Proportional Payoffs in a Model of Two-Stage Bargaining with Reversible Coalitions

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Abstract

This paper studies coalition formation and payoff division in majority games under the following assumptions: first, payoff division can only be agreed upon after the coalition has formed (two-stage bargaining); second, negotiations in the coalition can break down, in which case a new coalition may be formed (reversible coalitions). Under the most natural bargaining protocol, both ex ante and ex post payoff division are proportional to the voting weights. Other bargaining rules may generate counterintuitive comparative statics.

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1 Introduction

Noncooperative models of coalition formation usually assume that players can agree on payoff division at the time they form a coalition (see e.g. Selten (1981), Baron and Ferejohn (1989), Chatterjee et al. (1993), Okada (1996)). However, there are situations in which players form coalitions before agreeing on payoff division: in coalition governments, parties cannot negotiate effectively on all issues before the formation of the government, partly because one does not know exactly which issues will come up (see Aumann and Myerson (1988)). A coalition is then understood as a *negotiation group*: if coalition S forms, it means that players in S bargain over the division of $v(S)$. In this context, it is natural to think that negotiations may break down, resulting in the dissolution of the coalition and possibly in the formation of a new one (see the discussion in Aumann and Drèze (1974)). We will refer to the first class of models (in which players agree on payoff division at the time the coalition is formed) as one-stage models, and to the second class (in which player agree on payoff division after the coalition has formed) as two-stage models. Coalition formation in two-stage models can be thought of as reversible or irreversible, depending on whether coalitions can be broken up once formed. We will be interested in two-stage reversible processes.

The two-stage models studied in the literature are hybrid: the process of coalition formation is modeled as a noncooperative game, but payoff division is determined by a cooperative solution concept (Hart and Kurz (1983), Aumann and Myerson (1988)).[1] A fully noncooperative approach has been missing, with the important exception of the models of bargaining in markets, in which only two-player coalitions can form and matching is random (see Osborne and Rubinstein (1990) and Muthoo (1999)).

In this paper, we study a fully noncooperative game of coalition formation and payoff division. The formation of a coalition means the beginning of negotiations between the players in the coalition over the division of the

[1] Aumann and Myerson (1988) clearly have irreversible coalitions in mind. In contrast, Hart and Kurz (1983) seem to see coalitions as reversible.

coalitional value. Negotiations can break down, in which case the coalition is dissolved and a new coalition may be formed. Proposers are randomly chosen both at the coalition formation and at the payoff division stage. The question then arises of what should be the probability of being a proposer. It seems natural to assume that each of the players is chosen with a probability proportional to his number of votes. It then turns out that expected payoffs are proportional both before playing the game (for all players) and after a coalition is formed (for the members of the coalition) provided that the game is constant-sum and homogeneous. While this is a very natural result it is not present in the literature to the best of my knowledge. Von Neumann and Morgenstern's main simple solution and other cooperative solution concepts like the aspiration core (Cross, 1967) predict a proportional division of payoffs inside the coalition that actually forms, but have nothing to say about ex ante expected payoffs. Ex ante solution concepts like the Shapley value do not predict proportional payoffs; the nucleolus assigns proportional payoffs to the grand coalition but not to all minimal winning coalitions. Something similar can be said of noncooperative models: they either make no prediction of ex ante expected payoffs (as Morelli 1999) or they assign a disproportionate share to the proposer (as Montero 2001).

While the model makes intuitive predictions for the proportional bargaining rule, it can make surprising predictions for other rules. Consider the case of apex games. In apex games there is a major player (the apex player) and $n-1 \geq 3$ minor players. In order for a coalition to be winning, it must contain the major player together with a minor player, or all the minor players together. The major player seems to be stronger than the minor players, and it could be reasonable to adopt bargaining rules that reflect this strength (for example, selecting this player more often to be proposer). However, it can be shown that the major player can actually be better-off under more egalitarian rules. Ex post payoff division remains proportional over a range of bargaining rules, and this is actually the reason why ex ante expected payoffs behave in a perverse way. We also show that coalitions larger than minimal winning may form for some bargaining rules.

The remainder of the paper is organized as follows. Section 2 introduces the two-stage game and some general properties of the equilibria. Section 3 refers to general majority games. Section 4 studies the case of apex games in detail, and section 5 concludes with some further remarks on the comparison with the literature and on possible extensions.

2 The game

Let $N = \{1, 2, ..., n\}$ be the set of players (parties) and (N, v) a proper simple characteristic function game ($v(S) = 0$ or 1 for all $S \subset N$, $v(N) = 1$, $v(\varnothing) = 0$ and $v(S) + v(T) \leq 1$ for all $T \subseteq N \backslash S$). In order to make things interesting we will assume $v(i) = 0$ for all i. We will interpret the characteristic function as referring to a budget of size 1 that has to be divided by majority rule. The characteristic function v indicates which coalitions of parties have a majority (and thus can divide the budget between themselves). We will assume that all players in N are risk-neutral and discount future payoffs with a discount factor $\delta < 1$.[2]

Given the underlying characteristic function game, bargaining proceeds as follows:

