Nash and Limit Equilibria of Games with a Continuum of Players*

Guilherme Carmona†

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Abstract

We show that a strategy is a Nash equilibrium in a game with a continuum of players if and only if there exists a sequence of finite games such that its restriction is an ε_n-equilibria, with ε_n converging to zero. In our characterization, the sequence of finite games approaches the continuum game in the sense that the set of players and the distribution of characteristics and actions in the finite games converge to those of the continuum game. These results render approximate equilibria of large finite economies as an alternative way of obtaining strategic insignificance. Also, they suggest defining a refinement of Nash equilibria for games with a continuum of agents as limit points of equilibria of finite games. This allows us to discard those Nash equilibria that are artifacts of the continuum model, making limit equilibrium a natural equilibrium concept for games with a continuum of players.

Keywords: Nash equilibrium, limit equilibrium, noncooperative games, continuum of players.

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†Faculdade de Economia, Universidade Nova de Lisboa, Campus de Campolide, 1099-032 Lisboa, Portugal. Email: gcarmona@fe.unl.pt

1 Introduction

Many economic situations involve a large number of participants, each of which has a negligible influence on the aggregate outcome. As Aumann [1] has convincingly argued, the ideal situation of strategic insignificance can only be obtained in models featuring a continuum of agents. Likewise, equilibrium concepts that depend on the idea of strategic insignificance make good sense only in those models. This makes models with a continuum of agents an appealing framework for economic analysis.

Of course, real economies have a finite number of agents. Hence, conclusions obtained by studying economies with a continuum of agents will, typically, hold only approximately for real, finite economies. It is, thus, of central importance to know the extent to which continuum-of-agents economies are good approximations of large finite economies. For equilibrium analysis, in particular, it is important to know whether the action profiles that a given equilibrium concept singles out as reasonable outcomes in a continuum-of-agents economy are also reasonable in some sense in large finite economies.[1]

Moreover, models with a finite number of agents are more intuitive and therefore easier to understand than models with a continuum of agents. The same is true regarding equilibria of those models. Thus, we ask: can we relate equilibria of models with a continuum of agents with the more intuitive notion of equilibria of finite models?

We provide an answer to the above question for normal form games in which the payoff of each player depends on his choice and on the distribution of actions (see Mas-Colell [22]). First, we characterize Nash equilibria of games with a continuum of players in terms of approximate equilibria of games with a finite number of players. This result shows, in particular, that any Nash equilibrium of the continuum-of-players game is an approximate equilibrium in some sequence of similar finite games. Second, we provide an alternative characterization for games with a continuum of players in which there is a bound on the diversity of payoffs;[2] in particular, this second characterization shows that to any Nash equilibrium of the continuum-of-players game we can associate another, equivalent Nash equilibrium which is an approximate equilibrium in all sequences of similar finite games. Third, we show (under stronger assumptions) that there exist Nash equilibria that

[1] This is just one possible approximation question. In fact, others have been considered in the literature, and will be revised in Section 2.

[2] Formally, players' payoff functions form an equicontinuous family.

are limit points (in a particular sense described below) of Nash equilibria of finite games.

The two characterization results have the potential to make games with a continuum of players accessible to researchers that are not familiar with the measure theoretical tools needed to analyze such games. This is especially the case when there are a finite number of different payoff functions and possible actions, as we show below. In that case the only tools needed are the usual notions of convergence in the real line and approximate equilibrium in a finite normal form game.

Furthermore, our results suggest two ways of expressing the idea that conclusions obtained in games with a continuum of players hold approximately in some close finite games: first, conclusions reached using Nash equilibrium hold in approximate equilibria; second, those reached by limit equilibria also hold approximately in similar strategies that are Nash equilibria of finite games.

Despite the above interpretation of our characterization results, we point out that they do not imply that all Nash equilibria of games with a continuum of players will have properties similar to those of large finite games. In fact, we present two simple examples in which some Nash equilibria of games with a continuum of players have very different properties compared to those of Nash equilibria of similar large finite games. This difficulty with Nash equilibria can, in principle, be corrected by focusing on limit equilibria, which appears to be a better notion of equilibrium for games with a continuum of players.

A more precise description of our results, which we give below, will allow us to make some additional remarks. Our first result shows that a strategy is a Nash equilibrium in a game with a continuum of players if and only if there exists a sequence of finite games such that its restriction is an ε_n-equilibria, with ε_n converging to zero. Thus, Nash equilibria of games with a continuum of players are exactly the strategies that are approximate equilibria in some games obtained from the original one by selecting a finite number of players. In particular, this result gives us a sense in which approximate equilibria of large finite games is an alternative way of obtaining strategic insignificance. This interpretation is strengthened in equicontinuous games: in those games, every Nash equilibrium can be changed in a set of measure zero to obtain another Nash equilibrium for which the above characterization holds for all approximating sequences of finite games.

The sequence of finite games in the above results approximates the strate-

gic situation described by the given strategy in the game with a continuum of agents in the following sense: the set of players and distribution of characteristics and actions in the finite games converge to those of the continuum game.[3] The notion of approximation of games is important because, although there are many finite economies which we can associate with a continuum game, only those finite games that can approximately describe the same economic situation as the continuum game should be regarded as reasonable approximations. Since we obtain a complete characterization of equilibria, we can indeed interpret this approximation as convergence of the strategic situation in the finite games to the one in the continuum game.

This convergence also allows us to define a refinement of Nash equilibrium for games with a continuum of players in the same spirit of Selten's [32] perfect equilibrium: we say that a strategy is a limit equilibrium if it is the limit point in the above sense of a sequence of equilibria in finite games. Intuitively, limit equilibria are those that inherit the properties of equilibria of finite games, and so, by using it, we can in principle discard those Nash equilibria that are an artifact of the continuum construction. This point is illustrated by two examples, all of which having a continuum of Nash equilibria but a single, and plausible, limit equilibrium.

The interest of limit equilibrium would, nevertheless, be limited without an existence result. We show that in games in which the action space is finite, any strict equilibrium[4] is a limit equilibrium, thus providing a sufficient condition that guarantees the existence of limit equilibria. We also show that a limit equilibrium distribution exists in quasi-concave games; finally, we show that a limit equilibrium distribution can be represented as a limit equilibrium whenever it has a countable support.

In summary, we show that the conclusions obtained by studying Nash equilibria of games with a continuum of players hold in approximate equilibria in some similar large finite games. Despite this result, simple examples show that at least some Nash equilibria of games with a continuum of players can lead to very different conclusions than those obtained in large finite games. As the construction of limit equilibrium is designed to overcome this problem, it makes it a more natural notion of equilibrium for games with a continuum

[3]Instead of requiring that the set of players in the finite games converges to that of the continuum game, we can require only that the number of players in the finite games converges to infinity.

[4]I.e., a Nash equilibrium in which almost all players strictly prefer their action to any other.

of players.

2 Related Literature

Our results establish a relationship between equilibria of a continuum game with approximate equilibria of finite games. They are thus related to the paper of Fudenberg and Levine [10] which characterizes equilibria of a game as the limit points of games with smaller strategy space. As typically these smaller strategy spaces are finite whereas they are infinite in the game of interest, their results also establish a relationship between equilibria of a continuum game with approximate equilibria of finite games.

A related question is whether the limit of a converging sequence of equilibria of finite economies is an equilibrium in the continuum economy. This question was studied by Hildenbrand and Mertens [19] for pure exchange economies, by Dubey et al. [8] for strategic market games, and by Green [12] for normal form games. As those authors pointed out, this question amounts to asking whether the equilibrium correspondence is upper hemicontinuous when viewed as a (set-valued) function on the set of measures on the set of economic agents; general theorems on the upper hemicontinuity of the equilibrium correspondence were established in those papers.

Our work is also related to the question of whether or not the equilibrium correspondence is lower hemicontinuous. In general, the equilibrium correspondence will not have that property. However, the papers by Mas-Colell [21] and Novshek and Sonnenschein [24] show that regular Walrasian equilibria of a continuum economy can be approximated by the noncooperative Cournot equilibria for the tail of any approximating sequence of finite economies, a property that is close to lower hemicontinuity.

Postlewaite and Schmeidler [26] and Hildenbrand [18] showed that large finite games have properties that are approximate versions of those of continuum economies. For example, Postlewaite and Schmeidler [26] showed that Nash equilibria of large finite market games are approximately efficient, a property that equilibria of market games with a continuum of players have. Also, Rashid [27] and Wooders, Selten and Cartwright [36] showed that large finite games in a certain class have Nash equilibria in which all but a small fraction of players play pure strategies; this result clearly parallels Schmeidler's [31] Theorem 2, asserting that any game with a continuum of players of the same class has a Nash equilibrium in which almost all players play a

pure strategy.[5]

In Barlo and Carmona [3] we propose a refinement of Nash equilibria which is similar to limit equilibria. There we consider perturbed games that have a continuum of players, and that differ from the original one because every player believes that he alone has a small, but positive, impact on the societal choice. We define a strategic equilibrium as a limit point of Nash equilibria of these perturbed games when players' impact on the societal choice goes to zero. Although there seems to be a relationship between the two approaches, its precise nature is still unknown to us.

In Carmona [4], we consider a more specialized framework in which each player's payoff functions depend only on his action and on the average choice of the others. There we obtain similar characterization results for different notions of approximation of games, which can be thought of as alternative ways of describing the convergence of the economic situation. In Carmona [5] we use similar tools to those used here to characterize Nash equilibrium distributions of games with a continuum of players in terms of symmetric, approximate equilibrium distributions with finite support of similar continuum-of-players games.

On the technical side, we have gained much form the suggestion by Hildenbrand [18], Hart, Hildenbrand and Kolhberg [16] and Mas-Colell [22] that any game with a continuum of agents is easily analyzed by studying the distributions it induces on the space of action and on the space of players's characteristics. Such an analytical tool is helpful because the distribution induced by a game and an equilibrium strategy is an equilibrium distribution, and conversely, if an equilibrium distribution is the distribution induced by a game and a strategy, then this strategy is an equilibrium of that game. Also, as in Hildenbrand [17] and Debreu [7], respectively, we use the weak convergence of measures and the convergence of closed sets with respect to the Hausdorff distance in order to define our notion of convergence of games.

3 Large Games

Let A be a non-empty, compact metric space of *actions* and $\mathcal{M}$ be the set of Borel probability measures on A endowed with the weak convergence topol-

[5]Rashid [27] and Wooders, Selten and Cartwright [36] showed a stronger result: all Nash equilibria of sufficiently large finite games can be purified in ε-equilibria. See also Carmona [6].

ogy. By Parthasarathy [25, Theorem II.6.4], it follows that $\mathcal{M}$ is a compact metric space. We use the following notation: we write $\mu_n \Rightarrow \mu$ whenever $\{\mu_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ converges to μ and ρ denote the Prohorov metric on $\mathcal{M}$, which is known to metricize the weak convergence topology. We let d_A denote the metric on A.

Let $\mathcal{U}$ denote the space of continuous utility functions $u : A \times \mathcal{M} \to \mathbb{R}$ endowed with the supremum norm. The set $\mathcal{U}$ represents the space of *players' characteristics*; it is a complete, separable metric space.

A *game with a continuum of players* is characterized by a measurable function $U : [0,1] \to \mathcal{U}$, where the unit interval $[0,1]$ is endowed with the Lebesgue measure λ on the Lebesgue measurable sets and represents the set of *players*. We represent such game by $G = (([0,1], \lambda), U, A)$.

A *game with a finite number of players* is characterized by a function $U : T \to \mathcal{U}$, where T is a finite subset of $[0,1]$. The set T represents the set of players and it is endowed with the uniform measure ν: if T has N elements, then the measure ν on T satisfies $\nu(\{t\}) = 1/N$ for all $t \in T$. We represent such game by $G = ((T, \nu), U, A)$.

We are especially interested in games with a finite number of players that are derived from a given game $G = (([0,1], \lambda), U, A)$ with a continuum of players. Those are games $H = ((T, \nu), U_{|T}, A)$ where $U_{|T}$ denotes the restriction of U to T.

In all the cases above, a game is defined as a measurable function from a measure space of players into $\mathcal{U}$. Although we will focus exclusively on the particular cases mentioned above, we present the following definition in this general case.

Let $(X, \mathcal{X}, \mu)$ be a measure space and $G = ((X, \mathcal{X}, \mu), U, A)$ be a game. A strategy is a measurable function $f : X \to A$. Given a strategy f, $y \in A$, and $t \in T$, let $f \setminus_t y$ denote the strategy g defined by $g(t) = y$, and $g(\tilde{t}) = f(\tilde{t})$, for all $\tilde{t} \neq t$.

For any $\varepsilon \geq 0$ and strategy f let

$$\begin{aligned} E(f, \varepsilon, \mu) = \big\{ & t \in \operatorname{supp}(\mu) : U(t)(f(t), \mu \circ f^{-1}) \geq \\ & U(t)(a, \mu \circ (f \setminus_t a)^{-1}) - \varepsilon \text{ for all } a \in A \big\} . \end{aligned} \tag{1}$$

The set $E(f, \varepsilon, \mu)$ is the set of players in the support of μ that are within ε of their best response by playing according to f. When $\varepsilon = 0$, we will write $E(f, \mu)$ instead of $E(f, 0, \mu)$.

