Defining rules in cost spanning tree problems through the canonical form*

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Abstract

We define the canonical form of a cost spanning tree problem. The canonical form has the property that reducing the cost of any arc, the minimal cost of connecting agents to the source is also reduced. We argue that the canonical form is a relevant concept in this kind of problems and study a rule using it. This rule satisfies much more interesting properties than other rules in the literature. Furthermore we provide two characterizations. Finally, we present several approaches to this rule without using the canonical form.

1 Introduction

Many problems involving network formation have been studied in the operations research and the economic literature. In operations research two issues have been extensively explored: the design of efficient algorithms and the computational complexity. The economic literature focuses on aspects such

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like the cost sharing of the network and the design of mechanisms trying to explain the way in which the network forms.

In this paper we focus in the cost sharing aspect. Our contribution can be considered in the well-known literature of cost allocation. We assume that there are no external forces (for example, the market) which determine the final allocation. Agents can achieve agreements directly among themselves, or indirectly by letting the final decision to a neutral referee. In both cases the important issue is to find a "fair allocation" of the cost.

In particular we study cost spanning tree problems (*cstp*). Consider that a group of agents, located at different geographical places, want some particular service which can only be provided by a common supplier, called the source. Agents will be served through connections which entail some cost. However, they do not care whether they are connected directly or indirectly to the source.

There are many economic situations that can be modeled in this way. For instance, several towns may draw power from a common power plant, and hence have to share cost of the distribution network. This example appears in Dutta and Kar (2002). Bergantiños and Lorenzo (2003) study a real situation where villagers should pay the cost of constructing pipes from their respective houses to a water supplier.

The literature on *cstp* starts by defining algorithms for constructing minimal cost spanning trees (*mcst*). We can mention, for instance, the papers of Kruskal (1956) and Prim (1957). But constructing an *mcst* is only a part of the problem. Other important issue is how to allocate the cost associated to the *mcst* among agents.

Bird (1976) studies it using game theory. Bird associates to any *cstp* a cooperative game and proposes a cost allocation rule called the Bird's rule (we call it B). This paper has generated more literature. For instance, Grannot and Huberman (1981, 1984) study the core and the nucleolus of the game and Kar (2002) the Shapley value (we call it K). Recently, Dutta and Kar (2002) propose a new rule, which we call DK.

Feltkamp, Tijs, and Muto (1994) introduce a rule for *cstp* called Equal Remaining Obligations. Later, Branzei, Moretti, Norde, and Tijs (2003) call it the P-value, which is the name we use, and give an axiomatic characterization of it.

We associate to each *cstp* a canonical form, which is a *cstp* with the property that by reducing the cost of any arc, the total cost of connecting agents to the source is also reduced. We argue that the canonical form is a relevant concept for this class of problems and, that it, can be used for defining nice rules.

We prove that in canonical problems B and K coincide. Then, for each *cstp* we define a rule φ as K applied to the canonical form associated.

Surprisingly, φ coincides with P. When we start to study this problem we did not know the unpublished paper of Feltkamp *et al* (1994). In January 2004 we come across the paper of Branzei *et al* (2003), which was available in the web in December 2003. Through this paper we knew the first one. Then, the originality of our paper is not in the rule we study; it is in the way we introduce it, which has no relation with the way in which is defined P, and the results obtained.

This rule satisfies much more interesting properties that B, K, and DK. We also give two characterizations of φ, which are completely unrelated with the characterization given in Branzei *et al* (2003).

We also present four approachs to this rule without using the canonical form. In the first approach, we prove that P coincides with φ.

The other three approachs are defined in this paper. In the second approach we prove that φ is the Shapley value of a game defined in a similar way to Bird (1976) . In the third approach, we give a definition of φ considering only the original problem. In the fourth approach, we obtain that φ can be obtained considering the costs as indivisible goods.

Moreover, Bergantiños and Vidal-Puga (2004) prove that φ can be obtained as the equilibrium payoff of a non-cooperative mechanism.

The paper is organized as follows. In Section 2 we give an overview of the paper. In Section 3 we introduce the *cstp*, along with the rules and properties considered in the paper. In Section 4 we introduce the canonical form and the rule φ. In Section 5 we study the properties satisfied by φ and provide two axiomatic characterizations. The other approaches to φ are presented in Section 6. All the proofs are in the Appendix.

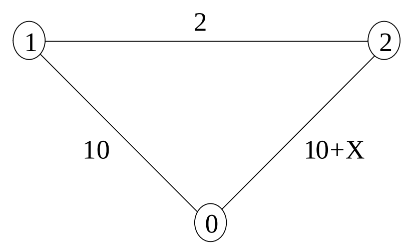
2 An overview of the paper

In this section we present, in an informal way, our most relevant results.

We first discuss, in a very simple example, the allocation proposed by B, K, and DK.

The first non-trivial case in *cstp* occurs when two agents want to be connected to the source and the optimal choice is that one of the agents connect through the other. The next example is a particular case of this situation.

Example 1. There are two agents. The connection cost between agent 1 and the source is 10, between agent 1 and 2 is 2, and between agent 2 and the source is $10 + x$, where $x \geq 0$. This situation can be represented as follows:



We now discuss how these rules work in this example for each $x \geq 0$. If $x = 0$, the problem is symmetric. The three rules coincide with the symmetric allocation $(6, 6)$.

If $x > 0$, the problem is not symmetric. The unique *mcst* is $\{(0, 1), (1, 2)\}$. We can proceed in two different ways. First, we ignore x because the arc $(0, 2)$ will not be constructed. Second, since the problem is asymmetric we use the information given by x. Next table shows the proposal given by each of the three rules.

	Agent 1	Agent 2
B	10	2
DK	2	10
K	$6 - \frac{x}{2}$	$6 + \frac{x}{2}$

The three rules use the information given by x. B and DK propose the same allocation indendently of x. Moreover, small changes in the cost of arc $(0, 2)$ produce large changes in the proposal. We think that this is unfair. What we are saying is that these rules are not continuous functions on C.

K is a continuous function on C. Take $x = 100$, then $K = (-44, 56)$. This means that agent 2 pays 44 units to agent 1 plus the cost of the network. Again we believe that this allocation is unfair. In this case we claim that the rule must be positive, *i.e.* agents cannot get beneficts.

Our conclusion, in Example 1, is that it is better not to take into account the information given by x and always propose $(6, 6)$, even the problem is not symmetric. This example suggests that in some cases it could be better to ignore part of the cost of some arcs if we want to obtain fair rules. In this paper we generalize this idea to the class of all *cstp*.

A *cstp* can be characterized through a matrix C where c_{ij} represents the connection cost between agents i and j. We say that a matrix is canonical if reducing the cost of any arc, the cost of connecting all agents to the source is also reduced. Notice that in Example 1, if $x = 0$ the matrix is canonical but if $x > 0$ the matrix is not canonical.

We define an algorithm for associating to each arbitrary matrix C a unique canonical matrix C^* (Proposition 2). Then, we define the canonical form of a problem as the problem obtained when we change the matrix C by its canonical matrix C^*. In Example 1 the matrix of the problem $x = 0$ is the canonical matrix of each problem with $x > 0$.

Canonical matrices are "nice" mathematical objects as we prove in propositions 1 and 3. Most of the proofs of the paper are based in the results obtained in these propositions.

In Example 1 the three rules coincide in the canonical form ($x = 0$). Even though this result is not true in general; we prove that B and K coincide in canonical matrices and DK is different. Then, we define a rule φ for a general matrix C as the rule K (or B) applied to its canonical matrix C^*.

Our next step is to explain why φ is a suitable rule for these problems. We first study which properties are satisfied by φ and compare it with the other three rules.

In Example 1 we criticise the proposals of B, DK, and K in terms of the properties of continuity (CON) and positivity (POS). Nevertheless, these rules satisfy other interesting properties. B satisfies core selection (CS). However B fails to satisfy cost monotonicity (CM), which says that if the connection cost between agents i and j increases, none of these agents should pay less. K satisfies CM but fails to satisfy CS and DK satisfies both.

We also consider other properties as strong cost monotonicity (SCM), population monotonicity (PM), equal sharing of extra cost ($ESEC$), separability (SEP), equal contributions (EC), and independence of irrelevant trees (IIT).

SCM means that if the connection cost between agents i and j increases, no agent should pay less. PM says that if we add new agents to the problem no agent is worse.

Consider a problem where, for any agent, his most expensive cost is to the source. Moreover, the connection cost to the source is the same for each agent. Assume that this connection cost increases x. $ESEC$ says that if agent i pays f_i in the original problem and n is the number of agents, he must pay $f_i + \frac{x}{n}$ when the cost increases.

Two subsets of agents, S and $N \setminus S$, can connect to the source separately or can connect jointly. If there are no savings when they connect jointly, SEP says that agents must pay the same in both circunstances.

EC says that the impact of the connection of agent j on agent's i cost coincides with the impact of the connection of agent i on agent's j cost. IIT says that if two *cstp* have a common *mcst*, both problems must have the same solution.

The next table summarize the properties satisfied by the four rules.

	B	K	DK	φ
CON	no	YES	no	YES
POS	YES	no	YES	YES
CS	YES	no	YES	YES
CM	no	YES	YES	YES
SCM	no	no	no	YES
PM	no	no	no	YES
$ESEC$	YES	YES	no	YES
SEP	YES	no	YES	YES
EC	no	no	no	YES
IIT	no	no	no	YES

These results suggest two comments. Firstly, φ satisfies much more interesting properties than the other three rules.

Secondly, φ and K coincide in canonical matrices. In a general problem C, φ and K are different. φ does not use the information of C which is not in its canonical form and then $\varphi(C) = K(C^*)$. K uses this information and thus $K(C)$ can be different to $K(C^*)$. If we consider the properties satisfied by both rules, it seems better not to use this information. A similar comment can be applied to φ and B.

Even though we are considering two particular rules, the above suggests that we may define rules in the general problem through its canonical form. A natural question is whether this is a good strategy.

In Corollary 2 (a) we characterize the rules defined through its canonical form as the only rules satisfying IIT. In Corollary 2 (b) we prove that if a rule is not defined through the canonical form then it does not satisfy SCM. This means that if we do not use the canonical form we miss at least an important property. Is it possible to gain anything? If we focus on B, DK, K, and φ and the properties mentioned here the answer is no. In general, we do not know. Although we believe that this is an interesting question, it is out of the scope of the paper. Here, we only want to point out that the canonical form is an important concept, which should be studied carefully because it can be very influencial in the problem.

Later, we present two characterizations of φ. The first one is with EC and the second one is with SCM, PM, and $ESEC$.

Let us comment the second characterization. SCM and PM are quite standard properties, which are used often in many economic problems. We believe that these properties are very natural and any fair rule should satisfy both. $ESEC$ is a property defined explicetely for *cstp*. We believe that is a natural property with a clear meaning but we do not feel that every fair rule should satify it. Thus, we can see $ESEC$ as a property selecting a rule among the set of "fair rules" (rules satisfying SCM and PM).

Moreover, the definition of these three properties is unrelated to canonical form. We believe that this is other argument supporting the canonical form as a good way for defining rules.

Finally, we argue that φ is a "focal point" of this class of problems because it can be achieved from different approaches. We mention four. In three of these approaches we do not use canonical matrices. In the fourth we use it but in a different way.

The first approach is through the $P - value$. As we said before the $P - value$ was introduced in Feltkamp *et al* (1994) in a very different way. In this paper we prove that φ coincides with P.

Second approach. Bird (1976) associated to each problem C a characteristic function v where $v(S)$ denotes the cost of connecting agents in S to the source assuming that agents of $N \setminus S$ are not present. Since the Shapley value is a suitable concept for TU games we can use it in *cstp*. Kar (2002) gives additional arguments supporting the Shapley value of v. He characterizes it considering only the problem C.

Following Bird's approach we associate to each problem C a characteristic function v^+ where $v^+(S)$ denotes the cost of connecting agents in S to the source *assuming that agents of* $N \setminus S$ *are already connected*. Surprisingly, the Shapley value of v^{+} coincides with φ.

The third approach is looking only to the problem C. We explain it with an example. Assume that $\{(0,1),(1,2),(2,3)\}$ is the unique $mcst$. Moreover, the cost of its arcs are 12, 4, and 6, respectively. We address the problem of how to divide the cost of each arc between the agents. We define a rule through an algorithm in which, at each step, we select for each agent both an arc and the part of the cost of this arc he must pay for.

Step 1. Every agent selects the cheapest arc to which he belongs. Thus agents 1 and 2 select $(1,2)$ and agent 3 selects $(2,3)$. Since agents 1 and 2 select arc $(1,2)$ this cost is equally divided between them. Agent 3 pays half of the cost of arc $(2,3)$.

Step 2. Agents 1 and 2 select the cheapest non-paid arc to which one of them belong. Thus, both select $(2,3)$. Agent 3 selects the cheapest non-paid arc to which he belongs, *i.e.* $(2,3)$. Then, the non-paid cost of arc $(2,3)$ is equally divided among the three agents.

Step 3. The three agents select arc $(0,1)$, the only whose cost was not paid. This cost is divided equally among agents.

Then, agents 1 and 2 pay $\frac{3}{6}4+\frac{1}{6}6+\frac{2}{6}12=7$ and agent 3 pays $\frac{3}{6}6+\frac{1}{6}6+\frac{2}{6}12=8$.

Fourth approach. In B and DK we first select a tree minimizing the cost of connecting agents to the network. Then, we assign the whole cost of each arc to one agent following some specific protocol. Thus, we can consider B and DK as rules assigning indivisible goods (cost of the arcs).

In the case of B we can interpret it assuming that agents sequentially connect to one of the cheapest available options. Each agent pays his connection cost. This protocol can provoque unfair allocations as we can see in Example 1 when x is very small.

A classical way for recovering fairness from an order-depending allocation is to take the average over the set of all permutations. In general problems, this approach is incompatible with efficiency. Nevertheless, if C is a general problem and C^{*} is its canonical matrix, it is possible to generate an efficient and fair allocation for C averaging over the permutations in C^{*}. This is the fourth approach.

3 The cost spanning tree problem

This section is divided in three subsections. In the first subsection, we introduce the problem. In the second subsection, we introduce some rules of the literature. Finally, in the third subsection, we present some properties of the rules.

3.1 The problem

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. We are interested in networks whose nodes are elements of a set $N_0 = N \cup 0$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1, ..., n\}$. Our interest lies on networks where each node in N is (directly or indirectly) connected to the source.

Let Π_N be the set of all permutations over the finite set N. Given $\pi \in \Pi_N$, let $Pre(i, \pi)$ denote the set of elements of N which come before i in the order given by π, *i. e.* $Pre(i, \pi) = \{j \in N \mid \pi(j) < \pi(i)\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on N represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we will work with undirected arcs, *i.e* $(i, j) = (j, i)$.

We denote by $\mathcal{C}^N$ the set of all cost matrices on N. Given $C, C' \in \mathcal{C}^N$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *cost spanning tree problem*, briefly *cstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is the set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

A *network* g over N_0 is a subset of $\{(i, j) \text{ such that } i, j \in N_0\}$. The elements of g are called *arcs*.

Given a network g and a pair of nodes i and j, a *path* form i to j in g is a sequence of arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$ and $j = i_l$. We say that $i, j \in N$ are *linked* in g if there exists a path from i to j which does not include the source. If $(i, j) \in g$, we say that i and j are *directly linked* in g. We say that the node i is *connected* in the network g if there exists a path from i to the source. Otherwise we say that i is *unconnected* in g.

Given a network g and $S \subset N$ we denote by g_S the network induced by g among agents in S, *i.e.*

$$g_S = \{(i, j) \in g \text{ such that } \{i, j\} \subset S\}.$$

A *tree* is a network satisfying that for all $i \in N$ there is a unique path from i to the source. If t is a tree we usually writte $t = \{(j, j^0)\}_{j \in N}$ where j^0 represents the first agent in the unique path in t from j to 0.

We denote by $\mathcal{G}^N$ the set of all networks over N_0 and by $\mathcal{G}_0^N$ the set of networks such that every agent in N is connected to the source.

Given a *cstp* (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* associated to g as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there are no ambiguities, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum cost spanning tree* for (N_0, C), briefly a *mcst*, is a tree $t \in \mathcal{G}_0^N$ such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. It is well-known in the literature about *cstp* that there exists a *mcst*, even though it does not need to be unique. Given a *cstp* (N_0, C) we denote by $m(N_0, C)$ the cost associated to any *mcst* t in (N_0, C).

Given a *cstp* (N_0, C), we define the *cstp* induced by C in $S \subset N$ as (S_0, C).

Bird (1976) associated to each *cstp* (N_0, C) a *TU* game (N, v_C) where for each coalition $S \subset N$,

$$v_C(S) = m(S_0, C).$$

Usually, we write v instead of v_C.

We now introduce some well-known results of *TU* games, which will be used through the paper. We introduce them considerating the *TU* game as a cost game.

We define the *core* of the *TU* game (N, w) as

$$core(N, w) = \left\{ x \in R^N \mid \sum_{i \in N} x_i = w(N) \text{ and } \forall S \subset N, \ \sum_{i \in S} x_i \leq w(S) \right\}.$$

We say that (N, w) is *concave* if, for all $S, T \subset N$ and $i \in N$ such that $S \subset T$ and $i \notin T$,

$$w(S \cup i) - w(S) \geq w(T \cup i) - w(T).$$

We denote by $Sh(N, w)$ the *Shapley value* (Shapley, 1953) of the *TU* game (N, w). It is well-known that the Shapley value belongs to the core when the game is concave.

3.2 Rules

One of the most important issues addressed in the literature about *cstp* is how to divide the cost of connecting agents to the source among them. We now introduce, briefly, some of the rules studied in the literature.

A *(cost allocation) rule* is a function ψ such that $\psi(N_0, C) \in \mathbb{R}^N$ for each *cstp* (N_0, C) and $\sum_{i \in N} \psi_i(N_0, C) = m(N_0, C)$. As usually, $\psi_i(N_0, C)$ represents the cost assigned to agent i.

Notice that we implicitly assume that agents built a *mcst*. As far as we know, all the rules proposed in the literature make this assumption.

Given a *cstp*, Prim (1957) provides an algorithm for solving the problem of connecting all agents to the source such that the total cost of creating the network is minimal. The idea of this algorithm is quite simple: starting from the source we construct a network by consecutively adding arcs with the lowest cost, without introducing cycles.

Formally, Prim's algorithm is defined as follows. We start with $S^0 = \{0\}$ and $g^0 = \emptyset$.

Stage 1: Take the arc $(0, i)$ such that $c_{0i} = \min_{i \in N} \{c_{0i}\}$. If there are several arcs $(0, i)$ satisfying this condition, select one of them. Now, $S^1 = \{0, i\}$ and $g^1 = \{(0, i)\}$.

