Nonanonymity and sensitivity of computable simple games*

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Abstract

This paper investigates algorithmic computability of simple games (voting games). It shows that (i) games with a finite carrier are computable, (ii) computable games have both finite winning coalitions and cofinite losing coalitions, and (iii) computable games violate any conceivable notion of anonymity, including finite anonymity and measure-based anonymity. The paper argues that computable games are excluded from the intuitive class of "nice" infinite games, employing the notion of "insensitivity"—-equal treatment of any two coalitions that differ only on a finite set.

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1 Introduction

A *coalitional game* is a function that assigns a worth (number) to each coalition of players. If there are only *finitely* many players, *computability* of the function is not a problem. In that case, there are only finitely many coalitions and one can just construct a finite table listing all coalitions and their worths. If there are *infinitely* many players, computability becomes a problem. First, there are uncountably many coalitions, but one can observe (or describe in English) only countably many of them. We have to restrict observable coalitions in a natural way. Second, we must describe each coalition in order for an algorithm to recognize it. Third, the process of computing the worth of a coalition from a description of it, has to be algorithmic.

This paper investigates algorithmic computability (Turing computability) of a particular class of coalitional games, called *simple games* (voting games). Simple games assign either 0 or 1 to each coalition of players. In the setting of players who face a choice problem from a set of two alternatives, a coalition intuitively describes those players that prefer a certain alternative to the other. Simple games are characterized by its *winning coalitions*—those whose worth is 1. Winning coalitions are understood to be those coalitions whose preferences count. The class of simple games has close connections with social choice theory. The paper by Banks, Duggan, and Le Breton (2003) is a recent example of a successful application of infinite simple games to political theory.

In the setting of Arrow's Theorem, Kirman and Sondermann (1972) and Armstrong (1980, 1985) prove that underlying a social welfare function satisfying certain properties (Unanimity and Independence) is an *ultrafilter*, a special case of a simple game. But ultrafilters (viewed as a simple game) have a particular form—either they have no finite winning coalitions or they are *dictatorial* in the sense that only one player's preferences matter. I show (Mihara, 1997) that among those ultrafilter-based social welfare functions, only dictatorial functions are computable. This implies that an ultrafilter, viewed as a simple game, is computable only if it has finite winning coalitions.

But what happens to simple games other than ultrafilters? Some simple games are not dictatorial but has a *finite carrier*—this means that the games are in effect finite, or almost all players' preferences are ignored. Some simple games are not dictatorial and do not have a finite carrier, but have *finite winning coalitions.* The above result (Mihara, 1997) from computability analysis of social choice leaves open the question about computability of such simple games (that are not ultrafilters). One may get an impression that "games that depend on infinitely many players' preferences" are not computable. A result characterizing computable simple games is called for.

In this paper, I give a sufficient condition and necessary conditions for computability of simple games. As criteria for such characterization, I pay particular attention to finite carriers and finite coalitions. First, I must suitably define the notion of computability. I discard a notion (σ-computability) that excludes even dictatorial games from the class of computable games. I argue that the failure of σ-computability is due to the lack of descriptive power of indices describing each coalition. Instead, I introduce the notion of δ-computability, which uses sufficiently descriptive indices.

With δ-computability, the results are as follows: (i) Proposition 4 shows that *games with a finite carrier are computable.* In particular, dictatorial games are. (ii) Corollary 9 shows that *computable games have both finite winning coalitions and cofinite losing coalitions.* An implication of these two results is Corollary 10 saying that a filter, viewed as a simple game, is computable iff it has a finite carrier. (iii) Section 3.3 (Corollary 11 in particular) argues that *computable games violate any conceivable notion of anonymity, including finite anonymity and measure-based anonymity.*

Result (i) is intuitive since games with a finite carrier are in effect finite, and our intuition tells finite games are computable. This is the class of games that one would safely exclude from the intuitive class of "games that depend on infinitely many players' preferences."