- Nature selects a proposer according to a probability distribution θ ($\theta_i \geq 0$ for all i and $\sum_{i \in N} \theta_i = 1$).
- The selected proposer i proposes a coalition S such that $S \ni i$.
- Players in S accept or reject the proposal sequentially. If one of them rejects, a period elapses and Nature selects a new proposer according to the probability distribution θ.
- If all players in S accept, coalition S is formed. If S is a losing coalition, the game ends and all players get 0. If S is a winning coalition, players in S bargain over the division of the budget. The "internal" game, played only by players in S, is a bargaining game with random proposers (Nature follows a probability distribution θ^S with $\theta_i^S \geq 0$ for all $i \in S$ and $\sum_{i \in S} \theta_i^S = 1$) and breakdown probability. A proposal

[2] Not all the results require players to discount payoffs.

x^S is a division of the budget between the players in S ($\sum_{i\in S} x_i^S = 1$). Every time a responder rejects a proposal, coalition S is dissolved with probability $1-p$ ($0 < p < 1$).

- If coalition S is dissolved, Nature selects a proposer again according to the probability distribution θ.

We will be interested in stationary subgame perfect equilibria (SSPE).

Since coalition formation occurs before payoff division, we can think of the extensive game described above as a two-stage game. Of course, both the coalition formation "stage" and the payoff division "stage" are complicated objects and the play may bring the players back from the second to the first stage.

We will refer to the probability distribution θ as the *protocol*, and to θ^S as the *internal protocol*. Given a SSPE σ^* we will denote by y the expected equilibrium payoff vector computed before Nature starts the game, and by y^S the expected equilibrium payoff vector computed after S has formed and before Nature starts the internal game. Let z^S be the vector of continuation values (i.e., expected payoffs after a proposal has been rejected) in the internal game. Notice that because of stationarity, y, y^S and z^S depend only on σ^* but not on the history of the game. We start by computing the equilibrium of the internal game.

2.1 The equilibrium of the internal game

Suppose we have a SSPE of the game with associated expected equilibrium payoff y. We now show that the internal game has a unique stationary subgame perfect equilibrium payoff y^S.

If a player rejects a proposal in the internal game, with probability p Nature starts the internal game again (so that player i expects to get y_i^S) and with probability $1-p$ coalition S breaks apart and Nature starts the coalition formation game again (so that player i expects to get y_i). We have the following equation for the continuation value of player i

$$z_i^S = py_i^S + (1-p)y_i$$

As for y_i^S, it is given by the probability i is selected to be a proposer in the internal game times his expected payoff as a proposer plus the probability that he is selected to be the responder (which is $1-\theta_i^S$ because bargaining is unanimous in the internal game) times his continuation value.[3]

$$y_i^S = \theta_i^S[1 - \sum_{j\in S\setminus\{i\}} z_j^S] + (1-\theta_i^S)z_i^S$$

From this system of equations (and taking into account that $\sum_{j\in S} y_j^S = 1$) we see that

$$y_i^S = \theta_i^S(1 - \sum_{j\in S} y_j) + y_i. \tag{1}$$

This is a well-known result in bargaining games with breakdown probability: player i's expected payoff equals the breakdown payoff (in this case, y_i) plus a share of the surplus proportional to the probability of being proposer (cf. Binmore (1987) and Binmore et al. (1986)).

Equation (1) resembles an allocation rule in that each player receives a share of the available surplus and this share is determined by the internal protocol. However, there is an important difference: the payoff a player gets from being in a coalition is not fully determined by the rules of the internal game because the breakdown outcome is *endogenous*.

Let s denote $|S|$. The first possibility that comes to mind regarding the internal protocol θ^S is the *egalitarian protocol*, $\theta_i^S = \frac{1}{s}$ for all $i \in S$, which implies $y_i^S = \frac{1}{s}(1-\sum_{j\in S} y_j) + y_i$. If the underlying proper simple game is a weighted majority game with weights $(w_1, ..., w_n)$, the *proportional protocol* $\theta_i^S = \frac{w_i}{\sum_{j\in S} w_j}$ seems reasonable as well.

[3] The payoff for player i as a proposer is $1-\sum_{j\in S\setminus\{i\}} z_j^S$ regardless of whether agreement is immediate. There are two possible cases:

1) If $\sum_{i\in S} z_i^S < 1$ the proposer strictly prefers making acceptable proposals; since in any SPE the proposer must offer exactly z_j to each responder j the results holds.

2) If $\sum_{i\in S} z_i^S = 1$ acceptable and unacceptable proposals give the same payoffs to the players. Notice that this second case is only possible if $\sum_{i\in S} y_i^S = 1$ and $\sum_{i\in S} y_i = 1$.

2.2 The equilibrium of the game

Lemma 1 *If for any player i we have $\theta_i > 0$ and $\theta_i^S > 0$ for all winning coalitions $S \ni i$, then in any SSPE all proposals to form a winning coalition are accepted, and a winning coalition forms immediately.*

Proof. Suppose that we had an equilibrium in which a player i proposes coalition S with positive probability and some player $j \in S$ rejects the proposal with positive probability. When receiving the proposal, player j compares the payoff from accepting, $y_j + \theta_j^S(1 - \sum_{k \in S} y_k)$,[4] and the payoff from rejecting, δy_j. Player j must accept if $y_j > 0$ thus rejection requires $y_j = 0$ and $\sum_{k \in S} y_k = 1$. But if rejection occurs with positive probability, then $\sum_{k \in N} y_k < 1$, contradicting $\sum_{k \in S} y_k = 1$ (no y_i can be negative because a player can guarantee himself a payoff of 0).

