

# Information and Its Value in Zero-Sum Repeated Games

Ehud Lehrer\* and Dinah Rosenberg<sup>†</sup>

November 18, 2003

## Abstract

Two players play an unknown zero-sum repeated game. Before the game starts one player may receive signals, whose nature is specified by an information structure, regarding the game actually played. We characterize when one information structure is better for the maximizer than another. We also characterize those functions defined on partitions that determine the equilibrium payoff when one player is informed about the cell of the partition that contains the realized state.

---

\*School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel; e-mail: lehrer@post.tau.ac.il; home page: [www.math.tau.ac.il/~lehrer](http://www.math.tau.ac.il/~lehrer)

<sup>†</sup>LAGA Institut Galilée, Université Paris 13, avenue Jean Baptiste Clément 93430 Villetaneuse, France; e-mail: [dinah@math.univ-paris13.fr](mailto:dinah@math.univ-paris13.fr)

# 1 Introduction

In a Bayesian game players have a prior distribution regarding the game played. Any information that the players may receive about this game changes their posterior distribution and hence their behavior. Thus, the information structure of the game determines its equilibrium payoffs. The purpose of this paper is to examine the role of information structures in repeated Bayesian games.

We study two issues: comparison of different information structures and measurement of the contribution of the information received. The first issue was addressed by Blackwell (1951, 1953) in a model of the one-person decision problem. A state of nature is randomly chosen with a given probability. Then, a decision maker obtains a signal that depends on the state selected. This signal is stochastically determined by an information structure. Upon receiving the signal, the decision maker takes an action and receives a payoff that depends on the action chosen as well as on the state of nature. It is said that the information structure, say,  $I$ , is *better than* another, say,  $I'$  if, regardless of the payoff function,  $I$  provides the decision maker with the ability to obtain at least as high a payoff as he could obtain had the signal been determined by  $I'$ . Blackwell showed that  $I$  is better than  $I'$  if and only if by garbling the signals obtained from  $I$ , the decision maker can reproduce a signal that appears like the signals received through  $I'$ .

A lot of effort has been made to extend Blackwell's notion to interactive models. It is well known that in some games, players might prefer dropping payoff-relevant information, because their equilibrium payoff would then be higher (see Kamien et al. (1990)). Bassan et al. (1999) presented conditions under which having more information always improves all players' payoffs. Neyman (1991) pointed out that a player might prefer not receiving information because other players would know that he was receiving this information. Gossner (2000) compared information structures through the correlated equilibrium distribution they induce.

As a first step towards a more comprehensive understanding of the role of information, we consider zero-sum games. Examining the role of information is easier in zero-sum games for two reasons. First, such games have a unique equilibrium payoff. Second, receiving more information in such games can never be harmful: the equilibrium payoff of a player cannot decrease as a result of obtaining more information.

As in Blackwell's setting, in order to separate the impact of the information struc-

ture from the idiosyncracies of a particular interaction, we examine the characteristics of information structures that hold regardless of the payoff function. In this way we focus on informational aspects rather than on utility aspects. As in one-player decision problems, the question arises as to when one information structure yields as high a maximizer payoff as another information structure, regardless of the payoff function.

Our result that concerns comparison of different structures applies to repeated games with one-sided information (see Aumann and Maschler, 1995). It states that the equilibrium payoff of the maximizer is as high in a game with the information structure  $I$  as with the information structure  $I'$  if  $I$  is more informative than  $I'$ .

The second issue discussed here is that of measuring the contribution of information. This issue was first addressed by Gilboa and Lehrer (1991) who dealt with one-player decision problems. The information structure is called *partitional*, if there is a partition of the state space such that the information a player receives about the realized state is the cell of the partition containing that state. In a decision problem with a given partitional structure the decision maker has an optimal strategy which yields the optimal payoff. The function that attaches the corresponding optimal payoff to any partition is called a *value-of-information function*. Gilboa and Lehrer (1991) characterized those functions (defined on partitional information structures) that can be realized as value-of-information functions.

We extend the model of Gilboa and Lehrer (1991) and determine what functions (of partitional structures) can be realized as value-of-information functions in zero-sum repeated games. That is, we characterize those functions,  $f$ , for which there exists a repeated game such that for any partition,  $\mathcal{P}$ , the value of the corresponding game (with one-sided partitional information) coincides with  $f(\mathcal{P})$ . Since getting more information in zero-sum games is beneficial to the informed player, any value-of-information should be monotonic (with respect to refinement of partitions).

It turns out that unlike the case of one decision maker, in repeated games with one-sided information any monotonic function defined over partitions is also a value-of-information function. In other words, our second result states that there is no further condition beyond the obvious one, monotonicity, that characterizes the value-of-information functions.

In a companion paper (Lehrer and Rosenberg, 2003) we considered one-shot zero-sum games. In that paper we extended the characterization of Gilboa and Lehrer (1991) to one-sided and symmetric information structures.

The main tool used to investigate the issues described above is the value function of a one-shot zero-sum Bayesian game in which players receive no information beyond the prior. The value of such a Bayesian game can be viewed as a function from priors to numbers called the value of the game with no information. These functions play a crucial role in particular in the characterization of the values of repeated games with incomplete information (Aumann and Maschler (1995), Mertens et al. (1994)). The problem, which is of interest in its own right, is to characterize the real functions defined over the set of priors that are the values of some game with no information. This question is fully answered when there are two states of nature and remains open in the general case. In the general case we provide the properties needed for the rest of the paper.

The paper is organized as follows. In the next section we describe the model. We explain the notion of information structure and then introduce repeated games with one-sided information. The main issues and results of this paper are described in Section 3. Section 4 elaborates on zero-sum repeated games. Section 5 is devoted to the tool of the main proofs: the value of games with no information. These proofs are carried out in Section 6. The paper ends with final comments and problems that still remain open.

## 2 The model

### 2.1 Information structures

We consider zero-sum two-player games with incomplete information. A state of nature  $k$  is drawn from a finite set  $K$  according to a known probability  $p$ . None of the players is directly informed of the realized state  $k$ . The maximizing player receives a signal that may depend on  $k$  through an information structure. Here we focus on one-sided information structures, where only player 1 is (fully or partially) informed of  $k$ .

Information structures which are the main subject of this paper are formally defined in what follows.

**Definition 1** *An information structure  $I = (S, \pi)$  consists of*

- a. a set of signals:  $S$  for player 1;*

b. a (stochastic) information function  $\pi$  from  $K$  to the set of probability distributions over  $S$ . Given the realized state  $k \in K$ ,  $\pi(s|k)$  denotes the probability that player 1 receives the signal  $s$  in  $S$ .

In words, after the state  $k$  is chosen, player 1 receives a signal that depends on  $k$ . A signal  $s \in S$  is selected according to a probability distribution  $\pi(\cdot|k)$ . Player 1 is then informed of  $s$  and player 2 is not. Therefore, the information that player 1 has on  $k$  is imbedded in  $\pi$ .

**Remark 1** *If the set  $S$  is a singleton, then no player receives information about  $k$ . If  $S = K$  and  $s = k$  with probability 1, then player 1 is fully informed of  $k$ .*

Another way to model an information structure is by partitions. Let  $\mathcal{P}$  be a partition of the parameter set  $K$ . Here the signalling structure is deterministic: when the state  $k \in K$  is realized, the signal that player 1 receives is the cell  $T$  in  $\mathcal{P}$  that contains  $k$ . If the information structure is determined by a partition, we say that it is a *partitional information structure*.

A partitional information structure can also be written as an information structure. On the other hand, let  $I = (S, \pi)$  be an information structure, and let  $\tilde{k}$  denote the pair  $(k, s)$  drawn from the set  $\tilde{K} = K \times S$  with probability  $p(k)\pi(s|k)$ . We can rewrite it as a partitional structure with  $\tilde{K}$  being the state space and player 1's signals determined by the partition whose cells are the subsets of  $\tilde{K}$  that have the form  $\{(k', s') \in \tilde{K} | s' = s\}$  for some  $s \in S$ . Therefore, the two set-ups are exchangeable.

For a given state there are only finitely many partitional information structures, while there is a continuum of information structures. When converting to a partitional structure, different information structures typically induce different probabilities over the state space. On the other hand, when converting from partitional information structures to information structures, there are only finitely many possible probability distributions over the state space.