Assume that we have defined S^p and g^p. We now define S^{p+1} and g^{p+1}. Take an arc (j, i) such that $c_{ji} = \min_{k \in S^p, l \in N_o \setminus S^p} \{c_{kl}\}$. If there are several arcs (j, i) satisfying this condition, select one of them. Now, $S^{p+1} = S^p \cup \{i\}$ and $g^{p+1} = g^p \cup \{(j, i)\}$.

This process finishes in n stages. We say that g^n is a tree obtained via Prim's algorithm. Notice that this algorithm leads to a tree, but not neccesarily unique.

We now introduce four rules of the literature: Bird's rule, P's rule, Kar's rule and Dutta-Kar's rule.

Bird's rule (Bird, 1976) and Dutta-Kar's rule (Dutta and Kar, 2002) are defined through Prim's algorithm. We first assume that there is a unique tree t obtained through Prim's algorithm.

Let i^0 be the first node in the unique path in t from i to the source. Then, given a *cstp* (N_0, C), *Bird's rule* (B) is defined for each $i \in N$ as

$$B_i(N_0, C) = c_{ii^0}.$$

The idea of this rule is quite simple. Agents connect sequentially to the source following Prim's algorithm and each agent pays the cost of his connection.

Dutta-Kar's rule is defined in a more elaborate way. Assume that agents, according with Prim's algorithm, connect in the order $1, 2, ..., n$. First agent 1 connects to the source. We define $p^1 = c_{10}$. Now agent 2 connects to 2^* where $c_{22^*} = min\{c_{20}, c_{21}\}$. We take $x_1 = min\{p^1, c_{22^*}\}$ and $p^2 = max\{p^1, c_{22^*}\}$. Now agent 3 connects to 3^* where $c_{33^*} = min\{c_{30}, c_{31}, c_{32}\}$. We take $x_2 = min\{p^2, c_{33^*}\}$ and $p^3 = max\{p^2, c_{33^*}\}$. This process continue until agent n. In this case we take $x_n = max\{p^{n-1}, c_{nn^*}\}$. See Duta and Kar (2002) for a formal definition.

Dutta-Kar's rule (DK) is defined for each $i \in N$ as

$$DK_i(N_0, C) = x_i.$$

Assume now that the tree associated to Prim's algorithm is not unique. In this case Bird's rule and Dutta-Kar's rule can be defined as an average over the trees associated to Prim's algorithm.

Duta and Kar (2002) proceed as follows. Given $\pi \in \Pi_N$ they define $DK^\pi(N_0, C)$ as the allocation obtained when we apply the previous protocol to (N_0, C) and solving the indiferences selecting the first agent given by π. Then, they define

$$DK(N_0, C) = \frac{1}{n!} \sum_{\pi \in \Pi_N} DK^\pi(N_0, C).$$

$B(N_0, C)$ can be defined in a similar way.

Kruskal (1956) introduced an algorithm for computing the *mcst* of a *cstp*. Feltkamp, Tijs, and Muto (1994) define *$P's$ rule* (P) through Kruskal's algorithm. This rule was called Equal Remaining Obligations (ERO) in this paper. Recently, Branzei *et al* (2003) call it the $P-value$. We give the idea of this rule following Branzei *et al* (2003). Initially, every agent has an obligation 1 and the network is empty. We now apply Kruskal's algorithm and the obligations of the agents decrease when we add an arc to the network. This obligation is $\frac{1}{n_i}$, where n_i is the number of agents connected directly or indirectly to agent i through the network. At each step of the algorithm each agent pays the part of the cost of the arc we add given by the difference between his obligation before adding the arc and his obligation after adding the arc. See Branzei *et al* (2003) for a formal definition.

Another approach for defining rules is using game theory. Bird (1976) associated a cooperative game with any *cstp*. Later, several authors defined rules to the *cstp* using this cooperative game. For instance, Grannot and Huberman (1981, 1984) studied the core and the nucleolus, and Kar (2002) studied the Shapley value.

Kar's rule (K) is defined as

$$K(N_0, C) = Sh(N, v).$$

3.3 Properties

We now introduce some properties of rules.

We say that a rule ψ satisfies:

Core selection (CS) if for all *cstp* (N_0, C) and all $S \subset N$,

$$\sum_{i \in S} \psi_i(N_0, C) \leq m(S_0, C).$$

This property says that no group of agents can be better constructing their own network instead of paying what the rule ψ proposes to them. Notice that core selection is equivalent to say that $\psi(N_0, C) \in core(N, v_C)$.

Cost monotonicity (CM) if for all *cstp* (N_0, C) and (N_0, C') such that $c_{ij} < c'_{ij}$ for some $i, j \in N_0$ and $c'_{kl} = c_{kl}$ otherwise,

$$\psi_i(N_0, C) \leq \psi_i(N_0, C').$$

Cost monotonicity implies that if the connection cost between agents i and j increases and the rest of costs are the same, then agents i and j cannot be better.

Strong cost monotonicity (SCM) if for all *cstp* (N_0, C) and (N_0, C') such that $c_{ij} < c'_{ij}$ for some $i, j \in N_0$ and $c'_{kl} = c_{kl}$ otherwise,

$$\psi_k(N_0, C) \leq \psi_k(N_0, C') \text{ for all } k \in N.$$

Strong cost monotonicity implies that if the connection cost between agents i and j increases and the rest of the costs remain the same, then no agent can be better.

Population monotonicity (PM) if for all *cstp* (N_0, C), all $S \subset N$, and all $i \in S$,

$$\psi_i(N_0, C) \leq \psi_i(S_0, C).$$

Population monotonicity says that if new agents joint a society no agent of the initial society can be worse.

Continuity (CON) if ψ is a continuous function of C. This property says that small changes in the connection costs of agents cannot provoque large changes in the amount they have to pay.

Positivity (POS) if for all *cstp* (N_0, C) and all $i \in N$

$$\psi_i(N_0, C) \geq 0.$$

This property says that agents cannot get benefits.

Separability (SEP) if for all *cstp* (N_0, C) and $S \subset N$ satisfying $m(N_0, C) = m(S_0, C) + m((N \setminus S)_0, C)$,

$$\psi_i(N_0, C) = \begin{cases} \psi_i(S_0, C) & \text{if } i \in S \\ \psi_i((N \setminus S)_0, C) & \text{if } i \in N \setminus S. \end{cases}$$

Two subset of agents, S and $N \setminus S$, can connect to the source either separately or jointly. If there are no savings when they connect jointly, separability says that agents must pay the same in both circunstances.

Grannot and Huberman (1981) prove that the core and the nucleolus of a *cstp* have the property of tree decomposition. Even sepatability has a different formulation is clearly inspired by tree decomposition. Moreover, it is easy to check that separability implies tree independence.

Equal Sharing of Extra Cost (ESEC). Let (N_0, C) and (N_0, C') be such that $c_{i0} = c_0$ and $c'_{i0} = c'_0$ for all $i \in N$, $c_0 < c'_0$, and $c_{ij} = c'_{ij} \leq c_0$ for all $\{i, j\} \subset N$. Then,

$$\psi_i(N_0, C') = \psi_i(N_0, C) + \frac{c'_0 - c_0}{n}.$$

A group of agents face a problem (N_0, C) in which all of them have the same connection cost to the source $(c_{i0} = c_0)$ and this cost is larger than the connection costs between agents $(c_{ij} \leq c_0)$. Under these circumstances, in all *mcst* one agent (any of them) connects to the source directly, and the rest connect to the source through this agent. Moreover, they agree that the right solution is $\psi(N_0, C)$. Assume that there was a mistake and the connection cost to the source is $c_0' > c_0$.. $ESEC$ says that agents must pay equally this extra cost $c_0' - c_0$.

The next properties are are more technical but still very reasonable.

We say that two *cstp* (N_0, C) and (N_0, C') are *directly equivalent* if there exists a tree t such that t is a *mcst* for both (N_0, C) and (N_0, C') and it satisfies $c_{ij} = c_{ij}'$ for all $(i, j) \in t$.

We say that (N_0, C) and (N_0, C') are *equivalent* if there exists a succession of cost matrices $\{C^0, C^1, ..., C^p\}$ with $C^0 = C$, $C^p = C'$, and (N_0, C^{q-1}) directly equivalent to (N_0, C^q) for each $q = 1, 2, ..., p$.

Notice that if (N_0, C) and (N_0, C') are equivalent then $m(N_0, C) = m(N_0, C')$.

A rule ψ satisfies *Independence of Irrelevant Trees (IIT)* if $\psi(N_0, C) = \psi(N_0, C')$ whenever (N_0, C) and (N_0, C') are equivalent.

Remark 1. Dutta and Kar (2002) define the property of *tree invariance.* This property says that the rule must depends only on the set of *mcst*. They prove that DK satisfies tree invariance.

Notice that if a rule satisfies IIT it also satisfies tree invariance. However, tree invariance does not imply IIT. It is easy to check it in Example 1 taking $x = 0$ and $x = 100$.

Given a *cstp* (N_0, C) and $i \in N$ we denote by $((N \setminus \{i\})_0, C^{+i})$ the *cstp* obtained from (N_0, C) when agents of $N \setminus \{i\}$ have to be connected, assuming that agent i is already connected. This means that for all $j \in N \setminus \{i\}$, $c_{j0}^{+i} = \min\{c_{ij}, c_{j0}\}$ and $c_{jk}^{+i} = c_{jk}$ when $k \neq 0$.

We say that ψ satisfies *Equal Contributions (EC)* if for all $i, j \in N$, $i \neq j$,

$$\psi_i(N_0, C) - \psi_i\left((N \setminus \{j\})_0, C^{+j}\right) = \psi_j(N_0, C) - \psi_j\left((N \setminus \{i\})_0, C^{+i}\right).$$

EC says that the impact of the connection of agent j on agent's i cost coincides with the impact of the connection of agent i on agent's j cost.

There are some relations among these properties. It is trivial that SCM implies CM whereas the reciprocal is false.

PM implies SEP. Let ψ be a rule satisfying PM and $S \subset N$ as in the definition of SEP. By PM we know that $\psi_i(S_0, C) \geq \psi_i(N_0, C)$ for all $i \in S$ and $\psi_i((N \setminus S)_0, C) \geq \psi_i(N_0, C)$ for all $i \in N \setminus S$. Since $m(N_0, C) = m(S_0, C) + m((N \setminus S)_0, C)$ it is easy to conclude that ψ satisfies SEP. The reciprocal is false.

PM implies CS. For all $S \subset N$, $\sum_{i \in S} \psi_i(N_0, C) \leq \sum_{i \in S} \psi_i(S_0, C) = m(S_0, C)$. The reciprocal is false.

In Section 4 (Corollary 2) we prove that SCM implies IIT.

4 The canonical form of a cost spanning tree problem

In this section we associate to each *cstp* a unique canonical matrix, which has the property that, if we reduce the cost of any arc, then the cost of connecting agents to the source is also reduced. We define the canonical form of a *cstp* as the problem obtained by replacing the initial cost matrix by its canonical matrix associated.

We prove in propositions 1 and 3 that canonical matrices have nice mathematical properties, which will be often used in the paper.

Moreover, Bird's rule and Kar's rule coincide in canonical matrices. This allows us to define the rule φ for general *cstp* as Kar's rule (or Bird's rule) of the canonical form of the original problem.

We also study the rules that only depend on the canonical form and we prove that they coincide with the rules satisfying IIT. Finally, we obtain that if a rule does not only depend on the canonical form, it does not satisfy strong cost monotonicity. This allows to argue that defining rules through the canonical form is a good approach to *cstp*.

Given a *cstp* (N_0, C), we say that C is *canonical* if we cannot decrease the cost of an arc without decreasing $m(N_0, C)$.

Remark 2. If C is canonical, for any $i, j \in N_0$ there exists a *mcst* t of (N_0, C) such that $(i, j) \in t$.

In the next proposition we characterize the canonical matrices.

Proposition 1. C is a canonical matrix if and only if there exists a *mcst* t in (N_0, C) satisfying the two following conditions:

$(A1)$ $t = \{(i_{p-1}, i_p)\}_{p=1}^n$ where $i_0 = 0$ (the source).

$(A2)$ Given $i_p, i_q \in N_0$, $p < q$ then, $c_{i_p i_q} = \max\limits_{p<r\leq q} \left\{c_{i_{r-1} i_r}\right\}$.

Proof. See the appendix.

This means that canonical matrices are uniquely determined by a *mcst* t.

Given a *cstp* (N_0, C) we say that C^* is a *canonical matrix associated to* C if C^* is a canonical matrix, $C \geq C^*$, and $m(N_0, C) = m(N_0, C^*)$. Later we will prove that any *cstp* has a unique associated canonical matrix.

We now introduce an algorithm for associating to each arbitrary matrix $C \in \mathcal{C}^N$ a canonical matrix C^*. This algorithm has $n+1$ stages and it is inspired by Prim's algorithm.

Let $t^0 = \{(i, i^0)\}_{i \in N}$ be a *mcst* in (N_0, C). Remember that i^0 is the first agent in the unique path in t from i to 0. Initially, take $C^0 = C$, $S^0 = \{0\}$, and $g^0 = \varnothing$.

Stage 1. Define $T_1 = \{i \in N \mid i^0 = 0\}$. Since t^0 is a tree, $T_1 \neq \emptyset$. Take an arc $(i_1, 0)$ such that $i_1 \in T_1$ and $c_{i_1 0} = \min\limits_{i \in T_1} \{c_{i0}\}$. If there are several arcs satisfying this condition, take one of them. Now $C^1 = C^0$, $S^1 = \{0, i_1\}$, and $g^1 = \{(i_1, 0)\}$.

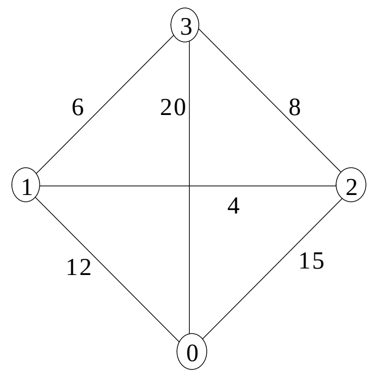
Assume that we have defined Stage r for all $r \leq p-1$. We now define Stage p when $p \leq n$.

Stage p. We define $T_p = \{i \in N \mid i \in N_0 \setminus S^{p-1} \text{ and } i^0 \in S^{p-1}\}$. Since t^0 is a tree, $T_p \neq \emptyset$. Take an arc $\left(i_p, i_p^0\right)$ such that $c_{i_p i_p^0} = \min\limits_{i \in T_p} \{c_{ii^0}\}$. If there are several arcs satisifying this condition, take one of them. We define C^p such that $c_{kj}^p = c_{kj}^{p-1}$ if $(k, j) \neq (i_p, i_{p-1})$ and $c_{i_p i_{p-1}}^p = c_{i_p i_p^0}$. Moreover, $S^p = S^{p-1} \cup \{i_p\}$ and $g^p = g^{p-1} \cup \{(i_p, i_{p-1})\}$.

Stage $n+1$. We define C^* such that $c_{i_p i_q}^* = \max\limits_{p<r\leq q} \left\{c_{i_{r-1} i_r}^n\right\}$ for all $i_p, i_q \in N_0$, $p < q$. Moreover $t = g^n$.

If we apply this algorithm to Example 1 we obtain that for all *cstp* where $x > 0$, the canonical matrix associated corresponds to the *cstp* where $x = 0$. We now apply this algorithm in a more complicate example.

Example 2. Consider the following *cstp* :

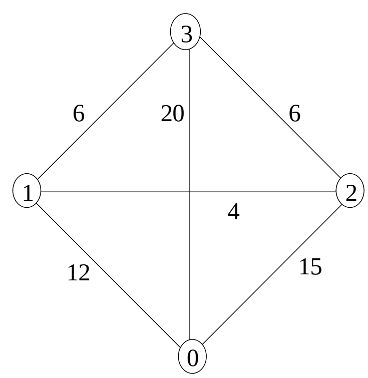


It is trivial to see that there is a unique $mcst$ $\{(0,1),(1,2),(1,3)\}$.

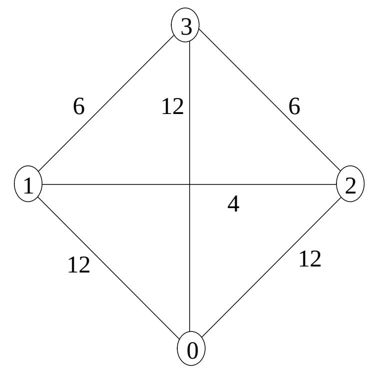
1. Stage 1. We select the arc $(0,1)$. Then, $g^1 = \{(0,1)\}$, $S^1 = \{0,1\}$, and $C^1 = C$.

2. Stage 2. We select the arc $(1,2)$. Then, $g^2 = \{(0,1),(1,2)\}$, $S^2 = \{0,1,2\}$, and $C^2 = C^1$.

3. Stage 3. We select the arc $(1,3)$. Then, $g^3 = \{(0,1),(1,2),(2,3)\}$, $S^3 = \{0,1,2,3\}$, and C^3 is given by



4. Stage 4. $t = \{(0,1),(1,2),(2,3)\}$. Moreover, C^* is given by



Proposition 2. C^* is a canonical matrix, $C \geq C^*$, C^* is equivalent to C, and C^* is the unique canonical matrix associated to C.

Proof: See the Appendix.

As a consequence of this proposition, for any *cstp* (N_0, C) we can define the canonical form (N_0, C^*) as the *cstp* where C^* is the unique canonical matrix associated to C.

If we compute the B, DK, and K in Example 2 we obtain

	Agent 1	Agent 2	Agent 3
Bird	12	4	6
Dutta-Kar	4	6	12
Kar	3.5	7.5	11

If we compute B, DK, and K in the canonical form associated to Example 2 we obtain

	Agent 1	Agent 2	Agent 3
Bird	7	7	8
Dutta-Kar	6	6	10
Kar	7	7	8

Notice B and K coincide in the canonical form associated to Example 1. Later we will prove that this is a general result.

In the next proposition we give a list of properties of canonical matrices. These properties will be used often in the rest of the paper. We will assume $t = \{(i-1, i)\}_{i=1}^{n}$ only to simplify the notation.

Proposition 3. Let (N_0, C) be a *cstp* where C is canonical and $t = \{(i-1, i)\}_{i=1}^{n}$ is a *mcst* in (N_0, C) satisfying $(A1)$ and $(A2)$. Take $S \subset N$. We can assume that $S = \left\{i_1, ..., i_{|S|}\right\}$, $i_0 = 0$, and $i_{p-1} \leq i_p$ for all $p = 1, ..., |S|$. Then,

(a) $t' = \{(i_{p-1}, i_p)\}_{p=1}^{|S|}$ is a *mcst* of (S_0, C) and $v(S) = \sum_{p=1}^{|S|} c_{i_{p-1}i_p}$.