Now consider player i as proposer. If he proposes a losing coalition with positive probability we have $\sum_{j \in N} y_j < 1$, thus a proposal of forming the grand coalition must be accepted and preferred by i to the losing coalition. ■

It follows from the proof of the above lemma that, in any equilibrium with $y_i > 0$ for all i, a proposer is sure that any proposal he may make of forming a winning coalition will be accepted, thus he will make a proposal that maximizes $\theta_i^S(1 - \sum_{k \in S} y_k)$.

Corollary 2 *Let λ_i^S be the probability that player i proposes coalition S. In an SPE with $y_i > 0$ for all i, the following conditions are satisfied for all i in N*

$$y_i = \theta_i \sum_{S \ni i} \lambda_i^S \left[\theta_i^S (1 - \sum_{k \in S} y_k) + y_i \right] + \sum_{j \in N \setminus \{i\}} \theta_j \sum_{S \supset \{i,j\}} \lambda_j^S \left[\theta_i^S (1 - \sum_{k \in S} y_k) + y_i \right]$$

$$\sum_{S \ni i} \lambda_i^S = 1$$

$$\lambda_i^S > 0 \textit{ implies } S \in \arg\max_{T : T \ni i} \theta_i^T (1 - \sum_{j \in T} y_j)$$

[4] Notice that equilibria in which several responders reject just because S is going to be rejected anyhow are ruled out by the fact that the players in S respond sequentially.

Remark 3 *If $\theta^S = \frac{1}{s}$ for all $S \subset N$, then only minimal winning coalitions form in an SPE.*

3 Constant-Sum Homogeneous Games

A simple game (N, v) is constant-sum if $v(S) + v(N \backslash S) = 1$ for all $S \subseteq N$. It is homogeneous if we can find weights $(w_1, ..., w_n)$ and a quota q such that

$$v(S) = 1 \Leftrightarrow \sum_{i \in S} w_i \geq q$$

$$\sum_{i \in S} w_i = q \text{ if } S \text{ is a minimal winning coalition.}$$

If there we impose $\sum_{i \in N} w_i = 1$ and $w_i = 0$ for any dummy player i, we can find a unique pair (q, w) satisfying the above properties. This pair is called the canonical representation of the game.

Proposition 4 *Let (N, v) be a constant-sum homogeneous game without dummies and (q, w) be its canonical representation. Let $\theta_i = w_i$, and $\theta_i^S = \frac{w_i}{\sum_{j \in S} w_j}$. Then there exists a SSPE of the game $G(v, \theta, (\theta^S)_{S \subseteq N}, \delta)$ in which, for any value of δ,*

(1) Only minimal winning coalitions are proposed.

(2) Expected payoffs are $y_i = w_i$ for all i in N.

(3) Ex post payoff division is $y_i^S = \frac{w_i}{q}$ for all minimal winning coalitions S.

Proof. Because players receive a share $\frac{w_i}{\sum_{j \in S} w_j}$ of the surplus of any coalition, the proposers will propose only minimal winning coalitions. It is easy to see that assuming $y_i = w_i$ for all i leads to $y_i^S = \frac{w_i}{q}$ for all minimal winning coalitions S. Indeed, $y_i + \theta_i^S(1 - \sum_{j \in S} y_j)$ becomes $w_i + \frac{w_i}{q}(1 - q) = w_i$. The question is whether we can construct an equilibrium of the game that generates $y_i = w_i$.

Let σ^* be the equilibrium we are going to construct. There are two conditions σ^* must satisfy: first, σ^* must prescribe an optimal behavior for the players (both as proposers and as responders) given that $y = w$; second,

$y = w$ should be the expected payoff vector corresponding to σ^*. We attend to these aspects in turn.

1. If $y = w$, responders will clearly accept a proposal of forming a minimal winning coalition since it will lead to a payoff of $\frac{w_i}{q} > \delta w_i$. They will accept or reject other proposals depending on whether $y_i^S \geq w_i$.

2. If $y = w$, proposers will always propose a minimal winning coalition and get $\frac{w_i}{q}$. A proposal of a larger coalition may be accepted but would lead to a lower payoff for the proposer; a proposal that would be rejected would lead to a payoff of δw_i.

3. We have fully specified the responders' behavior and part of the proposers' behavior (all minimal winning coalitions to which he belongs are optimal for a proposer; we have not yet chosen an exact probability distribution). Now all we need to do is to choose the mixed strategies in such a way that a player's expected payoff is indeed w.