On the other hand, those games with both finite winning coalitions and cofinite losing coalitions may not necessarily be excluded from this intuitive class. But one would expect that "nice" games "that depend on infinitely many players' preferences," are *insensitive* in the following sense: any two coalitions that differ only on a finite set have the same status as winning or losing. Result (ii) implies computable games are not insensitive (Corollary 12): they cannot ignore finitely many players. This result, combined with Result (iii), has a strong implication that computable games are excluded from the intuitive class of "nice" infinite games.

A complete characterization of computable simple games involves much more intricate arguments of recursion theory than this paper. It will be available in a collaborative work (Kumabe and Mihara (in preparation)). In the collaborative work, the connection of computability and well-known properties of simple games is also investigated. We will construct, for example, a computable simple game that is monotonic, proper, strong, nonweak, and has no finite carrier.

2 Framework

2.1 Simple games

Let $N = \mathbf{N} = \{0, 1, 2, \ldots\}$ be a countable set of (the names of) players. Any **recursive** (algorithmically decidable) subset of N is called a **(recursive) coalition**.

Intuitively, a simple game describes in a crude manner the power distribution among *observable* (describable) subsets of players. Since the cognitive ability of a human (or machine) is limited, it is not natural to assume that all subsets of players are observable, when there are infinitely many players. I therefore assume that only **recursive** subsets are observable. This is a natural assumption in the present context, where algorithmic properties of simple games are investigated. According to *Church's Thesis*, the recursive coalitions are the sets of players for which there is an algorithm that can decide for the name of each player whether she is in the set.[1] Note that **the class** REC **of recursive coalitions** forms a **Boolean algebra**; that is, it includes N and is closed under union, intersection, and complementation.

Formally, a **(simple) game** is a collection $\omega \subseteq \text{REC}$ of (recursive) coalitions. I often require that $N \in \omega$. The coalitions in ω are said to be **winning**. A coalition is said to be **losing** if it is not winning. One can regard a simple game as a function from REC to $\{0, 1\}$, assigning the value 1 or 0 to each coalition depending on whether it is winning or losing.

Remark 1. I assume that observable subsets of players are recursive, not just r.e. (*recursively enumerable*). I explain why. First, nonrecursive r.e. sets are observable in a very limited sense. An r.e. set is a set whose members can be enumerated by some algorithm. This does not mean in general that there is a method to tell *whether* a given player belongs to it. Second, the r.e. sets do not form a Boolean algebra, but certain properties of simple games implicitly assume that the observable coalitions form a Boolean algebra. For example, the property of strongness requires the complement of *any* losing coalition to be winning. In the setting where the complement of some losing coalition is not observable (indeed, the complement of an r.e. set is not necessarily r.e.), the notion, or any modification of it, is of limited interest. I give a third reason in Remark 2. ‖

I introduce from the theory of cooperative games a few basic notions of simple games (Peleg, 1984; Weber, 1994). A simple game ω is said to be **proper** if for all coalitions S, $S \in \omega$ implies $N \setminus S \notin \omega$. ω is **monotonic** if for all coalitions S and T, the conditions $S \in \omega$ and $T \supseteq S$ imply $T \in \omega$. ω is **strong** if for all coalitions S, $S \notin \omega$ implies $N \setminus S \in \omega$. ω is **weak** if the intersection $\bigcap \omega$ of the winning coalitions is nonempty. The members of $\bigcap \omega$ are called **veto players**; they are the players that belong to all winning coalitions. (The set $\bigcap \omega$ of veto players may or may not be observable.) ω is **dictatorial** if there exists some i_0 (called a **dictator**) in N such that $\omega = \{ S \in \text{REC} : i_0 \in S \}$. Note that a dictator is a veto player, but a veto player is not necessarily a dictator.