Lemma 1 *Let $\varepsilon \geq 0$, a game $G = (([0,1],\lambda), U, A)$, and a strategy f be given. Then, $E(f,\varepsilon,\lambda)$ is measurable.*

It is clear that $E(f,\varepsilon,\mu)$ is measurable when μ has finite support. Hence, $\mu(E(f,\varepsilon,\mu))$ is well-defined both when $\mu = \lambda$, and when μ has finite support. We then say that f is an $\varepsilon-$*equilibrium of a game* G if $\mu(E(f,\varepsilon,\mu)) \geq 1-\varepsilon$. Thus, in an $\varepsilon-$equilibrium, all but a small fraction of players are close to their optimum by choosing according to f. A strategy f is a *Nash equilibrium of* G if f is an $\varepsilon-$equilibrium of G for $\varepsilon = 0$.

4 Equilibrium Distributions

4.1 Games with a continuum of players

Instead of defining a game as a measurable function from players into characteristics, we could have started by describing the game as a probability measure μ on $\mathcal{U}$ as in Mas-Colell [21]. For our purpose, equilibrium distributions provide a useful device to study properties of equilibria in games with a continuum of players.

Given a Borel probability measure τ on $\mathcal{U} \times A$, we denote by $\tau_{\mathcal{U}}$ and τ_A the marginals of τ on $\mathcal{U}$ and A respectively. The expression $u(a,\tau) \geq u(A,\tau)$ means $u(a,\tau) \geq u(a',\tau)$ for all $a' \in A$.

Given a game μ, a Borel probability measure τ on $\mathcal{U} \times A$ is an *equilibrium distribution for* μ if

1. $\tau_{\mathcal{U}} = \mu$, and
2. $\tau(\{(u,a) \in \mathcal{U} \times A : u(a,\tau_A) \geq u(A,\tau_A)\}) = 1$.

We will use the following notation: $B_\tau = \text{supp}(\tau) \cap \{(u,a) \in \mathcal{U} \times A : u(a,\tau_A) \geq u(A,\tau_A)\}$. Note that B_τ is closed, and so a Borel set; hence $\tau(B_\tau)$ is well defined. Also, if (u,a) belong to B_τ, then a maximizes the function $\tilde{a} \mapsto u(\tilde{a},\tau_A)$. Thus, we are implicitly assuming that the choice of any player does not affect the distribution of actions. It is in this sense that the notions of this section describe a game with a continuum of players.

Any game $G = (([0,1],\lambda), U, A)$ and strategy f induces a Borel probability measure τ on $\mathcal{U} \times A$ by the formula $\tau = \lambda \circ (U,f)^{-1}$. Furthermore, as the next lemma shows, if f is a Nash equilibrium of G, then $\tau = \lambda \circ (U,f)^{-1}$

is an equilibrium distribution of $\lambda \circ U^{-1}$; conversely, if τ is an equilibrium distribution and $\tau = \lambda \circ (U, f)^{-1}$, then f is a Nash equilibrium of G.

Lemma 2 *A strategy f is a Nash equilibrium of a game $G = (([0,1], \lambda), U, A)$ if and only if $\tau = \lambda \circ (U, f)^{-1}$ is an equilibrium distribution of $\lambda \circ U^{-1}$.*

4.2 Games with a finite number of players

Similarly as for games with a continuum of players, any games with a finite number of players together with a strategy also induces a Borel probability measure on $\mathcal{U} \times A$, again by the formula $\tau = \nu \circ (U, f)^{-1}$. However, in such a game, the choice of a single player has an affect on the distribution of actions and the definition of an equilibrium distribution needs to be adapted accordingly.

Let $G = ((T, \nu), U, A)$ be a game with a finite number of players, f a strategy, $\tau = \nu \circ (U, f)^{-1}$ and $\varepsilon > 0$. Then τ is an $\varepsilon-$equilibrium distribution of $\nu \circ U^{-1}$ if $\tau(\{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(a', \tau_A^{u,a,a'}) - \varepsilon \text{ for all } a' \in A\}) \geq 1 - \varepsilon$, where $\tau_A^{u,a,a'} = \nu \circ g^{-1}$, g is defined by $g(t') = a'$, and $g(t) = f(t)$ for all $t \neq t'$ and $t' \in T$ is such that $(U(t), f(t)) = (u, a)$.

Note first that the distribution $\tau_A^{u,a,a'}$ is independent of the choice of t'. This is the distribution on the action space A that will arise if one player with characteristic u and playing a deviates and plays a'. In fact, we can simply define $\tau_A^{u,a,a'}$ as the marginal on A of $\tau^{u,a,a'}$, which is defined from τ as follows: $\tau^{u,a,a'}(\{(u, a)\}) = \tau(\{(u, a)\}) - 1/|T|$, $\tau^{u,a,a'}(\{(u, a')\}) = \tau(\{(u, a')\}) + 1/|T|$ and $\tau^{u,a,a'}(\{(\tilde{u}, \tilde{a})\}) = \tau(\{(\tilde{u}, \tilde{a})\})$ for all $(\tilde{u}, \tilde{a})$ different from (u, a) and from (u, a'). Note that this definition allows us to define an equilibrium distribution of a finite game without the explicit knowledge of a strategy, which will be useful in section 6.2.

The following lemma show that for large finite games $\tau_A^{u,a,a'}$ is close to τ_A.

Lemma 3 *Let $G = ((T, \nu), U, A)$ be a game with a finite number of players and f a strategy. If g is another strategy that differs from f in at most one point, then*

$$\rho(\nu \circ f^{-1}, \nu \circ g^{-1}) \leq \frac{1}{|T|}.$$

Note also that if the game G had a continuum of players then $\tau_A^{u,a,a'} = \tau_A$ for all u, a, a' and so in fact this definition coincides with the one given before. Similar to that case we have:

Lemma 4 *A strategy f is an $\varepsilon-$equilibrium of a game $G = ((T, \nu), U, A)$ if and only if $\tau = \nu \circ (U, f)^{-1}$ is an $\varepsilon-$equilibrium distribution of $\nu \circ U^{-1}$.*

We will use the following notation: $B_\tau^\varepsilon = \text{supp}(\tau) \cap \{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(a', \tau_A^{u,a,a'}) - \varepsilon \text{ for all } a' \in A\}$, where $\tau_A^{u,a,a'}$ is as before.

5 Characterizations of Nash Equilibria

Our first main result is a characterization result. It characterizes Nash equilibria of games with a continuum of players in terms of approximate equilibria of games with a finite number of players. As the finite games in the characterization converge (in the sense of properties 2 and 3 below) to the game with a continuum of players, Theorem 1 is also a limit result. Before we state it, we need the following notion of convergence of closed sets, which will be used in our notion of convergence of games: If C is a non-empty closed set and $\{C_n\}$ is a sequence of non-empty closed sets then $C_n \to C$ means convergence with respect to the Hausdorff distance (see Hildenbrand [18, p. 16]).

Theorem 1 *A strategy f is a Nash equilibrium of a game $G = ([0, 1], \lambda), U, A)$ with a continuum of players if and only if there exists a sequence $\{\nu_n\}_{n=1}^\infty$ of measures, and a sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive real numbers such that:*

1. *ν_n is the uniform measure on T_n, a finite subset of $[0, 1]$,*
2. *$\nu_n \circ (U_{|T_n}, f_{|T_n})^{-1} \Rightarrow \lambda \circ (U, f)^{-1}$,*
3. *$\text{supp}(\nu_n) \to \text{supp}(\lambda)$*
4. *$\varepsilon_n \searrow 0$, and*
5. *$f_{|T_n}$ is an ε_n-equilibrium of $G_n = ((T_n, \nu_n), U_{|T_n}, A)$ for all $n \in \mathbb{N}$.*

Theorem 1 provides a natural interpretation of Nash equilibria of games with a continuum of players: if f is such a strategy, then we can find a finite game, similar to the original continuum one, in which f is close to being a Nash equilibrium. Conversely, if a strategy f can be made as close to being a Nash equilibrium as we want in some finite game similar to the original continuum one, then f will be a Nash equilibrium of the continuum game.

Given this equivalence, it is quite natural that approximate equilibria of large finite games have approximately the same properties of Nash equilibria of games with a continuum of players, as has been shown by many authors.

This result also confirms Fudenberg and Levine's [10] conclusions on the appropriate definition of equilibria in games that are defined as limits. As in their paper, Theorem 1 shows that in order to describe all equilibria of a game with a continuum of players it is necessary to take limits, not only of equilibria of converging finite games, but of $\varepsilon-$equilibria with ε converging to zero.

Remark 1 *Theorem 1 holds if we replace the condition* $\text{supp}(\nu_n) \to \text{supp}(\lambda)$ *by the weaker condition that* $|T_n| \to \infty$.

Since $|T_n| \to \infty$ is weaker than $\text{supp}(\nu_n) \to \text{supp}(\lambda)$, the remark is clear regarding the necessary condition. Furthermore, it can be easily verified that the proof of the sufficiency part only requires $|T_n| \to \infty$.

An important implication of our two characterization results, especially in the format of the above remark, is that they allow us to determine whether a given strategy is a Nash equilibrium of a game with a continuum of agents without necessarily having to deal with the technical difficulties involved in such games. Consider, for instance, a game with a continuum of agents in which there is a finite number of actions and a finite number of possible payoff functions, a typical assumption in applications. In this case, all the tools we need to analyze such a game are standard: we need to determine what the minimal ε is that makes a given strategy an $\varepsilon-$equilibrium in a finite normal form game and we need to guarantee that $\nu_n \circ (U_{|T_n}, A)^{-1}(\{(u,a)\})$ converges to $\lambda \circ (U,A)^{-1}(\{(u,a)\})$ (in $\mathbb{R}$) for all pairs (u,a) in $U([0,1]) \times A$.

We illustrate the above comment with the following simple example. Let $G = (([0,1],\lambda), U, A)$ be described by: $A = \{a,b\}$ and $U(t) = u$ for all $t \in [0,1]$, where $u(a,\mu) = \mu(\{a\})$ and $u(b,\tau) = 1 - \mu(\{a\})$. It is clear that the only equilibrium distributions of this game are τ_1, τ_2 and τ_3 satisfying $\tau_1(\{(u,a)\}) = 1/2$, $\tau_2(\{(u,a)\}) = 1$ and $\tau_3(\{(u,a)\}) = 0$, and where $\tau_i(\{(u,b)\}) = 1 - \tau_i(\{(u,a)\})$ for $i = 1,2,3$. Thus, a strategy f defined by

$$f(t) = \begin{cases} a & \text{if } 0 \leq t \leq \frac{1}{2}, \\ b & \text{otherwise} \end{cases} \tag{2}$$

is a Nash equilibrium. This fact can be inferred by Theorem 1 as follows: for each $n \in \mathbb{N}$, let $t_n^1 = 1/2 - 1/(2n)$, $t_n^2 = 1/2 + 1/(2n)$, $T_n^1 = \{t_1^1, \ldots, t_n^1\}$,

$T_n^2 = \{t_1^2, \ldots, t_n^2\}$ and $T_n = T_n^1 \cup T_n^2$. Letting $\tau_n = \nu_n \circ (U_{|T_n}, f_{|T_n})^{-1}$, we have that $\tau_n(\{(u,a)\}) = \tau_n(\{(u,b)\}) = 1/2$ and so, obviously, $\tau_n \Rightarrow \tau$. For $t \in T_n^1$, we have $u(f(t), \tau_{A,n}) = 1/2$ if player t plays $f(t)$; if she chooses b, then she changes the distribution of actions to $\tau_{A,n}^{a,b}(\{a\}) = 1/2 - 1/(2n)$, thus receiving $u(b, \tau_{A,n}^{a,b}) = 1/2 + 1/(2n)$. Defining $\varepsilon_n = 1/2n$, we conclude that player t is ε_n-optimizing. Since a similar result holds for any $t \in T_n^2$, it follows that $f_{|T_n}$ is an ε_n-equilibrium of $G_n = ((T_n, \nu_n), U_{|T_n}, A)$ for all $n \in \mathbb{N}$. Finally, since $\varepsilon_n \searrow 0$, $\tau_n \Rightarrow \tau$ and $|T_n| \to \infty$, then f is an equilibrium of G.

We showed in Theorem 1 that for any Nash equilibrium f we can find a sequence of finite games such that f is an approximate equilibrium in those games. The following question arises naturally: if we are given a Nash equilibrium f and arbitrary sequence $\{G_n\}$ of finite games converging in the sense of 1–3 of Theorem 1, when is it the case that f is a ε_n-equilibrium of G_n with $\varepsilon_n \to 0$? There are essentially two difficulties with this question: first, players' characteristics may be too diverse; second, some players that are not optimizing in the limit game by playing according to f may be players in all finite games. We can solve the first problem by adding an equicontinuity assumption; we solve the second by replacing f by an equivalent strategy.

Let K be a subset of $\mathcal{U}$. Then K is equicontinuous if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|u(a,\tau) - u(b,\mu)| < \varepsilon$$

whenever $\max\{d_A(a,b), \rho(\tau,\mu)\} < \delta$, $a, b \in A$, $\tau, \mu \in \mathcal{M}$ and $u \in K$ (see Rudin [30, p. 156]). In our framework, equicontinuity can be interpreted as placing "a bound on the diversity of payoffs," as pointed out by Khan, Rath and Sun [13].

