 = E_1[a|y_{11}, y_{21}]$. Hence the proof.

The extension of results 2.1.1 and 2.1.2 to the multiperiod case is straightforward – we begin with the last period and work backwards. Since the distribution of income in any period is independent of the exact realizations of the prior period incomes, lemma 2.1 obtains.

Proof of Proposition 2

We prove the following two lemmas, which lead to proposition 2.

Lemma 2.2 *Given lemma 2.1, the optimal first period consumption choice and the maximum expected lifetime utility are*

$$c_1^F = 0.5z_1 - \frac{\ln U_2^F}{2\gamma} \quad \textit{and} \quad E_0[U^F] = -2\sqrt{k^F}E_0^F[e^{-0.5\gamma z_1}], \qquad (53)$$

where $k^F = \frac{e^{-2\gamma I}}{F_2^2}$, *with* F_2 *being defined in proposition 2.*

Lemma 2.3 *The optimal first period input choice, given lemma 2.2, is $\tilde{a}_1 = E_1[a] = a_0$ and the expected lifetime utility is given by proposition 2.*

Proof of Lemma 2.2

Lemma 2.1 shows that $E_1^F[e^{-\gamma z_2}|Y_1]$ is independent of Y_1. Similarly, it can be established that $E_1^D[e^{-\gamma z_2}|Y_1]$ is also independent of Y_1. Let k^F and k^D denote the expected second period utilities under optimal input choice. Thus,

$$k^F = \max_{\tilde{a}_2} E_1^F[e^{-\gamma z_2}|Y_1] = \frac{e^{-2\gamma I}}{(1-2\gamma I\sigma_e^2)^{\frac{1}{2}}(1-2\gamma I(\sigma_e^2+2\sigma_u^2+2\sigma_2^2))^{\frac{1}{2}}} \tag{54}$$

$$k^D = \max_{\tilde{a}_2} E_1^D[e^{-\gamma z_2}|Y_1] = \frac{e^{-2\gamma I}}{1-2\gamma I(\sigma_e^2+\sigma_u^2+\sigma_1^2)}. \tag{55}$$

The agent's lifetime utility under focus then becomes

$$E_0[U^F] = -E_0[e^{-\gamma c_1}] - E_0[k^F.e^{-\gamma(Y_1-c_1)}] \tag{56}$$

Choosing c_1 to maximize this expression leads to the following solution:

$$c_1^F = \frac{z_1}{2} - \frac{lnk^F}{2\gamma} \quad \text{and} \quad E_0[U^F] = -2E_0^F[e^{-\gamma c_1^F}] = -2\sqrt{k^F}E_0^F[e^{-\gamma\frac{z_1}{2}}].$$

Similarly, the solution for the case of diversification is

$$c_1^D = \frac{z_1}{2} - \frac{lnk^D}{2\gamma} \quad \text{and} \quad E_0[U^D] = -2E_0^D[e^{-\gamma c_2^F}] = -2\sqrt{k^D}E_0^D[e^{-\gamma\frac{z_1}{2}}].$$

Hence the proof.

Proof of Lemma 2.3

Following a similar argument to the one used in lemma 2.1, we can show that lemma 2.2 implies that the optimal first period input choice is $a_1 = E_0[a]$. Hence the proof.

Proof of Proposition 3

Since $E_0[U^F]$ and $E_0[U^D]$ are negative, $E_0[U^F] > E_0[U^D]$ if $F_1F2 > D_1D_2$, or equivalently if $\left(\frac{F_1F_2}{D_1D_2}\right)^4 > 1$. This occurs if $\left(\frac{F_1}{D_1}\right)^4 > \frac{1}{k}$ and $\left(\frac{F_2}{D_2}\right)^4 > k$ for some $k > 0$.