(b) $v(S) - v(S \setminus \{i_q\}) = \min\left\{c_{i_{q-1}i_q}, c_{i_q i_{q+1}}\right\}$ if $i_q \neq i_{|S|}$ and $v(S) - v\left(S \setminus \left\{i_{|S|}\right\}\right) = c_{i_{|S|-1}i_{|S|}}$.

(c) (N, v) is concave.

Proof. See the Appendix.

Corollary 1. Let (N_0, C) be a *cstp* where C is canonical. Then, $K(N_0, C) = B(N_0, C)$.

Proof. See the Appendix.

This corollary allows the following definition.

Definition 1. Given a *cstp* (N_0, C) we define the rule φ as

$$\varphi(N_0, C) = K(N_0, C^*) = B(N_0, C^*)$$

where C^* is the canonical matrix associated to C.

As we said in the initial discussion of Example 1 we define the rule through the canonical form. Moreover, we define it as K and B because the fact that these rules, which are very different in general, coincide in canonical matrices could mean something. In the next section we will give more arguments justifying this approach.

But now we discuss an importan issue of the paper, which is defining rules through the canonical form. We first characterize the rules which depends only on the canonical form, *i.e.* $\psi(N_0, C) = \psi(N_0, C^*)$ for all *cstp* (N_0, C).

Corollary 2. (a) A rule ψ depends only on the canonical form if and only if ψ satisfies IIT.

(b) SCM implies IIT.

Proof. See the Appendix.

The reciprocal of part (b) is false because there exist rules satisfying IIT but not SCM. For instance,

$$\psi_i(N_0, C) = \frac{c_{i0}^*}{\sum_{j=1}^{n} c_{j0}^*} m(N_0, C) \text{ for all } i \in N$$

satisfies IIT but not SCM.

One of the main messages of the paper is that defining rules through the canonical form is a very good approach for obtaining nice rules in *cstp*. Corollary 2 (b) is a strong argument supporting this approach. If we consider a rule that does not depend on the canonical form this rule will not satisfy SCM. Something will be missing for sure.

An interesting question from our point of view is whether it is possible to define a rule which does not depend on the canonical form, it satisfies interesting properties, but no rule depending only on the canonical form satisfies these properties. In the next section we will see that this is not a trivial question because this is not the case of the rules studied in the literature.

5 Properties of the new rule

In this section we prove that the rule φ satisfies all the properties stated in the paper. If we compare it with the other rules we realize that φ satisfies much more interesting properties than the other rules.

This support the idea of defining rules through canonical matrices.

Finally, we obtain two axiomatic characterizations of φ. The second characterization says that φ is the only rule satisfying strong cost monotonicty, population monotonicity, and equal sharing of extra costs. We believe that this is a nice characterization because we use properties based in clear economic principles.

In the next theorem we prove that the new rule φ satisfies all the properties mentioned in the paper.

Theorem 1. φ satisfies CS, CM, SCM, PM, CON, POS, SEP, $ESEC$, IIT, and EC.

Proof. See the Appendix.

Remark 3. Sprumont (1990) introduced the Population Monotonic Allocation Schemes ($PMAS$) for TU games. Let (N_0, C) be a *cstp* and let C^* be its canonical matrix. By Proposition 3 (c), v_{C^*} is concave and hence it is possible to find a $PMAS$ for v_{C^*}. Since $C \geq C^*$, $v_C(S) \geq v_{C^*}(S)$ for all $S \subset N$ and hence, the $PMAS$ of v_{C^*} are also $PMAS$ in v_C. Since φ satisfies PM we conclude that $\{\varphi(S_0, C)\}_{S \subset N}$ is a $PMAS$ of v.

In the next table we summarize the properties satisfied by the rules mentioned in the paper (in the Appendix we prove that this table is correct).

	B	K	DK	φ
CS	YES	no	YES	YES
CM	no	YES	YES	YES
SCM	no	no	no	YES
PM	no	no	no	YES
CON	no	YES	no	YES
POS	YES	no	YES	YES
SEP	YES	no	YES	YES
$ESEC$	YES	YES	no	YES
IIT	no	no	no	YES
EC	no	no	no	YES

This table shows clearly that φ satisfies much more interesting properties that the other rules mentioned in the paper except of course $P's$ rule.

This table together with Corollary 2 (b) suggest that we must take a lot of care when we define rules which do not depend only on the canonical form. K and B use information which is not contained in the canonical form. Nevertheless this table suggests that they use this information in a wrong way because they loss many properties. A similar argument can be applied to DK.

Our feeling is that the canonical form is a good environment for defining rules in *cstp*.

Remark 4. A natural subclass of *cstp* is the one which arises when we assume that for any $i, j \in N_0$, c_{ij}, the direct cost of connecting i and j, is not larger that the cost of connecting i and j indirectly through other agents. Formally, for all $i, j \in N_0$,

$$c_{ij} = \min_{g \in G_{ij}} c(N_0, C, g)$$

where G_{ij} is the set of paths connecting i and j.

It is not difficult to check that in this subclass of problems the only change in the table is that now K satisfies POS.

In the next theorem we give two characterizations of φ.

Theorem 2.(a) φ is the only rule satisfying IIT, SEP, and $ESEC$.
(b) φ is the only rule satisfying EC.
Proof. See the Appendix.

Since PM implies SEP, SCM implies IIT, and φ satisfies PM and SCM next corollary is a trivial consequence of Theorem 2.

Corollary 3. φ is the only rule satisfying SCM, PM, and $ESEC$.

Even though Corollary 3 is a trivial consequence of Theorem 3 (a) we state it explicitly because we believe that PM and SCM are more appealling properties that SEP and IIT.

Remark 5. Corollary 3 is a tight characterization result (the proof is in the Appendix). Thus, the characterization of Theorem 3 (a) is also tight.

6 Other approaches to the problem

In this section we present four alternative definitions for φ. The first one is, of course, the definition of the P-value given by Feltkamp *et al* (1994). In the second one we write φ as the Shapley value of a game. In the third we define it looking only to the problem (N_0, C) in a intuitive way. The fourth is made considering the costs as indivisible goods.

We believe that these results are very interesting because we arrive to it from very different approachs. In some way we can consider φ as a "focal point" of *cstp*.

First approach: The P-value.

Feltkamp *et al* (1994) introduce a rule for *cstp* called Equal Remaining Obligations. Later, Branzei *et al* (2003) study it more and call it P-value, which is the name we use. As we say before when we start to study this problem we didn't know the paper of Feltkamp *et al* (1994), which is an unpublished paper. In January 2004 we come across the paper of Branzei *et al* (2003), which was available in the web in December 2003. Through this paper we know the first one.

In this section we prove that the $P-value$, which has been defined in a complitely different way that φ, coincides with φ.

Second approach: The Shapley value.

We already know that φ is the Shapley value of (N, v_{C^*}). This approach is interesting because we prove that φ is the Shapley value of a game which is independent of the canonical form.

Given a *cstp* (N_0, C) we denote by $\left(S_0, C^{+T}\right)$ the *cstp* obtained from (N_0, C) assuming that agents of S have to be connected and agents of T are already connected. This means that $c_{ij}^{+T} = c_{ij}$ for all $i, j \in S$ and $c_{i0}^{+T} = \min_{j \in T_0} c_{ij}$ for all $i \in S$.

Notice that if $S = N \setminus \{i\}$ and $T = \{i\}$ this definition coincides with $\left((N \setminus \{i\})_0, C^{+i}\right)$, which has been defined before property of EC.

Now, we associate to each *cstp* (N_0, C) a TU game $\left(N, v_C^+\right)$ where for each $S \subset N$,

$$v_C^+(S) = m\left(S_0, C^{+(N \setminus S)}\right).$$

Usually, we write v^+ instead of v_C^+.

Given $S \subset N$, $v^+(S)$ is the minimal cost of connecting all agents of S to the source assuming that agents of $N \setminus S$ are already connected. Note that $v(S)$ is the minimal cost of connecting all agents of S to the source assuming that agents in $N \backslash S$ are not participating.

If we compute v and v^+ in Example 1 with $x = 100$ we obtain

S	v	v^+
{1}	10	2
{2}	100	2
{1,2}	12	12

This example shows that v and v^+ are very different. Nevertheless, for all *cstp* (N_0, C) we have that $v(N) = v^+(N) = m(N_0, C)$.

Definition 2. For all *cstp* (N_0, C), we define

$$\varphi^1(N_0, C) = Sh\left(N, v^+\right).$$

In the next proposition we give some results about v^+ and φ^1 which will be useful to prove that φ^1 coincides with φ. Moreover, we also prove that φ^1 satisfies EC.

Proposition 4. (a) If C is a canonical matrix then, $v(S) + v^+(N \setminus S) = m(N_0, C)$ for all $S \subset N$.
(b) φ^1 satisfies IIT.
(c) φ^1 satisfies EC.

Proof. See the Appendix.

Third approach: Looking to the problem.

We now introduce a rule to *cstp* looking only to the problem. Assume that there is a unique *mcst* $t = \{(i, i^0)\}_{i \in N}$ in (N_0, C). The problem that agents face is how to divide the cost induced by t among agents. We believe that a reasonable approach is to decide, for each arc of t, the part of the cost every agent has to pay. Notice that B and DK decide to assign (with different criteria) the cost of any arc to a single agent. We allow to divide the cost of an arc among several agents.

If we back to Example 1 with $x = 100$ we realize that the unique *mcst* is $t = \{(0, 1), (1, 2)\}$. We argue that both agents have rights over the cost of arc $(1, 2)$. In the case of agent 2 this is clear because he connects to the source through this arc. We also believe that agent 1 has rights over this cost. Agent 2 can connect through agent 1 only because agent 1 connects to the source. We believe that a fair rule must assign some of the beneficts that agent 2 obtains by connecting through agent 1, which can be identified with the small cost of arc $(1, 2)$, to agent 1.

Since both agents are using the arc $(0, 1)$ for connecting to the source, we believe that any fair rule must assign this cost between both agents.

We then define a multi-step procedure to compute the part of the cost of any arc of t that every agent has to pay. The idea is the following. We ask agents by the cost of the arc of t they want to pay for. In order to take into account the structure of t, each agent must select an arc which he belongs to. We assume each agent chooses the arc with the lowest cost. If several agents select the same arc we divide this cost equally among the agents who selected it. Each agent pays the same proportion of the cost of the arc he selected. It is trivial to see that in the first step this proportion should be 0.5 or 1.

In the next steps we apply the same idea but now agents can also select other arcs. Agent i can select arc (j,k) if there is a path in t connecting i with j or k through arcs whose cost have been already paid.

In Example 1 each agent selects the arc $(1,2)$. Since both agents select the same arc this is divided equally between both. Then, agents 1 and 2 pay half of the cost of arc $(1,2)$. In step 2 both agents select the arc $(0,1)$ and the cost is divided equally among them. Agent 2 selects $(0,1)$ because 2 is conected to 1 through arc $(1,2)$, which has been paid in step 1.

If there are several *mcst* we select any of them. Later we will prove that the final allocation is independent of the selected *mcst*.

We now present this procedure in a formal way. Let t be a *mcst* in (N_0, C).

Step 0. We consider:
$a^0(i) = \emptyset$ for all $i \in N$. In general $a^0(i)$ belongs to t.
$p^0 = 0$.
$\varrho^0(i,j) = 0$ for all $(i,j) \in t$.
$A^0 = \{(i,j) \in t \text{ such that } \varrho^0(i,j) < 1\}$. Notice that $A^0 = t$.
$P^0 = \{P_1^0, ..., P_n^0\}$ is a partition of N such that $P_i^0 = \{i\}$ for all $i \in N$.
$f_i^0 = 0$ for all $i \in N$.

Assume we have defined Step r for all $r \leq s-1$. We now define Step s. In order to clarify this definition we will compute Step 1 of the process when applied to a *cstp* (N_0, C) with *mcst* $t = \{(0,1), (1,2), (2,3)\}$ and $c_{01} = 12$, $c_{12} = 4$, and $c_{23} = 6$.

Given $P_q^{s-1} \in P^{s-1}$, agents of P_q^{s-1} select, among the non-paid arcs, an arc connecting them with agents outside P_q^{s-1} whose cost they want to pay. Formally, let $(i_q, j_q) \in A^{s-1}$ be such that $i_q \in P_q^{s-1}$, $j_q \in N_0 \setminus P_q^{s-1}$ and

$$c_{i_q j_q} = \min\left\{c_{ij} : (i,j) \in A^{s-1},\ i \in P_q^{s-1},\ \text{and } j \in N_0 \setminus P_q^{s-1}\right\}.$$

If there exist many $(i_q, j_q) \in A^{s-1}$ we select any of them. We define $a^s(i) = (i_q, j_q)$ for all $i \in P_q^{s-1}$.

Example. *For* $P_1^0 = \{1\}$ *there are two possibilities:* $(0,1)$ *and* $(1,2)$. *Then,* $(i_1, j_1) = (1,2)$. *For* $P_2^0 = \{2\}$ *there are two possibilities:* $(1,2)$ *and* $(2,3)$. Then, $(i_2, j_2) = (1,2)$. For $P_3^0 = \{3\}$ *there is only one possibility* $(2,3)$. *Then,* $(i_3, j_3) = (2,3)$. *Moreover,* $a^1(1) = a^1(2) = (1,2)$, *and* $a^1(3) = (2,3)$.

We now compute p^s, the proportion (of the arc he selected) that any agent pays at this step. For each arc $(i,j) \in A^{s-1}$ we define

$$\begin{aligned} N_{ij}^s &= \{k \in N : a^s(k) = (i,j)\} \text{ and} \\ p^s &= \min_{(i,j)\in A^{s-1}} \left\{ \frac{1 - \varrho^{s-1}(i,j)}{\left|N_{ij}^s\right|} \right\}. \end{aligned}$$

We define $\varrho^s(i,j) = \varrho^{s-1}(i,j) + \left|N_{ij}^s\right| p^s$ for each $(i,j) \in A^{s-1}$. Then, $\varrho^s(i,j) \leq 1$ for each $(i,j) \in A^{s-1}$ and there exists at least one $(i',j') \in A^{s-1}$ such that $\varrho^s(i',j') = 1$. Notice that $\varrho^s(i,j)$ denotes the proportion of the cost of arc (i,j) paid at the end of Step s.

Example. $N_{01}^1 = \emptyset$, $N_{12}^1 = \{1,2\}$, *and* $N_{23}^1 = \{3\}$. *Then,* $p^1 = \frac{1}{2}$ *(we take* $\frac{1}{0} = +\infty$*).* $\varrho^1(0,1) = 0$, $\varrho^1(1,2) = 1$, *and* $\varrho^1(2,3) = \frac{1}{2}$.

We now compute the arcs whose cost has not been paid.

$$A^s = \left\{(i,j) \in A^{s-1} \text{ such that } \varrho^s(i,j) < 1\right\}.$$

We define P^s joining elements of P^{s-1} in the following way. Two elements P_q^{s-1} and $P_{q'}^{s-1}$ of P^{s-1} are in the same element of P^s if $(i_{q'}, j_{q'}) = (i_q, j_q)$ and $\varrho^s(i_q, j_q) = 1$.

Example. $A^1 = \{(0,1), (2,3)\}$ and $P^1 = \{\{1,2\}, \{3\}\}$

Finally, for each $i \in P_q^{s-1}$ we compute the cost that agent i pays in Step s.

$$f_i^s = p^s c_{i_q j_q}.$$

Example. $f_1^1 = 2$, $f_2^1 = 2$, *and* $f_3^1 = 3$.

This process ends when $A^s = \emptyset$. Since $A^0 = t$, $A^s \subset A^{s-1}$ and $A^s \neq A^{s-1}$, this process ends in a finite number of steps (at most n), say γ.

Since each tree has n arcs it is trivial to see that $\sum_{s=1}^{\gamma} p^s = 1$.

Definition 3. For all *cstp* (N_0, C) and $i \in N$ we define $\varphi_i^3(N_0, C) = \sum_{s=1}^{\gamma} f_i^s$.

We must prove that φ^3 is well defined (it does not depend on things such as the tree t, the arc (i_q, j_q) ...). We will do it later.

Fourth approach: Indivisible costs.

To clarify the motivation of this approach we assume that there is a unique *mcst* in (N_0, C). We can think in B and DK as rules assigning indivisible goods (cost of the arcs) because the cost of any arc of t is paid only by one agent. Of course, the indivisible goods are not really indivisible.

Given $\pi \in \Pi_N$ assume that the agents connect sequentially in the order given by π to one of the cheapest available options. Following this procedure we can associate a tree (not necessarily unique) to any permutation. We say that the tree t^π is induced by the permutation $\pi \in \Pi_N$ if:

$$t^\pi = \left\{ \begin{array}{c} (i, i^\pi) \text{ such that } i \in N,\ i^\pi \in Pre(i, \pi) \cup \{0\} \\ \text{and } c_{ii^\pi} \leq c_{ij} \text{ for all } j \in Pre(i, \pi) \cup \{0\} \end{array} \right\}.$$

Let us suppose that any agent pays his connection cost in accordance to t^π, *i.e.* agent i pays c_{ii^π} for all $i \in N$. It is self-evident that if t^π and t'^π are two different trees induced by $\pi \in \Pi_N$, then agents pay the same under t^π as under t'^π. Thus it makes sense to define the cost allocation induced by the permutation $\pi \in \Pi_N$ as the vector $c^\pi = \{c_{ii^t}\}_{i \in N}$.

Assume that $\pi \in \Pi_N$ satisfies $c(N_0, C, t^\pi) = m(N_0, C)$. Then, t^π is a *mcst* and $B(N_0, C)$ and $DK(N_0, C)$ are obtained from c^π. It is trivial to see that $B_i(N_0, C) = c_{ii^\pi}$ and $DK_i(N_0, C) = c_{i_j i_j^\pi}$ for all $i \in N$. Then, B and DK could be very unfair (see, for instance Example 1 with $x > 0$).

A classical way for recovering fairness from an allocation depending on the permutations is taking the average over the set of all permutations. In *cstp* problems this approach is incompatible with efficiency because for some $\pi \in \Pi_N$, $c(N_0, C, t^\pi) > m(N_0, C)$.

Nevertheless, if C is a canonical matrix, for all $\pi \in \Pi_N$, $c(N_0, C, t^\pi) = m(N_0, C)$ (we will prove this statement later). Then, in canonical matrices it is possible to generate an efficient and fair allocation taking the average over the permutations.

We then define the rule φ^3 as follows.

Definition 4. For all *cstp* (N_0, C) and $i \in N$ we define

$$\varphi_i^3(N_0, C) = \frac{1}{n!} \sum_{\pi \in \Pi_N} c^*_{ii^\pi}$$

where $c^*_{ii^\pi}$ is the connection cost among i and i^π in the canonical form C^* associated to C.

In this approach the canonical form appears in a natural way. Nevertheless, this approach is completely different from the original definition of φ.

We now prove that all the rules defined in this section coincide with φ.
Theorem 3. $P = \varphi^1 = \varphi^2 = \varphi^3 = \varphi$.