Let f and g be strategies in a game $G = (([0,1], \lambda), U, A)$. We say that g is equivalent to f, and write $f \sim g$, if $\lambda(\{t \in [0,1] : g(t) \neq f(t)\}) = 0$ (see Kolmogorov and Fomin [15, 28.3, p. 288]). Note that if f is a Nash equilibrium of G and $f \sim g$, then g is a Nash equilibrium of G as well.

Theorem 2 *Assume that $U([0,1])$ is equicontinuous. Then, a strategy f is a Nash equilibrium of a game $G = (([0,1], \lambda), U, A)$ with a continuum of players if and only if there exists a strategy $g \sim f$ for which the following property holds:*

For any sequence $\{\nu_n\}_{n=1}^{\infty}$ of measures satisfying

1. *ν_n is the uniform measure on T_n, a finite subset of $[0,1]$,*

2. $\nu_n \circ (U_{|T_n}, g_{|T_n})^{-1} \Rightarrow \lambda \circ (U, g)^{-1}$,

3. $\text{supp}(\nu_n) \to \text{supp}(\lambda)$

there exists a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ *of positive real numbers such that* $\varepsilon_n \to 0$, *and* $g_{|T_n}$ *is an* ε_n-*equilibrium of* $G_n = ((T_n, \nu_n), U_{|T_n}, A)$ *for all* $n \in \mathbb{N}$.

Again, we remark that Theorem 2 holds if we replace the condition $\text{supp}(\nu_n) \to \text{supp}(\lambda)$ by the weaker condition that $|T_n| \to \infty$. Together with the following remark, it makes Theorem 2 a useful tool to analyze games with a continuum of agents with a finite number of actions and a finite number of possible payoff functions.

Remark 2 *If both* $U([0,1])$ *and* A *are finite (or more generally, if* $\text{supp}(\lambda \circ (U, f)^{-1})$ *is finite), then we can let* $g = f$ *in the statement of Theorem 2.*

We illustrate how Theorem 2 together with Remark 2 can be used to show that a strategy is not a Nash equilibrium. In the example following Theorem 1, let strategy g be defined by

$$g(t) = \begin{cases} a & \text{if } 0 \leq t \leq \frac{1}{4}, \\ b & \text{otherwise} \end{cases} \tag{3}$$

One easily sees that g is a not a Nash equilibrium. This fact can also be inferred via Theorem 2 as follows: for each $n \in \mathbb{N}$, let $t_n^1 = 1/4 - 1/(4n)$, $t_n^2 = 1/4 + 1/(4n)$, $T_n^1 = \{t_1^1, \ldots, t_n^1\}$, $T_n^2 = \{t_1^2, \ldots, t_{3n}^2\}$ and $T_n = T_n^1 \cup T_n^2$. Letting $\mu_n = \nu_n \circ (U_{|T_n}, f_{|T_n})^{-1}$, we have that $\mu_n(\{(u,a)\}) = 1/4$ and $\mu_n(\{(u,b)\}) = 3/4$; obviously, $\mu_n \Rightarrow \mu$, where $\mu = \lambda \circ (U, g)^{-1}$. For $t \in T_n^1$, we have $u(f(t), \mu_{A,n}) = 1/4$ if player t plays $f(t)$; if he chooses b, then he changes the distribution of actions to $\mu_{A,n}^{a,b}(\{a\}) = 1/4 - 1/(4n)$, thus receiving $u(b, \mu_{A,n}^{a,b}) = 3/4 + 1/(4n)$. Since $\nu_n(\{t \in T_n : u(f(t), \mu_{A,n}) < u(a', \mu_{A,n}^{a,a'}) - 1/2 \text{ for } a' \neq a\} \geq 1/4$ it follows that $g_{|T_n}$ is not a $1/4-$equilibrium of G_n for all $n \in \mathbb{N}$. Hence, g is not a Nash equilibrium of G.

Theorem 2 strengthens the idea that games with a continuum of players can be useful in order to infer properties about large finite games. In fact, any property that the Nash equilibrium g has, will hold in approximate equilibrium in all close finite games.

The following theorem builds upon this view and presents an asymptotic result on the existence of pure strategy Nash equilibria. The problem can

be stated as follows: consider a finite game $G = ((T, \nu), A, U)$, where A is a set of mixed strategies. That is, letting Δ_m denote the unit simplex in $\mathbb{R}^m$, $m \in \mathbb{N}$, we have that $A = \Delta_m$ with the usual interpretation: there are m pure strategies which are identified with the vertices $\{e_1, \ldots, e_m\}$ of Δ_m, and players can randomize over it. Also, we assume the following expected utility hypothesis: if $a = \sum_{i=1}^m a_i e_i$, then $U(t)(a, \mu) = \sum_{i=1}^m a_i U(t)(e_i, \mu)$ for all $t \in T$ and $\mu \in \mathcal{M}$.

To any such game G, with $T = \{t_1, \ldots, t_{|T|}\}$, we associate to it its $n^{th}-$replica, $n \in \mathbb{N}$: we say that a game G_n is an $n^{th}-$replica of G if the set of players has $n|T|$ elements (and is endowed with the uniform measure), each player has A as his action space and there are n players with playoff function $U(t_i)$ for all $1 \leq i \leq |T|$. We regard all such G_ns to be equivalent and we refer to that equivalence class as the $n^{th}-$replica of G.

Given a finite game as above, and the sequence $\{G_n\}$ of its replicas, we can define a game with a continuum of players that can be thought of as the limit of $\{G_n\}$. In such a limit game, we can restrict players to play pure strategies and we still get a Nash equilibrium. We can then use such a Nash equilibrium to construct an approximate equilibrium for the replica games in which a large fraction of players play pure strategies.

Theorem 3 *Let $G = ((T, \nu), A, U)$ be a game with T finite and $A = \Delta_m$ for some $m \in \mathbb{N}$. Let G_n be the $n^{th}-$replica of G. Then, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that G_n has an $\varepsilon-$equilibrium in which all but a fraction of players less than ε plays a pure strategy.*

Theorem 3 provides an example of a property which, because it holds for any game with a continuum of agents, will hold for sufficiently large finite games. Since it is in part a consequence of Theorem 1, it justifies the claim that the characterization given in Theorem 1 makes some asymptotic results available become quite natural.

It is also useful to compare Theorem 3 to Rashid's [27] Theorem. Theorem 3 does not require that payoffs depend only on the average choice of the other players as in Rashid's Theorem, and so it is more general in this respect. However, it requires replication, which is not required by Rashid's Theorem.[6]

[6]Note that, although not explicitly stated, Rashid's Theorem requires that players' payoff functions are selected from an equicontinuous family. For details, see Carmona [6].

6 Limit Points of Equilibria of Finite Games

6.1 Strict Equilibria

A natural question that arises from Theorem 1 is what kind of equilibria of a game with a continuum of players can be described as equilibria of a sequence of converging finite games. In other words, for what kind of equilibria can we take $\varepsilon_n = 0$ for all $n \in \mathbb{N}$ in Theorem 1? Theorem 4 below provides a partial answer by showing that in games with a finite action space, strict equilibria satisfy that property.

A strategy f in a game $G = (([0,1], \lambda), U, A)$ is a *strict* equilibrium if

$$U(t)(f(t), \lambda \circ f^{-1}) > U(t)(a, \lambda \circ f^{-1})$$

for all $a \neq f(t)$ and a.e. $t \in [0,1]$.

Theorem 4 *Let f be a strict equilibrium of a game $G = (([0,1], \lambda), U, A)$ with A finite. Then, there exists a sequence $\{\nu_n\}_{n=1}^{\infty}$ of measures such that:*

1. *ν_n is the uniform measure on T_n, a finite subset of $[0,1]$,*
2. *$\nu_n \circ (U_{|T_n}, f_{|T_n})^{-1} \Rightarrow \lambda \circ (U, f)^{-1}$,*
3. *$\text{supp}(\nu_n) \to \text{supp}(\lambda)$ and*
4. *$f_{|T_n}$ is an equilibrium of $G_n = ((T_n, \nu_n), U_{|T_n}, A)$ for all $n \in \mathbb{N}$.*

Thus, Theorem 4 shows that strict equilibria of games with a finite action space have the appealing property of being limit points of Nash equilibria of converging finite games. Unfortunately, not every game has a strict equilibrium. In the following, we present an example of a game with a strict equilibrium.

Consider two bridges that start at the same location on the north bank of a river but end in different places on the south bank. Denote this different location on the east bank by 0 and 1. There is a continuum of people living on the east bank, each of which is indexed by his address. That is, player $t \in [0,1]$ lives at t. Players' preference about which bridge to take from the north bank depends on how many people is using each bridge, and on the distance from their ends to their house.

Formally, we have a game with a continuum of players $G = (([0,1],\lambda), U, A)$ where $A = \{0,1\}$ and 0 stands for taking the bridge that ends at 0. We assume that 1. U is continuous, 2. player 0 always prefers bridge 0 (i.e., $U(0)(0,\mu_A) > U(0)(1,\mu_A)$ for all μ_A), 3. player 1 always prefers bridge 1, 4. $t \mapsto U(t)(0,\mu_A)$ is strictly decreasing for all μ_A, 5. $t \mapsto U(t)(1,\mu_A)$ is strictly increasing for all μ_A, 6. $U(t)(0,\cdot)$ is strictly decreasing in $\mu_A(\{0\})$ and 7. $U(t)(1,\cdot)$ is strictly increasing in $\mu_A(\{0\})$.

Given the above assumptions, for every μ_A there exists a unique $t \in (0,1)$ such that $U(t)(0,\mu_A) = U(t)(1,\mu_A)$. Let $g : \mathcal{M} \to [0,1]$ denote this dependence and, for every $t \in [0,1]$, let $\varphi(t) \in \mathcal{M}$ be defined by $\varphi(t)(\{0\}) = t$. Then the unique fixed point t^* of $t \mapsto g \circ \varphi(t)$ induces the unique Nash equilibrium f of G: $f(t) = 0$ for $t < t^*$ and $f(t) = 1$ for $t > t^*$. Also, one easily sees that f is a strict equilibrium.

6.2 Limit Equilibria

If f is a strict equilibrium in a game $G = (([0,1],\lambda), U, A)$ in which A is finite, then f is a limit equilibrium in the sense that it is the limit of a sequence of Nash equilibria of finite games, which converge to G (in this case, every equilibrium in the sequence can be chosen to be f). This property is appealing insofar as we like to regard the ideal model of a game with a continuum of players as a limit model build upon the notion of games with a finite number of players. Furthermore, we may hope that limit equilibria, despite being Nash equilibria of games in which players are strategically insignificant, will preserve the properties of equilibria of large finite games.[7] Thus, we are led to use the above property in order to define a refinement of Nash equilibrium in games with a continuum of players as follows. A strategy f in a game $G = (([0,1],\lambda), U, A)$ is a *limit equilibrium* if there exists a sequence $G_n = ((T_n,\nu_n), U_{|T_n}, A)$ of games and a sequence of strategies $f_n : T_n \to A$ of G_n such that:

1. ν_n is the uniform measure on T_n, a finite subset of $[0,1]$,
2. $\nu_n \circ (U_{|T_n}, f_n)^{-1} \Rightarrow \lambda \circ (U,f)^{-1}$,
3. $\mathrm{supp}(\nu_n) \to \mathrm{supp}(\lambda)$ and

[7] As the examples of Section 6.3 illustrate, this may not be the case with any arbitrary Nash equilibrium.

4. f_n is an equilibrium of G_n for all $n \in \mathbb{N}$.

It follows from Theorem 1 that any limit equilibrium is a Nash equilibrium.

Theorem 5 *Let $G = (([0,1], \lambda), A, U)$ be a game with a continuum of players and f be a limit equilibrium of G. Then, f is a Nash equilibrium of G.*

Similarly, Theorem 4 shows that in games with a finite action space, any strict equilibrium is a limit equilibrium. In particular, limit equilibria exist in games with a finite action space that have a strict equilibrium. However, if the action space is uncountable, then in general there are no Nash equilibria and so, there are no limit equilibria (see Khan, Rath and Sun [13]). In contrast, as Mas-Colell [21] as shown, an equilibrium distribution always exists. Hence, it is natural to start by asking when is it that there exists a limit equilibrium distribution.

The definition of a limit equilibrium distribution is analogous to the one for limit equilibrium: A distribution τ on $\mathcal{U} \times A$ satisfying $\tau_{\mathcal{U}} = \lambda \circ U^{-1}$ is a *limit* equilibrium distribution for a game $G = (([0,1], \lambda), U, A)$ if there exists a sequence $G_n = ((T_n, \nu_n), U_{|T_n}, A)$ of games and a sequence of equilibrium distributions τ_n of G_n such that:

1. ν_n is the uniform measure on T_n, a finite subset of $[0,1]$,

2. $\tau_n \Rightarrow \tau$, and

3. $\text{supp}(\nu_n) \to \text{supp}(\lambda)$.

Note that property 2 implies that $\nu_n \circ U_{|T_n}^{-1} \Rightarrow \lambda \circ U^{-1}$.

We first provide sufficient conditions for the existence of a limit equilibrium distribution (Theorem 6) and then conditions for the existence of a limit equilibrium strategy (Theorem 7). In both results, the assumptions of Section 3 are still in place, i.e., A is a compact metric space, and players have payoff function in $\mathcal{U}$, the space of continuous function on $A \times \mathcal{M}$.