## 2.2 The repeated game

In this section we describe how the repeated game proceeds after the maximizer receives his information. This game is closely related to the incomplete information game defined by Aumann and Maschler (1995).

The  $n$ -stage game with incomplete information, denoted by  $\Gamma_n(p, I)$ , where  $I = (S, \pi)$ , is defined by an integer  $n$ , an information structure  $I$ , a probability  $p$  over

$K$ , a finite set of actions for each player  $i \in \{1, 2\}$ ,  $A_i$  and a payoff function  $g$  from  $K \times A_1 \times A_2$  to the reals where the payoff associated with  $(k, a_1, a_2)$  is denoted by  $g_k(a_1, a_2)$ . The game is played as follows. At stage 0 nature chooses an element  $k$  of  $K$  with probability  $p$  and a signal in  $S$  according to probability  $\pi(\cdot|k)$ . Player 1 is then informed of the signal  $s$ .

Then the game is played in  $n$  stages. At stage  $m = 1, \dots, n$  players 1 and 2 simultaneously choose actions according to probability distributions that may depend on the history of previous actions and signals. If the realized state is  $k$  and the chosen action is  $(a_1^m, a_2^m)$  the payoff at stage  $m$  is  $g_k(a_1^m, a_2^m)$ . The pair of the chosen action (and not the payoff) is then announced to both players and the game proceeds to its next stage. The payoff in  $\Gamma_n(p, I)$  is the expected average of the  $n$ -stage payoffs received during the game.

A *behavior strategy* of player 1 is a sequence  $\tau_1 = (\tau_1^1, \tau_1^2, \dots, \tau_1^m, \dots)$ , where  $\tau_1^m$  is a function from his information at stage  $m$ ,  $S \times (A_1 \times A_2)^{m-1}$  to the set<sup>1</sup>  $\Delta(A_1)$  of probability distributions over his set of actions. A behavior strategy of player 2 is a sequence  $\tau_2 = (\tau_2^1, \tau_2^2, \dots, \tau_2^m, \dots)$  where  $\tau_2^m$  is a function from his information at stage  $m$ ,  $(A_1 \times A_2)^{m-1}$  to the set  $\Delta(A_2)$ . When applied to the game  $\Gamma_n(p, \pi)$ , all  $\tau_i^m$ ,  $m > n$ , will be payoff irrelevant.

The probability distribution  $p, \pi$  and pair of strategies  $\tau_1, \tau_2$  induce a probability over the set of histories of length  $n$ ,  $H^n = K \times S \times (A_1 \times A_2)^n$ . The expectation with respect to this probability will be denoted by  $\mathbf{E}_{\tau_1 \tau_2}^{p, \pi}$  or simply by  $\mathbf{E}$  when no confusion can arise. If the players use the strategies  $\tau_1, \tau_2$ , the associated payoff in the  $n$ -stage game is  $\gamma_n^\pi(\tau_1, \tau_2, p) = \mathbf{E}_{\tau_1 \tau_2}^{p, \pi} \left[ \frac{1}{n} \sum_{m=1}^n g_k(a_1^m, a_2^m) \right]$ . This is the expected average payoff received along the  $n$  stages of the game.

The game  $\Gamma_n(p, I)$  is a finite game and therefore, by the minmax theorem, has a value denoted by  $v_n^I(p, (g_k)_{k \in K})$ . A basic property of the sequence of values is given by the following proposition. Its proof is postponed to the appendix.

**Proposition 1** *The sequence  $v_n^I(p, (g_k)_{k \in K})$  has a limit denoted by  $v^I(p, (g_k)_{k \in K})$ .*

The function  $v^I(p, (g_k)_{k \in K})$  will be referred to as the long-run value of the game. The reason this value is of special interest is that in long games there is no time constraint and therefore players may use their information in an unhindered fashion. This seems to be a relevant notion in measuring the value of an information structure.

---

<sup>1</sup>Throughout this paper  $\Delta(X)$  denotes the set of probability distributions over a set  $X$ .

**Remark 2** One could also define the infinitely repeated discounted version of this game. The strategy sets would be the same but the  $\lambda$ -discounted payoff would then be  $\gamma_\lambda^\pi(\tau_1, \tau_2, p) = \mathbf{E}_{\tau_1 \tau_2}^{p, \pi} [\sum_{m=1}^{\infty} \lambda(1-\lambda)^{m-1} g_k(a_1^m, a_2^m)]$ . For the same reason as above this game has a value that is denoted by  $v_\lambda^I(p, (g_k)_{k \in K})$  and  $\lim_{\lambda \rightarrow 0} v_\lambda^I(p, (g_k)_{k \in K})$  exists and is equal to  $v^I(p, (g_k)_{k \in K})$ .

### 3 The main results

Our results deal with two issues: comparison of information structures and the value-of-information. We start with the first.

#### 3.1 Comparison of information structures

The first issue is concerned with the **comparison** of information structures. Since in a zero-sum game there may be only one equilibrium payoff (the value), it is fairly easy to define when an information structure is better for the maximizing player than another information structure.

**Definition 2** An information structure  $I$  is better for player 1 than  $I'$  if for any payoff function  $(g_k)_{k \in K}$  and for any distribution  $p$ ,  $v^I(p, (g_k)_{k \in K}) \geq v^{I'}(p, (g_k)_{k \in K})$ .

By requiring that the value related to  $I$  is as high as that related to  $I'$  for **every**  $p$  and  $(g_k)_{k \in K}$  we isolate the role of the information structure. The first issue of this paper is to characterize in some interactive models when one structure is better for player 1 than another. In the case of one player decision problems this question has been solved by Blackwell who characterized better information structures as the ones that are more informative.

We now want to clarify the notion of quantity of information available to a player. To present Blackwell's result we need the following definition.

**Definition 3** An information structure  $I = (S, \pi)$  is more informative than  $I' = (S', \pi')$  if for each  $s \in S$  there is a probability distribution  $Q(s)$  over  $S'$  such that  $\pi'(s'|k) = \sum_{s \in S} \pi(s|k)Q(s)(s')$  for every  $s' \in S'$  and  $k \in K$ .

This definition states that  $I$  is more informative than  $I'$  if by garbling the signals released through  $I$  a statistician can reproduce a signal that appears, in terms of the

induced distributions, as a signal produced by  $I'$ . The garbling is done by choosing an  $S'$  signal according to the distribution  $Q(s)$  when receiving the signal  $s$ .

The following proposition translates the definition of a more informative signal to the case of partitional information structures, and will be useful in the sequel. Its proof is postponed to the appendix.

**Proposition 2** *A one-player partition information structure  $\mathcal{P}$  is more informative than  $\mathcal{P}'$  if  $\mathcal{P}$  is a refinement of  $\mathcal{P}'$  (namely, for any set  $T \in \mathcal{P}$  there is a set  $T' \in \mathcal{P}'$  such that  $T \subset T'$ ).*

The following theorem laid the foundation of the field dealing with the comparison of information structures.

**Theorem 1** *(Blackwell, 1953) An information structure  $I$  is better than  $I'$  in a one-player decision problem if and only if it is more informative.*

Our first result extends Blackwell's result to the case of two-player repeated games with one-sided information.

**Theorem 2** *An information structure  $I$  is better than  $I'$  for player 1 in a repeated game with one-sided information if and only if it is more informative.*

The proof of this theorem relies on the result of Section 5 and is therefore postponed to Section 6.

## 3.2 The value of information

For a fixed payoff function  $(g_k)_{k \in K}$  and a fixed distribution  $p$ , define the value of the information structure  $I$  to be  $v^I(p, (g_k)_{k \in K})$ . The second issue of this paper is the **axiomatic characterization** of such functions. Specifically, let  $V(I)$  be a function over information structures  $I$ . The question arises as to when there exist payoff functions  $(g_k)_{k \in K}$  and a distribution  $p$  over  $K$  such that for any  $I$ ,  $V(I) = v^I(p, (g_k)_{k \in K})$ . Answering this question amounts to giving axioms that characterize functions that measure the value of information.

This question is easier to analyze in the partitional structure case because there are only finitely many partitions of  $K$ . The problem then amounts to characterizing functions over the finite set of partitions of  $K$  that may be value-of-information functions.

The same problem in the setting of a one-player decision problem has been analyzed by Gilboa and Lehrer (1991). Gilboa and Lehrer characterized the value-of-information functions in partitional information structures. The one-player case is a particular case of a zero-sum game (with player 2 having only one action).