   Let r_i be the probability that i receives a proposal in equilibrium. If an equilibrium with $y = w$ exists, the following system of equations must have a solution

   $$w_i = [\theta_i + r_i] \frac{w_i}{q} \text{ for all } i \in N.$$

   As in Montero (2001), I use the fact that the set of minimal winning coalitions $\mathbf{W}^m$ is balanced. Let $(\lambda_S)_{S \in \mathbf{W}^m}$ be a collection of balancing weights, and set $\lambda_S = 0$ for all coalitions that are not minimal winning. Let each player in S propose coalition S with probability λ_S; balancedness ensures that $\sum_{S \ni i} \lambda_S = 1$ for all i in N. Then the equation above becomes (taking into account that $\theta_i = w_i$)

   $$w_i = [\theta_i + \sum_{j \neq i} \sum_{S \supseteq \{i,j\}} \theta_j \lambda_S] \frac{w_i}{q} \text{ for all } i \in N$$

Taking into account that $\theta = w$, this becomes

$$w_i = [w_i + \sum_{j \neq i} \sum_{S \supseteq \{i,j\}} w_j \lambda_S] \frac{w_i}{q} = [w_i + \sum_{S \ni i} \sum_{j \in S \setminus \{i\}} w_j \lambda_S] \frac{w_i}{q} =$$

$$[w_i + \sum_{S \ni i} \lambda_S \sum_{j \in S \setminus \{i\}} w_j] \frac{w_i}{q}.$$

Homogeneity implies that $\sum_{j \in S \setminus \{i\}} w_j = q - w_i$ for all minimal winning coalitions (we don't have to worry about other coalitions because they have $\lambda_S = 0$). We thus obtain

$$[w_i + \sum_{S \ni i} \lambda_S (q - w_i)] \frac{w_i}{q} = [w_i + (q - w_i) \sum_{S \ni i} \lambda_S] \frac{w_i}{q} = w_i.$$

■

If there are dummy players, they may be included in a coalition but only if they receive a payoff of zero.

4 The case of apex games

Apex games are a special class of weighted majority games with one major player (the apex player) and $n-1$ minor players (also called base players). If player 1 is the apex player, $v(S) = 1$ if either $\{1\} \subseteq S$ and $S \setminus \{1\} \neq \varnothing$, or $S = N \setminus \{1\}$. There are two types of minimal winning coalitions: the apex player together with one of the minor players, and all the minor players together. Apex games have received a lot of attention in the literature, both in theory (Davis and Maschler (1967), Horowitz (1973), Hart and Kurz (1984), Aumann and Myerson (1988), Bennett and van Damme (1991), Montero (2002)) and in experiments (Selten and Schuster (1968), Horowitz and Rapoport (1974), Albers (1978), Rapoport et al. (1978), Rapoport et al. (1979), Miller (1980), Komorita and Tumonis (1980)). They are also empirically common, especially in parliaments with a small number of parties. Below are two examples.

	Total	*SPD*	*CDU*	*FDP*	*Grüne*
Seats	231	102	88	24	17

The *Landtag* in North Rhine-Westphalia (2000)

Assuming that decisions are taken by simple majority, the SPD can form a winning coalition with any of the other three parties. In Sevilla we also have an apex player (PSOE-A) and three minor players (PP, PA, IULV-CA).

	Total	$PSOE-A$	PP	PA	$IULV-CA$
Seats	33	14	12	4	3

Local elections in Sevilla (2003)

We now analyze in detail the equilibrium in apex games for some protocols. There are three conclusions that emerge from the analysis: First, the internal protocol θ^S is more important than the protocol θ in determining the players' expected payoffs. Second, counterintuitive comparative statics may occur: even though a player benefits from being selected to be the proposer, he may be hurt by bargaining rules that select him more often. Third, coalitions larger than minimal winning are possible under some bargaining rules, unlike in one-stage bargaining models.

4.1 Apex games when the internal game has an egalitarian protocol

Suppose $\theta_i^S = \frac{1}{s}$ for all $S \subseteq N$ and for all $i \in S$. As we have seen, this implies that in the internal game expected payoffs are given by $y_i^S = \frac{1}{s}(1 - \sum_{j \in S} y_j) + y_i$.

As for the probability vector θ, we will assume that all minor players are treated equally, i.e. $\theta_i = \theta_j$ for any minor players i and j, and that $\theta_i > 0$ for all i in N. Then the protocol θ is characterized by one parameter, the probability that the apex player is selected to be the proposer, which can take any value strictly between 0 and 1.

The first thing to notice is that, since the internal protocol is egalitarian, only minimal winning coalitions can form in equilibrium. There are two types of minimal winning coalitions: coalitions consisting of the apex player and one minor player, and the coalition of all minor players. In an SSPE all minor players must have the same expected payoffs (see lemma 5), and thus the apex player must be indifferent between all the minimal winning coalitions containing him.

Lemma 5 *If $\theta_i = \theta_j$ for any minor players i and j, then in any SSPE $y_i = y_j$.*

Proof. Suppose $y_i > y_j$. If i is selected to be the proposer, his payoff will be $y_i + \Delta$ (where $\Delta = \frac{1}{s}[v(S) - \sum_{k \in S} y_k]$ for some coalition S which is optimal for player i). If he is not selected to be the proposer but receives a proposal to be in a coalition, his average payoff will be $y_i + \Delta'$ (where $\Delta' \leq \Delta$). Let r_i the probability that player i receives a proposal from other players (computed before Nature starts the game). Then

$$y_i = \theta_i \left[y_i + \Delta \right] + r_i \left[y_i + \Delta' \right] \tag{2}$$