Theorem 6 *Let $G = (([0,1], \lambda), U, A)$ be a game satisfying 1. A is a convex subset of a metric vector space and 2. $a \mapsto U(t)(a, \tau)$ is quasi-concave for all $t \in [0,1]$ and all $\tau \in \mathcal{M}$. Then G has a limit equilibrium distribution.*

We remark that the convexity of A and the quasi-convexity of $a \mapsto U(t)(a, \tau)$ are needed to guarantee the existence of Nash equilibria in finite games. Clearly, it can be replaced by a condition requiring that all finite game with a sufficiently large set of players have a Nash equilibrium.[8] This may be helpful since there are games that have equilibria if the set of players is large, although they may fail to have equilibria if played by a small number of players. An example of this case can be found in Bamón and Frayssé's [2] analysis of Cournot markets.

Alternatively, we could allow for mixed strategies in the finite games, and so allow for a limit equilibrium $f : [0, 1] \to A$ to be the limit point of a sequence of equilibria $f_n : T_n \to \mathcal{M}$. Since A is homeomorphic to the closed subset $D = \{p_a : a \in A\}$ of $\mathcal{M}$, where p_a is the measure degenerated at a, the definition would still make sense. Then, in order to prove the existence of a limit equilibrium (or, of a limit equilibrium distribution) it would be enough to show the existence of Nash equilibria of the finite games in the converging sequence with the property that only a vanishing fraction of players use mixed strategies. Such a result exists for $\varepsilon-$equilibrium, as shown by Rashid [27] and Wooders, Selten and Cartwright [36], but, to our knowledge, not for Nash equilibria.

We will illustrate a third approach for establishing the existence of a limit equilibrium distribution with the following example: let $G = (([0, 1], \lambda), A, U)$ be such that $A = \{a, b\}$ and $U(t) = u$ for all $t \in [0, 1]$, where $u(a, \mu) = 1 - \mu(\{a\})$ and $u(b, \mu) = \beta\mu(\{a\})$ and $\beta > 0$. One easily sees that the unique equilibrium distribution is τ satisfying $\tau(\{(u, a)\}) = 1/(1 + \beta)$ and $\tau(\{(u, b)\}) = 1 - \tau(\{(u, a)\})$. We claim that τ is also a limit equilibrium distribution. In order to show it, consider a game $G_n = ((T_n, \nu), A, U_{|T_n})$ where $|T_n| = n$. An equilibrium distribution τ_n in G_n has to satisfy three conditions. First, players playing a must prefer to play a to playing b:

$$1 - \tau_{A,n}(\{a\}) \geq \beta\tau_{A,n}(\{a\}) - \frac{\beta}{n}. \tag{4}$$

Second, players playing b must prefer to play b to playing a:

$$\beta\tau_{A,n}(\{a\}) \geq 1 - \tau_{A,n}(\{a\}) - \frac{1}{n}. \tag{5}$$

[8]Formally, we require that there is $N \in \mathbb{N}$ such that $T \subseteq [0, 1]$, finite and satisfying $|T| \geq N$ implies that $((T, \nu), U_{|T}, A)$ has a Nash equilibrium.

Finally, the distribution must be generated by a game with n players:

$$\tau_{A,n}(\{a\}) = \frac{x}{n}, \tag{6}$$

where $x \in \{0, \ldots, n\}$. Rearranging the first two inequalities, one obtains

$$\frac{n-1}{n(1+\beta)} \leq \tau_{A,n}(\{a\}) \leq \frac{n+\beta}{n(1+\beta)}, \tag{7}$$

and so one needs to find $x \in \{0, \ldots, n\}$ such that

$$0 \leq \frac{n-1}{1+\beta} \leq x \leq \frac{n+\beta}{1+\beta} \leq 1. \tag{8}$$

Since $\frac{n+\beta}{1+\beta} - \frac{n-1}{1+\beta} = 1$, such an x exists. Finally, by (7), $\tau_{A,n}(\{a\}) \to \tau_A(\{a\})$.

Note that if we write $a_1 = a$, $a_2 = b$, $\pi_1 = \tau(\{(u,a)\})$ and $\pi_2 = \tau(\{(u,b)\})$, then $\tau = \pi$ is an equilibrium distribution if and only if

$$\sum_{i=1}^{2} \pi_i u(a_i, \pi) \geq \sum_{i=1}^{2} \tilde{\pi}_i u(a_i, \pi), \tag{9}$$

for all $\tilde{\pi}$. Letting $v(a, a', \pi) = u(a', \pi)$, which measures the gains from changing from action a to a', this last condition holds if and only if for all $i = 1, 2$, $\pi_i > 0$ implies that $v(a_i, a_i, \pi) \geq v(a_i, a_j, \pi)$ for $j \neq i$. In the example $\pi = (1/(1+\beta), \beta/(1+\beta))$ is such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $||v - w|| = \sup_{a,a',\pi'} |v(a, a', \pi') - w(a, a', \pi')| < \delta$ then there exists $\tilde{\pi}$ such that

$$|\pi - \tilde{\pi}| < \varepsilon \text{ and} \tag{10}$$

$$\text{for all } i = 1, 2, \tilde{\pi}_i > 0 \text{ implies } w(a_i, a_i, \tilde{\pi}) \geq w(a_i, a_j, \tilde{\pi}) \text{ for } j \neq i, \tag{11}$$

a condition that resembles the definition of an essential Nash equilibrium (see Wen-Tsün and Jia-He [34] and also van Damme [33]). A particular case of interest for the function w is $w(a, a', \pi) = v(a', \pi^{n,a,a'})$, where $\pi_a^{n,a,a'} = \pi_a - 1/n$ and $\pi_{a'}^{n,a,a'} = \pi_{a'} + 1/n$. Thus, this suggests that those distributions that satisfy both this essentiality property, strengthened so that $\tilde{\pi}$ also satisfies condition (6), will be limit equilibrium distributions in games with finite action spaces and finite characteristics.

We now turn to the question of the existence of a limit equilibrium strategy. Our result will be in the form of a representation result, that is, we ask

when is it that there is a strategy that represents a given limit equilibrium distribution. A sufficient condition is provided in Theorem 7, which is an immediate consequence of Lemma 6 in the Appendix.

Theorem 7 *Let* $G = (([0,1],\lambda), U, A)$ *be a game. Suppose that* G *has a limit equilibrium distribution* τ *satisfying* $\mathrm{supp}(\tau)$ *is countable. Then* G *has a limit equilibrium.*

6.3 Two Examples

In this section we present two examples of games in which the set of Nash equilibria differs substantially from the set of limit equilibria.[9] We shall argue that in all of those games, their unique limit equilibrium is the only plausible Nash equilibrium.

6.3.1 A Game Solvable by Strict Dominance

Consider a game in which players have to choose a number between 1 and 10, their average then being given to the players. Formally, $A = \{1, \ldots, 10\}$ and $U(t) = u$ for all $t \in [0,1]$, where $u(a,\tau) = \sum_{i=1}^{10} i\tau(\{i\})$ for all $a \in A$ and $\tau \in \mathcal{M}$.

If this game is played by a finite number of players, then clearly each will choose 10 — this is the unique Nash equilibrium of any finite version of the above game. It then follows that the strategy in which all players chose 10 is the unique limit equilibrium of the continuum game.[10] In contrast, the set of its Nash equilibria equals the set of all strategies: since each player cannot influence the average when there are a continuum of players, he is indifferent between any action. In particular, a strategy in which every player chooses 1 is a Nash equilibrium, but an implausible outcome.

6.3.2 A Symmetric Cournot Market

Consider a Cournot market in which the price of the homogeneous good depends on the average (or per capita) supply $p = 1 - \int_T f \mathrm{d}\mu$ when the set of players is (T,μ). Players are firms who choose a quantity in $[0,2]$; all have

[9] These examples originally appeared in Barlo and Carmona [3], although analyzed with a different equilibrium concept.

[10] As usual, two strategies that differ on a set of measure zero are considered the same.

a marginal cost of production of zero and no fixed cost. Hence, $A = [0, 2]$ and $U(t) = u$ for all $t \in [0, 1]$, where $u(a, \tau) = \left(1 - \int_A i \mathrm{d}\tau\right) a$ and $i : A \to A$ denotes the identity function.

Consider first the case in which there is a continuum of firms. The set of Nash equilibria in this game equals the set of all strategies f satisfying $\int_{[0,1]} f \mathrm{d}\lambda = 1$. The reason is that $\int_{[0,1]} f \mathrm{d}\lambda = 1$ implies $p = 0$, and so any firm is indifferent between any quantity to produce. In particular, we have asymmetric equilibria, such as $f(t) = 0$ for $0 \leq t \leq 1/2$ and $f(t) = 2$ otherwise, in spite of the symmetry of the firms.

If we have a finite version of this game, where the set of players has $n \in \mathbb{N}$ elements, there is a unique equilibrium, in which all firms produce $n/(n+1)$ goods. This implies that there is a unique limit equilibrium as well: all firms produce 1. Thus, unlike the case for Nash equilibrium, the symmetry of the economic situation is preserved in the limit equilibrium.

7 Concluding Remarks

The main objective of this paper is to relate equilibria of games with a continuum of players with equilibria of games with a finite number of players. This is done in two ways: First, by characterizing Nash equilibria in terms of approximate equilibria of games with a finite number of players. Second, by defining and proving the existence of equilibria of continuum games as limit points of equilibria of finite games.

Our characterization result shows that approximate equilibria of finite games provides an alternative way for obtaining strategic insignificance of players, which is the main motivation of games with a continuum of players. In this way, they render as natural all the results that show that approximate equilibria of finite games have the same, or approximately the same, properties as equilibria of continuum games.

Our notion of limit equilibria defines an equilibrium concept in games with a continuum of players that refines Nash equilibrium. Its main objective is to eliminate Nash equilibria that can be thought of as artifacts of the continuum model; indeed, for the examples presented in Section 6.3, that seems to be the case. It is important to emphasize that players are still strategically insignificant in limit equilibrium as they are in Nash equilibrium. However, limit equilibria of a continuum game are those that can be related with Nash equilibria (in opposition of approximate equilibria) of finite games.

A Appendix

A.1 Lemmata

In this section, we present some technical results that we use and for which we were unable to find a reference. Lemma 5 below extends the well-known fact that the set of measures with finite support is dense in the set of all Borel measures in a separable metric space (see Parthasarathy [25, Theorem II.6.3]).

Lemma 5 *Let X be a separable metric space, $K \subseteq X$ be compact and $\mu \in \mathcal{M}(X)$. If $\mu = \lambda \circ h^{-1}$, where $h : [0,1] \to X$ is measurable, and λ is the Lebesgue measure on $[0,1]$, then there exists a sequence $\{\mu_n\}$ in $\mathcal{M}(X)$ such that*

1. $\mu_n \Rightarrow \mu$,
2. $\lim_n \mu_n(K) = \mu(K)$
3. *for all $n \in \mathbb{N}$, $\mu_n = \nu_n \circ h^{-1}_{|T_n}$ where T_n is a finite subset of $[0,1]$ and ν_n is the uniform measure on T_n and*
4. $\mathrm{supp}(\nu_n) \to \mathrm{supp}(\lambda)$.

Proof. Let $n \in \mathbb{N}$. Since K is compact, then we can write $K = \cup_{j=1}^{J_n} B_{1/2n}(y_{n,j})$ for some $y_{n,j} \in K$, $j = 1, \ldots, J_n$. Hence, we can write $K = \cup_{j=1}^{J_n} B_{n,j}$, where $B_{n,1} = B_{1/2n}(y_{n,1})$, and $B_{n,j} = B_{1/2n}(y_{n,j}) \setminus \cup_{i=1}^{j-1} B_{1/2n}(y_{n,i})$. Thus, $\{B_{n,j}\}_j$ is a disjoint collection, each of its members being a Borel set with a diameter no greater than $1/n$.

Since X is separable, we can write $K^c = X \setminus K = \cup_{i=1}^{\infty} A_{n,i}$ where $\{A_{n,i}\}$ is a disjoint collection, each of its members being a Borel set with a diameter no greater than $1/n$.

Let $I_n \in \mathbb{N}$ be such that $\sum_{i=I_n}^{\infty} \mu(A_{n,i}) < 1/n$. Let $\{\tilde{q}_{n,j}\}_{j=1}^{J_n} \subset \mathbb{Q}$ and $\{\tilde{p}_{n,i}\}_{i=1}^{I_n} \subset \mathbb{Q}$ be such that

$$\begin{aligned} &|\tilde{q}_{n,j} - \mu(B_{n,j})| < 1/(nJ_n), j = 1, \ldots, J_n, \\ &|\tilde{p}_{n,i} - \mu(A_{n,i})| < 1/(nI_n), i = 1, \ldots, I_n - 1, \text{ and} \\ &\sum_{j=1}^{J_N} \tilde{q}_{n,j} + \sum_{i=1}^{I_n} \tilde{p}_{n,i} = 1. \end{aligned} \tag{12}$$

Also, if $\mu(B_{n,j}) = 0, j = 1, \ldots, J_n$, let $\tilde{q}_{n,j} = 0$; if $\mu(A_{n,i}) = 0, i = 1, \ldots, I_n - 1$, let $\tilde{p}_{n,i} = 0$, and if $\sum_{i=I_n}^{\infty} \mu(A_{n,i}) = 0$ let $\tilde{p}_{n,I_n} = 0$. We remark that the above construction can always be done: if there is just one such set with positive measure this is clear, as its measure will be 1, a rational number. If there are $k > 1$ such sets with positive measure, then approximate the measure of $k-1$ of those sets by rational points in a way that the difference is smaller than $\zeta > 0$, where $\zeta(I_n + J_n) < \min\{1/(nI_n), 1/(nJ_n)\}$, and set the rational approximation for the remaining set using the formula $\sum_{j=1}^{J_N} \tilde{q}_{n,j} + \sum_{i=1}^{I_n} \tilde{p}_{n,i} = 1$.