There are more zero-sum games than one-player decision problems. Therefore, there are more value-of-information functions of zero-sum games than of a one-player decision problems. Thus, the conditions that characterize value-of-information functions of zero-sum games are less restrictive than those characterizing value-of-information functions of one-player decision problems.

The value-of-information functions of repeated games with one-sided information are formally defined in what follows.

**Definition 4** *A function  $f$  defined over all the partitions of  $K$  is the value of information of a one-sided incomplete information game with partitional information structure if there is a distribution  $p$  over  $K$  and if there are payoff functions  $(g_k)_{k \in K}$  such that for any partition  $\mathcal{P}$  over  $K$ ,  $f(\mathcal{P}) = v^{\mathcal{P}}(p, (g_k)_{k \in K})$ .*

We restrict our attention to partitional structures and characterize the value-of-information functions. By Theorem 2 and Proposition 2, one partitional information structure is more informative than another if and only if it refines it. However, when one partition refines another, the value of the game that corresponds to the first is higher than the value of the game that corresponds to the second. Therefore, the following monotonicity condition is clearly necessary.

**Definition 5** *A function  $v$  from the set of partitions of a finite set  $K$  to the real numbers is said to be monotonic if for any two partitions  $\mathcal{P}$  and  $\mathcal{P}'$ , the fact that  $\mathcal{P}$  refines  $\mathcal{P}'$  (i.e., for any  $T \in \mathcal{P}$  there is an  $T' \in \mathcal{P}'$  such that  $T \subset T'$ ) implies  $v(\mathcal{P}) \geq v(\mathcal{P}')$ .*

Theorem 3 states that this condition is also sufficient.

**Theorem 3** *Let  $V$  be a function from the set of partitions of  $K$  to the real numbers. The function  $V$  is a value-of-information function of a repeated game with state space  $K$ , one-sided information, and partitional signaling if and only if it is monotonic.*

The proof of this theorem relies on the results proven in Section 5 and is postponed to Section 6.

## 4 More on games with one-sided information

In this section we prove a general result on repeated games with one-sided information that will be used in the proof of the main results. A particular case of the model presented in Section 2.2 is that in which  $S = K$  and  $\pi(s = k|k) = 1$ . It means that player 1 knows the realized  $k$  whereas player 2 does not. Aumann and Maschler (1995) characterized the value  $v^I(p, (g_k)_{k \in K})$  in this case.

**Notation 1** Let  $u(p)$  denote the value of the one-shot zero-sum game with the action sets  $A_1, A_2$  and the payoff is  $\sum_{k \in K} p(k)g_k(a_1, a_2)$  when the pair of actions  $(a_1, a_2)$  is played.

$u(p)$  is the value of the one-shot game where no player has any information beyond the prior  $p$ . This function plays a central role in the theory of repeated games with incomplete information. The next section elaborates on it.

**Notation 2** Let  $f$  be a real valued function defined on  $\Delta(K)$ . Then  $\text{cav}(f)$  denotes the minimal concave function that is above  $f$ .

**Theorem 4** (Aumann and Maschler, 1995) In a repeated one-sided information game where player 1 is fully informed of the state, i.e.  $S = K$  and  $\pi(k|k) = 1$ , the value of information,  $v^I(p, (g_k)_{k \in K})$ , is equal to  $\text{cav}(u)(p)$ .

Our first goal is to extend this characterization. We denote by  $\pi_p(\cdot|s)$  the conditional probability over  $K$  given the signal  $s$ , that is,  $\pi_p(k|s) = \frac{p(k)\pi(s|k)}{\sum_{l \in K} p(l)\pi(s|l)}$ .

**Notation 3** For an information structure  $I = (S, \pi)$  and  $s \in S$  denote by  $r^I(s)$  the probability  $\sum_{k \in K} p(k)\pi(s|k)$ .

$r^I(s)$  is the probability of obtaining the signal  $s$  under the information structure  $I$ .

**Notation 4** For an information structure  $I = (S, \pi)$  denote by  $\mathcal{M}(I)$  the set of matrices  $M = (m_{is})_{\substack{i \leq D \\ s \in S}}$  with  $D$  lines ( $D$  can be any positive integer) and  $|S|$  columns such that for any  $i, s$ ,  $m_{is} \geq 0$ ,  $\sum_{s \in S} m_{is} = 1$  and for any  $s$ ,  $\sum_{i \leq D} m_{is} = r^I(s)$ . For such a matrix we denote by  $m_i$  the quantity  $\sum_{s \in S} m_{is}$ , and by  $p_i(M)$  the probability distribution over  $K$  defined by,

$$p_i(M)(k) = \frac{\sum_{s \in S} m_{is} \pi_p(k|s)}{m_i}.$$

**Proposition 3** *The value of the game with one-sided information is*

$$v^I(p, (g_k)_{k \in K}) = \max \left\{ \sum_{i \leq D} m_i u(p_i(M)) \mid M \in \mathcal{M}(I) \right\}.$$

The definition of  $M$  implies, in particular, that  $\sum_{i \leq D} m_i p_i(M) = p$ . Thus, the value  $v^I(p, (g_k)_{k \in K})$  is a kind of local concavification of  $u$ . Note, however, that the constraints on  $M$  are more restrictive than just  $\sum_{i \leq D} m_i p_i(M) = p$ .

**Proof.** We define first an auxiliary game with one-sided information in which player 1 is fully informed of the state of nature. The set of states of nature in this game is  $S$ , the sets of actions of the players remain unchanged and the payoff for a pair of actions  $a_1, a_2$  in state  $s$  is  $\bar{g}_s(a_1, a_2) = \sum_{k \in K} \pi_p(k|s) g_k(a_1, a_2)$ . The probability of the state  $s$  is  $r^{(S, \pi)}(s) = \sum_{k \in K} p(k) \pi(s|k)$ .

The original game is now viewed as a game with incomplete information as defined in Aumann and Maschler (1995), where the states are player 1's signals. The payoff function in the auxiliary game is naturally defined as the appropriate expected payoff of the original game. It is easily seen that the values of the  $n$ -stage repetitions of both games, the original and the auxiliary, are the same.

Thus,  $r^{(S, \pi)}$  is the distribution over the new set of states. By Theorem 4 the long run value of this game is  $\text{cav}(\bar{u})(r^{(S, \pi)})$ , where  $\bar{u}$  denotes the value of the auxiliary game in which player 1 gets no information about  $S$ .

For any  $q$  in  $\Delta(S)$ , let  $p_q \in \Delta(K)$  satisfy  $p_q(k) = \sum_{s \in S} q(s) \pi_p(k|s)$ . We prove first that  $\bar{u}(q) = u(p_q)$ . Indeed,  $\bar{u}(q)$  is the value of the game with the following payoff:

$$\begin{aligned} \sum_{s \in S_1} q(s) \bar{g}_s(a_1, a_2) &= \sum_{s \in S} \left( \sum_{k \in K} \pi(k|s) g_k(a_1, a_2) \right) q(s) \\ &= \sum_{k \in K} \left( \sum_{s \in S} q(s) \pi(k|s) \right) g_k(a_1, a_2) = \sum_{k \in K} p_q(k) g_k(a_1, a_2). \end{aligned}$$

Therefore, the value of this game is indeed  $u(p_q)$ . We now compute

$$\text{cav}(\bar{u})(r^{(S, \pi)}) = \max \left\{ \sum_i m_i \bar{u}(q_i) \mid \begin{array}{l} m_i \geq 0, \sum_i m_i = 1 \\ q_i \in \Delta(S_1), \sum_i m_i q_i = r^{(S, \pi)} \end{array} \right\}.$$

For any  $q_i$  in  $\Delta(S_1)$ , denote  $m_{is} = m_i q_i(s)$  and define the matrix  $M = (m_{is})$ . Since,  $\sum_{i, s} m_{is} = \sum_i m_i = 1$  and  $\sum_i m_{is} = r^{(S, \pi)}$ , we conclude that  $M \in \mathcal{M}(I)$ . Moreover,

$$p_i(M)(k) = \frac{\sum_{s \in S_1} m_{is} \pi_p(k|s)}{m_i} = \sum_{s \in S_1} q_i(s) \pi_p(k|s) = p_{q_i}(k). \quad (1)$$

Hence,  $\bar{u}(q_i) = u(p_i(M))$  and

$$\text{cav}(\bar{u})(r^{(S,\pi)}) \leq \max \left\{ \sum_{i \leq D} m_i u(p_i(M)) \mid M \in \mathcal{M}(I) \right\}. \quad (2)$$

On the other hand, for any  $M \in \mathcal{M}(I)$ , due to (1),  $\sum_i m_i u(p_i(M)) = \sum_i m_i \bar{u}(q_i)$ , with  $q_i(s) = \frac{m_i s}{m_i}$ . Therefore,  $u(p_i(M)) = \bar{u}(q_i)$  and

$$\max \left\{ \sum_{i \leq D} m_i u(p_i(M)) \mid M \in \mathcal{M}(I) \right\} \leq \text{cav}(\bar{u})(r^{(S,\pi)}). \quad (3)$$

Equations (2) and (3) yield the desired result. ■

## 5 The main tool: the value of games with no information

The previous section demonstrates the significance of the function  $u(p)$ . It turns out that knowing the properties of this function is the key to proving our main results. This section is devoted to the study of  $u(p)$ . It seems that the importance of this study lies beyond the current context. Therefore, the subject is presented in a self-contained manner.