A **carrier** of a simple game ω is a coalition $S \subseteq N$ such that

$$T \in \omega \iff S \cap T \in \omega$$

[1] Soare (1987) gives a more precise definition of *recursive sets* as well as detailed discussion of recursion theory. My earlier papers (Mihara, 1997; Mihara, 1999) contain short reviews of recursion theory.

for all coalitions T.

Finally, I introduce a few notions from the theory of Boolean algebras (Koppelberg, 1989); they can be regarded as properties of simple games. A monotonic simple game ω is called a **filter** if it is closed with respect to finite intersection: if S, $S' \in \omega$, then $S \cap S' \in \omega$. We may think of a filter as a family of "large" sets. Note that a filter has the *finite intersection property* (the intersection of *finitely* many winning coalitions is nonempty), but it may or may not have a veto player. A filter ω is called an **ultrafilter** if it is a strong simple game. If ω is an ultrafilter, then $S \cup S' \in \omega$ implies that $S \in \omega$ or $S' \in \omega$. A **free** ultrafilter is an ultrafilter that is not dictatorial; a free ultrafilter does not contain any finite coalitions.

2.2 Indicators for simple games

To define the notions of computability for simple games, I introduce below two indicators for them. In order to do that, I first represent each recursive coalition by a natural number: either by an r.e. index (Σ_1-index) or by a characteristic index (Δ_0-index). A characteristic index gives more information about the coalition that it represents than an r.e. index does. The indicators then assign the value 0 or 1 to each number representing a recursive coalition, depending on whether the coalition is winning or losing. When a number does not represent a recursive coalition, the value is undefined.

Given a simple game ω, its **σ-indicator** is the partial function σ_ω on $\mathbf{N}$ defined by

$$\sigma_\omega(e) = \begin{cases} 1 & \text{if } W_e \text{ is recursive and } W_e \in \omega, \\ 0 & \text{if } W_e \text{ is recursive and } W_e \notin \omega, \\ \uparrow & \text{if } W_e \text{ is nonrecursive,} \end{cases} \tag{1}$$

where $W_e = \{\, x : \varphi_e(x) \downarrow \,\}$ is the domain of the eth partial recursive function. Note that $\sigma_\omega(e) = \sigma_\omega(e')$ if $W_e = W_{e'}$. Using r.e. indices (Σ_1-indices) for r.e. sets, the σ indicator σ_ω can be rewritten as follows:

$$\sigma_\omega(e) = \begin{cases} 1 & \text{if } e \text{ is an r.e. index for a recursive set in } \omega, \\ 0 & \text{if } e \text{ is an r.e. index for a recursive set not in } \omega, \\ \uparrow & \text{if } e \text{ is not an r.e. index for any recursive set.} \end{cases} \tag{2}$$

Given a simple game ω, its **δ-indicator** is the partial function δ_ω on $\mathbf{N}$ defined by

$$\delta_\omega(e) = \begin{cases} 1 & \text{if } e \text{ is a characteristic index for a recursive set in } \omega, \\ 0 & \text{if } e \text{ is a characteristic index for a recursive set not in } \omega, \\ \uparrow & \text{if } e \text{ is not a characteristic index for any recursive set.} \end{cases} \tag{3}$$

Note that δ_ω is well-defined since each $e \in \mathbf{N}$ can be a characteristic index (Δ_0-index) for at most one set.

2.3 Computability notions

I now introduce notions of σ-computable simple games and δ-computable simple games. These notions of computability are weaker than Exact Computability and Strong Computability in Appendix A. The latter notions and their variants are too strong, easily leading to impossibility results.

σ-Computability σ_ω has an extension to a partial recursive function.

δ-Computability δ_ω has an extension to a partial recursive function.

The following lemma states that σ-Computability implies δ-Computability.

Lemma 1 *If a simple game ω is σ-computable, then it is δ-computable.*

Proof. Suppose that ω is σ-computable. Then σ_ω has an extension σ' to a partial recursive function.