Since $\sum_{i=I_n}^{\infty} \mu(A_{n,i}) = 1 - \sum_{j=1}^{J_N} \mu(B_{n,j}) - \sum_{i=1}^{I_n-1} \mu(A_{n,i})$, it follows that $|\tilde{p}_{n,I_n} - \sum_{i=I_n}^{\infty} \mu(A_{n,i})| < 2/n$ and that $\sum_{j=1}^{J_n} \tilde{q}_{n,j} + \sum_{i=1}^{I_n} \tilde{p}_{n,i} = 1$. Furthermore, there exists $N_n \in \mathbb{N}$, $\{q_{n,j}\}_{j=1}^{J_n} \subset \mathbb{N}$ and $\{p_{n,i}\}_{i=1}^{I_n} \subset \mathbb{N}$ such that $\tilde{q}_{n,j} = q_{n,j}/N_n$, $j = 1, \ldots, J_n$ and $\tilde{p}_{n,i} = p_{n,i}/N_n$, $i = 1, \ldots, I_n$.

Let $i \in \{1, \ldots, I_n - 1\}$. Since $\mu(A_{n,i}) = \lambda(\{t \in [0,1] : h(t) \in A_{n,i}\})$, select $p_{n,i}$ points from $\{t \in [0,1] : h(t) \in A_{n,i}\}$. By the above convention, $p_{n,i} > 0$ implies $\mu(A_{n,i}) > 0$, and so we can indeed select such points from $\{t \in [0,1] : h(t) \in A_{n,i}\}$. Similarly, select p_{n,I_n} points from $\{t \in [0,1] : h(t) \in \cup_{i=I_n}^{\infty} A_{n,i}\}$ and $q_{n,j}$ points from $\{t \in [0,1] : h(t) \in B_{n,j}\}$, $j = 1, \ldots, J_n$. Let $\{t_l^{n,1}\}_{l=1}^{L_{n,1}}$ denote this collection of points from $[0,1]$, where $L_{n,1} = N_n$ and where $t_1^{n,1} < \ldots < t_{L_{n,1}}^{n,1}$.

We want to define T_n in such a way that the distance between any two consecutive points is less than $1/n$, and also that the difference from the minimum (resp. maximum) point from zero (resp. 1) is less than $1/n$; this will imply $\mathrm{supp}(\nu_n) \to \mathrm{supp}(\lambda)$. We proceed by induction.

Let $\mathcal{P}$ be the collection of those sets from

$$\mathcal{N} = \{A_{n,1}, \ldots, A_{n,I_n-1}, \cup_{i=I_n}^{\infty} A_{n,i}, B_{n,1} \ldots, B_{n,J_n}\}$$

that have strictly positive measure. Suppose that $\{t_l^{n,k}\}_{l=1}^{L_{n,k}}$ is defined for $k \in \mathbb{N}$ with $t_1^{n,k} < \ldots < t_{L_{n,k}}^{n,k}$ and $h(t_l^{n,k})$ belongs to a set $h^{-1}(P)$ with P in $\mathcal{P}$ for all l. Define $t_0^{n,k} = 0$, and $t_{L_{n,k}+1}^{n,k} = 1$. For $l = 0, \ldots, L_{n,k}$, define

$$\tilde{t}_l^{n,k} = \frac{t_l^{n,k} + t_{l+1}^{n,k}}{2} \tag{13}$$

if $h(\tilde{t}_l^{n,k})$ belongs to a set $h^{-1}(P)$ with P in $\mathcal{P}$; otherwise, let $\tilde{t}_l^{n,k}$ be such that it belongs to a set $h^{-1}(P)$ with P in $\mathcal{P}$ and

$$\left|\tilde{t}_l^{n,k} - \frac{t_l^{n,k} + t_{l+1}^{n,k}}{2}\right| < \frac{1}{2^k}. \tag{14}$$

Since the union of the sets in $\mathcal{N}$ equals X, then $\{h^{-1}(N)\}_{N\in\mathcal{N}}$ partitions $[0,1]$. If $N \in \mathcal{N}$ is such that $(t_l^{n,k}+t_{l+1}^{n,k})/2 \in h^{-1}(N)$, then since $\mu(N) = \lambda(h^{-1}(N))$, it follows that any neighborhood of $(t_l^{n,k}+t_{l+1}^{n,k})/2$ intercepts a set $h^{-1}(P)$ with $P \in \mathcal{P}$; otherwise a neighborhood of $(t_l^{n,k} + t_{l+1}^{n,k})/2$ would be contained in $\cup_{N\in\mathcal{N}\setminus\mathcal{P}} h^{-1}(N)$ implying that this set has positive Lebesgue measure — but this is impossible as $\cup_{N\in\mathcal{N}\setminus\mathcal{P}} h^{-1}(N)$ is the union of finitely many sets of Lebesgue measure zero. Hence, we can construct $\tilde{t}_l^{n,k}$ in the way described above. Finally define $\{t_l^{n,k+1}\}_{l=1}^{L_{n,k+1}}$ as the union of $\{t_l^{n,k}\}_{l=1}^{L_{n,k}}$ and $\{\tilde{t}_l^{n,k}\}_{l=0}^{L_{n,k}}$ with $t_1^{n,k+1} < \ldots < t_{L_{n,k+1}}^{n,k+1}$.

Let $M^{n,k} = \max_{0\le l\le L_{n,k+1}+1}(t_{l+1}^{n,k+1} - t_l^{n,k+1})$, with $t_0^{n,k+1} = 0$, and $t_{L_{n,k+1}+1}^{n,k+1} = 1$. By construction, we have that

$$M^{n,k+1} < \frac{M^{n,k}}{2} + \frac{1}{2^k}, \tag{15}$$

which implies, by induction, that

$$M^{n,k+1} < \frac{M^{n,1}}{2^k} + \frac{k}{2^k}. \tag{16}$$

Hence, $\lim_{k\to\infty} M^{n,k} = 0$, and so let K_n be such that $M^{n,K_n} < 1/n$.

Let $\gamma_n \in \mathbb{N}$ be such that

$$\begin{aligned} &\gamma_n \min\{p_{n,i} : p_{n,i} > 0, i = 1, \ldots I_n\} > |\{t_l^{n,K_n}\}_{l=1}^{L_{n,K_n}}| \text{ and} \\ &\gamma_n \min\{q_{n,j} : q_{n,j} > 0, j = 1, \ldots J_n\} > |\{t_l^{n,K_n}\}_{l=1}^{L_{n,K_n}}|. \end{aligned} \tag{17}$$

Then select $\gamma_n q_{n,j}$ points from $\{t \in [0,1] : h(t) \in B_{n,j}\}$, $j = 1, \ldots, J_n$, $\gamma_n p_{n,i}$ points from $\{t \in [0,1] : h(t) \in A_{n,i}\}$, $i = 1, \ldots I_n - 1$, and $\gamma_n p_{n,I_n}$ points from $\{t \in [0,1] : h(t) \in \cup_{i=I_n}^{\infty} A_{n,i}\}$, in a way that $\{t_l^{n,K_n}\}_{l=1}^{L_{n,K_n}}$ is contained in that collection of $\gamma_n N_n$ points. This last requirement is possible to fulfil because of (17). Then we define T_n as the collection of these $\gamma_n N_n$ points, and we define $\mu_n = \nu_n \circ h_{|T_n}^{-1}$. By construction, $\inf_{z\in T_n} |t - z| < 1/n$ for all $t \in [0,1] = \text{supp}(\lambda)$, and so $\text{supp}(\nu_n) \to \text{supp}(\lambda)$.

We claim that $\lim_n \mu_n(K) = \mu(K)$. We have that

$$\begin{aligned}
\mu_n(K) &= \nu_n \circ h^{-1}_{|T_n}(K) \\
&= \nu_n(\{t \in T_n : h(t) \in K\}) \\
&= \sum_{j=1}^{J_n} \nu_n(\{t \in T_n : h(t) \in B_{n,j}\}) \\
&= \sum_{j=1}^{J_n} \frac{\gamma_n q_{n,j}}{\gamma_n N_n} = \sum_{j=1}^{J_n} \tilde{q}_{n,j}.
\end{aligned} \tag{18}$$

Hence, $|\mu_n(K) - \mu(K)| \leq \sum_{j=1}^{J_n} |\tilde{q}_{n,j} - \mu(B_{n,j})| < 1/n \to 0$ as $n \to \infty$.

Finally, we show that $\mu_n \Rightarrow \mu$. Let g be a bounded, uniformly continuous real-valued function on X, and let g be bounded by M. Since

$$\begin{aligned}
\left| \int g \mathrm{d}\mu_n - \int g \mathrm{d}\mu \right| \leq & \left| \sum_{j=1}^{J_n} \left(\int_{B_{j,n}} g \mathrm{d}\mu_n - \int_{B_{j,n}} g \mathrm{d}\mu \right) \right| \\
& + \left| \sum_{i=1}^{I_n - 1} \left(\int_{A_{i,n}} g \mathrm{d}\mu_n - \int_{A_{i,n}} g \mathrm{d}\mu \right) \right| \\
& + \left| \sum_{i=I_n}^{\infty} \left(\int_{A_{i,n}} g \mathrm{d}\mu_n - \int_{A_{i,n}} g \mathrm{d}\mu \right) \right|,
\end{aligned} \tag{19}$$

it is enough to show that each of the three terms on the right side of the above inequality converges to zero as n converges to infinity.

We have that $\left| \sum_{i=I_n}^{\infty} \int_{A_{i,n}} g \mathrm{d}\mu_n \right| \leq M \sum_{i=I_n}^{\infty} \mu_n(A_{i,n}) = M \mu_n(\cup_{i=I_n}^{\infty} A_{i,n}) = M \tilde{p}_{n,i} < M \left(\sum_{i=I_n}^{\infty} \mu(A_{i,n}) + 2/n \right) < 3M/n \to 0$ as $n \to \infty$. Similarly, $\left| \sum_{i=I_n}^{\infty} \int_{A_{i,n}} g \mathrm{d}\mu \right| \leq M \sum_{i=I_n}^{\infty} \mu(A_{i,n}) < M/n \to 0$ as $n \to \infty$. Hence

$$\left| \sum_{i=I_n}^{\infty} \left(\int_{A_{i,n}} g \mathrm{d}\mu_n - \int_{A_{i,n}} g \mathrm{d}\mu \right) \right| \to 0$$

as $n \to \infty$.

Let $\alpha_{n,j} = \inf_{x \in B_{n,j}} g(x)$ and $\beta_{n,j} = \sup_{x \in B_{n,j}} g(x)$, $j = 1, \ldots, J_n$. Also, let $\alpha_{n,i} = \inf_{x \in A_{n,i}} g(x)$ and $\beta_{n,i} = \sup_{x \in A_{n,i}} g(x)$, $i = 1, \ldots, I_n - 1$. Since g is uniformly continuous, and the diameters of $A_{n,i}$, and $B_{n,j}$ converge to

zero as n converges to infinity uniformly on i and j respectively, it follows that $\sup_i(\beta_{n,i} - \alpha_{n,i})$ and $\sup_j(\beta_{n,j} - \alpha_{n,j})$ converge to zero as n converges to infinity.

Let $j \in \{1, \ldots, J_n\}$, and let $\{x_{n,j}^m\}_{m=1}^{\gamma_n q_{n,j}} = h(T_n) \cap B_{n,j}$. Then

$$\int_{B_{n,j}} g \mathrm{d}\mu_n = \sum_{m=1}^{\gamma_n q_{n,j}} \frac{g(x_{n,j}^m)}{\gamma_n N_n}. \tag{20}$$

We have that

$$\begin{aligned} \int_{B_{n,j}} g \mathrm{d}\mu - \sum_{m=1}^{\gamma_n q_{n,j}} \frac{g(x_{n,j}^m)}{\gamma_n N_n} &\leq \beta_{n,j}\mu(B_{n,j}) - \alpha_{n,j}\frac{\gamma_n q_{n,j}}{\gamma_n N_n} \\ &= \mu(B_{n,j})(\beta_{n,j} - \alpha_{n,j}) + \alpha_{n,j}(\mu(B_{n,j}) - \tilde{q}_{n,j}) \\ &\leq \mu(B_{n,j}) \sup_{j'}(\beta_{n,j'} - \alpha_{n,j'}) + M\left|\mu(B_{n,j}) - \tilde{q}_{n,j}\right|. \end{aligned} \tag{21}$$

Similarly,

$$\sum_{m=1}^{\gamma_n q_{n,j}} \frac{g(x_{n,j}^m)}{\gamma_n N_n} - \int_{B_{n,j}} g \mathrm{d}\mu \leq \mu(B_{n,j}) \sup_{j'}(\beta_{n,j'} - \alpha_{n,j'}) + M\left|\mu(B_{n,j}) - \tilde{q}_{n,j}\right|.$$

Thus,

$$\left|\int_{B_{j,n}} g \mathrm{d}\mu_n - \int_{B_{j,n}} g \mathrm{d}\mu\right| < \mu(B_{n,j}) \sup_{j'}(\beta_{n,j'} - \alpha_{n,j'}) + \frac{M}{nJ_n},$$

and so

$$\left|\sum_{j=1}^{J_n}\left(\int_{B_{j,n}} g \mathrm{d}\mu_n - \int_{B_{j,n}} g \mathrm{d}\mu\right)\right| < \sup_j(\beta_{n,j} - \alpha_{n,j}) + \frac{M}{n} \to 0 \tag{22}$$

as $n \to \infty$.