Consider $\left(\frac{F_2}{D_2}\right)^4$ first. Denote γI by g. Let $\sigma_e^2 = x\sigma_0^2$ and $\sigma_u^2 = y\sigma_0^2$. From (6), we have

$$\begin{aligned}
\sigma_1^2 &= \frac{1}{\tau_0 + \tau_u + e} = \frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_u^2+\sigma_e^2}} = \frac{\sigma_0^2(x+y)}{(1+x+y)} \quad \text{and} \\
\sigma_2^2 &= \frac{1}{\tau_0 + 2\tau_u + e} = \frac{\sigma_0^2(x+y)}{2+x+y}.
\end{aligned}$$

Hence, $\left(\frac{F_2}{D_2}\right)^4 = \frac{(1-2gx\sigma_0^2)(1-2g\sigma_0^2(x+2y+\frac{2(x+y)}{2+x+y}))}{\left[1-2g\sigma_0^2(x+y+\frac{x+y}{1+x+y})\right]^2} = \frac{1+4g^2\sigma_0^4x(x+2y+\frac{2(x+y)}{2+x+y})-2g\sigma_0^2(2x+2y+\frac{2(x+y)}{2+x+y})}{\left[1-2g\sigma_0^2(x+y+\frac{x+y}{1+x+y})\right]^2}$.

Let idiosyncratic risk be vanishingly small - then $x \to 0$. Also, let $g\sigma_0^2 = 1$. Then, $\left(\frac{N_2}{D_2}\right)^4 - k$ simplifies to $\frac{-[4ky^5+(4+20k)y^4+(19+29k)y^3+(24+4k)y^2+(7-11k)y+2k-2]}{(2+y)(1-2y^2-3y)^2}$. It is easy to see that if y is small and k close to 1, the above expression is positive. For example, if $k = 1+\varepsilon$ and $y = \varepsilon$,

$\left(\frac{N_2}{D_2}\right)^4 - k = \frac{2\varepsilon-17\varepsilon^2-52\varepsilon^3-53\varepsilon^4-24\varepsilon^5-4\varepsilon^6}{(2+\varepsilon)(1-3\varepsilon-2\varepsilon^2)^2} > 0$ for ε sufficiently small.

Next, for $k = 1+\varepsilon, y = \varepsilon$, $\left(\frac{N_1}{D_1}\right)^4 - \frac{1}{k} = \frac{1+5\varepsilon+8\varepsilon^2+4\varepsilon^3-\varepsilon^4}{\varepsilon^4(1+\varepsilon)} > 0$ for small ε.

Thus, for $k = 1+\varepsilon, y = \varepsilon$, $\left(\frac{N_1}{D_1}\right)^4 > \frac{1}{k}$ and $\left(\frac{N_2}{D_2}\right)^4 > k \Rightarrow \frac{N_1N_2}{D_1D_2} > 1$. Further, by continuity of both $\left(\frac{N_1}{D_1}\right)$ and $\left(\frac{N_2}{D_2}\right)$ with respect to x, $\left(\frac{N_1 N_2}{D_1 D_2}\right) > 1$ even for some $x > 0$.

Hence, there exists a set of parameters γ,I,σ_0^2, σ_u^2, and σ_e^2 such that focus is better than diversification.

Proof of Proposition 4

We prove that there exists a set of parameters such that $\frac{d(U^F/U^D)}{d\sigma_u^2}$ is negative. Since U^F and U^D are negative, this implies that for any element of this set, as σ_u^2 increases, $U^F - U^D$ increases as type-specific risk increases.

$\left(\frac{U^F}{U^D}\right) = \frac{D_1 D_2}{F_1 F_2} \Rightarrow \frac{d(U^F/U^D)}{d\sigma_u^2} < 0$ if $\frac{d(F_1 F_2/D_1 D_2)}{d\sigma_u^2} > 0$ which is also if $\frac{d\left(\frac{F_1}{D_1}\right)^4 \cdot \left(\frac{F_2}{D_2}\right)^4}{d\sigma_u^2} > 0$.