Let φ_e be the eth partial recursive function. Using the Parameter Theorem (see the proof of Proposition 2 for details), define a recursive function f by

$$\varphi_{f(e)}(u) = \begin{cases} 1 & \text{if } \varphi_e(u) = 1, \\ \uparrow & \text{otherwise.} \end{cases}$$

We claim that $\sigma' \circ f$ is an extension of δ_ω. To show this, suppose that e is a characteristic index for a recursive B. Then $f(e)$ is an r.e. index for B. Now, if B is in ω, then by (2), $\sigma'(f(e)) = \sigma_\omega(f(e)) = 1$. Similarly, if B is not in ω, then $\sigma'(f(e)) = \sigma_\omega(f(e)) = 0$. ∎

3 The Main Results

3.1 σ-Computability

The following results suggest that σ-Computability is not a good notion of computability. I will come back to this point in Section 3.2.

Proposition 2 *Suppose that $N \in \omega$ and $\emptyset \notin \omega$. Then the simple game ω is not σ-computable. In particular, proper simple games violate the condition.*

Proof. Suppose that σ-Computability is satisfied with an extension σ' of $\sigma = \sigma_\omega$. As shown below, we can define a recursive function f such that

$$W_{f(e)} = \begin{cases} N & \text{if } e \in K, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4}$$

where $K = \{\, e : e \in W_e \,\}$. K is known (Soare, 1987, I.4.3 and I.4.4, pp. 18–9) to be a nonrecursive r.e. set.

Details. Define a partial function θ by

$$\theta(e,u) = \begin{cases} 0 & \text{if } e \in K, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then θ is partial recursive since its graph g is r.e. ($(e,u,y) \in g$ iff $y = 0$ & $e \in K$.) Hence, for some z, $\theta = \varphi_z^{(2)}$. By the Parameter Theorem (*s-m-n* Theorem), there is a recursive function $s = s_1^1$ such that

$$\varphi_{s(z,e)}(u) = \varphi_z^{(2)}(e,u) = \theta(e,u).$$

Let $f(e) = s(z,e)$. Then f is recursive and

$$\varphi_{f(e)}(u)\downarrow \iff \theta(e,u)\downarrow \iff e \in K.$$

Hence (4) follows. $\diamondsuit$

Since $N \in \omega$ and $\emptyset \notin \omega$, and these sets are recursive, we have, by (4) and (1),

$$e \in K \Longrightarrow W_{f(e)} = N \Longrightarrow \sigma'(f(e)) = \sigma(f(e)) = 1,$$

and

$$e \notin K \Longrightarrow W_{f(e)} = \emptyset \Longrightarrow \sigma'(f(e)) = \sigma(f(e)) = 0.$$

This gives an algorithm to decide whether $e \in K$, contradicting the fact that K is nonrecursive. ∎

Corollary 3 *If a simple game ω is σ-computable, then it is not proper. Furthermore, if it is monotonic, then $\omega =$ REC; that is, all coalitions are winning.*

3.2 δ-Computability

The following proposition asserts that games that are in effect finite are δ-computable, as expected.

Proposition 4 *Suppose that a simple game ω has a finite carrier. Then ω is δ-computable.*

Proof. Suppose that ω has a finite carrier $S = \{s_0, \ldots, s_m\}$, where $s_0 < \cdots < s_m$. Let $\omega_S = \{\, S \cap T : T \in \omega \,\}$ be the collection of all the intersections of a winning coalition with S. (ω_S is the collection of all winning coalitions in S.) Then ω_S is finite and the collection of all subsets of S is also finite.