Proof. See the Appendix.

In the next corollary, which is a trivial consequence of Theorem 3 and Proposition 3 (c), we prove that φ satisfies EC.
Corollary 4. φ satisfies EC.

We believe that Theorem 3 gives more support to this rule because we can justify it from different approaches to the problem. This rule is in some sense a focal point of the problem.

7 Appendix

In this section we prove the results stated in the paper.

7.1 Proof of Proposition 1

We first prove that if C satisfies $A1$ and $A2$ then C is canonical. By simplicity we assume that $i_p = p$ for all $p = 0, 1, ..., n$. This means that $t = \{(i-1, i)\}_{i=1}^{n}$.

Take $i, j \in N_0$ and C' such that $c'_{ij} < c_{ij}$ and $c'_{kl} = c_{kl}$ otherwise. We need to prove that $m(N_0, C') < m(N_0, C)$. It is enough to prove that there exists a tree t such that $c(N_0, C', t) < m(N_0, C)$.

Assume without loss of generality that $i < j$. By $(A2)$, $c_{ij} = c_{(k-1)k}$ where $i < k \leq j$. We define $t' = (t \setminus \{(k-1, k)\}) \cup \{(i, j)\}$. Then, t' is a tree and

$$
\begin{aligned}
c(N_0, C', t') &= m(N_0, C) - c_{(k-1)k} + c'_{ij} \\
&= m(N_0, C) - c_{ij} + c'_{ij} \\
&< m(N_0, C).
\end{aligned}
$$

We now prove that if C is canonical then C satisfies $A1$ and $A2$.

Let (N_0, C) be a *cstp* and $t = \{(i, i^0)\}_{i \in N}$ a *mcst* in (N_0, C). If $(j, k) \notin t$ there exists a unique path g in t connecting j and k. Take $(l, l^0) \in g$ such that $c_{ll^0} = \max_{(i,i^0) \in g} \{c_{ii^0}\}$.We now prove two claims.

Claim 1. $c_{jk} \geq c_{ll^0}$.

Assume that $c_{jk} < c_{ll^0}$. Take $t' = (t \setminus \{(l, l^0)\}) \cup \{(j, k)\}$. Then, t' is a tree and

$$
c(N_0, C, t') = c\left(N_0, C, t^2\right) - c_{ll^0} + c_{jk} < c(N_0, C, t)
$$

which is a contradiction because t is a *mcst* in (N_0, C).

Notice that Claim 1 holds also when C is not canonical.

Claim 2. $c_{jk} \leq c_{ll^0}$.

Assume that $c_{jk} > c_{ll^0}$.By Remark 2, we can find a *mcst* t' such that $(j, k) \in t'$. Let T_j be the set of nodes in N_0 which are connected to j in $t' \setminus \{(j, k)\}$. We can define T_k in a similar way. Notice that $j \in T_j$ and $k \in T_k$. Since t' is a tree we can find $(p, p^0) \in g$ such that $(p, p^0) \notin t'$ and $t'' = (t' \setminus \{(j, k)\}) \cup \{(p, p^0)\}$ is a tree. Then,

$$
c(N_0, C, t'') = c(N_0, C, t') - c_{jk} + c_{pp^0} < c(N_0, C, t')
$$

which is a contradiction because t' is a *mcst* in (N_0, C).

We proceed by induction on n (the number of agents). If $n = 1$ the result is trivial. Assume that the result holds for $n \leq s$ and we prove it when $n = s + 1$.

We first prove that we can find a *mcst* satisfying $(A1)$.

Let t' be a *mcst* obtained following Prim's algorithm. Then, t' can be obtained assuming that agents connect sequentially in some order $\pi \in \Pi_N$. By simplicity we take $\pi = \{1, 2, ..., n\}$.

Because of the definition of Prim's algorithm we know that $t'_{N\setminus\{n\}}$ is a *mcst* in $\left((N \setminus \{n\})_0, C\right)$. Since $N \setminus \{n\}$ has $n-1$ agents, by induction hypothesis, we can find $t^1 = \{(i_{p-1}, i_p)\}_{p=1}^{n-1}$ satisfying $(A1)$ and $(A2)$. Moreover,

$$c\left((N \setminus \{n\})_0, C, t'_{N\setminus\{n\}}\right) = c\left((N \setminus \{n\})_0, C, t^1\right).$$

Let $i_q \in \{0, 1, ..., n-1\}$ be the agent to whom n connects according with t'. We take $t^2 = t^1 \cup (i_q, n)$. It is easy to see that $c(N_0, C, t^2) = c(N_0, C, t)$. Thus, t^2 is a *mcst* in (N_0, C).

For all $p = q+1, ..., n-1$, $g_p = \left\{(n, i_q), \{(i_{r-1}, i_r)\}_{r=q+1}^{p}\right\}$ is the only path in t^2 connecting n and i_p. By claims 1 and 2, $c_{i_p n} = \max\left\{\max_{r=q+1,...,p}\left\{c_{i_{r-1}i_r}\right\}, c_{i_q n}\right\}$.

Assume that there exist $r \in \{q+1, ..., n-1\}$ such that $c_{i_{p-1}i_p} \leq c_{i_q n}$ for all $p = q+1, ..., r$ and $c_{i_q n} \leq c_{i_r i_{r+1}}$. It is trivial to see that the tree

$$t = \{(i_{p-1}, i_p)\}_{p=1}^{r} \cup (i_r, n) \cup (n, i_{r+1}) \cup \{(i_{p-1}, i_p)\}_{p=r+2}^{n-1}$$

is a *mcst* satisfying $(A1)$ because $c_{i_r n} = c_{i_q n}$ and $c_{n i_{r+1}} = c_{i_r i_{r+1}}$.

Assume that $c_{i_{p-1}i_p} \leq c_{i_q n}$ for all $p = q+1, ..., n-1$. Then, the tree

$$t = \{(i_{p-1}, i_p)\}_{p=1}^{n-1} \cup (i_q, n)$$

is a *mcst* satisfying $(A1)$ because $c_{i_{n-1}n} = c_{i_q n}$.

We have proved that we can find a *mcst* $t = \{(i_{p-1}, i_p)\}_{p=1}^{n}$ satisfying $(A1)$. We now prove that t also satisfies $(A2)$.

For all $i_p, i_q \in N$, $p < q$ we have that $g = \{(i_{r-1}, i_r)\}_{r=p+1}^{q}$ is the unique path in t connecting i_p and i_q. By claims 1 and 2 we conclude that t also satisfies $(A2)$.

7.2 Proof of Proposition 2

We prove several claims.

Claim 1. For all $p = 1, ..., n$,

$$c_{i_p i_p^0} = \min_{k \in N_0 \setminus S^{p-1}, l \in S^{p-1}} \left\{c_{kl}^{p-1}\right\}.$$

Because of the definition of C^{p-1}, $c_{kl}^{p-1} = c_{kl}$ when $k \in N_0 \setminus S^{p-1}$ and $l \in S^{p-1}$. Since $i_p \in N_0 \setminus S^{p-1}$ and $i_p^0 \in S^{p-1}$ we conclude that $c_{i_p i_p^0} \geq \min\limits_{k \in N_0 \setminus S^{p-1}, l \in S^{p-1}} \{c_{kl}\}$.

Assume that $c_{i_p i_p^0} > \min\limits_{k \in N_0 \setminus S^{p-1}, l \in S^{p-1}} \{c_{kl}\} = c_{k_p l_p}$. We take $t' = \left(t^0 \setminus \left\{\left(i_p, i_p^0\right)\right\}\right) \cup \{(k_p, l_p)\}$. Then, t' is a tree and

$$c(N_0, C, t') = c\left(N_0, C, t^0\right) - c_{i_p i_p^0} + c_{k_p l_p} < c\left(N_0, C, t^0\right)$$

which is a contradiction because t is a *mcst* in (N_0, C).

Claim 2. For all $p = 1, ...n$, $C^{p-1} \geq C^p$ and C^{p-1} and C^p are directly equivalent.

We prove that $C^{p-1} \geq C^p$. By definition $c_{jk}^p = c_{jk}^{p-1}$ if $\{j,k\} \neq \{i_{p-1}, i_p\}$ it is enough to prove that $c_{i_{p-1} i_p}^{p-1} \geq c_{i_{p-1} i_p}^p$. Since $i_{p-1} \in S^{p-1}$, by Claim 1, $c_{i_{p-1} i_p}^{p-1} \geq c_{i_p i_p^0} = c_{i_{p-1} i_p}^p$.

Assume that t' is a tree obtained through Prim's algorithm in (N_0, C^{p-1}). By Claim 1 it is straightforward to prove that t' can be obtained also through Prim's algorithm in (N_0, C^{p-1}). Now is easy to conclude that C^{p-1} and C^p are directly equivalent.

Claim 3. $C^n \geq C^*$.

We must prove that $c_{i_p i_q}^n \geq c_{i_p i_q}^* = \max\limits_{p < r \leq q} \left\{c_{i_{r-1} i_r}^n\right\}$ for all $i_p, i_q \in N_0$, $p < q$. Suppose not. Then, there exists $i_p, i_q, i_{r-1}, i_r \in N_0$, $p < r \leq q$ such that $c_{i_{r-1} i_r}^n > c_{i_p i_q}^n$.

As a consequence of Prim's algorithm, $t = \{(i_{p-1}, i_p)\}_{p=1}^n$ is a *mcst* of (N_0, C^n). We consider the tree $t' = (t \setminus \{(i_{r-1}, i_r)\}) \cup \{(i_p, i_q)\}$. Then,

$$c(N_0, C^n, t') = c(N_0, C^n, t) - c_{i_{r-1} i_r}^n + c_{i_p i_q}^n < c(N_0, C^n, t)$$

which is a contradiction because t is a *mcst* of (N_0, C^n).

Claim 4. C^n and C^* are equivalent.

If $C^n = C^*$ it is trivial.

Assume that $C^n \neq C^*$. Since $C^n \geq C^*$ (Claim 3) we can find $i_p, i_q \in N_0$ such that $p < q$ and $c_{i_p i_q}^n > c_{i_p i_q}^*$. We define C'^1 as

$$c_{ij}'^1 = \begin{cases} c_{i_p i_q}^* & \text{if } (i,j) = (i_p, i_q) \\ c_{ij}^n & \text{otherwise.} \end{cases}$$

Notice that $C^n \geq C'^1 \geq C^*$. As a consequence of Prim's algorithm, t is a *mcst* in (N_0, C'^1) and then C^n and C'^1 are directly equivalent.

If $C'^1 = C^*$ Claim 4 holds.

Assume that $C'^1 \neq C^*$. Since $C'^1 \geq C^*$ we can find $i_{p_1}, i_{q_1} \in N_0$ such that $p_1 < q_1$ and $c'^1_{i_{p_1} i_{q_1}} > c^*_{i_{p_1} i_{q_1}}$. We define C'^2 as

$$c'^2_{ij} = \begin{cases} c^*_{i_{p_1} i_{q_1}} & \text{if } (i,j) = (i_{p_1}, i_{q_1}) \\ c'^1_{ij} & \text{otherwise.} \end{cases}$$

Notice that $C' \geq C'^1 \geq C'^2 \geq C^*$. Using arguments similar to those used with C'^1, we can prove that C'^1 and C'^2 are equivalent and t is a *mcst* in (N_0, C'^2).

If we repeat this procedure we can find l such that

$$C' \geq C'^1 \geq C'^2 \geq ... \geq C'^{l-1} \geq C'^l = C^*$$

where for all $k = 2, ..., l : C'^{k-1} \neq C'^k$ and C'^{k-1} and C'^k are equivalent.

Then, C^n and C^* are equivalent.

Claim 5. C^* is a canonical matrix, $C \geq C^*$, C and C^* are equivalent, and C^* is a canonical matrix associated to C.

By Proposition 1 C^* is a canonical matrix.

By Claim 2, $C = C^0 \geq C^1 \geq ... \geq C^n$. By Claim 3, $C^n \geq C^*$. Then, $C \geq C^*$.

Since C^{p-1} and C^p are directly equivalent for all $p = 1, ..., n$ (Claim 2) and $C^0 = C$, we conclude that C and C^n are equivalent. By Claim 4, C^n and C^* are equivalent. Then, C and C^* are equivalent.

Since C^* is a canonical matrix, $C \geq C^*$, and $m(N_0, C) = m(N_0, C^*)$ (C and C^* are equivalent) we conclude that C^* is a canonical matrix associated to C.

Claim 6. C^* is the unique canonical matrix associated to C.

Since C^* is a canonical matrix associated to C we must prove that if C' is another canonical matrix associated to C then, $C' = C^*$. Because of the definition of canonical matrices it is enough to prove that $C' \geq C^*$.

Let t be the tree obtained in Stage $n+1$ of the algorithm. Assume, without loss of generality, that $t = \{(i-1, i)\}_{i=1}^n$.

Since C^* and C' are canonical matrices associated to C and t^0 is a *mcst* in (N_0, C), t^0 is also a *mcst* in (N_0, C^*) and (N_0, C'). Moreover, $c'_{ii^0} = c^*_{ii^0} = c_{ii^0}$ for all $i \in N$.

We prove that given $i, j \in N_0$, $c'_{ij} \geq c^*_{ij}$. Assume that $i < j$ (otherwise is similar). Since $c_{kk^0} = c^*_{(k-1)k}$ (it is a consequence of the definition of t),

$$c^*_{ij} = \max_{i<k\leq j} \left\{ c^*_{(k-1)k} \right\} = \max_{i<k\leq j} \left\{ c_{kk^0} \right\}.$$

Let $k \in N$ be such that $i < k \leq j$. We consider two cases:

1. k is in the unique path in t^0 from j to 0. Then, $t' = t^0 \backslash \{(k, k^0)\} \cup \{(i, j)\}$ is a tree. Suppose that $c'_{ij} < c_{kk^0}$. Thus

$$c(N_0, C', t') = c\left(N_0, C', t^0\right) - c_{kk^0} + c'_{ij} < c\left(N_0, C', t^0\right)$$

which is a contradiction because t^0 is a *mcst* in (N_0, C'). Then $c'_{ij} \geq c_{kk^0}$.

2. k is not in the unique path in t^0 from j to 0. Let l be such that $k < l \leq j$, l is in the unique path in t^0 from j to 0, and for all $r \in \{k, k+1, ..., l-1\}$, r is not in the unique path in t^0 from j to 0. Because of the definition of the algorithm we know that $\{l^0, k^0\} \in S^{k-1}$, $\{l, k\} \in N_0 \setminus S^{k-1}$, and

$$c'_{kk^0} = c_{kk^0} = \min_{p \in S^{k-1}, q \in N_0 \setminus S^{k-1}} \{c_{pq}\} \leq c_{ll^0} = c'_{ll^0}.$$

By case 1, $c'_{ij} \geq c'_{ll^0}$ and then, $c'_{ij} \geq c_{kk^0}$.

Now $c'_{ij} \geq \max\limits_{i<k\leq j} \{c_{kk^0}\}$ is a trivial consequence of cases 1 and 2.

7.3 Proof of Proposition 3

We first prove (a). We compute t' following Prim's algorithm. Since t satisfies $(A2)$ and $i_{p-1} \leq i_p$ for all $p = 1, ..., s$ we conclude that $c_{0i_1} \leq c_{0i_p}$ for all $p = 1, ..., |S|$. Then, the arc $(0, i_1)$ could be the first arc in Prim's algorithm.

Since t satisfies $(A2)$ we conclude that $c_{0i_2} \leq c_{0i_p}$ and $c_{i_1 i_2} \leq c_{i_1 i_p}$ for all $p = 2, ..., |S|$. Then, $\min\{c_{0i_2}, c_{i_1 i_2}\} \leq \min\left\{c_{0i_p}, c_{i_1 i_p}\right\}$ for all $p = 2, ..., |S|$. By $(A2)$, $c_{i_1 i_2} \leq c_{0i_2}$. Thus, $c_{i_1 i_2} = \min\{c_{0i_2}, c_{i_1 i_2}\}$ and, hence, we can choose the arc (i_1, i_2) as the second arc in Prim's algorithm.

If we continue with this process we obtain that $t' = \{(i_{p-1}, i_p)\}_{p=1}^{|S|}$ is a tree obtained following Prim's algorithm. This means that t' is a *mcst* in (S_0, C) and hence,

$$v(S) = m(S_0, C) = c(S_0, C, t') = \sum_{p=1}^{|S|} c_{i_{p-1}i_p}.$$

We now prove (b). Assume that $i_q \neq i_s$. By (a),

$$\begin{aligned} v(S) - v(S \setminus \{i_q\}) &= \sum_{p=1}^{|S|} c_{i_{p-1}i_p} - \left(\sum_{p=1}^{q-1} c_{i_{p-1}i_p} + c_{i_{q-1}i_{q+1}} + \sum_{p=q+1}^{|S|} c_{i_{p-1}i_p} \right) \\ &= c_{i_{q-1}i_q} + c_{i_q i_{q+1}} - c_{i_{q-1}i_{q+1}}. \end{aligned}$$

Since t satisfies $(A2)$ we conclude that $c_{i_{q-1}i_{q+1}} = \max\left\{c_{i_{q-1}i_q}, c_{i_r i_{q+1}}\right\}$. Then,

$$v(S) - v(S \setminus \{i_q\}) = \min\left\{c_{i_{q-1}i_q}, c_{i_q i_{q+1}}\right\}.$$

Assume that $i_q = i_{|S|}$. By (a),

$$v(S) - v(S \setminus \{i_q\}) = \sum_{p=1}^{|S|} c_{i_{p-1}i_p} - \sum_{p=1}^{|S|-1} c_{i_{p-1}i_p} = c_{i_{|S|-1}i_{|S|}}.$$

We now prove (c). Let S' be such that $S \subset S' \subset N$ and $S' = \{i'_1, ..., i'_{s'}\}$ such that $i'_{p-1} \leq i'_p$ for all $p = 1, ..., |S'|$. Notice that $|S'| \geq |S|$ because $S \subset S'$. We prove that for all $i \in S$,

$$v(S) - v(S \setminus \{i\}) \geq v(S') - v(S' \setminus \{i\}).$$

We consider three cases:

- $i \neq i_{|S|}$. Then, $i = i_q = i'_{q'}$ and $i'_{q'} \neq i'_{|S'|}$. By (b),

$$\begin{aligned} v(S) - v(S \setminus \{i\}) &= \min\left\{c_{i_{q-1}i}, c_{i i_{q+1}}\right\} \\ v(S') - v(S' \setminus \{i\}) &= \min\left\{c_{i'_{q'-1}i}, c_{i i'_{q'+1}}\right\}. \end{aligned}$$

  Since $S \subset S'$ we conclude that $i_{q-1} \leq i'_{q'-1}$ and $i'_{q'+1} \leq i_{q+1}$. Then, $c_{i_{q-1}i} \geq c_{i'_{q'-1}i}$ and $c_{i i_{q+1}} \geq c_{i i'_{q'+1}}$ because t satisfies $(A2)$. Thus,

$$v(S) - v(S \setminus \{i\}) \geq v(S') - v(S' \setminus \{i\}).$$

- $i = i_{|S|}$ and $i \neq i'_{|S'|}$. Then $i = i_{|S|} = i'_{q'} < i'_{|S'|}$. By (b),
$$\begin{aligned} v(S) - v(S \setminus \{i\}) &= c_{i_{s-1}i} \\ v(S') - v(S' \setminus \{i\}) &= \min\left\{c_{i'_{q'-1}i}, c_{ii'_{q'+1}}\right\}. \end{aligned}$$
Since $S \subset S'$ we conclude that $i_{s-1} \leq i'_{q'-1}$. Then, $c_{i_{s-1}i} \geq c_{i'_{q'-1}i}$ because t satisfies $(A2)$. Thus,
$$v(S) - v(S \setminus \{i\}) \geq v(S') - v(S' \setminus \{i\}).$$
- $i = i_{|S|} = i'_{|S'|}$. By (b),
$$\begin{aligned} v(S) - v(S \setminus \{i\}) &= c_{i_{s-1}i} \\ v(S') - v(S' \setminus \{i\}) &= c_{i'_{s'-1}i}. \end{aligned}$$
Since $S \subset S'$ we conclude that $i_{s-1} \leq i'_{s'-1}$. Then, $c_{i_{s-1}i} \geq c_{i'_{s'-1}i}$ because t satisfies $(A2)$.