Again, let $x = 0; g\sigma_0^2 = z$. Then, $\left(\frac{F_1}{D_1}\right)^4 \cdot \left(\frac{F_2}{D_2}\right)^4 = \frac{(1-2\varepsilon^2 y-2\varepsilon^2)^2(2+y-12\varepsilon^2 y-4\varepsilon^2 y^2)(1+y^2)}{(2+y)(1-\varepsilon^2-\varepsilon^2 y)^4(1+y-4\varepsilon^2 y-2\varepsilon^2 y^2)^2}$.

$\left(\frac{F_1}{D_1}\right)^4 \cdot \left(\frac{F_2}{D_2}\right)^4 = \frac{[1-2z(y+1)]^2[2+y-12zy-4zy^2](1+y)^2}{(2+y)(1-z(1+y))^4(1+y-4zy-2zy^2)^2}$

Differentiating this with respect to y and setting y to be vanishingly small, we get:

$$\frac{d(.)}{dy} = \frac{2(1-2z)z(2z^2-5z+1)}{(1-z)^5}$$

Clearly, we need $1-2z > 0$ for the utilities to be real. Hence $z < 0.5$. Thus, $\frac{d(.)}{dy} > 0$ if $\exists y < 0.5$ such that $2z^2 - 5z + 1 > 0$. The roots of $2z^2 - 5z + 1 = 0$ are $\frac{5 \pm \sqrt{17}}{4}$. Since $\frac{5-\sqrt{17}}{4} \approx 0.2192$, $\frac{d(.)}{dy} > 0$ if $0 < z < \frac{5-\sqrt{17}}{4}$. Again, by continuity of U^F and U^D with respect to x, y, g, and σ_0^2, this will also hold in the neighbourhood of the parameter values chosen here.

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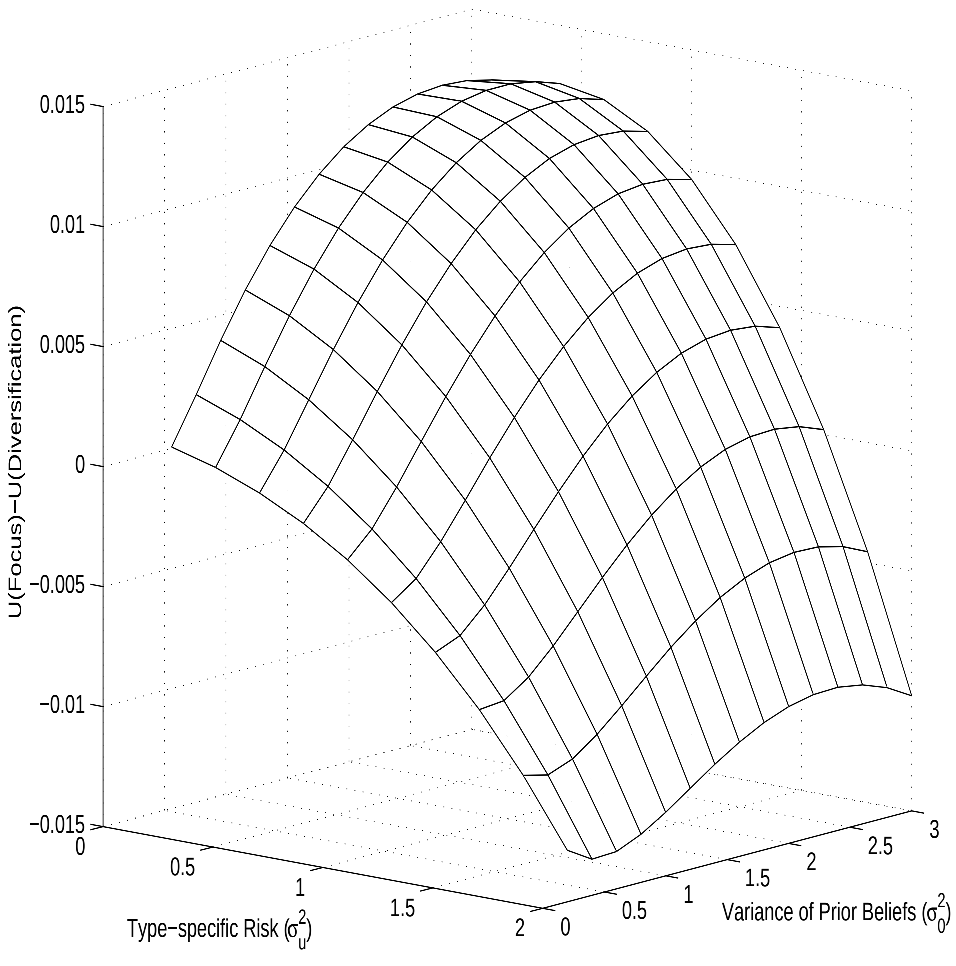