A zero-sum Bayesian game is defined by a set  $K$  of states, a zero-sum game  $G_k$  for every  $k \in K$  and a probability  $p$  over  $K$ . A game  $G_k$  is chosen with probability  $p(k)$  and is played once. Denote by  $\bar{G}(p)$  the average game,  $\sum_{k \in K} p(k)G_k$ , and its value by  $u(p)$ .

The set  $K$  and the games  $G_k$ ,  $k \in K$ , induce a function  $u(p)$  whose domain is the set of probability distributions over  $K$ .

**Definition 6** *Let  $\Delta(K)$  be the set of probability distributions over the finite set  $K$ . We say that a function  $u$  defined on  $\Delta(K)$  is realizable if there are  $G_k$ ,  $k \in K$ , such that the value of  $\bar{G}(p)$  is  $u(p)$  for every  $p$  in  $\Delta(K)$ .*

Our task in this section is to study the set of realizable functions. The results obtained in this framework will be used to prove the theorems stated in the previous sections.

**Example 1** Let  $G_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  and  $G_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ . When the distribution is  $(q, 1 - q)$ , the average game is  $\begin{pmatrix} q & 0 \\ 2q - 1 & q \end{pmatrix}$ , and the value is  $q^2$ . Therefore,  $u(q) = q^2$ . By adding the matrix  $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$  to  $G_1$ , one obtains the matrix  $\begin{pmatrix} 4 & 3 \\ 4 & 4 \end{pmatrix}$ . Referred to as  $G_1$  of another game, together with the original  $G_2$ , one obtains a game whose value function is  $u(q) = q^2 + 3q$ . Finally, if the matrix  $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$  is added to these  $G_1$  and  $G_2$ , one obtains a game whose value function is  $u(q) = q^2 + 3q + 5$ .

**Definition 7** A function  $u$  is piecewise rational, if it is continuous and if there are finitely many rational functions such that at any point  $p$ ,  $u$  coincides with one of them.

The next proposition holds for any finite set of states.

**Proposition 4** If  $u$  is realizable, then it is piecewise rational.

**Proof.** Let  $G$  be the incomplete information game that realizes  $u$ . For any  $p$  let  $I'(p) \subseteq I$  (resp.  $J'(p) \subseteq J$ ) be a minimal set (with respect to inclusion) of player 1's (resp. player 2's) actions which have been assigned a positive probability by one of his optimal strategies. The value of  $G(p)$  when the players are restricted to  $I'$  and to  $J'$  is equal to  $u(p)$ .

Fix  $I' \subseteq I$  and  $J' \subseteq J$ . For any  $p$  such that  $I' = I'(p)$  and  $J' = J'(p)$ , an optimal strategy of player 1,  $x(p)$ , satisfies the following system of linear equations with  $x_i(p)$ ,  $i \in I'$ , being the variables:

$$\begin{aligned} \sum_{i \in I'} \sum_{k \in K} x_i(p) p^k g_k(i, j) &= \sum_{i \in I'} \sum_{k \in K} x_i(p) p^k g_k(i, j') \quad j, j' \in J'; \\ \text{and } \sum_{i \in I'} x_i(p) &= 1. \end{aligned} \quad (4)$$

Let  $S(p)$  be the set of solutions of system (4). Typically,  $S(p)$  contains more than one point, and moreover, not all the points in  $S(p)$  are optimal strategies of player 1.

Let  $d$  be the dimension of  $S(p)$ .  $S(p)$  may be obtained by the diagonal method. It is, therefore, the image of  $f : \mathbb{R}^d \rightarrow \mathbb{R}^{|J'|}$ ,  $f = (f_1, \dots, f_{|J'|})$ , where  $f_i$  is linear for every  $i \in I'$ . Moreover, the coefficients of  $f_i$  are rational functions of the numbers in (4). The optimal strategies are the intersection of  $S(p)$  with the non-negative orthant. This is the image of a polygon in  $\mathbb{R}^d$ , say  $D(p)$ , by  $f$ . The polygon  $D(p)$  is defined by finitely many linear inequalities whose coefficients are all rational functions of the numbers in (4).

The value of  $G(p)$  is the maximum of a linear functional over  $D(p)$ . This linear functional is also rational in the coefficients of (4). The maximum is attained in one of the extreme points of  $D(p)$ , defined by a set of linear inequalities (over  $\mathbb{R}^d$ ) whose coefficients are rational in the coefficients appearing in (4). Any specific extreme point induces a rational function in  $p$ . Thus, the value may take the value of one of the rational functions, depending on the extreme point of  $D(p)$ , where the maximum is attained.

To conclude, for each pair  $I', J'$  there is a polygon  $D_{I',J'}(p)$  defined by inequalities whose coefficients are rational in the numbers of (4). The maximum attained by the linear functional (which provides the value of the game) over  $D_{I',J'}(P)$  is a rational function, denoted by  $u_{I',J'}(p)$ . Thus, for each  $I'$  and  $J'$  there is a piecewise rational function  $u_{I',J'}$ . For any  $p$  there is a pair  $I'$  and  $J'$  such that the rational function  $u_{I',J'}(p)$  is the value of the game. Since there are finitely many pairs  $I'$  and  $J'$ ,  $u$  is piecewise rational. ■

## 5.1 The case of two states

In this subsection we restrict the discussion to the case where there are only two states ( $|K| = 2$ ). Thus,  $\Delta(K)$  is actually the interval  $[0, 1]$ .

**Lemma 1** *If  $u$  is realizable, then so are  $cu$ , for  $c > 0$ , and  $-u$ .*

**Proof.** Suppose that  $u$  is realizable by  $A = G_1$  and  $B = G_2$ . Then  $cu$  is realizable by  $cA$  and  $cB$ , and  $-u$  is realizable by  $-A^T - B^T$ . ■

**Lemma 2** *If  $u$  and  $v$  are realizable, then so is  $u + v$ .*

**Proof.** Suppose that  $u$  is realizable by  $A = G_1$  and  $B = G_2$ . Furthermore,  $v$  is realizable by  $C = G_1$  and  $D = G_2$ . Denote by  $A_{ij}$  the matrix  $A + c_{ij}$ , where  $c_{ij}$  is the  $ij$ -th entry of the matrix  $C$ . Similarly, let  $B_{ij} = B + d_{ij}$ , where  $d_{ij}$  is the  $ij$ -th entry of the matrix  $D$ .

Define  $\tilde{G}_1$  to be the matrix  $(A_{ij})_{ij}$ , whose  $ij$ -th entry is the matrix  $A_{ij}$ . Similarly, let  $\tilde{G}_2$  be the matrix  $(B_{ij})_{ij}$ . The value function of the game with matrices  $\tilde{G}_1$  and  $\tilde{G}_2$  is  $u + v$ . ■

**Lemma 3** *If  $u$  is realizable, then so is  $u^2$ .*

**Proof.** It is sufficient to show it when  $u$  is bounded between 0 and 1. This is so because when  $au + b$ , where  $a$  and  $b$  are scalars, is realizable, then due to the previous lemmas, by subtracting the realizable function  $2abu + b^2$ , and then dividing by  $a^2$ , one obtains a realizable function.