7.4 Proof of Corollary 1

By simplicity we assume that the tree t satisfying $(A1)$ is $\{(i-1, i)\}_{i=1}^{n}$.

We know that for all $i \in N$,
$$\begin{aligned} B_i(N_0, C) &= \frac{1}{n!} \sum_{\pi \in \Pi_N} B_i^{\pi}(N_0, C) \text{ and} \\ K_i(N_0, C) &= \frac{1}{n!} \sum_{\pi \in \Pi_N} \left(v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi))\right). \end{aligned}$$

We prove that for all $\pi = \{i_1, ..., i_n\} \in \Pi_N$ and $i \in N$,
$$B_i^{\pi}(N_0, C) = v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi)).$$

Assume that i_p is the first agent selected in $B^{\pi}(N_0, C)$. Then, $B_{i_p}^{\pi}(N_0, C) = c_{i_p 0} = \min_{j \in N} \{c_{j0}\}$ and $c_{i_q 0} > c_{i_p 0}$ for all $q = 1, ..., p-1$. Since C is a canonical matrix $i_p < i_q$ for all $q = 1, ..., p-1$. Let q' be such that $q' \leq p-1$ and $i_{q'} \leq i_q$ for all $q = 1, ..., p-1$. Then, $c_{i_p i_{q'}} = c_{i_{q'} 0} \geq c_{i_p 0}$.

By Proposition 3 (c) we conclude that $v(Pre(i_p, \pi) \cup \{i_p\}) - v(Pre(i_p, \pi)) = c_{i_p 0} = B_{i_p}^{\pi}(N_0, C)$.

Applying an induction argument it is easy to conclude that $B_i^{\pi}(N_0, C) = v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi))$ for all $i \in N$.

7.5 Proof of Corollary 2

(a) Assume that ψ satisfies IIT. By Proposition 2, C is equivalent to C^*. Then, $\psi(N_0, C) = \psi(N_0, C^*)$.

Assume that ψ depends only on the canonical form. Because of the definition of equivalent problems, it is enough to prove that if C and C' are two directly equivalent problems then $\psi(N_0, C) = \psi(N_0, C')$.

Assume that C and C' are directly equivalent problems. There exists a $mcst$ $t = \{(i, i^0)\}_{i \in N}$ in (N_0, C) and (N_0, C') such that $c_{ii^0} = c'_{ii^0}$ for all $i \in N$. It is straightforward to prove, because of the definition of the algorithm, that $C^* = C'^*$. Since ψ depends only on the canonical form, $\psi(N_0, C) = \psi(N_0, C')$.

(b) If ψ satisfies SCM, $\psi(N_0, C) \geq \psi(N_0, C^*)$ because $C \geq C^*$ (Proposition 2). Since $m(N_0, C) = m(N_0, C^*)$, $\psi(N_0, C) = \psi(N_0, C^*)$, by part (a) we conclude that ψ satisfies IIT.

7.6 Proof of Theorem 1

Let (N_0, C) be a $cstp$ and (N_0, C^*) its canonical form. We know that $\varphi(N_0, C) = K(N_0, C^*) = Sh(N, v_{C^*})$. We now prove that φ satisfies the following properties:

Independence of irrelevant trees (IIT). It is a consequence of Corollary 2 (a).

Strong cost monotonicity (SCM). Let (N_0, C) and (N_0, C') be two $cstp$ such that there exist $j, k \in N$ satisfying $c'_{jk} = c_{jk} + a$ where $a > 0$ and $c'_{il} = c_{il}$ otherwise. We must prove that $\varphi_i(N_0, C) \leq \varphi_i(N_0, C')$ for all $i \in N$.

Since $C \leq C'$ we conclude that $m(N_0, C) \leq m(N_0, C')$.

Assume there exists a $mcst$ t in (N_0, C) such that $(j, k) \notin t$. Then,

$$c(N_0, C', t) = c(N_0, C, t) = m(N_0, C).$$

Thus, $m(N_0, C) = m(N_0, C')$ and t is a $mcst$ in (N_0, C'). Moreover, $c_{il} = c'_{il}$ for all $(i, l) \in t$. This means that (N_0, C) and (N_0, C') are directly equivalent. Then, $\varphi(N_0, C) = \varphi(N_0, C')$ because φ satisfies IIT.

Assume that for all $mcst$ t in (N_0, C), $(j, k) \in t$. Let G be the set of trees which do not contain the arc (j, k), *i.e.*

$$G = \left\{ t \in \mathcal{G}_0^N \mid t \text{ is a tree and } (j, k) \notin t \right\}.$$

Let t^1 be such that

$$c\left(N_0, C, t^1\right) = \arg\min_{t \in G} \left\{c\left(N_0, C, t\right)\right\}.$$

Since $(j,k) \notin t^1$, we conclude that $c\left(N_0, C, t^1\right) = c\left(N_0, C', t^1\right)$.

We define $b = c\left(N_0, C, t^1\right) - m\left(N_0, C\right)$. Notice that $b \geq 0$ because t^1 is a tree in $\left(N_0, C\right)$.

We now distinguish two cases:

Case 1: $a \leq b$. Let $t^0 = \left\{\left(i, i^0\right)\right\}_{i \in N}$ be a *mcst* in $\left(N_0, C\right)$. We assume that $j^0 = k$ (the case $k^0 = j$ is similar). Then, $(j,k) \in t^0$ and t^0 is a *mcst* in $\left(N_0, C'\right)$. We can assume without loss of generality that t, the tree obtained when we apply the algorithm to $\left(N_0, C\right)$, is $\left\{(i-1, i)\right\}_{i \in N}$.

Let $t' = \left\{\left(i_{p-1}, i_p\right)\right\}_{p=1}^{n}$ be the tree obtained when we apply the algorithm to $\left(N_0, C'\right)$. It is trivial to see that $i_p = p$ for all $p < j$.

We consider two subcases:

Subcase 1.1: $t = t'$.

We first prove that $C^* \leq C'^*$.

When $i \neq j$, $c^*_{(i-1)i} = c_{ii^0} = c'_{ii^0} = c'^*_{(i-1)i}$. Moreover, $c^*_{(j-1)j} = c_{jj^0} < c_{jj^0} + a = c'^*_{(j-1)j}$.

Let $i, l \in N_0$ such that $i < l$. Because of the definition of C^* and C'^* we have that $c^*_{il} = \max\limits_{i < p \leq l} \left\{c^*_{(p-1)p}\right\}$ and $c'^*_{il} = \max\limits_{i < p \leq l} \left\{c'^*_{(p-1)p}\right\}$. Then, $c^*_{il} \leq c'^*_{il}$.

We now prove that $\varphi\left(N_0, C\right) \leq \varphi\left(N_0, C'\right)$.

We know that $\varphi\left(N_0, C\right) = Sh\left(N, v_{C^*}\right)$ and $\varphi\left(N_0, C'\right) = Sh\left(N, v_{C'^*}\right)$.

Take $S = \left\{j_1, ..., j_{|S|}\right\}$ such that $j_{p-1} < j_p$ for all $p = 2, ..., |S|$ and $j_q \in S$. By Proposition 3 (b) we conclude that

$$v_{C^*}(S) - v_{C^*}(S \setminus \{j_q\}) \leq v_{C'^*}(S) - v_{C'^*}(S \setminus \{j_q\})$$

because $C^* \leq C'^*$.

Take $i \in N$. Then,

$$\begin{aligned} Sh_i\left(N, v_{C^*}\right) &= \frac{1}{n!} \sum_{\pi \in \Pi_N} \left(v_{C^*}\left(Pre\left(i, \pi\right) \cup \{i\}\right) - v_{C^*}\left(Pre\left(i, \pi\right)\right)\right) \\ &\leq \frac{1}{n!} \sum_{\pi \in \Pi_N} \left(v_{C'^*}\left(Pre\left(i, \pi\right) \cup \{i\}\right) - v_{C'^*}\left(Pre\left(i, \pi\right)\right)\right) \\ &= Sh_i\left(N, v_{C'^*}\right). \end{aligned}$$

Subcase 1.2: $t \neq t'$.

It is trivial to see that $c_{jj^0} \leq c_{i_j i_j^0} < c_{jj^0} + a = c'_{jj^0}$. This means that $j = i_q$ with $q > j$. Because of the definition of the algorithm we can take t' such that $c'^{*}_{i_{p-1} i_p} < c'^{*}_{i_{q-1} i_q}$ for all $p = j, ..., q-1$. Then, $c'^{*}_{i_{q-1} i_q} = c'_{jj^0} = c_{jj^0} + a$.

Take $C'^{(j-1)} = C$. For all $p = j, ..., q$ we define C'^p such that $c'^p_{jj^0} = c'^{(p-1)}_{jj^0} + \max \left\{ c_{i_p i_p^0} - c'^{(p-1)}_{jj^0}, 0 \right\}$ and $c'^p_{il} = c'^{(p-1)}_{il}$ otherwise. Let t'^p be the tree obtained when we apply the algorithm to (N_0, C'^p).

Notice that

$$C' = C'^q \geq C'^{(q-1)} \geq ... \geq C'^j \geq C'^{(j-1)} = C.$$

and t^0 is a $mcst$ in (N_0, C'^p) for all $p = j, ..., q$.

We prove that for all $p = j, ..., q$ we can take $t'^{(p-1)}$ and t'^p such that $t'^{(p-1)} = t'^p$. We prove it when $p = j$, the other cases are similar.

We take $t'^{(j-1)} = t$. If we apply the algorithm to C'^j, for all $i = 1, ..., j-1$, we can select the arc (i, i^0).

Since $c'^{(j-1)}_{jj^0} = c_{jj^0}$ and $c_{jj^0} \leq c_{i_j i_j^0}$,

$$c'^j_{jj^0} = c'^{(j-1)}_{jj^0} + \max \left\{ c_{i_j i_j^0} - c'^{(j-1)}_{jj^0}, 0 \right\} = c_{i_j i_j^0}.$$

Then, in Stage j of the algorithm we can select the arc (j, j^0). Now it is trivial to see that in Stage i, $i > j$, we can select the arc (i, i^0). Thus, $t'^j = t$.

Take $p \in \{j, ..., q\}$. Since $t'^{(p-1)} = t'^p$ and $\left(N_0, C'^{(p-1)}\right)$ and (N_0, C'^p) are in subcase 1.1 we conclude that $\varphi(N_0, C'^p) \geq \varphi\left(N_0, C'^{(p-1)}\right)$.

Thus,

$$\varphi(N_0, C') = \varphi(N_0, C'^q) \geq ... \geq \varphi\left(N_0, C'^{(j-1)}\right) = \varphi(N_0, C).$$

Case 2: $a > b$. Then, t^1 is a $mcst$ in (N_0, C') but t is not.

Let C'' such that $c''_{jk} = c_{jk} + b$ and $c''_{il} = c_{il}$ otherwise. Notice that $C \leq C'' \leq C'$.

We have proved before that $\varphi_i(N_0, C) \leq \varphi_i(N_0, C'')$ for all $i \in N$.

We know that t^1 is a $mcst$ of (N_0, C') and (N_0, C'') and $(j, k) \notin t^1$. Then, $c'_{il} = c''_{il}$ for all $(i, l) \in t^1$.

This means that (N_0, C') and (N_0, C'') are directly equivalent. Since φ satisfies IIT we conclude that $\varphi(N_0, C'') = \varphi(N_0, C')$. Thus, $\varphi(N_0, C) \leq \varphi_i(N_0, C')$.

Cost monotonicity (CM). Since φ satisfies SCM we conclude that φ also satisfies CM.

Population monotonicity (PM). We must prove that for all *cstp* (N_0, C), all $S \subset N$, and all $i \in S$

$$\varphi_i(N_0, C) \leq \varphi_i(S_0, C).$$

It is enough to prove it when $S = N \setminus \{j\}$ and $j \in N$.

We prove it in several claims.

Claim 1. Let (N_0, C) be such that there exists $j \in N$ satisfying: $c_{j0} = a$, $c_{ji} = b$ for all $i \in N \setminus \{j\}$, and $b > a > \max\limits_{i,k \in N_0 \setminus \{j\}} \{c_{ik}\}$. Then,

$$\varphi_i(N_0, C) = \begin{cases} a & \text{if } i = j \\ \varphi_i\left((N \setminus \{j\})_0, C\right) & \text{otherwise.} \end{cases}$$

By simplicity we assume that $j = n$. Let t'^0 be a *mcst* in $\left((N \setminus \{n\})_0, C\right)$. Then, $t^0 = t'^0 \cup \{(n, 0)\}$ is a *mcst* in (N_0, C).

Let t' be the tree obtained when we apply the algorithm to $\left((N \setminus \{n\})_0, C\right)$ starting in t'^0. By simplicity we assume that $t' = \{(i-1, i)\}_{i=1}^{n-1}$. It is trivial to see that $t = t' \cup \{(n-1, n)\}$ is the tree obtained when we apply the algorithm to (N_0, C) starting in t^0 and for all $i = 1, ..., n-1$, the cost of the arc $(i-1, i)$ is the same in the canonical form associated to $\left((N \setminus \{n\})_0, C\right)$ and (N_0, C). Moreover, the cost of arc $(n-1, n)$ in (N_0, C^*) is a.

We denote $v = v_{C^*}$ and by v_{-n} the characteristic function associated to the *cstp* $\left((N \setminus \{n\})_0, C^*\right)$.

By Proposition 1 (a), $v(S) = v_{-n}(S \setminus \{n\}) + a$ if $n \in S$ and $v(S) = v_{-n}(S)$ if $n \notin S$.

Consider the TU games (N, w_1) and (N, w_2) where for all $S \subset N$

$$\begin{aligned} w_1(S) &= v_{-n}(S \setminus \{n\}) \text{ and} \\ w_2(S) &= \begin{cases} a & \text{if } n \in S \\ 0 & \text{if } n \notin S. \end{cases} \end{aligned}$$

It is trivial to see that

$$\begin{aligned} Sh_i(N, w_1) &= Sh_i(N \setminus \{n\}, v_{-n}) = \varphi_i\left((N \setminus \{j\})_0, C\right) \text{ if } i \neq n, \\ Sh_n(N, w_1) &= 0, \\ Sh_i(N, w_2) &= 0 \text{ if } i \neq n, \text{ and} \\ Sh_j(N, w_2) &= a. \end{aligned}$$

Since $v(S) = w_1(S) + w_2(S)$ for all $S \subset N$ and the Shapley value is additive on the characteristic function, Claim 1 holds.

Claim 2. Given $j \in N$, $\varphi_i(N_0, C) \leq \varphi_i\left((N \setminus \{j\})_0, C\right)$ for all $i \in N \setminus \{j\}$.

By simplicity assume that $j = n$. Let $a \in \mathbb{R}$ be such that $a > \max_{i,j \in N_0} \{c_{ij}\}$. Take $b = a + 1$. Let C'^0 be such that $c'^0_{n0} = a$ and $c'^0_{ij} = c_{ij}$ otherwise, Let C'^1 be such that $c'^1_{n1} = b$ and $c'^1_{ij} = c'^0_{ij}$ otherwise. In general, for all $k = 2, ..., n-1$ we define $c'^k_{nk} = b$ and $c'^k_{ij} = c'^{k-1}_{ij}$ otherwise.

Take $i \in N \setminus \{j\}$. Since φ satisfies SCM,

$$\varphi_i(N_0, C) \leq \varphi_i\left(N_0, C'^0\right) \leq \varphi_i\left(N_0, C'^1\right) \leq .... \leq \varphi_i\left(N_0, C'^{n-1}\right).$$

By Claim 1, $\varphi_i(N_0, C'^{n-1}) = \varphi_i\left((N \setminus \{j\})_0, C'^{n-1}\right)$. Since $c'^{n-1}_{kl} = c_{kl}$ for all $k, l \in (N \setminus \{j\})_0$ we conclude that

$$\varphi_i\left((N \setminus \{j\})_0, C'^{n-1}\right) = \varphi_i\left((N \setminus \{j\})_0, C\right).$$

Then, $\varphi_i(N_0, C) \leq \varphi_i\left((N \setminus \{j\})_0, C\right)$.

Continuity (CON). Since C^* is a continuous function of C, v_{C^*} is a continuous function of C^*, and Sh is a continuous function of v_{C^*}, φ is a continuous function of C.

Separability (SEP). We know that φ satisfies PM. Since PM implies SEP, the result holds.

Core selection (CS). We know that φ satisfies PM. Since PM implies CS, the result holds.

Equal Sharing of Extra Costs $(ESEC)$. Let (N_0, C) and (N_0, C') be as in the definition of $ESEC$. Assume that t^0 is a $mcst$ in (N_0, C). Then, t^0 is also a $mcst$ in (N_0, C').

If $t = \{(i_{p-1}, i_p)\}_{p=1}^n$ is the tree given by the algorithm when we apply to (N_0, C), t is also the tree given by the algorithm when we apply to (N_0, C'). Moroever, $c^*_{0i_1} = c_0$, $c'^*_{0i_1} = c'_0$, and $c^*_{i_{p-1}i_p} = c'^*_{i_{p-1}i_p}$ for all $p = 2, ..., n$.

By Proposition 1 (a), $v_{C'^*}(S) = v_{C^*}(S) + (c'_0 - c_0)$ for all $S \subset N$.