Similarly, we can show that

$$\left|\sum_{i=1}^{I_n - 1}\left(\int_{A_{i,n}} g \mathrm{d}\mu_n - \int_{A_{i,n}} g \mathrm{d}\mu\right)\right| < \sup_i(\beta_{n,i} - \alpha_{n,i}) + \frac{M}{n} \to 0$$

as $n \to \infty$. Hence, we conclude that $\left|\int g \mathrm{d}\mu_n - \int g \mathrm{d}\mu\right| \to 0$ as $n \to \infty$ and, since g is arbitrary, that $\mu_n \Rightarrow \mu$. ■

The following result, Lemma 6, is a representation theorem for measures with a countable support. In the particular context of a game with a continuum of players, it says that any distribution with a countable support can be thought of as the distribution induced by a strategy together with the function assigning preferences to players.

Lemma 6 *Let μ be a distribution on $\mathcal{U} \times A$ satisfying $\mu_{\mathcal{U}} = \lambda \circ U^{-1}$, where $U : [0,1] \to \mathcal{U}$ is measurable and λ denotes the Lebesgue measure on $[0,1]$.*

If $\mathrm{supp}(\mu)$ is countable, then there exists a measurable function $f : [0,1] \to A$ such that $\mu = \lambda \circ (U, f)^{-1}$.

Proof. Let $\{a_i\}_{i=1}^{\infty} \subseteq A$ and $\{b_j\}_{j=1}^{\infty} \subseteq \mathcal{U}$ be such that $\mathrm{supp}(\mu) \subseteq \{a_i\}_i \times \{b_j\}_j$.

Let $j \in \mathbb{N}$ be fixed. We have that

$$\sum_{i=1}^{\infty} \mu(\{b_j\} \times \{a_i\}) = \mu_{\mathcal{U}}(\{b_j\}) = \lambda(U^{-1}(\{b_j\})). \tag{23}$$

In particular, $0 \leq \mu(\{b_j\} \times \{a_i\}) \leq \lambda(U^{-1}(\{b_j\}))$. Hence, by Liapunov's Theorem (see Hildenbrand [18, prop. 15, p. 45]), there exists a Borel measurable set $\tilde{T}_{j,1} \subseteq U^{-1}(\{b_j\})$ satisfying $\lambda(\tilde{T}_{j,1}) = \mu(\{b_j\} \times \{a_1\})$. By induction, we can find a partition $\{T_{j,i}\}_{i=1}^{\infty}$ of $U^{-1}(\{b_j\})$ satisfying $T_{j,i}$ is measurable and $\lambda(T_{j,i}) = \mu(\{b_j\} \times \{a_i\})$ for all i: if $k > 1$ and we have constructed a disjoint collection $\{\tilde{T}_{j,i}\}_{i=1}^{k-1}$ of Borel measurable sets, we find $\tilde{T}_{j,k} \subseteq U^{-1}(\{b_j\}) \setminus \cup_{i=1}^{k-1} \tilde{T}_{j,i}$ satisfying $\lambda(\tilde{T}_{j,k}) = \mu(\{b_j\} \times \{a_k\})$, which is possible by Liaponov's Theorem since

$$\lambda(U^{-1}(\{b_j\})) = \sum_{i=1}^{\infty} \mu(\{b_j\} \times \{a_i\}) \geq \sum_{i=1}^{k} \mu(\{b_j\} \times \{a_i\}) \tag{24}$$

implies that

$$\begin{aligned} &\lambda(U^{-1}(\{b_j\}) \setminus \cup_{i=1}^{k-1} \tilde{T}_{j,i}) = \lambda(U^{-1}(\{b_j\})) - \sum_{i=1}^{k-1} \lambda(\tilde{T}_{j,i}) = \\ &\lambda(U^{-1}(\{b_j\})) - \sum_{i=1}^{k-1} \mu(\{b_j\} \times \{a_i\}) \geq \\ &\mu(\{b_j\} \times \{a_k\}). \end{aligned} \tag{25}$$

Clearly, $\lambda(\cup_{i=1}^{\infty}\tilde{T}_{j,i}) = \lambda(U^{-1}(\{b_j\}))$, and so define $T_{j,i} = \tilde{T}_{j,i}$ for $i > 1$ and $T_{j,1} = \tilde{T}_{j,1} \cup \left(U^{-1}(\{b_j\}) \setminus \cup_{i=1}^{\infty}\tilde{T}_{j,i}\right)$.

Then, we define $f : [0,1] \to A$ by $f(t) = a_i$ if $t \in T_{j,i}$ for some $j \in \mathbb{N}$ and $i \in \mathbb{N}$. For $i, j \in \mathbb{N}$, we have that $\lambda \circ (U,f)^{-1}(\{b_j\} \times \{a_i\}) = \lambda(T_{j,i}) = \mu(\{b_j\} \times \{a_i\})$. This implies that $\lambda \circ (U,f)^{-1} = \mu$ as follows: Denote by

$$C = \bigcup_{i,j\in\mathbb{N}} \{(b_j, a_i)\}. \tag{26}$$

Then, $\mu(C^c) = \lambda \circ (U,f)^{-1}(C^c) = 0$. If B is a Borel set in $\mathcal{U} \times A$, and if $L = \{(j,i) \in \mathbb{N} \times \mathbb{N} : (b_j, a_i) \in B\}$, then

$$\begin{aligned} \mu(B) = \sum_{(j,i)\in L} \mu(\{b_j\} \times \{a_i\}) = \\ \sum_{(j,i)\in L} \lambda \circ (U,f)^{-1}(\{b_j\} \times \{a_i\}) = \lambda \circ (U,f)^{-1}(B). \end{aligned} \tag{27}$$

Finally, we claim that f is measurable. If $C \subseteq A$ is Borel measurable, then $f^{-1}(C) = \cup_{a\in C\cap\{a_i\}_i} f^{-1}(\{a\})$, and so it is enough to show that $f^{-1}(\{a_i\})$ is measurable for all $i \in \mathbb{N}$. Since for every i we have $f^{-1}(\{a_i\}) = \cup_{j=1}^{\infty} T_{j,i}$, where $\{T_{j,i}\}_{j=1}^{\infty}$ is a collection of measurable sets, it follows that f is measurable. ■

A.2 Proofs

In this section we present the proofs of all the results stated in the main text.

Proof of Lemma 1. Let $\{r_k\}_{k=1}^{\infty} \subseteq A$ be dense in A. Then, the continuity of $U(t)$ for all $t \in [0,1]$, implies that

$$E(f,\varepsilon,\lambda) = \bigcap_k \{t \in [0,1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\}. \tag{28}$$

Hence, it is enough to show that $\{t \in [0,1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\}$ is measurable for all k.

Let $k \in \mathbb{N}$, and $\eta > 0$. By remark 7 in Rath [28], $(t,a) \mapsto U(t)(a, \lambda \circ f^{-1})$ is measurable. Denote this function by $\hat{u}$. By changing it in a set of measure zero, we may assume that it is Borel measurable; similarly, assume that f is Borel measurable. Then, the functions $t \mapsto U(t)(f(t), \lambda \circ f^{-1})$ and

$t \mapsto U(t)(r_k, \lambda \circ f^{-1})$ are Borel measurable in F since they equal $\hat{u} \circ f$, and $\hat{u} \circ g$ respectively, where $f(t) = (t, f(t))$, and $g(t) = (t, r_k)$ are Borel measurable. Thus, by Lusin's Theorem, let $C \subseteq [0,1]$ be a compact set, $\lambda([0,1] \setminus C) < \eta$, be such that $t \mapsto U(t)(f(t), \lambda \circ f^{-1})$ and $t \mapsto U(t)(r_k, \lambda \circ f^{-1})$ are continuous in C. Since

$$\begin{aligned} &\{t \in [0,1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\} \setminus \\ &\{t \in C : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\} \\ &\subseteq [0,1] \setminus C, \end{aligned} \tag{29}$$

the outer measure of $\{t : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\} \setminus \{t \in C : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon\}$ is smaller than η. Hence, it is enough to show that $\{t \in C : u(t, f(t), \lambda \circ f^{-1}) \geq u(t, r_k, \lambda \circ f^{-1}) - \varepsilon\}$ is closed (see Wheeden and Zygmund [35, Lemma 3.22]). This follows easily from the fact that both $t \mapsto U(t)(f(t), \lambda \circ f^{-1})$ and $t \mapsto U(t)(r_k, \lambda \circ f^{-1})$ are continuous in C. ■

Proof of Lemma 2. For notational convenience let $h = (U, f)$. We have

$$\begin{aligned} &h^{-1}(B_\tau) = \\ &\{t : (U(t), f(t)) \in B_\tau\} = \\ &\{t : U(t)(f(t), \tau_A) \geq U(t)(A, \tau_A)\} = \\ &\{t : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})\}. \end{aligned} \tag{30}$$

Hence τ is an equilibrium distribution if and only if $\tau(\{(u,a) : u(a, \tau_A) \geq u(A, \tau_A)\}) = 1$ if and only if $\lambda(h^{-1}(B_\tau)) = 1$ if and only if $\lambda(\{t : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1})\}) = 1$ if and only if f is a Nash equilibrium. ■

Proof of Lemma 3. Let $\tau = \nu \circ f^{-1}$ and $\tau' = \nu \circ g^{-1}$. We have that

$$\tau(\{a\}) = \frac{\{t \in T : f(t) = a\}}{|T|}$$

for all $a \in A$, and similarly for τ'. Since there is just one $t \in T$ such that g and f differ, there are only two points in A such that $\tau(\{a\})$ differ from $\tau'(\{a\})$. Note that in that case $|\tau(\{a\}) - \tau'(\{a\})| = 1/|T|$. Denoting these points a' and a'' we see that for any Borel subset E of A $|\tau(E) - \tau'(E)| = 0$ if either a' and a'' belong to E or a' and a'' belong to E^c, while $|\tau(E) - \tau'(E)| = 1/|T|$ otherwise. This implies $\rho(\tau, \tau') \leq 1/|T|$. ■

Proof of Lemma 4. For notational convenience let $h = (U, f)$. We have

$$\begin{aligned}
&h^{-1}(B_\tau^\varepsilon) = \\
&\{t \in T : (U(t), f(t)) \in B_\tau^\varepsilon\} = \\
&\{t \in T : U(t)(f(t), \tau_A) \geq U(t)(a', \tau_A^{u,a,a'}) - \varepsilon \text{ for all } a'\} = \\
&\{t \in T : U(t)(f(t), \nu \circ f^{-1}) \geq U(t)(a', \nu \circ (f \setminus_t a')^{-1}) - \varepsilon \text{ for all } a'\}.
\end{aligned} \tag{31}$$

Hence, τ is an $\varepsilon-$equilibrium distribution if and only if $\tau(B_\tau^\varepsilon) \geq 1 - \varepsilon$ if and only if $\nu(h^{-1}(B_\tau^\varepsilon)) \geq 1 - \varepsilon$ if and only if $\nu(\{t \in T : U(t)(f(t), \nu \circ f^{-1}) \geq U(t)(a', \nu \circ (f \setminus_t a')^{-1}) - \varepsilon \text{ for all } a'\}) \geq 1 - \varepsilon$ if and only if f is an $\varepsilon-$equilibrium. ■

Proof of Theorem 1. (Necessity) Let $\tau = \lambda \circ (U, f)^{-1}$; by Lemma 2 τ is an equilibrium distribution on $\mathcal{U} \times A$. For notational convenience, let $h = (U, f)$. Let $n \in \mathbb{N}$ and define $\varepsilon_n = 1/n$. Since $\mathcal{U}$ is a complete separable metric space, and A is compact then τ is tight by Parthasarathy [25, Theorem II.3.2], as $\mathcal{U} \times A$ is also a complete separable metric space.

Since B_τ is closed, and so a Borel set, let $K_n \subseteq B_\tau$ be compact, and satisfy $\tau(B_\tau \setminus K_n) < 1/2n$. Since τ is an equilibrium distribution, it follows that $\tau(B_\tau) = 1$, and so $\tau(K_n) > 1 - 1/2n$. If π denotes the projection of $\mathcal{U} \times A$ into $\mathcal{U}$, then $\pi(K_n)$ is compact, and $K_n \subseteq \pi(K_n) \times A$. In particular, $\pi(K_n)$ is equicontinuous by the Ascoli-Arzela Theorem since A, and $\mathcal{M}$ are both compact metric spaces. Furthermore, denoting $C_n = \pi(K_n) \times A$, it follows $\tau(C_n \cap B_\tau) \geq \tau(K_n \cap B_\tau) = \tau(K_n) > 1 - 1/2n$.