Assume that  $u$  is realizable by  $A = G_1$  and  $B = G_2$ . Let  $\tilde{G}_1 = \begin{pmatrix} A & 0 \\ 2A - 1 & A \end{pmatrix}$  and  $\tilde{G}_2 = \begin{pmatrix} B & 0 \\ 2B - 1 & B \end{pmatrix}$ . The value function of this game is  $u^2$ . ■

**Lemma 4** *If  $u$  and  $v$  are realizable, then so is  $uv$ .*

**Proof.** By the previous lemma,  $(u + v)^2$  is realizable. Thus by subtracting the realizable functions  $u^2$  and  $v^2$  one obtains  $2uv$ . ■

An immediate consequence is,

**Corollary 1** *Any polynomial is realizable.*

**Lemma 5** *If  $u$  and  $v$  are positive and realizable, with no common root, then  $\frac{uv}{u+v}$  is realizable.*

**Proof.** Suppose that  $u$  is realizable by  $A = G_1$  and  $B = G_2$ . Furthermore,  $v$  is realizable by  $C = G'_1$  and  $D = G'_2$ . Define the game  $\tilde{G}_1 = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$  and  $\tilde{G}_2 = \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}$ . The value function of this game is  $\frac{uv}{u+v}$ . ■

**Lemma 6** *If  $u$  is a polynomial which does not vanish in  $0, 1$ , then  $\frac{1}{u}$  is realizable.*

**Proof.** Suppose that  $u$  is positive on the interval  $[0, 1]$ . Let the minimum of  $u$  be denoted by  $c$ . By applying the previous lemma to  $u - c$  and  $c$ , one obtains that  $\frac{(u-c)c}{u}$  is realizable. By subtracting  $c$  and multiplying by  $\frac{1}{-c^2}$  one obtains the desired result. ■

We therefore obtain,

**Proposition 5** *Any rational function defined on  $[0, 1]$  whose denominator does not vanish in the interval is realizable.*

**Proposition 6** *Any piecewise rational function is realizable.*

**Proof.** For convenience, let us say that the relevant rational function for the  $i$ -th interval is  $r_i$ . Without loss of generality,  $r_i$  is either greater or smaller than  $r_{i-1}$  in the  $i$ -th interval. (Otherwise, one can refine the original partition in order to obtain it.)

Proceeding, say, from left to right, we have the first rational function in the first interval,  $r_1$ . Now for the second interval, we define a function, called  $s_2$ . Suppose that  $r_2$  is greater than  $r_1$  on the second interval. The function  $s_2$  is: **a.** continuous; **b.** linear and smaller than  $r_1$  on the first interval; and **c.** equal to  $r_2$  on the rest. This can be done by taking the minimum of  $r_2$  and some, properly chosen, linear function. (If  $r_2$  is smaller than  $r_1$  on the second interval, the function  $s_2$  is: **a.** continuous; **b.** linear and greater than  $r_1$  on the first interval; and **c.** equal to  $r_2$  on the rest.) The function  $s_2$  is realizable as the minimum between  $r_2$  and a linear (thus realizable) function  $l_2$  that is smaller than  $r_1$  and  $r_2$  on the first interval and greater than  $r_2$  on the second (such a function exists).

After having  $r_1$  and  $s_2$ , we form a game whose payoff is the maximum of  $r_1$  and  $s_2$ . The payoff function of this game is called  $w_2(p)$ . (If  $r_2$  is smaller than  $r_1$  on the second interval, we take the minimum of  $r_1$  and  $s_2$ .)

**The general step:** Suppose that after stage  $i$  of the process we obtained  $w_i(p)$ , which coincides with  $r_j$  on the  $j$ -th interval,  $j = 1, \dots, i$ , and with  $r_i$  on the intervals  $i + 1, i + 2, \dots$ .

Suppose that  $r_{i+1}$  is greater than  $w_i$  on the  $i + 1$  interval (this is without loss of generality from above). The function  $s_{i+1}$  is: **a.** continuous; **b.** linear and smaller than  $w_i$  on the first  $i$  intervals; and **c.** equal to  $r_{i+1}$  on the rest. After obtaining  $s_{i+1}$ , we form a game whose payoff is the maximum of  $w_i$  and  $s_{i+1}$ . The payoff function of this game is called  $w_{i+1}(p)$ . Similarly to the first step,  $s_{i+1}$  is realizable as the minimum between  $r_{i+1}$  and  $l_{i+1}$ , a linear function that is smaller than all  $w_i$  on the first  $i$  intervals and greater than  $w_i$   $r_{i+1}$  on the rest (such a function exists).

At the end of the process we obtain the desired piecewise rational function. ■

The previous proposition and Proposition 4 imply the following

**Theorem 5** *A function is realizable if and only if it is piecewise rational.*

## 5.2 The general symmetric case

In this subsection we extend the discussion to any finite state space  $K$ .

**Lemma 7** *If  $u$  is realizable, then so is  $u^2$ .*

**Proof.** We use the same method as in the proof of Lemma 3. If  $u$  is realizable by the game  $G_1, \dots, G_{|K|}$  then the game whose  $i$ -th matrix game is  $\begin{pmatrix} G_i & 0 \\ 2G_i - 1 & G_i \end{pmatrix}$  realizes  $u^2$ . ■

**Proposition 7** *Any polynomial is realizable.*

**Proof.** By induction on  $|K|$ . By Corollary 1 the claim is true for the case of  $|K| = 2$ . Let  $f(q_1, \dots, q_{|K|-1})$  be a polynomial with  $|K| - 1$  variables. By the method of Lemma 2's proof, in order to show that  $f$  is realizable, it is sufficient to show that any monom is realizable.

Let  $r_1 = \prod_{i=1}^{|K|-1} q_i^{\ell_i}$  be a monom. By the induction hypothesis,  $\prod_{i=1}^{|K|-2} q_i^{\ell_i}$  is realizable as a game with  $|K| - 1$  states. By adding an identically zero matrix, one obtains a game with  $|K|$  states whose value function is  $r_1$ .

$r_2 = q_{|K|-1}^{\ell_{|K|-1}}$  is a polynomial with one variable and thus can be realized by a game with two states. Let  $G_1$  and  $G_2$  be the matrices corresponding to the two states. Using this game we will define a game with  $|K|$  states,  $G'_1, \dots, G'_{|K|}$ , as follows. Let  $G'_{|K|} = G_1$  and  $G'_i = G_2$ ,  $i \neq |K|$ . The value function of the new game is  $r_2$ .

Since  $2r_1r_2 = (r_1 + r_2)^2 - r_1^2 - r_2^2$ ,  $r_1r_2$  is realizable. Thus, every monom is realizable and therefore so is any polynomial. ■

This implies the following

**Corollary 2** *Given a finite number of pairs  $(x_\ell, y_\ell) \in \Delta(K) \times \mathbb{R}$ ,  $\ell = 1, \dots, L$ , there is a realizable function  $u$  such that  $u(x_\ell) = y_\ell$ ,  $\ell = 1, \dots, L$ .*

**Lemma 8** *If  $u$  and  $v$  are realizable then so are  $\max\{u, v\}$  and  $\min\{u, v\}$ .*

**Proof.** Suppose that  $f$  is realizable. Then by adding a row of 0's to each matrix game, one obtains a game that realizes  $\max\{f, 0\}$ . Thus,  $\max\{u, v\} = v + \max\{u - v, 0\}$  is realizable and so is  $\min\{u, v\} = -\max\{-u, -v\}$ . ■

**Definition 8** *A set  $C$  is semi-algebraic if it is a finite union of sets of form*

$$\left\{ x \in \mathbb{R}^k; f_1(x) = f_2(x) = \dots = f_\ell(x) = 0, r_1(x) > 0, r_2(x) > 0, \dots, r_m(x) > 0 \right\},$$

where  $f_1, \dots, f_\ell, r_1, \dots, r_m$  are polynomials.

**Proposition 8** *Let  $C$  be a closed semi-algebraic set. Then there is a realizable function  $u$  that satisfies  $u(x) \leq 0$  when  $x \in C$ , and  $u(x) > 0$ , otherwise.*

**Proof.** Let  $D$  be a set of form

$$\left\{x \in \mathbb{R}^k; f_1(x) = f_2(x) = \dots = f_\ell(x) = 0, r_1(x) \geq 0, r_2(x) \geq 0, \dots, r_m(x) \geq 0\right\},$$

where  $f_1, \dots, f_\ell, r_1, \dots, r_m$  are polynomials. By Proposition 7,  $f_1, \dots, f_\ell, r_1, \dots, r_m$  are realizable and by Lemma 8,  $u_D = \min\{f_1, \dots, f_\ell, -f_1, \dots, -f_\ell, r_1, \dots, r_m\}$  is realizable. Clearly,  $u_D(x) \geq 0$  when  $x \in C$  and  $u_D(x) < 0$ , otherwise. Since any closed semi-algebraic set is a finite union of such  $D$ 's (see Bochnak et al. (1988), p. 46), the desired  $u$  is  $u = -\max\{u_D\}_D$ , which is also realizable. ■

Since a closed polygon is in particular a closed semi-algebraic, we obtain the following corollary.