Then, for all $i \in N$

$$\begin{aligned} \varphi_i(N_0, C') &= Sh_i(N, v_{C'^*}) = Sh_i(N, v_{C^*}) + \frac{c'_0 - c_0}{n} \\ &= \varphi_i(N_0, C) + \frac{c'_0 - c_0}{n}. \end{aligned}$$

Positivity (POS). Given $S \subset N$, by Proposition 3 (b), $v_{C^*}(S) - v_{C^*}(S \setminus \{i\}) \geq 0$ for all $i \in N$. Then, $\varphi_i(N_0, C) = Sh_i(N, v_{C^*}) \geq 0$.

Equal contributions (EC). We will prove it in Corollary 4.

7.7 Proof of the table

About Bird's rule B.

- CS. YES. See Bird (1976).
- CM. NO. See Dutta and Kar (2002).
- SCM. NO because B does not satisfy CM.
- PM. NO. Let (N_0, C) be such that $N = \{1, 2, 3\}$, $c_{10} = 30$, $c_{20} = 100$, $c_{30} = 10$, $c_{12} = 10$, $c_{13} = 40$, and $c_{23} = 20$.

  Making some computations we obtain that $B_2(N_0, C) = 20$ but $B_2(\{1,2\}_0, C) = 10$.
- CON. NO. See Example 1.
- POS. YES. It is trivial.
- SEP. YES. Given $\pi \in \Pi_N$, let π_S denote the order induced by π in S. It is trivial to see that $B_i^{\pi}(N_0, C) = B_i^{\pi_S}(S_0, C)$ when $i \in S$ and $B_i^{\pi}(N_0, C) = B_i^{\pi_{N \setminus S}}((N \setminus S)_0, C)$ when $i \in N \setminus S$. Since

$$B_i(N_0, C) = \frac{1}{n!} \sum_{\pi \in \Pi_N} B_i^{\pi}(N_0, C)$$

  making some computations we can conclude that B satisfies SEP.
- We only make the proof when the *mcst* in (N_0, C) is unique. Let $t' = \{(i, i^0)\}_{i \in S}$ and $t'' = \{(i, i^0)\}_{i \in N \setminus S}$ be two *mcst* in (S_0, C) and $((N \setminus S)_0, C)$ respectively. Then, $t = t' \cup t''$ is a *mcst* in (N_0, C). Now it is easy to conclude that the result holds.

- $ESEC$. YES. Let (N_0, C) and (N_0, C') be as in the defintion os $ESEC$. It is sraightforward to see that for all $\pi \in \Pi_N$,

$$B_i^\pi (N_0, C') = \begin{cases} B_i^\pi (N_0, C) + (c_0' - c_0) & \text{if } i = \pi (1) \\ B_i^\pi (N_0, C) & \text{otherwise.} \end{cases}$$

  Now it is easy to conclude that B satisfies $ESEC$.

- IIT. NO. See Example 1 and Corollary 2 (a).

- EC. NO. It is a trivial consequence of Theorem 2.

About Kar's rule K.

- CS. NO. Let (N_0, C) be such that $N = \{1, 2, 3\}$, $c_{10} = 10$, $c_{20} = 18$, $c_{30} = 9$, $c_{12} = 10$, $c_{13} = 10$, and $c_{23} = 9$. Making some computations we obtain that $K(N_0, C) = (8.66, 12.16, 7.16)$ and $v(\{1, 2\}) = 20$.

- CM. YES. If c_{ij} increases, $v_C (S \cup i) - v_C (S)$ does not decrease. Then, $K_i (N_0, C) = Sh_i (N, v_C)$ does not decrease.

- SCM. NO. Later we will prove that K does not satisfy IIT and by Corollary 2 (b) SCM implies IIT.

- PM. NO. K does not satisfy CS and PM implies CS.

- CON. YES because $K(N_0, C) = Sh(N, v_C)$.

- POS. NO. See Example 1 with $x = 100$.

- SEP. NO. Let (N_0, C) be such that $N = \{1, 2, 3\}$, $c_{10} = 10$, $c_{20} = 100$, $c_{30} = 20$, $c_{12} = 10$, $c_{13} = 100$, and $c_{23} = 40$. Take $S = \{1, 2\}$. Then, $K_1 (S_0, C) = -35$ but $K_1 (N_0, C) = -15$.

- $ESEC$. YES. Let (N_0, C) and (N_0, C') be as in the definition of $ESEC$. It is easy to see that $v_{C'} (S) = v_C (S) + (c_0' - c_0)$ for all $S \subset N$. Then,

$$\begin{aligned} K_i (N_0, C) &= Sh_i (N, v_{C'}) = Sh_i (N, v_C) + \frac{c_0' - c_0}{n} \\ &= K_i (N_0, C) + \frac{c_0' - c_0}{n}. \end{aligned}$$

- IIT. NO. See Example 1 and Corollary 2 (b).
- EC. NO. It is a trivial consequence of Theorem 2.

About Dutta-Kar's rule DK.

- CS. YES. See Dutta and Kar (2002).
- CM. YES. See Dutta and Kar (2002).
- SCM. NO. Later we will prove that DK does not satisfy IIT and SCM implies IIT.
- PM. NO. Let (N_0, C) be such that $N = \{1, 2, 3\}$ and $c_{10} = 10$, $c_{20} = 100$, $c_{30} = 8$, $c_{12} = 2$, $c_{13} = 5$, and $c_{23} = 4$. Making some computations we obtain that $DK_1 (N_0, C) = 8 > 2 = DK_1 ((\{1, 2\})_0 , C)$. Then, DK does not satisfy PM.
- CON. NO. See Example 1.
- POS. YES. It is trivial.
- SEP. YES. Assume that when we apply Prim's algorithm in (N_0, C) the order induced is $\pi = \{i_1, ..., i_n\}$. Applying an induction argument over p^q it is easy to prove that if $i_q \in S$ and $x_q = \min \left\{p^q, c_{i_{q+1}i^*_{q+1}}\right\} = c_{i_{q+1}i^*_{q+1}} < p^q$ then, $i_{q+1} \in S$. This means that "switches costs" among an agent of S and an agent of $N \setminus S$ are not possible.

  Now it is easy to see that $DK_i^\pi (N_0, C) = DK_i^{\pi_S} (S_0, C)$ when $i \in S$ and $DK_i^\pi (N_0, C) = DK_i^{\pi_{N \setminus S}} ((N \setminus S)_0 , C)$ when $i \in N \setminus S$. Since

  $$DK_i (N_0, C) = \frac{1}{n!} \sum_{\pi \in \Pi_N} DK_i^\pi (N_0, C)$$

  making some computations we can conclude that DK satisfies SEP.
- $ESEC$. NO. Take (N_0, C) and (N_0, C') such that $c_{10} = c_{20} = c_{30} = 10$, $c'_{10} = c'_{20} = c'_{30} = 13$, $c_{12} = c_{13} = c'_{12} = c'_{13} = 10$, and $c_{23} = c'_{23} = 6$. Then, $B (N_0, C) = (10, 8, 8)$ and $B (N_0, C') = (12, 8.5, 8.5)$.
- IIT. NO. See Example 1 and Corollary 2 (a).
- EC. NO. It is a trivial consequence of Theorem 2.

7.8 Proof of Theorem 2

(a) By Theorem 1 we know that φ satisfies SEP, $ESEC$, and IIT.

We now prove the uniqueness. Let ψ be a rule satisfying these properties. We apply an induction argument over n. If $n = 1$, $\psi = \varphi$. Assume that the result holds for all $n \leq p - 1$. We prove it for $n = p$.

Since ψ satisfies IIT, by Corollary 2 (a), we can restrict to canonical matrices. By simplicity assume that the tree t associated to the canonical matrix C satisfying $(A1)$ is $\{(i-1,i)\}_{i=1}^{n}$. Let $j \in N$ be such that $c_{(j-1)j} = \max_{i \in N} \left\{c_{(i-1)i}\right\}$. We distinguish two cases:

Case 1. $j > 1$. Take $S = \{1, ..., j-1\}$. By Proposition 1 (a) we know that $t' = \{(i-1,i)\}_{i=1}^{j-1}$ is a $mcst$ in (S_0, C) and $t' = \{(0,j)\} \cup \{(i-1,i)\}_{i=j+1}^{n}$ is a $mcst$ in $((N \setminus S)_0, C)$. Then, $mcst\,(S_0, C) = \sum_{i=1}^{j-1} c_{(i-1)i}$ and $mcst\,((N \setminus S)_0, C) = c_{0j} + \sum_{i=j+1}^{n} c_{(i-1)i}$.

Since C is a canonical matrix, $c_{0j} = \max_{i \leq j} \left\{c_{(i-1)i}\right\} = c_{(j-1)j}$. Then,

$$mcst\,(S_0, C) + mcst\,((N \setminus S)_0, C) = \sum_{i=1}^{n} c_{(i-1)i} = mcst\,(N_0, C).$$

By SEP, $\psi_i(N_0, C) = \psi_i(S_0, C)$ when $i \in S$ and $\psi_i(N_0, C) = \psi_i((N \setminus S)_0, C)$ when $i \notin S$.

We know that $S \neq \emptyset$ and $N \setminus S \neq \emptyset$ because $j > 1$. By induction hypothesis, $\psi(S_0, C) = \varphi(S_0, C)$ and $\psi((N \setminus S)_0, C) = \varphi((N \setminus S)_0, C)$.

Since φ satisfies SEP we can conclude that $\psi(N_0, C) = \varphi(N_0, C)$.

Case 2. $j = 1$. Let $k \in N \setminus \{1\}$ be such that $c_{(k-1)k} = \max_{i>1} \left\{c_{(i-1)i}\right\}$. If $c_{(k-1)k} = c_{(j-1)j}$ using similar arguments to those used in Case 1 we can conclude that $\psi(N_0, C) = \varphi(N_0, C)$.

Assume that $c_{(k-1)k} < c_{(j-1)j}$.

Consider now the $cstp$ (N_0, C^1) where $c^1_{il} = c_{il}$ if $0 \notin \{i, l\}$ and $c^1_{i0} = c_{i0} - x$ where $x = c_{(j-1)j} - c_{(k-1)k} > 0$. Since $c_{i0} = c_{(j-1)j}$ for all $i \in N$ we can apply the property of $ESEC$ to problems (N_0, C^1) and (N_0, C). Then, for all $i \in N$,

$$\begin{aligned}
\psi_i(N_0, C) &= \psi_i\left(N_0, C^1\right) + \frac{x}{n} \text{ and} \\
\varphi_i(N_0, C) &= \varphi_i\left(N_0, C^1\right) + \frac{x}{n}.
\end{aligned}$$

It is straightforward to prove that C^1 is a canonical matrix satisfying that $c^1_{(k-1)k} = \max_{i \in N} \left\{ c^1_{(i-1)i} \right\}$. Applying Case 1 to C^1 we conclude that $\psi(N_0, C^1) = \varphi(N_0, C^1)$. Then, $\psi(N_0, C) = \varphi(N_0, C)$.

(b) By Theorem 1 we know that φ satisfies EC.

We now prove the uniqueness. Let ψ be a rule satisfying EC. We prove that $\varphi = \psi$ by induction on n. If $n = 1$ it is trivial to see that $\varphi = \psi$. Assume that the result holds when $n \leq p - 1$ and we prove it when $n = p$.

Given $i, j \in N$, by simplicity we writte $\varphi_i = \varphi_i(N_0, C)$, $\psi_i = \psi_i(N_0, C)$, $\varphi_i^{+j} = \varphi_i\left((N \setminus \{j\})_0, C^{+j}\right)$, and $\psi_i^{+j} = \psi_i\left((N \setminus \{j\})_0, C^{+j}\right)$.

EC can be written, for ψ, as

$$\psi_i - \psi_i^{+j} = \psi_j - \psi_j^{+i}.$$

Then,

$$\sum_{j \in N \setminus \{i\}} \psi_i - \sum_{j \in N \setminus \{i\}} \psi_i^{+j} = \sum_{j \in N \setminus \{i\}} \psi_j - \sum_{j \in N \setminus \{i\}} \psi_j^{+i}.$$

Since $\sum_{j \in N \setminus \{i\}} \psi_i = (n-1)\psi_i$ and $\sum_{j \in N \setminus \{i\}} \psi_j = m(N_0, C) - \psi_i$ (remember that any rule must satisfy $\sum_{j \in N} \psi_j = m(N_0, C)$),

$$n\psi_i = m(N_0, C) + \sum_{j \in N \setminus \{i\}} \psi_i^{+j} - \sum_{j \in N \setminus \{i\}} \psi_j^{+i}.$$

Since φ also satisfies EC,

$$n\varphi_i = m(N_0, C) + \sum_{j \in N \setminus \{i\}} \varphi_i^{+j} - \sum_{j \in N \setminus \{i\}} \varphi_j^{+i}.$$

By induction hypothesis, for all $i, j \in N$, $\psi_i^{+j} = \varphi_i^{+j}$ and $\psi_j^{+i} = \varphi_j^{+i}$. Then, $\varphi_i = \psi_i$ for all $i \in N$.

7.9 Proof of Remark 5

We prove that new rules appear if we remove some of the properties of Corollary 3.

SCM. Consider the following subset of permutations:

$$\Pi' = \{\pi \in \Pi_N \mid \pi(i) < \pi(j) \text{ when } c_{i0} < c_{j0}\}.$$

For each *cstp* (N_0, C) and $i \in N$ we define

$$\psi_i^1(N, C) = \frac{1}{|\Pi'|} \sum_{\pi \in \Pi'} \left(v_{C^*}\left(Pre(i, \pi) \cup \{i\}\right) - v_{C^*}\left(Pre(i, \pi)\right)\right).$$

It is not difficult to prove that ψ^1 satisfies PM and $ESEC$.

Nevertheless ψ^1 does not satisfy SCM. Let (N_0, C) and (N_0, C') be such that $N_0 = \{1, 2, 3\}$, $c_{10} = 2$, $c_{20} = 2$, $c_{30} = 15$, $c_{12} = 9$, $c_{13} = 1$, $c_{23} = 1$, $c'_{20} = 3$, and $c'_{ij} = c_{ij}$ if $(i, j) \neq (2, 0)$. Then, $\psi_2^1(N, C') = 1 < 1.5 = \psi_2^1(N, C)$.

PM. Let ψ^2 the egalitarian rule, *i.e.*

$$\psi_i^2(N, C) = \frac{m(N_0, C)}{n} \text{ for all } i \in N.$$

It is trivial to see that ψ^2 satisfies SCM and $ESEC$.

Nevertheless ψ^2 does not satisfy PM. Let (N_0, C) be such that $N_0 = \{1, 2\}$, $c_{10} = 2$, $c_{20} = 4$, and $c_{12} = 6$. Then, $\psi_1^2(\{1\}, C) = 2 < 3 = \psi_1^2(N, C)$.

$ESEC$. Given $S \subset \mathcal{N} = \{1, 2, ...\}$ we denote by π^S the order induced in S by the index of the agents, *i.e.* given $i, j \in S$, $\pi^S(i) < \pi^S(j)$ if and only if $i < j$.

For each *cstp* (N_0, C) and $i \in N$ we define

$$\psi_i^3(S, C) = v_{C^*}\left(Pre\left(i, \pi^S\right) \cup \{i\}\right) - v_{C^*}\left(Pre\left(i, \pi^S\right)\right).$$

It is not dificult to see that ψ^3 satisfies SCM and PM.

Nevertheless ψ^2 does not satisfy $ESEC$. Let (N_0, C) be as in Example 1 when $x = 0$ and C'such that $c'_{10} = 12$ and $c'_{20} = 12$. Then, $\psi^3(N, C) = (10, 2)$ whereas $\psi^3(N, C) = (12, 2)$.

7.10 Proof of Proposition 4

(a) Assume that $t = \{(i-1,i)\}_{i=1}^{n}$ is the tree associated to C satisfying $(A1)$. Take $S = \{i_1, ..., i_{|S|}\}$ where $i_{p-1} \leq i_p$ for all $p = 2, ..., |S|$.

For each $p = 1, ..., |S|$ we consider the following sets

$$S_p = \{i \in N \mid i_{p-1} < i < i_p\}.$$

$S_p \subset N \setminus S$ for all $p = 1, ..., |S|$, $\bigcup_{p=1}^{|S|} S_p = N \setminus S$, and $S_p \cap S_q = \emptyset$ for all $p \neq q$. Notice that S_p can be empty.

We know that $v^+(N \setminus S) = m\left((N \setminus S)_0, C^{+S}\right) = c\left((N \setminus S)_0, C^{+S}, t'\right)$ where t' is computed following Prim's algorithm. We now compute t'.

Assume that $i \in S_p$, $j \in S_q$ and $p < q$. Then, $i < i_p \leq i_{q-1} < j$. By Proposition 3 (b), $c_{i0}^{+S} = \min_{k \in S_0}\{c_{ik}\} = \min\{c_{i_{p-1}i}, c_{ii_p}\} \leq c_{ii_p}$: Moreover, $c_{ij}^{+S} = c_{ij}$. Since t satisfies $(A2)$ we conclude that $c_{ii_p} \leq c_{ij}$ and hence $c_{i0}^{+S} \leq c_{ij}^{+S}$. This means that we can construct t' such that there is no arc linking directly agents of S_p and S_q.

Then, t' can be expressed as $\bigcup_{p=1}^{s} t'_{(S_p)_0}$ where $t'_{(S_p)_0}$ is a *mcst* computed following Prim's algorithm in $\left((S_p)_0, C^{+S}\right)$. If $S_p = \emptyset$ we take $t'_{(S_p)_0} = \emptyset$.

We now apply Prim's algorithm to $\left((S_p)_0, C^{+S}\right)$. We first select an arc $(0,i)$ such that $c_{0i}^{+S} = \min_{j \in S_p}\{c_{0j}^{+S}\}$. Since t satisfies $(A2)$ we conclude that $c_{0i}^{+S} = \min\{c_{i_{p-1}i}, c_{ii_p}\}$.

Since t satisfies $(A2)$,

$$\min_{j \in S_p}\{c_{0j}^{+S}\} = \min\{c_{i_{p-1}(i_{p-1}+1)}, c_{(i_p-1)i_p}\} = \min\left\{c_{0(i_{p-1}+1)}^{+S}, c_{(i_p-1)0}^{+S}\right\}.$$

Assume that the arc selected is $(0, i_{p-1}+1)$ (the case where the arc selected is $(i_p - 1, 0)$ is similar). Then, $c_{0(i_{p-1}+1)}^{+S} = c_{i_{p-1}(i_{p-1}+1)}$, *i.e.* this arc corresponds, in (N_0, C), with the arc $(i_{p-1}, i_{p-1}+1)$.

Using arguments similar to those used before we can conclude that the second arc selected is the arc (i,j) such that $c_{ij}^{+S} = \arg\min\{c_{(i_{p-1}+1)(i_{p-1}+2)}, c_{(i_p-1)i_p}\}$.