Assign to each subcoalition $T \subseteq S$ the number

$$\langle T(s_0), \ldots, T(s_m) \rangle,$$

where T is identified with its characteristic function (so that $T(x) = 1$ iff $x \in T$, and $T(x) = 0$ iff $x \notin T$) and $\langle x_0, x_1, \ldots, x_m \rangle$ denotes the image of $(x_0, x_1, \ldots, x_m)$ under a certain one-to-one recursive function from $\mathbf{N}^{m+1}$ onto $\mathbf{N}$. The function can be constructed from the standard pairing function as in Soare (1987, I.3.6, p. 16). For example, if $T = \{s_0, s_1, s_3\}$ and $m = 4$, then

$$\langle T(s_0), \ldots, T(s_m) \rangle = \langle 1, 1, 0, 1, 0 \rangle.$$

Note that each subcoalition of S is identified with a unique number. Let

$$\mathrm{WIN} = \{\, \langle T(s_0), \ldots, T(s_m) \rangle : T \in \omega_S \,\}$$

be the set of numbers assigned to winning coalitions in S and let LOSE be the set of numbers assigned to losing coalitions in S. The sets WIN and LOSE are recursive since they are finite.

Let

$$\alpha(e) = \langle \varphi_e(s_0), \ldots, \varphi_e(s_m) \rangle$$

if all of $\varphi_e(s_0)$, ..., $\varphi_e(s_m)$ are either 0 or 1; otherwise, let $\alpha(e) \uparrow$. If e is a characteristic index for a recursive coalition T (which is not necessarily a subcoalition of S), then $\alpha(e)$ specifies all the elements of $S \cap T$. Define δ' by

$$\delta'(e) = \begin{cases} 1 & \text{if } \alpha(e) \in \mathrm{WIN}, \\ 0 & \text{if } \alpha(e) \in \mathrm{LOSE}, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then, clearly, δ' is partial recursive and an extension of δ_ω. ∎

In particular, *if a simple game ω is dictatorial, then ω is δ-computable.* Indeed, the coalition consisting of the dictator is a finite carrier for the dictatorial game ω. In contrast, Proposition 2 implies that a dictatorial game cannot be σ-computable.

The reason for the difference lies in the ways each coalition is described. To decide whether a particular coalition is winning or not in a dictatorial game, one has only to check whether the dictator is in the coalition. In the case of δ-Computability, each coalition is described by a characteristic index, which gives a computable characteristic function of the coalition. So, one can tell whether the dictator is in the coalition simply by putting his name into the algorithm. In the case of σ-Computability, each coalition is described by an r.e. index. But an r.e. index only gives an algorithm for a partial computable function whose domain is the coalition. If the dictator is in the coalition, the algorithm will eventually halt; but if not, then it will never halt. So, if the dictator is not in the coalition, one cannot generally be assured that the dictator is not.

Note that if a game has a finite carrier S and N is winning, then there exits a finite winning coalition, namely $S = N \cap S$. When there does not exist a finite winning coalition, it is a corollary (Corollary 6) of the following negative result that the computability condition is violated. (The number

k is identified with the set $\{0, 1, \ldots, k-1\}$, following the set-theoretic notation.)

Proposition 5 *Suppose that a simple game ω has an infinite winning coalition $S \in \omega$ such that for each $k \in \mathbf{N}$, its k-initial segment $S \cap k$ is losing. Then ω is not δ-computable.*

Proof. Let ω be a simple game satisfying the assumption. Suppose that ω is δ-computable. Then there exists a partial recursive function δ' which extends δ_ω.

Let $K = \{\, e : e \in W_e \,\}$. K is a nonrecursive r.e. set. Since K is r.e., there is (Soare, 1987, II.1.2, p. 28) a recursive set $R \subseteq \mathbf{N} \times \mathbf{N}$ such that

$$e \in K \iff \exists z R(e, z).$$

Using the Parameter Theorem, define a recursive function f by

$$\varphi_{f(e)}(u) = \begin{cases} 1 & \text{if } \neg\exists z \leq u \; R(e,z) \text{ and } u \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Details. The function h defined by

$$h(e,u) = \begin{cases} 1 & \text{if } \neg\exists z \leq u \; R(e,z) \text{ and } u \in S, \\ 0 & \text{otherwise.} \end{cases}$$