Let $\delta_n > 0$ be such that $d((a, \mu), (b, \nu)) := \max\{d_A(a, b), \rho(\mu, \nu)\} < \delta_n$ implies that $|u(a, \mu) - u(b, \nu)| < 1/4n$ for all $u \in \pi(K_n)$. By Lemma 5, there exists a sequence $\{\mu_j\}$ such that $\mu_j \Rightarrow \tau$, $\lim_j \mu_j(C_n \cap B_\tau) = \tau(C_n \cap B_\tau)$, $\mu_j = \nu_j \circ h_{T_j}^{-1}$ where ν_j is the uniform measure on some finite set $T_j \subset [0, 1]$, and $T_j = \text{supp}(\nu_n) \to \text{supp}(\lambda) = [0, 1]$. Hence, $\mu_{A,j} \Rightarrow \tau_A$, and let $J_n \in \mathbb{N}$ be such that $\rho(\mu_{A,J_n}, \tau_A) < \delta_n$, $|\mu_{J_n}(C_n \cap B_\tau) - \tau(C_n \cap B_\tau)| < 1/2n$ $\rho(\mu_{J_n}, \tau) < 1/n$ and $1/|T_{J_n}| < \delta_n$. Define $\tau_n = \mu_{J_n}$, $T_n = T_{J_n}$ and $\nu_n = \nu_{J_n}$.

By construction of $\{\tau_n\}_n$ we have $\tau_n \Rightarrow \tau$, and that, for every $n \in \mathbb{N}$, $\rho(\tau_n, \tau) < 1/n$, $\rho(\tau_{A,n}, \tau_A) < \delta_n$, $|\tau_n(C_n \cap B_\tau) - \tau(C_n \cap B_\tau)| < 1/2n$, $1/|T_n| < \delta_n$ and $\tau_n = \nu_n \circ h_{|T_n}^{-1}$ where T_n is a finite subset of $[0, 1]$ and ν_n is the uniform measure on T_n.

We have that $C_n \cap B_\tau \subseteq C_n \cap B_{\tau_n}^{1/n}$, since if $(u, a) \in C_n \cap B_\tau$ and $a' \in A$ then $u(a, \tau_{A,n}) > u(a, \tau_A) - 1/4n \geq u(a', \tau_A) - 1/4n > u(a', \tau_{A,n}) - 1/2n >$

$u(a', \tau_{A,n}^{u,a,a'}) - 1/n$ since $\rho(\tau_{A,n}, \tau_A) < \delta_n$ and $\rho(\tau_{A,n}, \tau_{A,n}^{u,a,a'}) \leq 1/|T_n| < \delta_n$ by Lemma 3. So $1 - 1/n < \tau(C_n \cap B_\tau) - 1/2n < \tau_n(C_n \cap B_\tau) \leq \tau_n(C_n \cap B_{\tau_n}^{1/n}) \leq \tau_n(B_{\tau_n}^{1/n})$. Hence, $\tau_n = \nu_n \circ h_{|T_n}^{-1} = \nu_n \circ (U_{|T_n}, f_{|T_n})^{-1}$ is an ε_n-equilibrium distribution of the game $\tau_{\mathcal{U},n} = \nu_n \circ U_{|T_n}^{-1}$. By Lemma 4 then $f_{|T_n}$ is an ε_n-equilibrium of $G_n = ((T_n, \nu_n), U_{|T_n})$.

(Sufficiency) Let $\tau = \lambda \circ (U, f)^{-1}$ and let $\tau_n \Rightarrow \tau$, where $\tau_n = \nu_n \circ (U, f)_{|T_n}^{-1}$ is an ε_n-equilibrium distribution, $\varepsilon_n \searrow 0$ and $\text{supp}(\nu_n) \to \text{supp}(\lambda)$. Then $\tau_{A,n} \Rightarrow \tau_A$; so, taking a subsequence if necessary, we may assume that $\rho(\tau_A, \tau_{A,n}) < 1/2n$ and $\rho(\tau_{A,n}, \tau_{A,n}^{u,a,a'}) < 1/2n$ for every $u \in \mathcal{U}$ and $a, a' \in A$; the second inequality is obtained via Lemma 3 by taking $1/|\text{supp}(\nu_n) < 1/2n$. Clearly, we have $\rho(\tau_A, \tau_{A,n}^{u,a,a'}) < 1/n$ for every $u \in \mathcal{U}$ and $a, a' \in A$.

Define, for each $u \in \mathcal{U}$,

$$\beta_n(u) = \sup_{a \in A, \nu \in \mathcal{M}} \{|u(a, \nu) - u(a, \tau_A)| : \rho(\nu, \tau_A) < 1/n\}.$$

Since u is continuous on $A \times \mathcal{M}$, which is compact, it follows that u is uniformly continuous. Thus, $\beta_n(u) \searrow 0$ as $n \to \infty$. We claim that β_n is continuous in $\mathcal{U}$.

Let $\eta > 0$. Define $\delta < \eta/2$. Then if $||u - v|| < \delta$, we have for any $a \in A$, and $\nu \in \mathcal{M}$ such that $\rho(\nu, \tau_A) < 1/n$

$$\begin{aligned} |v(a, \nu) - v(a, \tau_A)| \leq |v(a, \nu) - u(a, \nu)| + |u(a, \nu) + u(a, \tau_A)| + \\ + |v(a, \tau_A) - u(a, \tau_A)| < \delta + \beta_n(u) + \delta, \end{aligned} \tag{32}$$

and so $\beta_n(v) \leq 2\delta + \beta_n(u) < \eta + \beta_n(u)$. By symmetry, $\beta_n(u) < \eta + \beta_n(v)$, and so $|\beta_n(u) - \beta_n(v)| < \eta$. Hence, β_n is continuous as claimed.

Given the definition of β_n, we have that $B_{\tau_n}^{\varepsilon_n} \subseteq D_n := \{(u, a) : u(a, \tau_A) \geq u(A, \tau_A) - \varepsilon_n - 2\beta_n(u)\}$. Since β_n is continuous, we see that D_n is closed, and so Borel measurable. Thus, $\tau_n(D_n) \geq 1 - \varepsilon_n$. Also, $D_n \searrow B_\tau$. Hence, for fixed $j \in \mathbb{N}, j \geq n$, it follows that $\tau_j(D_n) \geq \tau_j(D_j) \geq 1 - \varepsilon_j \geq 1 - \varepsilon_n$, and so $\tau(D_n) \geq \limsup_j \tau_j(D_n) \geq 1 - \varepsilon_n$. Hence, $\tau(B_\tau) = \lim_n \tau(D_n) = 1$. Therefore, $\tau = \lambda \circ (U, f)^{-1}$ is an equilibrium distribution of $\lambda \circ U^{-1}$ and so f is an equilibrium of G. ■

We remark that Theorem 1 still holds if we replace the condition $\varepsilon_n \searrow 0$ by the weaker condition $\varepsilon_n \to 0$. Clearly, we only need to show how the proof of the sufficiency part changes. The condition $\varepsilon_n \searrow 0$ was only used

to show that $\lim_n \tau(D_n) = 1$; we will now show that this continues to hold even when we only have $\varepsilon_n \to 0$.

Note that D_n decreases (to B_τ). Let $\gamma > 0$ be given and let $j \in \mathbb{N}$ be such that $j \geq n$ and $\varepsilon_k < \gamma$ whenever $k \geq j$. Then, if $k \geq j$, it follows that $\tau_k(D_n) \geq \tau_k(D_k) \geq 1 - \varepsilon_k \geq 1 - \gamma$, and so $\tau(D_n) \geq \limsup_j \tau_j(D_n) \geq 1 - \gamma$. Hence, $\lim_n \tau(D_n) = 1$.

Proof of Theorem 2. (Sufficiency) By Theorem 1 and the above remark, g is a Nash equilibrium of G, and since $g \sim f$, so is f.

(Necessity) Define g by changing f only for those players that are not optimizing: if $t \in [0,1]$ is such that there exists $a^* \in A$ satisfying $U(t)(f(t), \lambda \circ f^{-1}) < U(t)(a^*, \lambda \circ f^{-1})$ and $U(t)(A, \lambda \circ f^{-1}) \leq U(t)(a^*, \lambda \circ f^{-1})$, then define $g(t) = a^*$. Since f is a Nash equilibrium of G, it follows that $f \sim g$ and that g is a Nash equilibrium.

Let $\tau = \lambda \circ (U, g)^{-1}$ and let $\tau_n = \nu_n \circ (U_{|T_n}, g_{|T_n})^{-1}$. Define

$$\alpha_n(u) = \sup_{a \in A, \nu \in \mathcal{M}} \{|u(a,\mu) - u(a,\phi)| : \rho(\mu,\phi) \leq \rho(\tau_{A,n}, \tau_A)\},$$

and $\alpha_n = \sup_{u \in U([0,1])} \alpha_n(u)$; similarly, define

$$\gamma_n(u) = \sup_{a \in A, \nu \in \mathcal{M}} \{|u(a,\mu) - u(a,\phi)| : \rho(\mu,\phi) \leq \rho(\tau_{A,n}, \tau_{A,n}^{u,a,a'})\},$$

and $\gamma_n = \sup_{u \in U([0,1])} \gamma_n(u)$.

As in the proof of Theorem 1 we can show that $B_\tau \subseteq B_{\tau_n}^{\varepsilon_n}$ if we define $\varepsilon_n = 2\alpha_n + \gamma_n$. Since $U([0,1])$ is equicontinuous and $\lim_{n\to\infty} \rho(\tau_{A,n}, \tau_A) = \rho(\tau_{A,n}, \tau_{A,n}^{u,a,a'}) = 0$, it follows that $\varepsilon_n \to 0$. Finally, note that $B_\tau = (U,g)([0,1]) \cap \mathrm{supp}(\tau) = \mathrm{supp}(\tau)$ since all players are optimizing by choosing according to g; hence, $\tau_n(B_{\tau_n}^{\varepsilon_n}) = 1$, and so $g_{|T_n}$ is an ε_n-equilibrium of G_n. ■

Proof of Remark 2. For each $n \in \mathbb{N}$, we define $\gamma_n = \inf\{\varepsilon \geq 0 : f_{|T_n}$ is an $\varepsilon-$ equilibrium of $G_n\}$. Note that the set $\{\varepsilon \geq 0 : f_{|T_n}$ is an $\varepsilon-$ equilibrium of $G_n\}$ is nonempty since if $B > 0$ is such that u is bounded by B for all $u \in U([0,1])$, then $f_{|T_n}$ is an $2B-$ equilibrium of G_n. Define $\varepsilon_n = \gamma_n + 1/n$. Thus, it is enough to show that $\gamma_n \to 0$.

Let $\eta > 0$ be given. Denote $\tau_n = \nu_n \circ (U_{|T_n}, f_{T_n})^{-1}$ and $\tau = \lambda \circ (U, f)^{-1}$. Let $\delta > 0$ be such that $d((a,\mu),(b,\nu)) := \max\{d_A(a,b), \rho(\mu,\nu)\} < \delta$ implies that $|u(a,\mu) - u(b,\nu)| < \eta/4$ for all $u \in U([0,1])$. Finally, let $N \in \mathbb{N}$ be such that $n \geq N$ implies that $1/|T_n| < \delta$, $\rho(\tau_{A,n}, \tau_A) < \delta$ and $|\tau_n(B_\tau) - \tau(B_\tau)| < \eta$.

This last inequality follows from the fact that $\text{supp}(\tau)$ is finite and $\text{supp}(\tau_n) \subseteq \text{supp}(\tau)$.

Let $n \geq N$. As in the proof of Theorem 1, we can show that $B_\tau \subseteq B^\eta_{\tau_n}$. Hence,

$$1 - \eta = \tau(B_\tau) - \eta < \tau_n(B_\tau) \leq \tau_n(B^\eta_{\tau_n}), \tag{33}$$

and $f_{|T_n}$ is an $\eta-$equilibrium of G_n. This implies that $\gamma_n \leq \eta$ and, since η is arbitrary, that $\gamma_n \to 0$. ■

Proof of Theorem 3. Let $T = \{t_1, \ldots, t_k\}$ and $u_i = U(t_i)$ for all $1 \leq i \leq k$. Note first that the names of the players are irrelevant in the following sense: if the conclusion of Theorem 3 holds for a game $\tilde{G} = ((\tilde{T}, \nu), A, U)$, with $|\tilde{T}| = k$ and $U(\tilde{t}_i) = u_i$ for all $1 \leq i \leq k$, then it will hold for G. In words, we are free to choose the name of the players.

Let $\varepsilon > 0$. Let P denote the set of vertices of $A = \Delta_m$. Consider the game $G_\lambda = ((\,[0,1], \lambda), P, V)$ where $V(t) = u_i$ if $t \in T_i = [\frac{i-1}{k}, \frac{i}{k})$ for $1 \leq i \leq k-1$ and $V(t) = u_k$ if $t \in T_k = [\frac{k-1}{k}, 1]$. Then G_λ has Nash equilibrium f in which all players optimize (this follows from Theorem 1 in Mas-Colell [22] and from Lemma 6 above; see also Khan and Sun [14, p. 9]). By the expected utility hypothesis ($a = \sum_{i=1}^m a_i e_i$ implies that $U(t)(a, \mu) = \sum_{i=1}^m a_i U(t)(e_i, \mu)$), f is also a Nash equilibrium of the game $((\,[0,1], \lambda), A, V)$. By Lusin's Theorem (see Wheeden and Zygmund [35, Theorem 4.20]) and Tietze-Urysohn's Extension Theorem (see Machado [20, 1.9.5, p. 110][11]), let for every $l \in \mathbb{N}$, $g_l : [0,1] \to A$ be continuous and satisfy $\lambda(\{g_l \neq f\}) < 1/l$. Then, $\lambda \circ g_l^{-1} \Rightarrow \lambda \circ f^{-1}$. So for every $\gamma > 0$ we can find l such that $\sup_{a \in A} |u_i(a, \lambda \circ g_l^{-1}) - u_i(a, \lambda \circ f^{-1})| < \gamma$ and $\lambda(\{g_l \neq f\}) < \gamma$, which implies that g_l is a $\gamma-$equilibrium. Therefore, we have shown that there exists a continuous function $g : [0,1] \to A$ satisfying $\lambda(\{g \neq f\}) < \varepsilon/2$ and g is an $\varepsilon/2-$equilibrium.