Figure 1: Impact of Prior Variance and Type-specific Risk

The figure shows the impact of variance of prior beliefs and type-specific risk on the difference in discounted lifetime utility between the *focus* and *diversification* strategies.

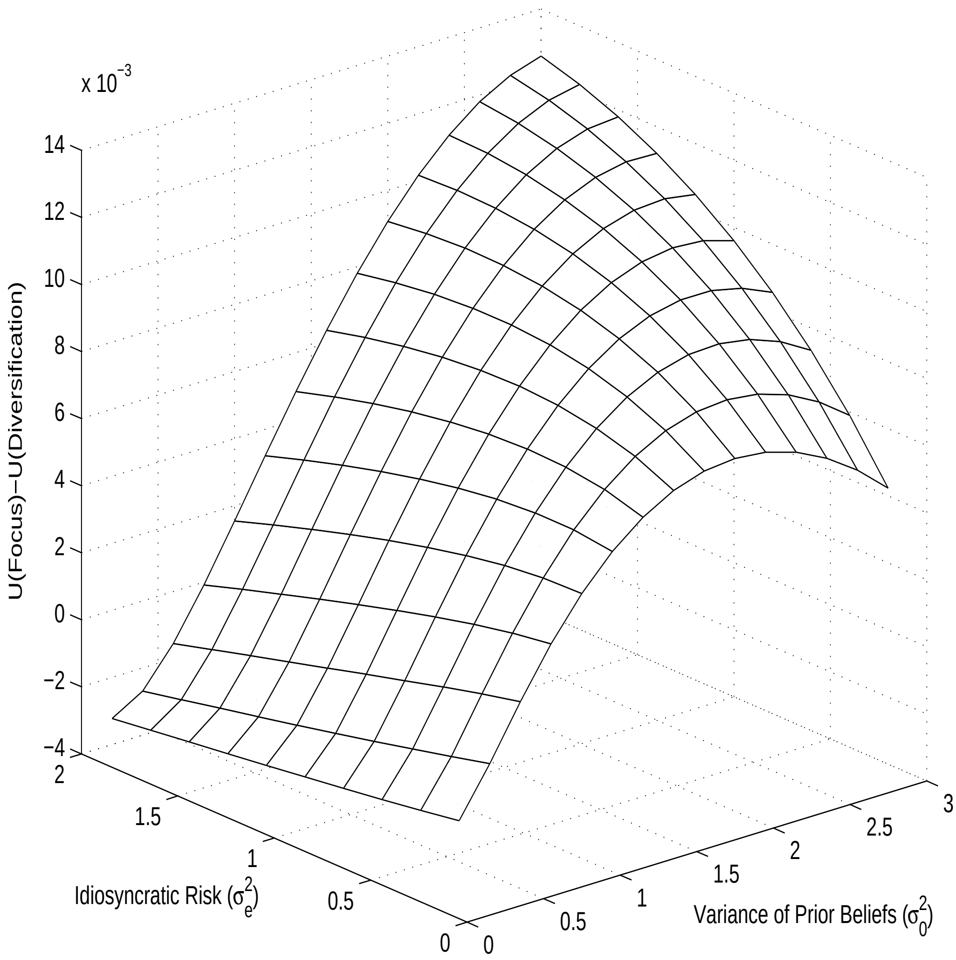

Figure 2: Impact of Prior Variance and Idiosyncratic Risk