**Corollary 3** *If  $C$  is a finite union of closed polygons, then there is a realizable function  $u$  that satisfies  $u(x) \leq 0$  when  $x \in C$ , and  $u(x) > 0$  otherwise.*

By taking the maximum with zero one obtains,

**Corollary 4** *If  $C$  is a closed polygon, then there is a realizable function  $u$  that satisfies  $u(x) = 0$  when  $x \in C$ , and  $u(x) > 0$  otherwise.*

**Corollary 5** *If  $C_1$  is finite,  $C_2$  is a union of closed polygons,  $C_1 \cap C_2 = \emptyset$  and  $c_1$  and  $c_2$  are two numbers such that  $c_1 > c_2$ , then there is a realizable function  $u$  that satisfies  $u(x) \leq c_2$  when  $x \in C_2$ ,  $u(x) = c_1$  when  $x \in C_1$ , and  $u(x) \leq c_1$  otherwise.*

**Proof.** A union of closed polygons is a closed semi-algebraic set. Therefore, by Proposition 8, there is a realizable function  $u$  which is less than or equal to 0 on  $C_2$  and greater than 0 otherwise. Let  $c$  be the minimum of  $u$  over the set  $C_1$ . By multiplying  $u$  with the constant  $\frac{c_1 - c_2}{c}$  and adding  $c_2$  one obtains a realizable function  $u'$  that satisfies  $u'(x) \leq c_2$  when  $x \in C_2$  and  $u'(x) \geq c_1$  when  $x \in C_1$ . Taking the minimum of  $u'$  and  $c_1$  would yield the desired realizable function. ■

**Proposition 9** *Let  $C_1$  and  $C_2$  be two disjoint closed semi-algebraic sets, and  $f_1$  and  $f_2$  two realizable functions. Then, there is a realizable function  $u$  that satisfies  $u(x) = f_1(x)$  when  $x \in C_1$ , and  $u(x) = f_2(x)$  when  $x \in C_2$ .*

**Proof.** By Proposition 8, for  $i = 1, 2$ , there is  $u'_i$  which is at least 0 on  $C_i$  and less than 0 otherwise. Consider  $u_i = \max\{\min\{u'_i + 1, 1\}, 0\}$ ;  $u_i$  is realizable, bounded between 0 and 1, and equal to 1 on  $C_i$  and less than 1 otherwise.

For any integer  $\ell_i$ ,  $u_i^{\ell_i}$  is also realizable. It is bounded between 0 and 1, and is equal to 1 on  $C_i$  and less than 1 otherwise. By adding a large positive number, say  $M$ , we may assume that the functions  $f_1$  and  $f_2$  are positive. There are sufficiently large  $\ell_i$ 's such that  $f_j > f_i u_i^{\ell_i}$ ,  $i \neq j$ , on  $C_j$ . Thus,  $\max\{f_1 u_1^{\ell_1}, f_2 u_2^{\ell_2}\}$  is realizable and (subtracting  $M$ , if necessary) satisfies the desired properties. ■

We do not know how to prove that if  $C_1$  and  $C_2$  are closed and semi-algebraic sets and if  $f_1$  and  $f_2$  are realizable functions such that  $f_1 = f_2$  on  $C_1 \cap C_2$  then the function  $u$  defined by  $u(p) = f_1(p)$  if  $p \in C_1$  and  $u(p) = f_2(p)$  if  $p \in C_2$  is realizable.

## 6 Proofs of the main theorems

In this section we prove the main results of the previous section.

**Proof of Theorem 2.** Assume first that  $I = (S, \pi)$  is more informative than  $I' = (S', \pi')$ , meaning that for any<sup>2</sup>  $s' \in S'$  and  $k \in K$ ,

$$\pi'(s'|k) = \sum_s \pi(s|k)Q(s)(s'),$$

where for each  $s \in S$ ,  $Q(s)$  is a probability distribution over  $S'$ .

The following auxiliary formula will be useful later:

$$r^{I'}(s') = \sum_k \pi'(s'|k)p(k) = \sum_{k,s} \pi(s|k)Q(s)(s')p(k) = \sum_s Q(s)(s')r^I(s).$$

For any  $M' = (m'_{is'}) \in \mathcal{M}(I')$  we define the matrix  $M = (m_{is})$  by  $m_{is} = \sum_{s' \in S'} \frac{r^I(s)}{r^{I'}(s')} m'_{is'} Q(s)(s')$ .

We prove that  $M \in \mathcal{M}(I)$ . Indeed,

$$\begin{aligned} \sum_{i,s} m_{is} &= \sum_{i,s,s'} \frac{r^I(s)}{r^{I'}(s')} m'_{is'} Q(s)(s') = \sum_{s,s'} \frac{r^I(s)}{r^{I'}(s')} \left( \sum_i m'_{is'} \right) Q(s)(s') \\ &= \sum_{s,s'} r^I(s) Q(s)(s') = \sum_s r^I(s) \left( \sum_{s'} Q(s)(s') \right) = \sum_s r^I(s) = 1. \end{aligned}$$

---

<sup>2</sup>In the following summations, if not specified otherwise,  $s \in S$ ,  $s' \in S'$ ,  $k \in K$  and  $i$  runs over a finite range.

Moreover,

$$\begin{aligned} \sum_i m_{is} &= \sum_{is'} \frac{r^I(s)}{r^{I'}(s')} m'_{is'} Q(s)(s') \\ &= \sum_{s'} \frac{r^I(s)}{r^{I'}(s')} \left( \sum_i m'_{is'} \right) Q(s)(s') = \sum_{s'} r^I(s) Q(s)(s') = r^I(s). \end{aligned}$$

We now show that  $m_i = m'_i$  and  $p_i(M) = p_i(M')$ . Indeed,

$$m_i = \sum_{s \in S} m_{is} = \sum_{s, s'} \frac{r^I(s)}{r^{I'}(s')} m'_{is'} Q(s)(s') = \sum_{s'} m'_{is'} = m'_i.$$

Moreover,

$$\begin{aligned} p_i(M)(k) &= \frac{\sum_s m_{is} \pi_p(k|s)}{m_i} = \frac{1}{m_i} \sum_{s, s'} \pi_p(k|s) \frac{r^I(s)}{r^{I'}(s')} m'_{is'} Q(s)(s') \\ &= \frac{1}{m'_i} \sum_{s, s'} \frac{r^I(s)}{r^{I'}(s')} m'_{is'} Q(s)(s') \frac{p(k) \pi(s|k)}{r^I(s)} \\ &= \frac{1}{m'_i} \sum_{s'} \frac{1}{r^{I'}(s')} m'_{is'} p(k) \pi'(s'|k) = \frac{1}{m'_i} \sum_{s'} m'_{is'} \pi'_p(k|s') = p_i(M')(k). \end{aligned}$$

This proves that for any  $M' \in \mathcal{M}(I')$  there is an  $M \in \mathcal{M}(I)$  such that  $\sum_i m'_i u(p_i(M')) = \sum_i m_i u(p_i(M))$  which implies that for any payoff function  $(g_k)_{k \in K}$ ,  $v^I(p, (g_k)_{k \in K}) \geq v^{I'}(p, (g_k)_{k \in K})$ .

As for the inverse direction of the theorem, let  $I = (S, \pi)$  and  $I' = (S', \pi')$  be two information structures such that for any payoff function  $(g_k)_{k \in K}$ ,  $v^I(p, (g_k)_{k \in K}) \geq v^{I'}(p, (g_k)_{k \in K})$ . In particular, for any payoff function that does not depend on the moves of player 2,  $v^I(p, (g_k)_{k \in K}) \geq v^{I'}(p, (g_k)_{k \in K})$ . However, such a game is a one-player decision problem since player 2 is dummy. Thus,  $I$  is better than  $I'$  for any one-player decision problem. By Blackwell (1953) this implies that  $I$  is more informative than  $I'$ . ■

**Proof of Theorem 3.** Let  $V$  be a monotonic function over partitions. Let  $p$  be  $(\frac{1}{|K|}, \dots, \frac{1}{|K|})$ . Order all partitions of  $K$ :  $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ , so that if  $j < i$  then  $V(\mathcal{P}_j) \leq V(\mathcal{P}_i)$ , and therefore  $\mathcal{P}_j$  does not refine  $\mathcal{P}_i$ .