If we continue with this procedure we obtain that the cost of the tree $t'_{(S_p)_0}$ in $\left((S_p)_0, C^{+S}\right)$ coincide with the cost of the network $t_{S_p \cup \{i_{p-1}, i_p\}} \setminus \{(i_{p'-1}, i_{p'})\}$ in (N_0, C) where $(i_{p'-1}, i_{p'})$ satisfies that

$$c_{i_{p'-1}i_{p'}} = \max_{(i,j) \in t_{S_p \cup \{i_{p-1}, i_p\}}} c_{ij}.$$

Since t satisfies $(A2)$ we conclude that $c_{i_{p'-1}i_{p'}} = c_{i_{p-1}i_p}$. Then,

$$m\left((S_p)_0, C^{+S}\right) = \sum_{j=i_{p-1}+1}^{i_p} c_{(j-1)j} - c_{i_{p-1}i_p}.$$

Notice that if $S_p = \emptyset$ and we compute $m\left((S_p)_0, C^{+S}\right)$ according with the previous formula we obtain that

$$m\left((S_p)_0, C^{+S}\right) = c_{i_{p-1}i_p} - c_{i_{p-1}i_p} = 0.$$

Thus,

$$\begin{aligned} v^+(N \setminus S) &= c\left((N \setminus S)_0, C^{+S}, t'\right) = \sum_{p=1}^{|S|} m\left((S_p)_0, C^{+S}\right) \\ &= \sum_{p=1}^{|S|} \left(\sum_{j=i_{p-1}+1}^{i_p} c_{(j-1)j} - c_{i_{p-1}i_p} \right) \\ &= \sum_{p=1}^{|S|} \sum_{j=i_{p-1}+1}^{i_p} c_{(j-1)j} - \sum_{p=1}^{|S|} c_{i_{p-1}i_p} \\ &= \sum_{j=1}^{n} c_{(j-1)j} - \sum_{p=1}^{|S|} c_{i_{p-1}i_p}. \end{aligned}$$

Since t is a *mcst* of (N_0, C), $\sum_{j=1}^{n} c_{(j-1)j} = m(N_0, C)$. Moreover, by Proposition 3 (a), $\sum_{p=1}^{|S|} c_{i_{p-1}i_p} = v(S)$. This concludes the proof of (a).

(b) In order to prove that φ^1 satisfies IIT it is enough to prove that if (N_0, C) and (N_0, C') are directly equivalents then, $\varphi^1(N_0, C) = \varphi^1(N_0, C')$. Assume that $t = \{(i, i^0)\}_{i=1}^n$ is a *mcst* of (N_0, C) and (N_0, C') satisfying that $c_{ii^0} = c'_{ii^0}$ for all $i = 1, ..., n$.

We proceed by induntion on n. If $n = 1$ it is trivial to see that $\varphi^1(N_0, C) = \varphi^1(N_0, C')$. Assume that the result holds when $n \leq \alpha - 1$. We now prove that it is true also when $n = \alpha$.

In order to simplify the notation we take $N_{-i} = N \setminus \{i\}$.

We prove it in several claims.

Claim 1. Given a *cstp* (N_0, C), $S \subset N$, and $j \in N \setminus S$, $\left(S_0, C^{+N\setminus S}\right) = \left(S_0, \left(C^{+j}\right)^{+(N_{-j}\setminus S)}\right)$.

Take $i, k \in S$ such that $i \neq 0$ and $k \neq 0$. Then,

$$c_{ik}^{+N\setminus S} = c_{ik} = \left(c_{ik}^{+j}\right)^{+(N_{-j}\setminus S)}.$$

Given $i \in S$,

$$\begin{aligned}
c_{i0}^{+N\setminus S} &= \min_{k\in(N\setminus S)_0} \{c_{ik}\} \\
&= \min\left\{\min_{k\in(N_{-j}\setminus S)} \{c_{ik}\}, \min\{c_{i0}, c_{ij}\}\right\} \\
&= \min\left\{\min_{k\in(N_{-j}\setminus S)} \left\{c_{ik}^{+j}\right\}, c_{i0}^{+j}\right\} \\
&= \min_{k\in(N_{-j}\setminus S)_0} \left\{c_{ik}^{+j}\right\} \\
&= \left(c_{i0}^{+j}\right)^{+(N_{-j}\setminus S)}.
\end{aligned}$$

Thus, $C^{+N\setminus S} = \left(C^{+j}\right)^{+(N_{-j}\setminus S)}$.

Claim 2. Assume that t^* is a *mcst* in (N_0, C) and $j \in N$. Let $g = \{(i_{p-1}, i_p)\}_{p=1}^{r}$ be the unique path in t^* from $0\ (= i_0)$ to $j\ (= i_r)$. Assume that $c_{i_{q-1}i_q} = \max_{p=1,\ldots,r} \left\{c_{i_{p-1}i_p}\right\}$. Then, $t^* \setminus \{(i_{q-1}, i_q)\}$ is a *mcst* in $\left((N_{-j})_0, C^{+j}\right)$.

We denote $t_q^* = t^* \setminus \{(i_{q-1}, i_q)\}$. Then, t_q^* can be identified with a tree in $\left((N_{-j})_0, C^{+j}\right)$ simply by changing the arc (k, j), when $(k, j) \in t_q^*$, by $(k, 0)$.

Suppose that t_q^* is not a *mcst* in $\left((N_{-j})_0, C^{+j}\right)$. Then, it exists a tree t' in $\left((N_{-j})_0, C^{+j}\right)$ such that

$$c\left((N_{-j})_0, C^{+j}, t'\right) < c\left((N_{-j})_0, C^{+j}, t_q^*\right).$$

Let S_j denote the set of agents of N_{-j} who are conected to the source in t' through agent j. We now define S_j formally. For each $i \in N_{-j}$, let $\{(0, l_1), (l_1, l_2), ..., (l_{p-1}, i)\}$ be the unique path in t' from the source to i. Then,

$$S_j = \left\{i \in N_{-j} \mid c_{l_1 0}^{+j} = c_{l_1 j} \text{ and } (l_1, j) \in t^*\right\}.$$

Notice that there is no arc $(i,k) \in t'$ such that $i \in S_j$, $k \in (N_{-j}) \setminus S_j$. Otherwise, t' have a cycle and could not be a tree in $\left((N_{-j})_0, C^{+j}\right)$. We can find $(i_{r-1}, i_r) \in g \subset t^*$ such that $i_{r-1} \in ((N_{-j}) \setminus S_j) \cup \{0\}$ and $i_r \in S_j \cup \{j\}$.

Now, $t' \cup \{(i_{r-1}, i_r)\}$ could be identified with a tree in (N_0, C) simply by changing the arcs $(l,0) \in t'$ such that $c_{l_1 0}^{+j} = c_{l_1 j}$ and $(l_1, j) \in t^*$ by the arc (l_1, j).

Since

$$\begin{aligned} c\left(N_0, C, t' \cup \{(i_{r-1}, i_r)\}\right) &= c\left((N_{-j})_0, C^{+j}, t'\right) + c_{i_{r-1} i_r}, \\ c\left(N_0, C, t^*\right) &= c\left((N_{-j})_0, C^{+j}, t_q^*\right) + c_{i_{q-1} i_q}, \text{ and} \\ c_{i_{r-1} i_r} &\leq c_{i_{q-1} i_q} \end{aligned}$$

we conclude that

$$c\left(N_0, C, t' \cup \{(i_{r-1}, i_r)\}\right) < c\left(N_0, C, t^*\right),$$

which is a contradiction because t^* is a *mcst* of (N_0, C).

Claim 3. Take $j \in N$. For all $i \in N_{-j}$

$$\varphi_i^1\left(N_{-j}, C^{+j}\right) = \varphi_i^1\left(N_{-j}, C'^{+j}\right).$$

Let $g = \{(i_{p-1}, i_p)\}_{p=1}^r$ be the unique path in t from 0 $(= i_0)$ to j $(= i_r)$. Take $c_{kk^0} = \max\limits_{p=1,\ldots,r} \left\{c_{i_{p-1} i_p}\right\}$. Since t is a *mcst* in (N_0, C) and (N_0, C') we conclude, by Claim 2, that $t \setminus \{(k, k^0)\}$ is a *mcst* of $\left((N_{-j})_0, C^{+j}\right)$ and $\left((N_{-j})_0, C'^{+j}\right)$ satisfying that $c_{ii^0} = c'_{ii^0}$ for all $i \in N \setminus \{k\}$. Then, $\left((N_{-j})_0, C^{+j}\right)$ and $\left((N_{-j})_0, C'^{+j}\right)$ are directly equivalents.

By induction hypothesis, $\varphi_i^1\left(N_{-j}, C^{+j}\right) = \varphi_i^1\left(N_{-j}, C'^{+j}\right)$.

Claim 4. Given a *cstp* (N_0, C) and $i, j \in N$ such that $i \neq j$,

$$\varphi_i^1\left(N_{-j}, C^{+j}\right) = Sh_i\left(N_{-j}, v^+\right).$$

For all $S \subset N_{-j}$,

$$\begin{aligned} v_C^+(S) &= m\left(S_0, C^{+N\setminus S}\right) \text{ and} \\ v_{C^{+j}}^+(S) &= m\left(S_0, \left(C^{+j}\right)^{+((N\setminus j)\setminus S)}\right). \end{aligned}$$

By Claim 1, $v_C^+(S) = v_{C^{+j}}^+(S)$. Then,

$$\varphi_i^1\left(N_{-j}, C^{+j}\right) = Sh_i\left(N_{-j}, v_{C^{+j}}^+\right) = Sh_i\left(N_{-j}, v_C^+\right).$$

Claim 5. For all $i, j \in N$, $i \neq j$,

$$Sh_i\left(N_{-j}, v_C^+\right) = Sh_i\left(N_{-j}, v_{C'}^+\right).$$

By Claim 4, $Sh_i\left(N_{-j}, v_C^+\right) = \varphi_i^1\left(N_{-j}, C^{+j}\right)$ and $Sh_i\left(N_{-j}, v_{C'}^+\right) = \varphi_i^1\left(N_{-j}, C'^{+j}\right)$. By Claim 3, $\varphi_i^1\left(N_{-j}, C^{+j}\right) = \varphi_i^1\left(N_{-j}, C'^{+j}\right)$. Thus, Claim 5 holds.

Claim 6. For all $i \in N$, $\varphi_i^1\left(N_0, C\right) = \varphi_i^1\left(N_0, C'\right)$.

In Pérez-Castrillo and Wettstein (2001) appears the following expression of the Shapley value:

$$Sh_i\left(N, w\right) = \frac{1}{n}\left[w\left(N\right) - w\left(N_{-i}\right) + \sum_{j \in N_{-i}} Sh_i\left(N_{-j}, w\right)\right].$$

Since $w\left(N_{-i}\right) = \sum_{j \in N_{-i}} Sh_j\left(N_{-i}, w\right)$ we conclude that

$$Sh_i\left(N, w\right) = \frac{1}{n}\left[w\left(N\right) - \sum_{j \in N_{-i}} Sh_j\left(N_{-i}, w\right) + \sum_{j \in N_{-i}} Sh_i\left(N_{-j}, w\right)\right].$$

Since $\varphi^1\left(N_0, C\right) = Sh\left(N, v_C^+\right)$ and $\varphi^1\left(N_0, C'\right) = Sh\left(N, v_{C'}^+\right)$,

$$\varphi_i^1\left(N_0, C\right) = \frac{1}{n}\left[v_C^+\left(N\right) - \sum_{j \in N_{-i}} Sh_j\left(N_{-i}, v_C^+\right) + \sum_{j \in N_{-i}} Sh_i\left(N_{-j}, v_C^+\right)\right] \text{ and}$$

$$\varphi_i^1\left(N_0, C'\right) = \frac{1}{n}\left[v_{C'}^+\left(N\right) - \sum_{j \in N_{-i}} Sh_j\left(N_{-i}, v_{C'}^+\right) + \sum_{j \in N_{-i}} Sh_i\left(N_{-j}, v_{C'}^+\right)\right].$$

Since $v_C^+\left(N\right) = v_{C'}^+\left(N\right) = m\left(N_0, C\right) = m\left(N_0, C'\right)$ and Claim 5 we conclude that $\varphi_i^1\left(N_0, C\right) = \varphi_i^1\left(N_0, C'\right)$.

(c) We must prove that for all $i, j \in N$, $i \neq j$,

$$\varphi_i^1\left(N_0, C\right) - \varphi_i^1\left(\left(N_{-j}\right)_0, C^{+j}\right) = \varphi_j^1\left(N_0, C\right) - \varphi_j^1\left(\left(N_{-i}\right)_0, C^{+i}\right).$$

Given a TU game (N, w) Myerson (1980) proved that the Shapley value satisfies that for all $i, j \in N$, $i \neq j$,

$$Sh_i\left(N, w\right) - Sh_i\left(N_{-j}, w\right) = Sh_j\left(N, w\right) - Sh_j\left(N_{-i}, w\right).$$

We know that $\varphi^1(N_0, C) = Sh(N, v^+)$. By Claim 4 of part (b), $\varphi_i^1\left((N_{-j})_0, C^{+j}\right) = Sh_i(N_{-j}, v^+)$ and $\varphi_j^1\left((N_{-j})_0, C^{+i}\right) = Sh_j(N_{-i}, v^+)$.

Applying Myerson's result to the TU game (N, v^+) we obtain that φ^1 satisfies EC.

7.11 Proof of Theorem 3

We prove several claims:

Claim 1. $\varphi^1 = \varphi$.

By definition, $\varphi(N_0, C) = K(N_0, C^*) = Sh(N, v_{C^*})$. By Proposition 4 (b) and Corollary 2 (a) we conclude that $\varphi^1(N_0, C) = \varphi^1(N_0, C^*) = Sh\left(N, v_{C^*}^+\right)$.

By Proposition 4 (a), $v_{C^*}(S) + v_{C^*}^+(N \setminus S) = m(N_0, C)$ for all $S \subset N$. Since $v(N) = v^+(N) = m(N_0, C)$ and the Shapley value of a TU game (N, w) can be expressed for all $i \in N$ as

$$Sh_i(N, w) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \left(w\left(Pre(i, \pi) \cup \{i\}\right) - w\left(Pre(i, \pi)\right)\right)$$

it is easy to conclude that $\varphi(N_0, C) = K(N_0, C)$.

Claim 2. $\varphi = \varphi^3$.

We must prove that for all $i \in N$, $\varphi_i(N_0, C) = \frac{1}{n!} \sum\limits_{\pi \in \Pi_N} c_{ii^\pi}^*$. Since φ satisfies IIT it is enough to prove that if C is a canonical matrix then, $\varphi_i(N_0, C) = \frac{1}{n!} \sum\limits_{\pi \in \Pi_N} c_{ii^\pi}$.

We proceed in several claims.

Claim 2a. Let (N_0, C) be a *cstp* where C is a canonical matrix and $t = \{(i_{p-1}, i_p)\}_{p=1}^n$ is the tree given by $(A1)$. Assume that $c_{i_{p-1}i_p} \neq c_{i_{q-1}i_q}$ for all p, q such that $p \neq q$. Given $\pi \in \Pi_N$ we can find a one-to-one application $f_\pi : N \to t$ such that $f_\pi(i) = (i_{p-1}, i_p)$ satisfies $v\left(Pre(i, \pi) \cup \{i\}\right) - v\left(Pre(i, \pi)\right) = c_{i_{p-1}i_p} = c_{ii^\pi}$.

By simplicity we take $\pi = (1, 2, ..., n)$. Assume that $1 = i_q$. By Proposition 3 (b) we know that

$$v\left(Pre(1, \pi) \cup \{1\}\right) - v\left(Pre(1, \pi)\right) = v(1) = c_{i_0 i_q}.$$

Since C is a canonical matrix we can find $(i_{p_1-1}, i_{p_1}) \in t$ such that $p_1 \leq q$ and $c_{i_0 i_q} = c_{i_{p_1-1} i_{p_1}}$. Moreover $1^{\pi} = 0$ and $c_{11^{\pi}} = c_{i_0 i_q} = c_{i_{p_1-1} i_{p_1}}$. We define $f_{\pi}(1) = (i_{p_1-1}, i_{p_1})$.

Assume that we have defined $f_{\pi}(j)$ for all $j \leq i-1$ satisfying that $f_{\pi}(j) = \left(i_{p_j-1}, i_{p_j}\right) \in t$ for all $j \leq i-1$ and $f_{\pi}(j) \neq f_{\pi}(k)$ for all $j \leq i-1$, $k \leq i-1$, $j \neq k$.

We now define $f_{\pi}(i)$. Assume that $i = i_q$ and $Pre(i, \pi) = \left\{i_{q_1}, i_{q_2}, ..., i_{q_{i-1}}\right\}$ where $q_{l-1} \leq q_l$ for all $l = 2, ..., i-1$. Notice that we order the agents of $Pre(i, \pi)$ according with t. Two cases can occur:

Firstly, $q < q_{i-1}$. By Proposition 3 (b),

$$v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi)) = \min\left\{c_{i_{q_{k-1}} i_q}, c_{i_q i_{q_k}}\right\}$$

where $q_{k-1} < q < q_k$ and $k \leq i-1$.

Since t satisfies $(A2)$ and $c_{i_{p-1} i_p} \neq c_{i_{q-1} i_q}$ for all p, q such that $p \neq q$ we conclude that $c_{i_{q_{k-1}} i_q} \neq c_{i_q i_{q_k}}$. Assume that $\min\left\{c_{i_{q_{k-1}} i_q}, c_{i_q i_{q_k}}\right\} = c_{i_q i_{q_k}}$ (the other case is similar).

By Proposition 3 (b), for all $j \leq i$

$$c_{i_q i_{q_k}} \neq v(Pre(j, \pi) \cup \{j\}) - v(Pre(j, \pi)).$$

Then, we can find $(i_{p_i-1}, i_{p_i}) \in t \setminus \left(\bigcup_{j=1}^{i-1} \left(i_{p_j-1}, i_{p_j}\right)\right)$ such that $q \leq p_i - 1 < p_i \leq q_k$ and $c_{i_{p_i-1} i_{p_i}} = c_{i_q i_{q_k}}$. Moreover, $c_{ii^{\pi}} = \min_{j \in Pre(i,\pi)} \{c_{ij}\} = c_{i_q i_{q_k}}$.

We now define $f_{\pi}(i) = (i_{p_i-1}, i_{p_i})$.

Secondly, $q > q_{i-1}$. By Proposition 3 (b),

$$v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi)) = c_{i_{q_{i-1}} i_q}.$$

By Proposition 3 (b) we can find $(i_{p_i-1}, i_{p_i}) \in t \setminus \left(\bigcup_{j=1}^{i-1} \left(i_{p_j-1}, i_{p_j}\right)\right)$ such that $q_{i-1} \leq p_i - 1 < p_i \leq q$ and $c_{i_{p_i-1} i_{p_i}} = c_{i_{q_{i-1}} i_q}$. Moreover, $c_{ii^{\pi}} = \min_{j \in Pre(i,\pi)} \{c_{ij}\} = c_{i_{q_{i-1}} i_q}$.