The figure shows the impact of variance of prior beliefs and idiosyncratic risk on the difference in discounted lifetime utility between the *focus* and *diversification* strategies.

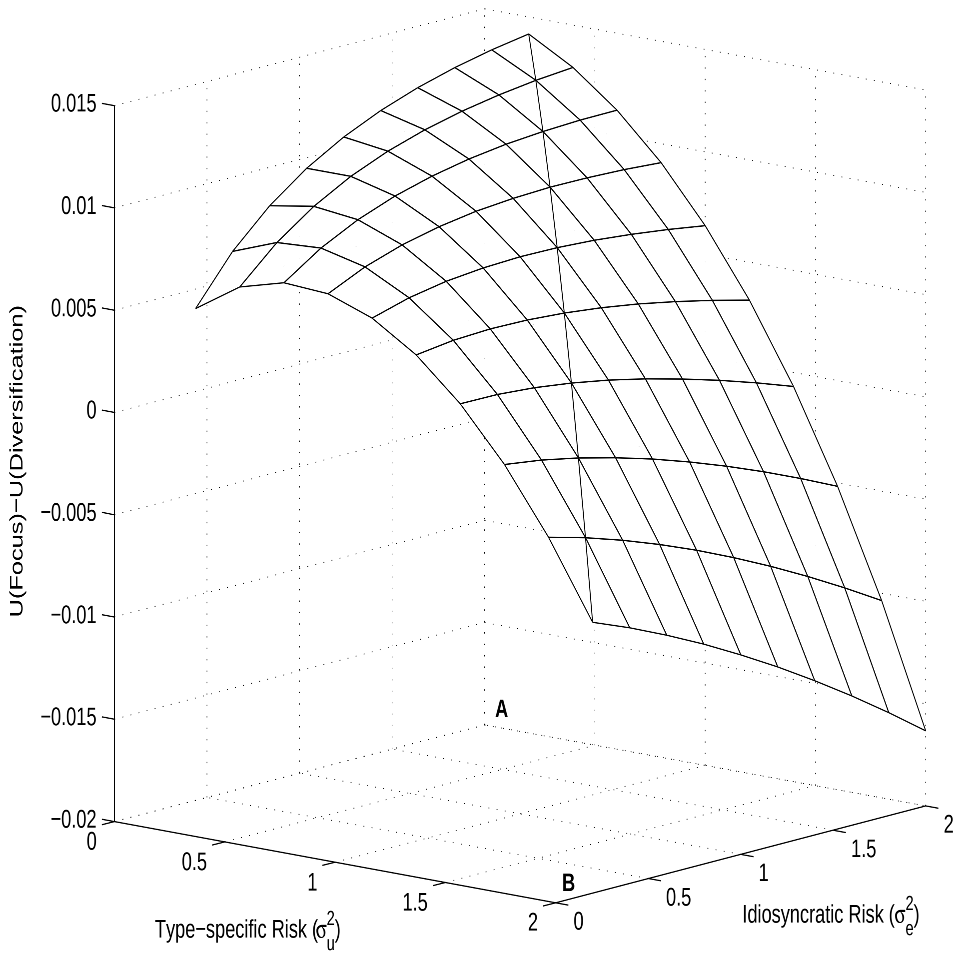

Figure 3: Impact of Type-specific and Idiosyncratic Risk

The figure shows the impact of idiosyncratic and type-specific risk on the difference in discounted lifetime utility between the *focus* and *diversification* strategies.