The payoff of player j as a proposer is at least $y_j + \Delta$ (if $j \in S$, then j can always propose coalition S; if $j \notin S$, then j can always propose coalition $\{S \backslash \{i\}\} \cup \{j\}$). Moreover, since $y_i > y_j$, whenever i receives a proposal j receives it as well. Thus

$$y_j \geq \theta_j \left[y_j + \Delta \right] + r_i \left[y_j + \Delta' \right] + (r_j - r_i)(y_j + \Delta'') \tag{3}$$

where $r_j \geq r_i$ and $\Delta'' \geq 0$. Substracting 3 from 2 and taking into account $\theta_i = \theta_j$ we obtain

$$(1 - \theta_i - r_i)(y_i - y_j) \leq -(r_j - r_i)(y_j + \Delta'')$$

Since the apex player will never propose to player i, $r_i < 1 - \theta_i$ and the left-hand side of the equation is strictly positive. Since the right-hand side is at most zero we have a contradiction. ■

We have shown that in any SSPE only minimal winning coalitions may form and that the apex player must be indifferent between all the coalitions he can propose. As for the minor players, there are three possible cases: they may prefer a coalition with the apex player, the minor player coalition, or they may be indifferent. We will examine each possibility in turn.

Let y_a denote the expected payoff for the apex player and y_m the expected payoff for a minor player. We will also denote the probability that the apex player is selected to be the proposer as θ_a.

Lemma 6 *There is no SSPE in which the apex player is always part of the coalition that forms.*

Proof. If all minor players propose to the apex player, the apex player is always part of a coalition and his expected payoff is given by the equation $y_a = y_a + \frac{1}{2}(1 - y_a - y_m)$. This equation can only hold if $y_a + y_m = 1$, which (since $y_a + (n-1)y_m = 1$) implies $y_a = 1$. But then it would not be optimal for a minor player to propose to the apex player. ■

Proposition 7 *If $\theta_a \geq 1 - \frac{1}{n}$, there is an SSPE in which all minor players propose the minor player coalition.*

Proof. If all minor players propose the minor player coalition, expected equilibrium payoffs are given by the following equations (taking into account that, in order for all minor players to have the same expected payoffs, the apex player must propose to each of them with equal probability)

$$y_a = \theta_a \left[y_a + \frac{1}{2}(1 - y_a - y_m) \right]$$
$$y_m = (1 - \theta_a)\frac{1}{n-1} + \theta_a \frac{1}{n-1} \left[y_m + \frac{1}{2}(1 - y_a - y_m) \right]$$

The solution to this system is $y_a = \frac{\theta_a(n-2)}{n(2-\theta_a)-2}$ and $y_m = \frac{2(1-\theta_a)}{n(2-\theta_a)-2}$. In order for this strategy combination to be an equilibrium we need $y_m + \frac{1}{2}(1 - y_a - y_m) \leq \frac{1}{n-1}$. This is the case if $\theta_a \geq 1 - \frac{1}{n}$. ■

Proposition 8 *If $\theta_a \leq 1 - \frac{1}{n}$, there is an SSPE in which the minor players are indifferent between proposing to the apex player or proposing the minor player coalition.*

Proof. The indifference condition for the minor players, $y_m + \frac{1}{2}(1 - y_a - y_m) = \frac{1}{n-1}$, together with equation $y_a + (n-1)y_m = 1$ implies $y_a = \frac{n-2}{n}$ and $y_m = \frac{2}{n(n-1)}$. In order to construct an equilibrium, we need to find mixed strategies that yield those expected payoffs.

For any minor player i, let λ_i be the probability that i proposes to the apex player and $\lambda := \frac{\sum_{i \in N \setminus \{1\}} \lambda_i}{n-1}$. Expected equilibrium payoffs for the apex player are given by

$$y_a = [\theta_a + (1-\theta_a)\lambda]\left[y_a + \frac{1}{2}(1 - y_a - y_m)\right].$$

Substituting for y_a and y_m, we find $\lambda = 1 - \frac{1}{n(1-\theta_a)}$. In order for λ to be nonnegative we need $\theta_a \leq 1 - \frac{1}{n}$.

There is a continuum of equilibria, all with the same value of λ. Let μ_i be the probability that the apex player proposes to player i. Expected payoffs for player i are given by

$$y_m = \left[(1-\theta_a)(1-\lambda) + \frac{1-\theta_a}{n-1}\lambda_i + \theta_a \mu_i\right]\frac{1}{n-1}$$

Substituting for the equilibrium values of y_a, y_m and λ, we obtain $\mu_i = \frac{1}{n\theta_a} - \frac{\lambda_i(1-\theta_a)}{\theta_a(n-1)}$ (notice that $\sum_{i \in N \setminus \{1\}} \mu_i = 1$). Any collection of λ_i's with average $1 - \frac{1}{n(1-\theta_a)}$ and such that $0 \leq \mu_i \leq 1$ for all i is part of an SSPE. The symmetric equilibrium has $\lambda_i = \lambda$ for any minor player i and $\mu_i = \frac{1}{n-1}$. ■

The conclusion that emerges from the analysis is that if the internal protocol is egalitarian expected equilibrium payoffs will be $y_a = \frac{n-2}{n}$ and $y_m = \frac{2}{n(n-1)}$ provided that $\theta_a \leq 1 - \frac{1}{n}$.