**Notation 5** Let  $\mathcal{P}$  be a partition. Denote by  $H(\mathcal{P})$  the space in  $\mathbb{R}^{|K|}$  spanned by  $\{\mathbb{1}_T; T \in \mathcal{P}\}$ , where  $\mathbb{1}_T$  denotes the characteristic vector of the set  $T$ .

Note that the partition  $\mathcal{P}$  refines  $\mathcal{P}'$  if and only if  $H(\mathcal{P}')$  is a subspace of  $H(\mathcal{P})$ . This implies in particular that  $\dim[H(\mathcal{P}_j) \cap H(\mathcal{P}_i)] < \dim H(\mathcal{P}_i)$  for  $i > j$ .

We would like to prove that for any  $i \leq \ell$  the following property, denoted by  $A_i$ , holds: there is a realizable function  $u$  such that for any  $j \leq i$ , and every  $p \in H(\mathcal{P}_j)$ ,  $u(p) \leq V(\mathcal{P}_j)$ . Furthermore, for any  $j \leq i$  there is a  $D_j \times l$  matrix<sup>3</sup>  $M_j \in \mathcal{M}(\mathcal{P}_j)$  such that for any  $r \leq D_j$ ,  $u(p_r(M_j)) = V(\mathcal{P}_j)$ .

We proceed by induction over  $i$ . Let  $i = 1$ . The partition  $\mathcal{P}_1$  is the trivial partition and  $u$  can be taken to be the constant function  $V(\mathcal{P}_1)$ . Now assume that  $A_{i-1}$  holds and denote by  $u_{i-1}$  the corresponding realizable function.

**Step 1: Definition of a class of matrices.**

Consider the following square matrix,  $M_1$ , with  $|\mathcal{P}_i|$  columns, all of whose entries except for those on the diagonal, are zero. The diagonal entry corresponding to  $T \in \mathcal{P}_i$  is  $p(T)$ . Note that for every row  $r$  of  $M_1$ ,  $p_r(M_1) \neq p$ . Define  $M_2$  as a matrix of the same dimension whose entries in the column corresponding to the cell  $T \in \mathcal{P}_i$  are all equal to  $\frac{p(T)}{D}$ . Obviously,  $M_1, M_2 \in \mathcal{M}(\mathcal{P}_i)$ . Define  $M = \alpha M_1 + (1 - \alpha)M_2$ . If  $\alpha$  is positive and sufficiently small, then  $M \in \mathcal{M}(\mathcal{P}_i)$  and for every row  $r$  of  $M$ ,  $p_r(M) \neq p$ . Furthermore, all entries of  $M$  are strictly positive.

Define  $H(\mathcal{P}_i)^0 = \{v; v = \sum \varepsilon_\sigma \pi_p(\cdot|T), \sum_{T \in \mathcal{P}_i} \varepsilon_T = 0\}$ . For every  $v \in H(\mathcal{P}_i)^0$  let  $a_{rT} = m_r \varepsilon_T \langle p_r(M) - p, v \rangle$ ,  $r \leq D$ ,  $T \in \mathcal{P}_i$ . Consider the matrix  $M^v$  whose  $(r, T)$ -th entry is  $m_{rT} + a_{rT}$ . If the  $\varepsilon_T$ 's that define  $v$  are small enough, then the entries of  $M^v$  are positive.

We show now that  $M^v$  is in  $\mathcal{M}(\mathcal{P}_i)$ . This is so since  $\sum_r m_{rT} + a_{rT} = p(T) + \sum_r m_r \varepsilon_T \langle p_r(M) - p, v \rangle = p(T) + \varepsilon_T \langle \sum_r m_r p_r(M) - p, v \rangle = p(T)$ . The last equality is due to the fact that  $\sum_r m_r p_r(M) = p$ .

**Step 2: Proof of the claim that there is  $v \in H(\mathcal{P}_i)^0$  such that for any row  $r$ ,  $p_r(M^v) \notin \cup_{j < i} H(\mathcal{P}_j)$ .**

Note that  $\sum_{T \in \mathcal{P}_i} a_{rT} = \sum_{T \in \mathcal{P}_i} m_r \varepsilon_T \langle p_r(M) - p, v \rangle = m_r \langle p_r(M) - p, v \rangle \sum_{T \in \mathcal{P}_i} \varepsilon_T = 0$ . Therefore,  $p_r(M^v) = p_r(M) + \langle p_r(M) - p, v \rangle \sum_{T \in \mathcal{P}_i} \varepsilon_T \pi_p(\cdot|T) = p_r(M) + \langle p_r(M) - p, v \rangle v$ . If the claim is incorrect, then there is a neighborhood of  $H(\mathcal{P}_i)^0$  around the origin, denoted  $W$ , such that for every  $v \in W$  there is  $j < i$  and a row  $r$  such that  $p_r(M^v) = p_r(M) + \langle p_r(M) - p, v \rangle v \in H(\mathcal{P}_j)$ . Define the set  $F_{rj}$  to be the set containing  $v \in \overline{W}$  such that  $p_r(M) + \langle p_r(M) - p, v \rangle v \in H(\mathcal{P}_j)$ , where  $\overline{W}$  is the closure (the relative one in  $H(\mathcal{P}_i)^0$ ) of  $W$ .  $F_{rj}$  is a closed set for every  $r$  and  $j$ .

<sup>3</sup>Recall Notation 4.  $\mathcal{P}_j$  in  $\mathcal{M}(\mathcal{P}_j)$  stands for the information structure that corresponds to the partition  $\mathcal{P}_j$ .

By assumption, the union of the closed sets  $F_{rj}$  contains  $\overline{W}$ . Since  $\overline{W}$ , as a complete space, is of category II, at least one of the  $F_{rj}$ 's contains an open set. Thus, there are  $j$  and  $r$  so that  $p_r(M) + \langle p_r(M) - p, v \rangle v \in H(\mathcal{P}_j)$  for  $v$ 's in an open (recall, in  $\overline{W}$ ) set.

Note that for every  $v \in W$ ,  $p_r(M) + \langle p_r(M) - p, v \rangle v \in H(\mathcal{P}_i) \cap \Delta(K)$ . Furthermore, since  $p_r(M) - p \neq 0$ , the map  $v \mapsto p_r(M) + \langle p_r(M) - p, v \rangle v$  is an open map. Thus,  $H(\mathcal{P}_j) \cap \Delta(K)$  contains an open set of  $H(\mathcal{P}_i) \cap \Delta(K)$ . Since both are intersections of linear spaces whose spanning vectors are in  $\Delta(K)$  with  $\Delta(K)$ , it implies that  $H(\mathcal{P}_j) \cap \Delta(K)$  contains  $H(\mathcal{P}_i) \cap \Delta(K)$ . However, when  $j < i$ ,  $\dim[H(\mathcal{P}_j) \cap H(\mathcal{P}_i)] < \dim H(\mathcal{P}_i)$ . This implies that  $H(\mathcal{P}_i) \cap \Delta(K)$  is not included in  $H(\mathcal{P}_j) \cap \Delta(K)$ . We therefore conclude that there exists a matrix, which we denote now by  $M_i$ , in  $\mathcal{M}(\mathcal{P}_i)$  that satisfies  $p_r(M_i) \notin \cup_{j < i} H(\mathcal{P}_j)$  for every row  $r$  of  $M_i$ .

### Step 3: Conclusion of the proof.

By Corollary 5, there is a realizable function  $g$  that satisfies:

- for every row  $r$  of  $M_i$ ,  $g$  attains its maximum,  $V(\mathcal{P}_i)$ , on  $p_r(M_i)$ ; and
- $g$  is smaller than or equal to  $\min_{1 \leq j \leq i-1} V(\mathcal{P}_j)$  on  $\cup_{1 \leq j \leq i-1} H(\mathcal{P}_j) \cap \Delta(K)$ .