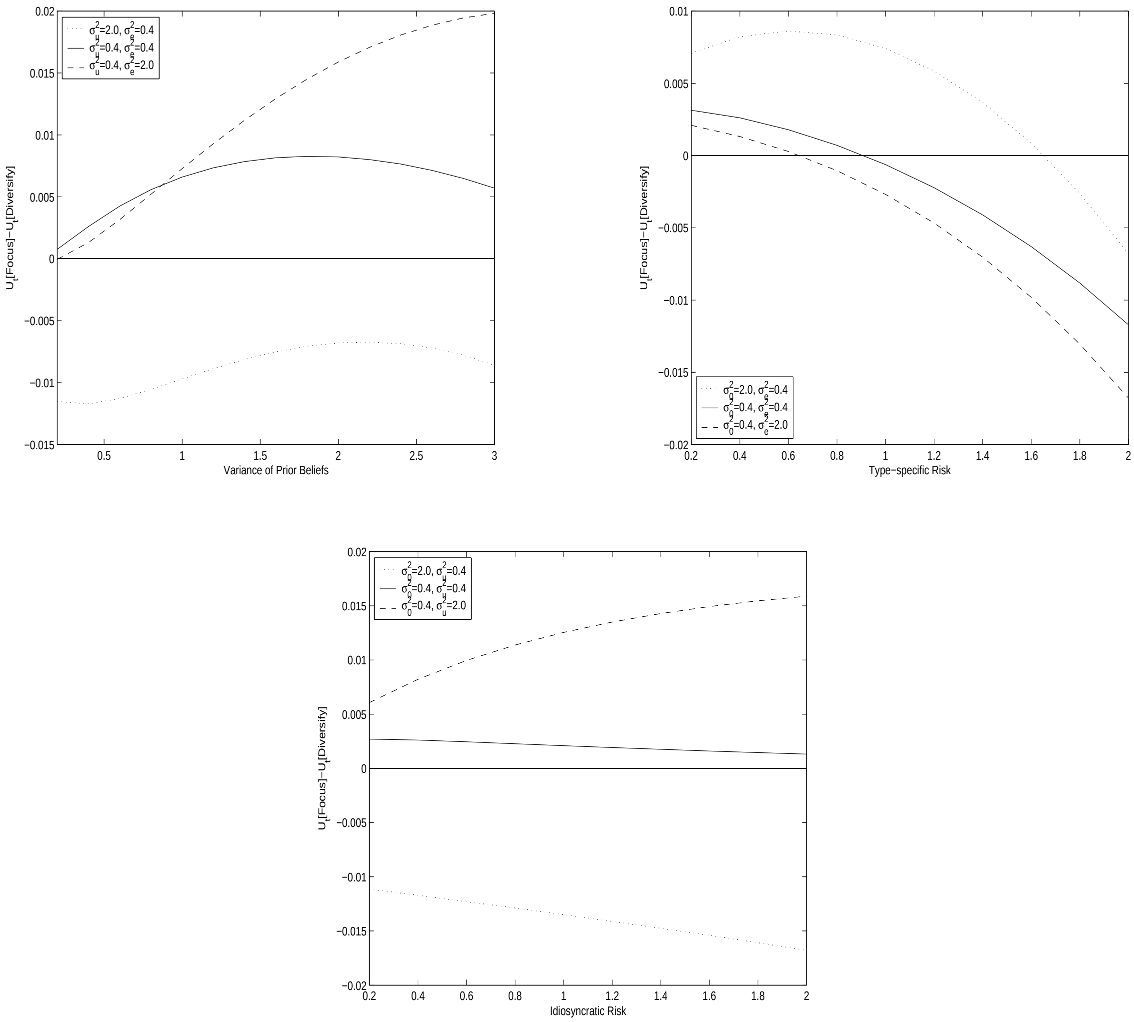

Figure 4: Effect of Learning and Risk on the Focus-Diversification Choice

The y-axis in all panels is the difference in discounted lifetime utility between *focus* and *diversification* for an agent who learns the technology of a target-input model over time. This is graphed against the learning and risk parameters. The panel on the top left shows the impact of variance of prior beliefs, the top-right panel the impact of type-specific risk and, the lower panel, the impact of idiosyncratic risk.