4.2 Apex games when the internal game has a proportional protocol

Weighted majority games have many representations. We will take the canonical representation of apex games with $w_1 = \frac{n-2}{2n-3}$ and $w_i = \frac{1}{2n-3}$ for $i \in N \setminus \{1\}$; this assumption is essential to the results.

Lemma 9 *If $\theta_i = \theta_j$ for any minor players i and j, then in any SSPE $y_i = y_j$.*

Proof. Similar to that of lemma 5. ■

As in the previous subsection, only minimal winning coalitions will form and the apex player will be indifferent between possible partners. As for the minor players, there are three possible cases. Again as in the previous section, the possibility that the apex player is always in the coalition that forms is discarded.

Proposition 10 *There is no SSPE in which the apex player is always part of the coalition that forms. An SSPE in which all minor players propose the minor player coalition requires* $\theta_a \geq \frac{n-1}{2n-3}$*. An SSPE in which the minor players are indifferent between proposing the minor player coalition and proposing to the apex player requires* $\theta_a \leq \frac{n-1}{2n-3}$*. Expected payoffs in this kind of equilibrium are* $y_a = \frac{n-2}{2n-3}$ *and* $y_m = \frac{1}{2n-3}$*.*

Proof. As in the previous subsection. ■

Comparing protocols gives rise to surprising comparative statics. Because $\frac{n-2}{n} > \frac{n-2}{2n-3}$, the apex player prefers bargaining rules to be egalitarian rather than proportional! A second surprise is that in both cases the payoff for the apex player *conditional on being in a coalition* is the same, $\frac{n-1}{n-2}$. What is the intuition for these results?

At the coalition formation stage, being the proposer ensures getting the highest possible payoff one can get from the game; at the payoff division stage, being the proposer is irrelevant in the limit when p tends to 1 and it benefits the player if $p < 1$. Thus, *actually being selected* as a proposer is a good thing. How about having a large *probability* of being selected as a proposer? If strategies were to remain constant after a change in the protocol the effect would be positive, but once we take into account the possible change of equilibrium strategies the effect is ambiguous. Once a coalition has formed, a large θ_i^S is clearly a good thing, since it increases the share of the surplus a player gets. On the other hand, a large θ_i^S may keep other players from proposing S. As for a large θ_i, an increase in θ_i will generally change the equilibrium strategies. Suppose θ_a increases and the minor players continue proposing to the apex player with the same probability. Then y_a will increase to y_a' and it may not be optimal anymore to propose to the apex player, which in turn is not consistent with y_a'. Since strategies must adjust in a new equilibrium, it is not clear if the effect on payoffs will be positive or negative.

Ex post payoffs in this game tend to remain at the "competitive equilibrium" where the apex player receives as much as $n-2$ minor players. Thus, we usually will have an equilibrium with $y_a^S = \frac{n-2}{2n-3}$ for all $S = \{1, i\}$. On the other hand, we know from section **??** that $y_a^S = y_a + \theta_a^S[1 - y_a - y_m]$. Because $y_m = \frac{1-y_a}{n-1}$, an increase in θ_a^S will need a decrease of y_a if y_a^S is to remain constant.

One can also look at comparative statics from the point of view of cooperative game theory. Common cooperative game theory concepts like the excess and the per capita excess have a natural role in the present bargaining game.

Suppose we are looking for a fair division of the value of the grand coalition. Given a possible division y, the difference between what a coalition can get by itself and what is getting at y can be thought of as a measure of the dissatisfaction of the coalition with S, or the temptation to defect from y. This difference is called the *excess* of S at y, denoted by $e(S, y) := v(S) - x(S)$. The *nucleolus* (Schmeidler, 1969) minimizes the largest excess over the set of imputations. Since the decision units are the players and not the coalitions, it makes sense to measure the temptation to defect by the *per capita excess*, $\overline{e}(S, y) := \frac{1}{s}(v(S) - y(S))$, instead of the excess. The *per capita nucleolus* (Wallmeier, 1983) minimizes the largest per capita excess over the set of imputations.

What is the relation between these normative solution concepts and the equilibrium of our extensive form game?

Consider the internal egalitarian protocol. When a player proposes a coalition, he gets $\frac{1}{s}(v(S) - y(S))$, precisely the per capita excess. Thus each proposer will choose the coalition containing him with higher per capita excess. As for the proportional protocol, only minimal winning coalitions will be proposed in equilibrium. Because apex games are homogeneous games and we have chosen a homogeneous representation as a basis for the proportional protocol, a player i gets a fixed share of the excess in all minimal winning coalitions. Thus, even if the proposer does not get the whole excess, he has an incentive to choose the coalition containing him with maximal excess.