For every $n \in \mathbb{N}$, partition $[0,1]$ in nk intervals of the same length:

$$[0,1] = \left[0, \frac{1}{nk}\right) \cup \left[\frac{1}{nk}, \frac{2}{nk}\right) \cup \ldots \cup \left[\frac{nk-1}{nk}, 1\right].$$

Let $T_{j,n} = [\frac{j-1}{nk}, \frac{j}{nk})$ and select a point $t_{j,n} \in T_{j,n}$ in the following way: if $T_{j,n} \cap \{g = f\} \neq \emptyset$ then let $t_{j,n} \in T_{j,n} \cap \{g = f\}$. Let ν_n be the uniform measure with

[11] The particular statement we need for our purposes is: if $F \subseteq [0,1]$ is closed and $h : F \to \Delta_m$ is continuous, then there exists a continuous function $\tilde{h} : [0,1] \to \Delta_m$ such that $\tilde{h}(t) = h(t)$ for all $t \in F$.

support equal to $\{t_{j,n}\}_{j=1}^{nk}$. It follows that $\nu_n(\{g \neq f\}) < \varepsilon/2$ since otherwise we would be able to find p sets $\{T_{j_l,n}\}_{l=1}^{p}$ such that $T_{j_l,n} \subseteq \{g \neq f\}$, with $p/nk \geq \varepsilon/2$; this would imply that $\lambda(\{g \neq f\}) \geq \lambda(\cup_{l=1}^{p} T_{j_l,n}) = p/nk \geq \varepsilon/2$, a contradiction. We also have that $\nu_n \Rightarrow \lambda$ by construction. Furthermore, if $h : T_i \to \mathbb{R}$ is bounded and uniformly continuous then

$$\lim_{n\to\infty} \int_{T_i} h \mathrm{d}\nu_n = \int_{T_i} h \mathrm{d}\lambda, \tag{34}$$

for all $1 \leq i \leq k$. We finally define $G_n = ((\{t_{j,n}\}_{j=1}^{nk}, \nu_n), A, V_n)$, where V_n is the restriction of V to $\{t_{j,n}\}_{j=1}^{nk}$. Note that G_n is a replica of G because of the way V was defined.

Let g_n denote the restriction of g to $\{t_{j,n}\}_{j=1}^{nk}$. We claim that $\nu_n \circ (V_n, g_n)^{-1} \Rightarrow \lambda \circ (V, g)^{-1}$. Let $h : \mathcal{U} \times A \to \mathbb{R}$ be a bounded and uniformly continuous function. Then, by the change-of-variable-formula (see Hildenbrand [18, prop. 36, p.50]),

$$\begin{aligned} &\int_{\mathcal{U}\times A} h \mathrm{d}\nu_n \circ (V_n, g_n)^{-1} = \int_{[0,1]} h \circ (V, g) \mathrm{d}\nu_n = \\ &\sum_{i=1}^{k} \int_{T_i} h \circ (V, g) \mathrm{d}\nu_n \to \sum_{i=1}^{k} \int_{T_i} h \circ (V, g) \mathrm{d}\lambda = \\ &\int_{[0,1]} h \circ (V, g) \mathrm{d}\lambda = \int_{\mathcal{U}\times A} h \mathrm{d}\lambda \circ (V, g)^{-1}, \end{aligned} \tag{35}$$

since $h \circ (U, g)$ is bounded and uniformly continuous on T_i for all $1 \leq i \leq k$. Clearly, we also have $\mathrm{supp}(\nu_n) \to \mathrm{supp}(\lambda)$. Hence, analogous to the proof of Theorem 2, we can show that $B_\tau^{\varepsilon/2} \subseteq B_{\tau_n}^{\varepsilon_n}$, where $\tau = \lambda \circ (V, g)^{-1}$, $\tau_n = \nu_n \circ (V_n, g_n)^{-1}$, $\varepsilon_n = \varepsilon/2 + 2\alpha_n + \gamma_n$ and α_n and γ_n are as in the proof of Theorem 2. Since $\varepsilon_n \to \varepsilon/2$, there is N such that $\varepsilon_n < \varepsilon$ for all $n \geq N$. Hence, for all $n \geq N$, we have

$$\begin{aligned} &\tau_n(B_{\tau_n}^{\varepsilon}) \geq \tau_n(B_{\tau_n}^{\varepsilon_n}) \geq \tau_n(B_\tau^{\varepsilon/2}) = \\ &\nu_n(\{t : (U, g)(t) \in B_\tau^{\varepsilon/2}\}) \geq \nu_n(\{g = f\}) \geq 1 - \varepsilon/2 > 1 - \varepsilon. \end{aligned} \tag{36}$$

Hence, g_n is an $\varepsilon-$equilibrium of G_n. Since f takes values on P and $\nu_n(\{g \neq f\}) < \varepsilon$, it follows that all but a fraction of players of G_n smaller than ε play a pure strategy in g_n. ■

Proof of Theorem 4. We may assume that $U(t)(f(t), \lambda \circ f^{-1}) > U(t)(a, \lambda \circ f^{-1})$ for all $a \neq f(t)$ and $t \in [0,1]$. Let $\gamma(t) = U(t)(f(t), \lambda \circ f^{-1}) - \max_{a \neq f(t)} U(t)(a, \lambda \circ f^{-1}) > 0$. Also, we denote $h = (U, f)$ and $\tau = \lambda \circ h^{-1}$.

Let $n \in \mathbb{N}$ be fixed. Since $\mathcal{U}$ is separable, we can write $\mathcal{U} = \cup_{i=1}^{\infty} A_{n,i}$ where $\{A_{n,i}\}$ is a disjoint collection, each of its members being a Borel set with a diameter no greater than $1/n$. Let $I_n \in \mathbb{N}$ be such that $\sum_{i=I_n}^{\infty} \tau_{\mathcal{U}}(A_{n,i}) < 1/n$. Denote by $\mathcal{P}_n$ be the collection of those sets from

$$\mathcal{N} = \{A_{n,1} \times \{a\}, \ldots, A_{n,I_n-1} \times \{a\}, \cup_{i=I_n}^{\infty} A_{n,i} \times \{a\}\}_{a \in A}$$

that have strictly positive measure.

For each $a \in A$, let $\{\tilde{p}_{n,i,a}\}_{i=1}^{I_n} \subset \mathbb{Q}$ be such that

$$\begin{aligned} &|\tilde{p}_{n,i,a} - \tau(A_{n,i} \times \{a\})| < \delta/(2I_n), i = 1, \ldots, I_n - 1, \text{ and} \\ &\sum_{a \in A} \sum_{i=1}^{I_n} \tilde{p}_{n,i,a} = 1. \end{aligned} \tag{37}$$

Also, if $\tau(A_{n,i} \times \{a\}) = 0, i = 1, \ldots, I_n - 1$, let $\tilde{p}_{n,i,a} = 0$, and if $\sum_{i=I_n}^{\infty} \tau(A_{n,i} \times \{a\}) = 0$ let $\tilde{p}_{n,I_n} = 0$.

It follows that $|\tilde{p}_{n,I_n} - \sum_{i=I_n}^{\infty} \mu(A_{n,i})| < \delta/2$ Furthermore, there exists $N_n \in \mathbb{N}$, and $\{p_{n,i}\}_{i=1}^{I_n} \subset \mathbb{N}$ such that $\tilde{p}_{n,i} = p_{n,i}/N_n$, $i = 1, \ldots, I_n$.

Let $K_n \in \mathbb{N}$ be such that $(K_n + 1)2^{-K_n} < 1/n$, and $\eta_n = \min\{\tau(P)\}_{P \in \mathcal{P}_n}$. Let C_n be a compact subset of $[0,1]$ satisfy $\lambda([0,1] \setminus C_n) < \min\{\eta_n, 2^{-K_n}\}$, $\{U(t)(a, \cdot)\}_t$ is equicontinuous at τ_A and $t \mapsto \gamma(t)$ be continuous in C_n. The existence of C_n follows from Lemma 6 in Carmona [4] and by Lusin's theorem.

Let $\gamma = \min_{t \in C_n} \gamma(t) > 0$ and let $\delta > 0$ be such that $\rho(\tau'_A, \tau_A) < \delta$ implies that $|U(t)(a, \tau'_A) - U(t)(a, \tau_A)| < \gamma$ for all $t \in C_n$.

We define ν_n and T_n similarly as in the proof of Lemma 5; however, we impose that $\text{supp}(\nu_n) \subset C_n$. This requirement can always be satisfied since $\lambda([0,1] \setminus C_n) < \min\{\tau(P)\}_{P \in \mathcal{P}_n} = \min\{\lambda(h^{-1}(P))\}_{P \in \mathcal{P}_n}$ implies that $\lambda(C_n \cap h^{-1}(P)) > 0$ for all $P \in \mathcal{P}$. Also, since $\lambda([0,1] \setminus C_n) < 2^{-K_n}$ and $\lambda(\cup_{P \in \mathcal{P}_n} h^{-1}(P)) = 1$, it follows that $\lambda(C_n \cap \cup_{P \in \mathcal{P}_n} h^{-1}(P)) > 1 - 2^{-K_n}$ we can define $\tilde{t}_l^{n,k}$ as in the proof of Lemma 5 (see (13) and (14)) in a way that $\tilde{t}_l^{n,k} \in C_n$ for all $k \leq K_n$, and $l = 1, \ldots, L_{n,K_n}$.

Note that $\nu_n \circ f_{|T_n}^{-1}(\{a\}) = \sum_{i=1}^{I_n} \tilde{p}_{n,i,a}$ and so

$$\begin{aligned}
&\left|\nu_n \circ f_{|T_n}^{-1}(\{a\}) - \lambda \circ f^{-1}(\{a\})\right| = \\
&\left|\sum_{i=1}^{I_n-1} (\tilde{p}_{n,i,a} - \tau_A(A_{n,i} \times \{a\})) + \tilde{p}_{n,I_n,a} - \tau_A(\cup_{i=I_n}^{\infty} A_{n,i} \times \{a\})\right| \\
&< \sum_{i=1}^{I_n-1} \frac{\delta}{2I_n} + \frac{\delta}{2} < \delta.
\end{aligned} \tag{38}$$

This implies that $\rho(\nu_n \circ f_{|T_n}^{-1}, \lambda \circ f^{-1}) < \delta$. Since $\text{supp}(\nu_n) \subset C_n$, it follows that $f_{|T_n}$ is an equilibrium of $G_n = ((T_n, \nu_n), U_{|T_n})$.

Since

$$M^{n,K_n+1} < \frac{M^{K_n,1}}{2^{K_n}} + \frac{K_n}{2^{K_n}} < \frac{K_n+1}{2^{K_n}} < \frac{1}{n}, \tag{39}$$

it follows that $\text{supp}(\nu_n) \to \text{supp}(\lambda)$. Also, by the same argument as in the proof of Lemma 5, we see that $\nu_n \circ (U_{|T_n}, f_{|T_n})^{-1} \Rightarrow \lambda \circ (U, f)^{-1}$. This completes the proof. ■

Proof of Theorem 5. Analogous to the proof of the sufficiency part of Theorem 1. ■

Proof of Theorem 6. Let $\{\nu_n\}$ and $\{T_n\}$ be such that ν_n is uniform on T_n, a finite subset of $[0,1]$, $T_n \to [0,1]$ and $\nu_n \circ U_{|T_n}^{-1} \Rightarrow \lambda \circ U^{-1}$ (existence is guaranteed by Lemma 5).

Since A is a compact, convex subset of metric vector space, $U(t)$ is continuous and $a \mapsto U(t)(a, \tau)$ is quasi-concave for all $t \in [0,1]$ and all $\tau \in \mathcal{M}$, the game $G_n = ((T_n, \nu_n), U_{|T_n}, A)$ has an equilibrium f_n (see Reny [29]).

Define $\tau_n = \nu_n \circ (U_{|T_n}, f_n)^{-1}$. Since $\tau_{\mathcal{U},n} \Rightarrow \lambda \circ U^{-1}$ and $\tau_{\mathcal{U},n}$ is tight for all n since $\mathcal{U}$ is a separable complete metric space, it follows that the family $\{\tau_{\mathcal{U},n}\}$ is tight (see Hildenbrand [18, Proposition 32, p. 49]). Clearly, the family $\{\tau_{\mathcal{M},n}\}$ is tight since $\mathcal{M}$ is compact. Hence, by Proposition 35 in Hildenbrand [18], the family $\{\tau_n\}$ is tight and so has a convergent subsequence τ (Hildenbrand [18, Proposition 31, p. 49]). Clearly, $\tau_{\mathcal{U}} = \lambda \circ U^{-1}$, and τ is a limit equilibrium distribution of $\lambda \circ U^{-1}$. ■

Proof of Theorem 7. Follows from Lemma 6. ■

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