We now define $f_{\pi}(i) = (i_{p_i-1}, i_{p_i})$.

Claim 2b. Let (N_0, C) be a *cstp* where C is a canonical matrix and $t = \{(i_{p-1}, i_p)\}_{p=1}^{n}$ is the tree given by $(A1)$. Assume that $c_{i_{p-1} i_p} = c_{i_{q-1} i_q}$ for some p, q such that $p \neq q$. Given $\pi \in \Pi_N$ we can find a one-to-one application $f_\pi : N \to t$ such that $f_\pi(i) = (i_{p-1}, i_p)$ satisfies $v(Pre(i,\pi) \cup \{i\}) - v(Pre(i,\pi)) = c_{i_{p-1}i_p} = c_{ii^\pi}$.

Let C^ε be a canonical matrix obtained from C modifying slighlitly the cost of the arcs $c_{i_{p-1}i_p}$ such that there exist $q \neq p$ with $c_{i_{p-1}i_p} = c_{i_{q-1}i_q}$. We modify these cost in such a way that if $c_{i_{p-1}i_p} < c_{i_{q-1}i_q}$ then, $c^\varepsilon_{i_{p-1}i_p} < c^\varepsilon_{i_{q-1}i_q}$ and if $c_{i_{p-1}i_p} = c_{i_{q-1}i_q}$ then $c^\varepsilon_{i_{p-1}i_p} \neq c^\varepsilon_{i_{q-1}i_q}$.

By Claim 2a, given the *cstp* (N_0, C^ε) and $\pi \in \Pi_N$ we can find a one-to-one application $f^\varepsilon_\pi = N \to t$ such that $f^\varepsilon_\pi(i) = (i_{p-1}, i_p)$ satisfy $v^\varepsilon(Pre(i,\pi) \cup \{i\}) - v^\varepsilon(Pre(i,\pi)) = c^\varepsilon_{i_{p-1}i_p} = c^\varepsilon_{ii^\pi}$.

Given the *cstp* (N_0, C) and $\pi \in \Pi_N$ we define $f_\pi = f^\varepsilon_\pi$. Of course, f_π is a one-to-one application. Since v is a continuous function of C we conclude that

$$v(Pre(i,\pi) \cup \{i\}) - v(Pre(i,\pi)) = c_{i_{p-1}i_p} = c_{ii^\pi}.$$

Claim 2c. If C is a canonical matrix, $\varphi_i(N_0, C) = \frac{1}{n!} \sum_{\pi \in \Pi_N} c_{ii^\pi}$ for all $i \in N$.

Since $\varphi(N_0, C) = Sh(N, v)$, for all $i \in N$,

$$\begin{aligned} \varphi_i(N_0, C) &= \frac{1}{n!} \sum_{\pi \in \Pi_N} \left(v(Pre(i,\pi) \cup \{i\}) - v(Pre(i,\pi)) \right) \\ &= \frac{1}{n!} \sum_{\pi \in \Pi_N} c_{ii^\pi}. \end{aligned}$$

Claim 3. $\varphi = \varphi^2$.

Before proving part Claim 3 we need an aditional result. Let (N_0, C) be a *cstp* where C is a canonical matrix and $t = \{(i_{p-1}, i_p)\}_{p=1}^n$ is the tree given by $(A1)$. For each $i \in N$ and $p = 1, ..., n$ we define

$$\Pi_i^p = \{\pi \in \Pi_N \mid f_\pi(i) = (i_{p-1}, i_p)\} \text{ and } \mu_i^p = \frac{|\Pi_i^p|}{n!}.$$

Claim 3a. Let (N_0, C) be a *cstp*. Given a *mcst* $t^0 = \{(j, j^0)\}_{j \in N}$, for all $i \in N$

$$\varphi_i(N_0, C) = \sum_{p=1}^n \mu_i^j c_{jj^0}.$$

where for all $i \in N$, $\sum_{j=1}^{n} \mu_i^j = 1$ and for all $j \in N$, $\sum_{i=1}^{n} \mu_i^j = 1$.

Let $t = \{(i_{p-1}, i_p)\}_{p=1}^{n}$ be the *mcst* associated to C^* given by the algorithm when we start with the tree t^0.

We define the one-to-one application $g : t^0 \to t$ such that $g(j, j^0) = (i_{p-1}, i_p)$ where $j = i_p$.

By claims $2a$, $2b$, and $2c$, for all $i \in N$, $\varphi_i(N_0, C) = \sum_{p=1}^{n} \mu_i^p c^*_{i_{p-1}i_p}$; $\sum_{p=1}^{n} \mu_i^p = 1$ for all $i \in N$; and $\sum_{i=1}^{n} \mu_i^p = 1$ for all $p = 1, ..., n$.

We proceed by induction on the number of agents n. If $n = 1$ then $\varphi = \varphi^2$. Assume that the result holds for all $n \leq p - 1$. We now prove for $n = p$.

Let $t = \{(i, i^0)\}_{i \in N}$ be a *mcst* in (N_0, C). We denote by $\varphi^2(N, t)$ the rule φ^2 computed when the set of agents is N and the tree starting the process is t. We will prove that $\varphi^2(N, t) = \varphi(N_0, C)$. Notice that this imply that φ^2 is well defined because φ is independent of t.

Take $j \in N$ such that $c_{jj^0} = \max_{i \in N} \{c_{ii^0}\}$. We assume that $c_{jj^0} > c_{ii^0}$ for all $i \in N \setminus \{j\}$. Otherwise we can achieve the same conclusion using similar ideas but in a more elaborate way.

We define $N' = \{i \in N \mid i^0 = 0\}$. We consider three cases:

Case 1. $|N'| \geq 2$. For any $i \in N'$ let F^i be the set of agents $j \in N$ such that $(i, 0)$ is in the unique path from j to 0. Then, $\{F^i\}_{i \in N'}$ is a partition of N satisfying that $\sum_{i \in N'} m((F_i)_0, C) = m(N_0, C)$ and t_{F^i} is a tree in (F^i, C).

Since φ satisfies SEP, for all $i \in N'$ and $k \in F^i$, $\varphi_k(N_0, C) = \varphi_k(F_0^i, C)$.

Because of the process defining φ^2 it is straightforward to prove that for all $i \in N'$ and $k \in F^i$, $\varphi_k^2(N, t) = \varphi_k^2(F^i, t_{F^i})$.

Since $|N'| \geq 2$ we can appy the induction hypotesis to each F^i. Then, for all $i \in N'$ and $k \in F^i$, $\varphi_k^2(F_0^i, t_{F^i}) = \varphi_k(F_0^i, C)$.

This means that φ^2 is well defined and $\varphi^2(N_0, C) = \varphi(N_0, C)$.

Case 2. $|N'| = 1$ and $j^0 \neq 0$. Let F be the set of agents $i \in N$ such that the arc (j, j^0) is in the unique path from i to 0 and $B = N \setminus F$. Notice that $F \neq \emptyset$ and $B \neq \emptyset$ because $j \in F$ and $j^0 \in B$.

Next statements are a trivial consequence of the process defining φ^2. If $a^s(i) = (j, j^0)$ for some step s and $i \in N$ then the only arc in $A^{s-1} = \{(j, j^0)\}$.

Since in t_B there is exactly $|B|$ arcs, $\sum_{s=1}^{\gamma} p^s = 1$, and the previous statements we conclude that the arcs of t_B are paid by agents in B and the arcs in $t \setminus t_B$ are paid by agents in F.

Now it is easy to conclude that $\varphi_i^2(N,t) = \varphi_i^2(B,t_B)$ for all $i \in B$. Moroever, if we take j^0 as the source in $(F, t \setminus t_B)$, $\varphi_i^2(N,t) = \varphi_i^2(F, t \setminus t_B)$ for all $i \in F$.

We can compute the canonical form C^* associated to (N_0, C) starting with the tree t. Then, we obtain that the tree $\{(i_{q-1}, i_q)\}_{q=1}^n$ associated to C^* satisfies that it exists $i_{|B|}$ such that $B = \{i_1, ..., i_{|B|}\}$, $F = \{i_{|B|+1}, ..., i_n\}$, $j^0 = i_{|B|}$, and $j = i_{|B|+1}$.

It is straightforward to prove that $m(N_0, C^*) = m(B_0, C^*) + m(F_0, C^*)$, t_B is a *mcst* in (B_0, C^*) and $t \setminus t_B$ is a *mcst* in (F_0, C^*).

Since φ satisfies SEP, $\varphi_i(N_0, C) = \varphi_i(B_0, C^*)$ for all $i \in B$ and $\varphi_i(N_0, C) = \varphi_i(F_0, C^*)$ for all $i \in F$.

By induction hypothesis we conclude that $\varphi_i^2(B, t_B) = \varphi_i(B_0, C^*)$ for all $i \in B$ and $\varphi_i^2(F, t \setminus t_B) = \varphi_i(F_0, C^*)$ for all $i \in F$.

Then, φ^2 is well defined and $\varphi^2(N_0, C) = \varphi(N_0, C)$.

Case 3. $|N'| = 1$ and $j^0 = 0$. Take $k \in N \setminus \{j\}$ such that $c_{kk^0} = \max_{i \in N \setminus \{j\}} \{c_{ii^0}\}$. We assume that $c_{kk^0} > c_{ii^0}$ for all $i \in N \setminus \{j, k\}$. Otherwise we can achieve the same conclusion using similar ideas but in a more elaborate way.

Let F be the set of agents $i \in N$ such that the arc (k, k^0) is in the unique path from i to 0 and $B = N \setminus F$. Notice that $F \neq \emptyset$ and $B \neq \emptyset$ because $k \in F$ and $k^0 \in B$.

We assume that $|B| \geq |F|$. The case $|B| < |F|$ can be proved in a similar way.

Next statements are a trivial consequence of the process defining φ^2. If $a^s(i) = (j, 0)$ for some Step s and $i \in N$ then $A^{s-1} = \{(j, 0)\}$. If $a^s(i) = (k, k^0)$ for some Step s and $i \in N$ then $A^{s-1} = \{(j, 0), (k, k^0)\}$. Moreover, the cost of the arcs in t_F are paid only by agents in F and the cost of arcs in $t_B \setminus \{(j, 0)\}$ are paid only by agents in B.

Thus, we can find α, $\alpha' < \gamma$ such that: In Step α, for all $(i, i') \in t_F$, $\varrho(i, i') = 1$, $\varrho(k, k^0) = \varrho(j, 0) = 0$, and $\sum_{s=1}^{\alpha} p^s = \frac{|F|-1}{|F|}$. In Stage α' for all $(i, i') \in t_B$, $\varrho(i, i') = 1$, $\varrho(j, 0) = 0$, and $\sum_{s=1}^{\alpha'} p^s = \frac{|B|-1}{|B|}$. Notice that $\alpha = \alpha'$ if and only if $|B| = |F|$.

We can compute the canonical form C^* startig with the tree t. Then we obtain that the tree $\{(i_{q-1}, i_q)\}_{q=1}^n$ associated to C^* satisfies that it exists $i_{|B|}$ such that $B = \left\{i_1, ..., i_{|B|}\right\}$, $F = \left\{i_{|B|+1}, ..., i_n\right\}$, $k^0 = i_{|B|}$, and $k = i_{|B|+1}$.

By Claim 3a, for all $i \in N$,

$$\varphi_i(N_0, C) = \sum_{q=1}^{n} \mu_i^q c_{qq^0}$$

where $\mu_i^q = \dfrac{|\Pi_i^q|}{n!}$ and

$$\Pi_i^q = \{\pi \in \Pi_N \mid v(Pre(i,\pi) \cup \{i\}) - v(Pre(i,\pi)) = c_{qq^0}\}.$$

Since φ^2 can be expressed as $\sum\limits_{q=1}^{n} \lambda_i^q c_{qq^0}$ we only need to prove that for all $(q, q^0) \in t$, $\lambda_i^q = \mu_i^q$ for all $i \in N$.

Consider the problem (F_0, C^*). For each $i, q \in F$ we denote by $\mu_i^q(F)$ the corresponding function associated to (F_0, C^*). It is straightforward to prove that for all $q \in F \setminus \{k\}$ and $i \in F$, $\mu_i^q(F) = \mu_i^q$. Moreover, for all $q \in F \setminus \{k\}$ and $i \in B$, $\mu_i^q = 0$.

We know that $t_F \cup \{(k, 0)\}$ is a *mcst* in (F_0, C^*) and $c^*_{k0} = c_{j0} > c_{kk^0}$. For each $i, q \in F_0$ we denote by $\lambda_i^q(F)$ the corresponding function associated to $\varphi^2(F, t_F \cup \{(k, 0)\})$. We saw before that when we compute $\varphi^2(N, t)$ the cost of the arcs of t_F are paid only by agents in F. Then, for all $q \in F \setminus \{k\}$ and $i \in F$, $\lambda_i^q(F) = \lambda_i^q$. Moreover, for all $q \in F \setminus \{k\}$ and $i \in B$, $\lambda_i^q = 0$.

Take $q \in F \setminus \{k\}$ and $i \in F$. By induction hypothesis we know that $\mu_i^q(F) = \lambda_i^q(F)$ and hence, $\mu_i^q = \lambda_i^q$. Then, for all $q \in F \setminus \{k\}$ and $i \in N$, $\mu_i^q = \lambda_i^q$.

Using similar arguments to those used with F we can conclude that for all $q \in B \setminus \{j\}$ and $i \in N$, $\mu_i^q = \lambda_i^q$.

We only need to prove that $\mu_i^q = \lambda_i^q$ when $q \in \{j, k\}$ and $i \in N$. It is straightforward to prove that for all $i \in N$, $\mu_i^j = \lambda_i^j = \dfrac{1}{n}$.

We now prove that for all $i \in N$, $\mu_i^k = \lambda_i^k$.

Take $i \in F$. Π_i^k corresponds to the permutations satisfying that, in the order given by π, the first agent belongs to B and i is the first agent of F.

It is straightforward to prove that $\mu_i^k = \dfrac{|B|}{n\,|F|}$. Applying a similar argument to B we obtain that $\mu_i^k = \dfrac{|F|}{n\,|B|}$ if $i \in B$.

Take $i \in F$. From Step α to Step α' we have that $a^s(i) = (k, k^0)$ for all $i \in F$ whereas $a^s(i) \neq (k, k^0)$ for all $i \in B$. This means that from Step α to Step α' each agent $i \in F$ pays $\left(\frac{|B|-1}{|B|} - \frac{|F|-1}{|F|}\right) c_{kk^0}$. Then,

$$\varrho^{\alpha'}\left(k, k^0\right) = \left(1 - |F|\left(\frac{|B|-1}{|B|} - \frac{|F|-1}{|F|}\right)\right)$$

In Step $\alpha' + 1$, $a^{\alpha'+1}(i) = (k, k^0)$ for all $i \in N$. Then, each agent $i \in N$ pays

$$\frac{1}{n}\left(1 - |F|\left(\frac{|B|-1}{|B|} - \frac{|F|-1}{|F|}\right)\right)$$

and $\varrho^{\alpha'+1}(k, k^0) = 1$, which means that the cost of this arc is already paid. Then,

$$\begin{aligned}
\lambda_i^k &= \frac{1}{n}\left(1 - |F|\left(\frac{|B|-1}{|B|} - \frac{|F|-1}{|F|}\right)\right) \text{ if } i \in B \text{ and} \\
\lambda_i^k &= \left(\frac{|B|-1}{|B|} - \frac{|F|-1}{|F|}\right) + \frac{1}{n}\left(1 - |F|\left(\frac{|B|-1}{|B|} - \frac{|F|-1}{|F|}\right)\right) \text{ if } i \in F.
\end{aligned}$$

Making some computations we obtain that $\lambda_i^k = \mu_i^k$ for all $i \in N$.

Then, φ^2 is well defined and $\varphi^2(N_0, C) = \varphi(N_0, C)$.

Claim 4. $\varphi = P$.

Branzei *et al* (2003) characterizes P as the only rule satisfying efficiency (EF), equal treatment (ET), upper bound contributions (UBC), and cone-wise positive linearity ($CWPL$). We prove that φ satisfies these four properties.

EF means that $\sum_{i \in N} \varphi_i(N_0, C) = m(N_0, C)$ for all *cstp* (N_0, C). Of course, φ satisfies it.

Given a *cstp* (N_0, C), i and j are $C-$*connected* if there exists a path g from i to j satisfying that $c_{kl} = 0$ for all $(k, l) \in g$. $S \subset N$ is a $C-$ *component* if two conditions hold. First, for all $i, j \in S$, i and j are $C-$*connected*. Second, if $S \subsetneq T$ there exists $i, j \in T$ such that i and j are not $C-$ *connected*.

UBC means that for all $C-component$ S, $\sum_{i\in S}\varphi_i(N_0,C)\leq\min_{j\in S}\{c_{j0}\}$.

Let $P=\{S_1,...,S_p\}$ be the partition induced in N by $C-component$ coalitions. φ satisfies ET if for all $S_q\in P$ and all $i,j\in S_q$, $\varphi_i(N_0,C)=\varphi_j(N_0,C)$.

Let C^* be the canonical form associated to C. Because of the definition of the algorithm it is trivial to see that if S is a $C-component$ and $t=\{(i_{p-1},i_p)\}_{p=1}^n$ is the tree satisfying $(A1)$ then, there exist $i_q\in N$ such that $S=\left\{i_q,i_{q+1},...,i_{q+|S|-1}\right\}$, $c^*_{i_q0}\leq\min_{j\in S}\{c_{j0}\}$, and $c^*_{i_{p-1}i_p}=0$ for all $p=q+1,...q+|S|-1$.

Take $\pi\in\Pi_N$ and $i\in S$, by Proposition 3 (b),

$$v_{C^*}(Pre(i,\pi)\cup\{i\})-v_{C^*}(Pre(i,\pi))=\begin{cases}c^*_{i_q0} & \text{if } Pre(i,\pi)\cap S=\varnothing\\ 0 & \text{otherwise.}\end{cases}$$

We now conclude that φ satisfies UBC and ET because $\varphi(N_0,C)=Sh(N,v_{C^*})$.

$CWPL$. Let (N_0,C) and (N_0,C') be two *cstp* such that given $i,j,k,l\in N_0$, $c_{ij}\leq c_{kl}$ if and only if $c'_{ij}\leq c'_{kl}$. Take $x,x'\geq 0$.φ satisfies $CWPL$ if $\varphi(N_0,xC+x'C')=x\varphi(N_0,C)+x'\varphi(N_0,C')$ where xC is the cost matrix where the connection cost between i and j is xc_{ij}.

It is straightforward to prove that for all $S\subset N$,

$$v^+_{xC+x'C'}(S)=x\ v^+_C(S)+x'\ v^+_{C'}(S).$$

Since $\varphi=\varphi^1$ ant the Shapley value is additive on the characteristic function we conclude that φ satisfies $CWPL$.

8 References

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