is recursive (since R and S are recursive). Hence, for some y, $h = \varphi_y^{(2)}$. By the Parameter Theorem, there is a recursive function s such that

$$\varphi_{s(y,e)}(u) = \varphi_y^{(2)}(e,u) = h(e,u).$$

Let $f(e) = s(y, e)$. Then f is recursive. $\diamondsuit$

Now,

$$\begin{aligned} e \in K &\implies \varphi_{f(e)}(u) = 1 \text{ iff } u \text{ is small and } u \in S \\ &\implies f(e) \text{ is a characteristic index for an initial segment of } S \\ &\implies \delta'(f(e)) = \delta_\omega(f(e)) = 0, \end{aligned}$$

but

$$\begin{aligned} e \notin K &\implies \varphi_{f(e)}(u) = 1 \text{ iff } u \in S \\ &\implies f(e) \text{ is a characteristic index for } S \\ &\implies \delta'(f(e)) = \delta_\omega(f(e)) = 1. \end{aligned}$$

This implies that K is recursive, contradicting the fact that it is not. ∎

The following Corollary states that any δ-computable simple game (such that N is winning) has a finite winning coalition.

Corollary 6 *Suppose that $N \in \omega$. If all finite coalitions are losing (that is, if there is no finite winning coalition), then the simple game ω is not δ-computable.*

I also state results that are close to the preceding proposition and corollary.

Proposition 7 *Suppose that $\emptyset \notin \omega$. Suppose that the simple game ω has an infinite coalition $S \in \omega$ such that for each $k \in \mathbf{N}$, its difference $S \setminus k = \{ s \in S : s \geq k \}$ from the initial segment is winning. Then ω is not δ-computable.*

Proof. Let ω be a simple game satisfying the assumption. Suppose that ω is δ-computable. Then there exists a partial recursive function δ' which extends δ_ω.

Let $K = \{ e : e \in W_e \}$. K is a nonrecursive r.e. set. Since K is r.e., there is (Soare, 1987, II.1.2, p. 28) a recursive set $R \subseteq \mathbf{N} \times \mathbf{N}$ such that $e \in K \iff \exists z R(e, z)$. Using the Parameter Theorem, define a recursive function f by

$$\varphi_{f(e)}(u) = \begin{cases} 1 & \text{if } \exists z \leq u \ R(e, z) \text{ and } u \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Now,

$$\begin{aligned} e \in K &\implies \varphi_{f(e)}(u) = 1 \text{ iff } u \text{ is large and } u \in S \\ &\implies f(e) \text{ is a characteristic index for the difference} \\ &\qquad \text{of } S \text{ from an initial segment} \\ &\implies \delta'(f(e)) = \delta_\omega(f(e)) = 1, \end{aligned}$$

but

$$\begin{aligned} e \notin K &\implies \varphi_{f(e)}(u) = 0 \text{ for all } u \\ &\implies f(e) \text{ is a characteristic index for } \emptyset \\ &\implies \delta'(f(e)) = \delta_\omega(f(e)) = 0. \end{aligned}$$

This implies that K is recursive, contradicting the fact that it is not. ∎

A *cofinite* set is the complement of a finite set. The following corollary is the dual of Corollary 6.

Corollary 8 *Suppose that $\emptyset \notin \omega$. If all cofinite coalitions are winning, then the simple game ω is not δ-computable.*

Combining Corollary 6 and Corollary 8, I get the following powerful criterion for checking computability of simple games.

Corollary 9 *Suppose that $N \in \omega$ and $\emptyset \notin \omega$. If the simple game ω is δ-computable, then some finite coalition is winning and some cofinite coalition is losing.*

Finally, the following result generalizes the earlier result (Mihara, 1997) that an ultrafilter, viewed as a simple game, is computable iff it is dictatorial.