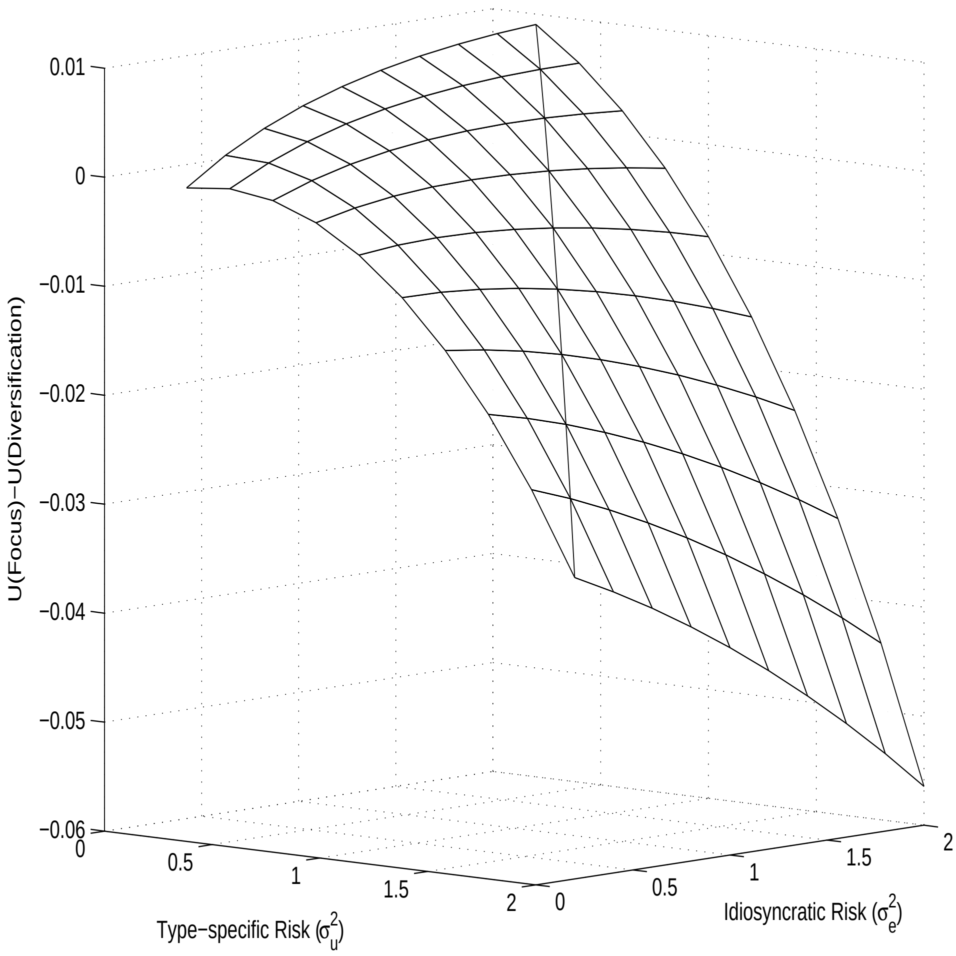

Figure 5: Impact of Type-specific and Idiosyncratic Risk with no borrowing or lending

The figure shows the impact of idiosyncratic and type-specific risk on the difference in discounted lifetime utility between the *focus* and *diversification* strategies. Borrowing or lending is prohibited in this model, which implies that consumption equals income in each period.