By taking the maximum of  $g$  and the function  $u_{i-1}$  we get a realizable function  $u_i$  that satisfies:

- (a) for any  $j \leq i$ ,  $u_i$  is smaller than or equal to  $V(\mathcal{P}_j)$  on  $H(\mathcal{P}_j) \cap \Delta(K)$ ; and
- (b) for any  $j \leq i$ , there is a  $D_j \times l$  matrix  $M_j \in \mathcal{M}(\mathcal{P}_j)$  such that for any  $1 \leq r \leq D_j$ ,  $u(p_r(M_j)) = V(\mathcal{P}_j)$ .

Property  $A_\ell$  is therefore proven by induction. Let  $(g_k)_{k \in K}$  be the payoff functions that realize  $u_\ell$ . (a) implies that  $v^{\mathcal{P}_i \mathcal{O}}(p, (g_k)_{k \in K}) \leq V(\mathcal{P}_i)$  for every  $i \leq \ell$ . Proposition 3 and (b) imply that  $v^{\mathcal{P}_i \mathcal{O}}(p, (g_k)_{k \in K}) = V(\mathcal{P}_j)$ , which completes the proof. ■

## 7 Comments and open problems

### 7.1 The value-of-information function in one-shot games

Theorem 3 deals with the value of information in a repeated one-sided game with partitional information. In Lehrer and Rosenberg (2003) we characterize the value-of-information functions in one-shot games with partitional information in two cases: one-sided and symmetric information.

## 7.2 The value of finite games with one-sided information

In Section 5 we analyzed the value of games with no information. The one-sided case remains open. Suppose that a game  $G_k$  is chosen with probability  $p(k)$ , is informed to the maximizing player and is played once. Denote the value of this game by  $w(p)$ .

It is clear that every such function  $w$  is a function  $u$  of another symmetric game. Moreover, any such function is concave. The question arises as to whether every concave function which is a value-of-information function of a symmetric game is a value-of-information function of a finite one-shot one-sided information game?

## 7.3 The symmetric case with a general state space

In Section 5 we could fully characterize the value of games with no information and only two states. In the case where there are more than two states we have only partial results. We could prove that every polynomial is realizable and that for every continuous piecewise rational function  $q$  and every  $\varepsilon > 0$  there is a realizable function  $u$  that coincides with  $q$  over a set whose Lebesgue measure is  $1 - \varepsilon$  (where the entire  $\Delta(K)$  has measure 1).

We conjecture that every continuous piecewise rational function over  $\Delta(K)$ , as in the two-state case, is realizable.

## 7.4 The value of games with bounded payoffs

In the discussion of Section 5 we imposed no restrictions on the payoffs. An interesting question concerns game with bounded payoffs. More specifically, what are the realizable functions with games whose payoffs are bounded between 0 and 1?

## 7.5 General information structures

In Theorem 3 we focus on partitional signaling structure. It would be interesting to characterize functions of a general signaling structure that are value-of-information functions.

## 8 References

- Aumann, R.J. and M. Maschler (1995), "*Repeated Games with Incomplete Information*," MIT Press.
- Bassan, B., O. Gossner, M. Scarsini and S. Zamir (1999), "A Class of Games with Positive Value of Information," to appear in *International Journal of Game Theory*.
- Blackwell, D. (1951), "Comparison of Experiments," *Proc. Second Berkeley Symposium on Math. Stat. and Prob.*, University of California Press, 93-102.
- Blackwell, D. (1953), "Equivalent Comparison of Experiments," *Ann. Math. Stat.*, **24**, 265-272.
- Bochnak, J. Coste M. and M-F. Roy (1988) "*Real Algebraic Geometry* ," Springer, Berlin ,Germany.
- Gilboa, I. and E. Lehrer (1991), "The Value of Information - An Axiomatic Approach," *Journal of Mathematical Economics* **20**, 443-459.
- Gossner, O. (1998), "Secure Protocols – or How Communication Generates Correlation," *Journal of Economic Theory*, **83**, 69-89.
- Gossner, O. (2000), "Comparison of Information Structures," *Games and Economic Behavior*, **30**, 44-63.
- Kamien, M., Y. Tauman and S. Zamir (1990), "Information Transmission," in T. Ichiishi, A. Neyman and Y. Tauman (eds.), *Game Theory and Applications*, Academic Press, 273-281.
- Lehrer, E. and D. Rosenberg (2003), "An Axiomatic Approach to the Value of Information in Games," mimeo.
- Mertens, J.-F. and S. Zamir (1971-1972), "The Value of Two-Person Zero-Sum Repeated Games with Lack of Information on Both Sides," *International Journal of Game Theory*, **1**, 39-64.
- Mertens, J.-F., S. Sorin and S. Zamir (1994), "*Repeated Games*," CORE Discussion Papers nos. 9420, 9421 and 9422.
- Neyman, A. (1991), "The Positive Value of Information," *Games and Economic Behavior*, **3**, 350-355.

## 9 Appendix

**Proof of Proposition 1.** We rewrite the game as a zero-sum game with incomplete information on one side as defined in Aumann and Maschler. Denote by  $\mu$  the probability on  $S$  defined by  $\mu(s) = \sum_{k \in K} p(k)\pi(s|k)$ . Consider a zero-sum game with incomplete information with state space  $S$ . Player 1 is informed of  $s$  and player 2 is not, the set of actions of the players is respectively  $A_1$  and  $A_2$ , and the stage payoff is  $\bar{g}_s(a_1, a_2) = \sum_{k \in K} p(k)\pi(s|k)g_k(a_1, a_2)$ .

The corresponding value of the  $n$ -th stage game is denoted by  $w_n(\mu)$ . It is easily seen that  $w_n(\mu) = v_n^I(p, (g_k)_{k \in K})$ . By Aumann and Maschler,  $w_n(\mu)$  converges as  $n$  goes to infinity to  $w(\mu)$  and therefore  $v_n^\pi(p, (g_k)_{k \in K})$  also converges to  $v^\pi(p, (g_k)_{k \in K}) = w(\mu)$ . ■

Recall that  $\pi_p(\cdot|s)$  denotes the conditional probability over  $K$  given the signal  $s$ . That is,  $\pi_p(k|s) = \frac{p(k)\pi(s|k)}{\sum_{l \in K} p(l)\pi(s|l)}$ . Also, denote  $\pi(s) = \sum_{k \in K} p(k)\pi(s|k)$ .

**Proof of Proposition 2.** Denote by  $\pi$  the information structure corresponding to the partition  $\mathcal{P}$  and by  $\pi'$  the information structure corresponding to  $\mathcal{P}'$ : for any  $T \in \mathcal{P}$ ,  $\pi(T|k) = 1$  if and only if  $k \in T$ , otherwise  $\pi(T|k) = 0$ . The definition of  $\pi'$  is similar.

Assume first that  $\mathcal{P}$  is a refinement of  $\mathcal{P}'$ . For any  $T \in \mathcal{P}$  and  $T' \in \mathcal{P}'$  let us define  $Q(T)(T')$  to be 1 if  $T \subset T'$  and 0 otherwise. This is a transition probability from  $\mathcal{P}$  to  $\mathcal{P}'$  because since both are partitions for any  $T$  there is a unique  $T'$  in  $\mathcal{P}'$  that contains  $T$ , and for any  $k$ ,  $\pi'(T'|k) = \sum_{T \in \mathcal{P}} Q(T)(T')\pi(T|k)$ .

Assume now that  $\pi$  and  $\pi'$  are derived from two partitional information structures  $\mathcal{P}$  and  $\mathcal{P}'$ , and that  $\pi$  is more informative than  $\pi'$ . This means that there is a probability distribution  $Q(T)$  for all  $T \in \mathcal{P}$  on  $\mathcal{P}'$ , such that for any  $k$ ,  $\pi'(T'|k) = \sum_{T \in \mathcal{P}} Q(T)(T')\pi(T|k)$ . Fix  $T \in \mathcal{P}$  and let  $k \in T$ . For any  $T' \in \mathcal{P}'$ ,  $\pi'(T'|k) = Q(T)(T')$ , so that  $Q(T)(T') \in \{0, 1\}$ . Hence, for any  $T$ , there is exactly one  $T'$  such that  $Q(T)(T') = 1$ , and  $\pi'(T'|k) = 1$ , so that  $k \in T'$ . This implies that  $\mathcal{P}$  is a refinement of  $\mathcal{P}'$ . ■