Corollary 10 *A filter, viewed as a simple game, is δ-computable iff it has a finite carrier.*

Proof. ($\Longleftarrow$). Immediate from Proposition 4.

($\Longrightarrow$). Suppose a filter is computable. Then, by Corollary 6, it has a finite winning coalition S. It is easy to show that S is in fact a carrier. ∎

3.3 Nonanonymity and sensitivity of computable games

As an application of the results in Section 3.2, I show that (under some weak conditions) δ-computable simple games violate any conceivable notion of anonymity as well as a related notion of insensitivity. The emphasis here is not on generality of assertions, but on illustrating the usefulness of the above results for obtaining nontrivial results about computable games.

Anonymity is usually defined by way of a permutation on the set of players. We say that a simple game ω is **anonymous** if for any permutation $\pi : N \to N$ and for any coalition S, if $\pi(S) = \{\pi(i) : i \in S\}$ is recursive, we have $S \in \omega$ iff $\pi(S) \in \omega$. (We may instead restrict π to those such that $\pi(S)$ is recursive for any recursive S.) This definition requires that any two coalitions of the same cardinality be treated equally. It is a rather strong requirement, violated by any simple game that has an infinite winning coalition with an infinite, losing complement. (Mihara (1997b) discusses a closely related problem in a different setting.)

If in the above definition we restrict permutations to those that permute only finitely many players, we say the game is **finitely anonymous**. In particular, finitely anonymous games treat any two coalitions with the same finite number of players equally. Finite anonymity is a notion much weaker than anonymity. For example, free (nondictatorial) ultrafilters are finitely anonymous.

The following corollary is a strong one, which shows that even *finite* anonymity is easily violated by computable simple games (we can actually drop the properness and monotonicity conditions—assuming $\emptyset \notin \omega$ instead—in the following corollary (Kumabe and Mihara (in preparation))):

Corollary 11 *Suppose that $N \in \omega$. Suppose that the simple game ω is proper, monotonic, and δ-computable. Then it is not finitely anonymous.*

Proof. Let ω be a simple game satisfying the conditions. Suppose ω is finitely anonymous. From Proposition 5, ω has a finite winning coalition $S = \{0, 1, \ldots, k-1\}$. By properness, $N \setminus S = \{k, k+1, \ldots\}$ is losing. By monotonicity, $S' = \{k, k+1, \ldots, 2k-1\} \subset N \setminus S$ is losing. Since S and S' are finite and have the same number of elements, they should be treated equally by the finitely anonymous ω. But S is winning and S' is not. ∎

Another way of defining anonymity is by way of a measure on the set of players. (Gomberg, Martinelli, and Torres (2002) study measure-based anonymity in the setting of preference aggregation rules.) A *coalition measure space* (N, REC, μ) is a list of the set N of players, the set REC of coalitions, and a finitely additive probability measure μ on REC. We say that a simple game ω is **μ-anonymous** if for any coalition S and S', $\mu(S) = \mu(S')$ implies $S \in \omega \iff S' \in \omega$.

If each one-player coalition $\{i\}$ is of measure zero (i.e., $\mu(\{i\}) = 0$), then any finite coalition is of measure zero. So, μ-anonymous games must treat any two finite coalitions equally. This implies that finite winning coalition S (Corollary 9) and $S' = \emptyset$ must be treated equally by a computable game, if it is μ-anonymous. But $S' = \emptyset$ is not winning under the assumption of Corollary 9, while S is. Therefore, the game is not μ-anonymous.

Since there is no permutation that maps $\emptyset$ to S, this argument is not the same as the one against permutation-based anonymity. Rather, it is based on a particular property of a μ-anonymous game: it is *insensitive* in the sense that it ignores finitely many players.