We have now argued that in an SSPE proposers will propose coalitions of maximum per capita excess (egalitarian internal protocol) or maximum excess (proportional internal protocol). Does it tell us anything about the expected equilibrium payoffs? When looking for an equilibrium with a given expected payoff vector y, we need to find strategies such that i) all players are playing a best response and ii) the strategies induce y as expected payoff vector. These two requirements are often incompatible: a high expected payoff vector for player i requires i to receive proposals relatively often, but precisely because y_i is high it may not be optimal for the other players to propose coalitions including i. Because strategies in which players strictly prefer one coalition tend to yield extreme expected payoffs that are often inconsistent with the strategies, there is a tendency for the equilibrium to be in mixed strategies. The nucleolus and the per capita nucleolus are then good candidates for an equilibrium payoff vector: they make the proposer indifferent between several coalitions, so that we have more degrees of freedom when constructing equilibrium strategies. The equilibrium payoffs for the egalitarian protocol are larger for the apex player because they correspond to the per capita nucleolus rather than the nucleolus, and the per capita nucleolus treats the apex player better since he is in smaller minimal winning coalitions.

In two-stage games the payoff a player gets in a coalition does not depend on who proposed the coalition. In the equilibria we have seen so far for apex games we can say something stronger: the payoff a player gets when entering a coalition is always the same, i.e., in equilibrium players only enter coalitions that offer them the maximal possible payoff. In other words, the equilibrium payoffs conditional on being in a coalition correspond to an aspiration (see Bennett (1983)). Furthermore, if we limit ourselves to parameters such that the equilibrium is in mixed strategies we see that the apex player gets $\frac{n-2}{n-1}$ and a minor player gets $\frac{1}{n-1}$ if they enter a coalition, regardless of whether the protocol is egalitarian or proportional. We have argued that this is due to the general tendency of equilibria to be "competitive" (i.e., in mixed strategies). This invariance is also shared by the nucleolus and the per capita prenucleolus: the per capita prenucleolus gives the players equal loses from $\frac{w_i}{q}$.

Proposition 11 *Let (N, v) be a constant-sum homogeneous majority game without dummy players and $(q; w_1, ..., w_n)$ a homogeneous representation. Let $E = \frac{\sum_{i \in N} w_i}{q} - 1$ and $e = \frac{E}{n}$. Then the per capita prenucleolus gives $\frac{w_i}{q} - e$ to player i.*

Proof. Suppose we have a game (N, v) satisfying the conditions above and its per capita prenucleolus is some other vector y. Take the smallest player with $y_i < \frac{w_i}{q} - e$. Consider a MWC including this player. If this MWC has a higher per capita excess than e, we are done. Otherwise, this means that on average players in $N \backslash S$ are getting at most $\frac{w_i}{q} - e$. If we now consider $N \backslash S \cup \{i\}$, this is a winning coalition because the game is constant-sum. Either it is a minimal winning coalition or there is a minimal winning coalition included in $N \backslash S \cup \{i\}$. In either case, we have found a coalition with a per capita excess higher than e. This is clear if $N \backslash S \cup \{i\}$ is a MWC, otherwise the fact that i was the smallest player getting less than $\frac{w_i}{q} - e$ plays a role: since the players that are left out of the MWC must have less votes than i, they were getting at least $\frac{w_i}{q} - e$, so the average per capita excess in the MWC is more than e. ■

Giving each player $\frac{w_i}{q} - e$ is not always individually rational. An example of a constant-sum homogeneous game in which the per capita prenucleolus is not individually rational is the game $[9; 5, 4, 4, 1, 1, 1, 1]$.

4.3 Apex games with an internal protocol related to pivotal power

Coalitions larger than minimal winning may arise in this model, though with rather extreme protocols. For illustration, we consider the case in which the probability that the apex player is selected to be the proposer in the internal game is $\frac{s-1}{s}$, where s is the total number of players in the coalition.[5] The payoff the apex player gets as a proposer is

$$[1 - y_a - (s-1)y_m] \frac{s-1}{s} + y_a$$

[5] The number $\frac{s-1}{s}$ is also the coalition structure Shapley value for the apex player.

Taking into account that $y_a = 1 - (n-1)y_m$, we obtain $\frac{y_m(n-s)(s-1)}{s}$. This expression is maximized for $s = \sqrt{n}$. For $n \geq 6$, it is optimal for the apex player to propose coalitions that are not minimal winning.

More generally, we can study the equilibria of the game in which the share of the surplus a player gets in a coalition depends on his pivotal power. For minimal winning coalition the surplus will be divided equally since all players are equally pivotal. As for larger coalitions, we will assume that the fraction of the surplus the apex player gets is a function ϕ of the number of players in the coalition.[6] Suppose we are in equilibrium, and let s be the size of the optimal coalition for the apex player. Then the expected payoff for the apex player is

$$y_a = \theta_a \left[(1 - y_a - (s-1)y_m)\, \phi(s) + y_a \right] + (1-\theta_a)\lambda \left[\frac{1 - y_a - y_m}{2} + y_a \right]$$

It is easy to check that, in any SSPE, all minor players propose the minor player coalition with positive probability. Thus, $\lambda < 1$. We look for two types of equilibria, depending on whether $\lambda = 0$ or $\lambda > 0$.

Set $\lambda = 0$. This is an equilibrium only if solving the equation above we obtain $y_a \geq \frac{n-2}{2}$. This is the case if $\theta_a \geq \frac{(n-1)(n-2)}{2\phi(s)(n-s)+(n-1)(n-2)}$.