When there are infinitely many players, a natural requirement for a simple game is that finitely many players do not count. (Banks, Duggan, and Le Breton (2003) require that measure zero of players do not count.) The following insensitivity criterion for simple games formalizes this idea. Here, the *symmetric difference* $S \triangle S'$ of sets S and S' is defined by $S \triangle S' = (S \setminus S') \cup (S' \setminus S)$. A simple game ω is **insensitive** if for any coalitions S and S', whenever they have a finite symmetric difference, we have $S \in \omega$ iff $S' \in \omega$. For example, free ultrafilters are insensitive. One can easily check that a μ-anonymous game is insensitive. Banks, Duggan, and Le Breton (2003) define "simple games" so that they are monotonic and insensitive (assuming that each player is of measure zero).

I conclude with a negative result, which implies that the δ-computable simple games are excluded from the intuitive class of "nice" infinite games.

Corollary 12 *Suppose that $N \in \omega$ and $\emptyset \notin \omega$. If the simple game ω is δ-computable, then it is not insensitive.*

Proof. From Corollary 9, we have a finite winning coalition S. But S and $S' = \emptyset$ have a finite symmetric difference, while S' is losing. ∎

Appendix A: Stronger Notions of Computability

There are several alternative conditions of computability that look more appealing on intuitive grounds than σ-Computability and δ-Computability introduced in Section 2.3. I show that such conditions are too strong to be satisfied.

Let σ_ω be the σ-indicator (1) of a simple game ω. Exact Computability below requires that there exist an algorithm that, given an r.e. index for a recursive coalition, tells whether the coalition is winning or not. When a number is given that is not an r.e. index for any recursive coalition, the algorithm must not give an output.

Exact Computability σ_ω is partial recursive.

Exact Computability is too stringent since it requires a certain partial function to be computable on a domain on which there is no computable partial function.

Proposition 13 *No simple game satisfies Exact Computability.*

Proof. Suppose the indicator σ_ω is partial recursive. Then its domain must be r.e. But the domain $\{ e : W_e \text{ is recursive} \}$ is known (Soare, 1987, p. 21) not to be r.e. ∎

Exact Computability requires an algorithm that does not give an output in the case that the input is not an r.e. index for any recursive coalition. It would be nice if such such an algorithm could instead give an output in this case too, indicating that the input is not legitimate. This leads to the following condition:

Strong Computability The extension σ' of σ_ω defined as follows is recursive: $\sigma'(e) = 2$ if e is not an r.e. index for any recursive set.

Unfortunately, this condition cannot be met by any simple game.

Proposition 14 *No simple game satisfies Strong Computability.*

Proof. Suppose that σ' is recursive. Then σ_ω is partial recursive. This means that Exact Computability is satisfied, contradicting Proposition 13. ∎

In the above definitions of computability, I used the indicator σ_ω, where legitimate inputs are the r.e. indices (Σ_1-indices) for recursive coalitions. Using the indicator δ_ω, where legitimate inputs are the characteristic indices (Δ_0-indices) for recursive coalitions, I can define conditions similar to Exact Computability and to Strong Computability for simple games. It turns out that *no* simple games satisfy these conditions. This can be proved similarly from the result (Mihara, 1997, Lemma 2) that the set of characteristic indices for a recursive set is not r.e.

Remark 2. Remark 1 gave two reasons why I do not assume that the observable subsets of players are the r.e. sets. The third reason is that no satisfactory notion of computability (which does not resort to nonrecursive oracles) can be defined if a simple game ω is defined on the domain of all r.e. sets. In that case, the only sensible indicator would be

$$\sigma_\omega^1(e) = \begin{cases} 1 & \text{if } W_e \in \omega, \\ 0 & \text{if } W_e \notin \omega. \end{cases}$$

And the only sensible computability condition would be to require σ_ω^1 to be recursive. It then follows that the index set $\{ e : W_e \in \omega \}$ is recursive. But Rice's Theorem in turn implies that ω consists of all r.e. sets (if $\omega \neq \emptyset$, as I assume). In other words, the only game satisfying the computability requirement is the one in which every subset of players is winning. $\|$

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