A strictly positive λ implies $y_a = \frac{n-2}{2}$ and $y_m = \frac{2}{n(n-1)}$. Substituting for y_a and y_m into the equations indeed yields $\lambda = \frac{(n-1)(n-2)(1-\theta_a)-2\phi(s)\theta_a(n-s)}{n(n-2)(1-\theta_a)}$. In order for this value to be larger than 0, we need $\theta_a \leq \frac{(n-1)(n-2)}{2\phi(s)(n-s)+(n-1)(n-2)}$. Notice that the critical value of θ_a is large (we can find a lower bound for it by setting $s = 0$ and $\phi(s) = 1$) and converges to 1 as n grows. This means that, for most values of the parameter θ_a, expected payoffs coincide with the per capita nucleolus. Thus, allowing for general internal protocols based on pivotal power does not change the qualitative results concerning equilibrium payoffs. The only difference is that coalitions other than minimal winning may form, and that the equilibrium outcomes no longer correspond to an

[6] The function ϕ could be interpreted as a reduced form of a more complicated internal game in which partial breakdown is possible, resulting in the formation of a smaller coalition. The possibility of partial breakdown makes the apex player profit from bargaining with several minor players rather than only one.

aspiration vector: the minor players may accept offers in which they receive less than their maximum possible payoff.

A very important assumption we have made is that coalitions cannot be enlarged once formed. If coalitions can be enlarged and the apex player profits from proposing coalitions larger than minimal winning, the coalition of the apex player with only one minor player is not possible: if formed, the apex player would invite new minor players and they would accept rather than get 0. Then the choice of the minor players is between the minor player coalition and a coalition of the optimal size for the apex player, and a mixed strategy equilibrium need not yield the per capita nucleolus. Interestingly, equilibrium payoffs can be *lower* for the apex player if coalitions can be enlarged. This result is in line with Aumann and Myerson (1988).

5 Concluding remarks

The results of two-stage bargaining for apex games are qualitatively similar to those of one-stage bargaining (see Montero (2002)) in terms of expected payoffs: there is a large region in which expected payoffs are constant, and the minor players randomize between proposing to the apex player and proposing the minor player coalition. With two-stage bargaining the division of payoffs inside a coalition does not depend on who proposed to form the coalition, whereas in one-stage bargaining the proposer always gets more than half of the total payoff. Perhaps surprisingly, the apex player has a larger expected payoff in the two-stage game, despite of the fact that the two-stage game might seem to protect the minor players by splitting the surplus equally. The two-stage model also provides some support for the formation of coalitions larger than minimal winning, something impossible in the one-stage model.

One-stage models with random proposers have provided noncooperative foundations for the nucleolus (see Montero (2001)). One may wonder whether two-stage models can provide noncooperative foundations for the per capita nucleolus. The answer is negative, at least for the present model. A difficulty is that players with zero payoffs in the per capita nucleolus may have a positive payoff in the two-stage game, even if they are never chosen to be proposers in the coalition formation stage.

The noncooperative approach to bargaining and coalition formation has often been criticized because of the sensitiveness of the results to the details of the extensive form game. However, this property may also be seen as a strength: institutions presumably matter, and different extensive form games may reflect different institutional environments. In this sense, the two-stage model with reversible coalitions presented here is a first step towards modelling situations in which player's commitment to coalitions is imperfect.

References

[1] Albers, W. (1978) Block forming tendencies as characteristics of bargaining behavior in different versions of apex games. In: Sauermann, H. (ed.) *Contributions to Experimental Economics, vol VIII: Coalition Forming Behavior.* J.C.B. Mohr, Tübingen

[2] Aumann, R. J., Drèze, J. H. (1974) Cooperative games with coalition structures. *International Journal of Game Theory* 3, 217-237

[3] Aumann, R. J., Myerson, R. (1988) Endogenous formation of links between players and of coalitions: an application of the Shapley value. In: Roth, A. (ed.) *The Shapley value.* Essays in Honor of Lloyd Shapley. Cambridge University Press, Cambridge

[4] Baron, D. P., Ferejohn, J. A. (1989) Bargaining in legislatures. *American Political Science Review* 83, 1181-1206

[5] Bennett, E. (1983) The aspiration approach to predicting coalition formation and payoff distribution in sidepayment games. *International Journal of Game Theory* 12, 1-28

[6] Bennett, E., van Damme, E. (1991) Demand commitment bargaining: the case of apex games. In: Selten, R. (ed.) *Game Equilibrium Models, Vol III: Strategic Bargaining.* Springer-Verlag, Berlin

[7] Binmore, K. (1987) Perfect equilibria in bargaining models. In: Binmore, K., Dasgupta, P. (eds.) *The Economics of Bargaining*. Blackwell, Oxford

[8] Binmore, K., Rubinstein, A., Wolinsky, A. (1986) The Nash bargaining solution in economic modelling. *Rand Journal of Economics* 17, 176-88

[9] Chatterjee, K., Dutta, B., Ray, D., Sengupta, K. (1993) A noncooperative theory of coalitional bargaining. *Review of Economic Studies* 60, 463-77

[10] Cross, J. (1967). Some theoretical characteristics of economic and political coalitions. *Journal of Conflict Resolution* **11**, 184-195.

[11] Davis, M., Maschler, M. (1965) The kernel of a cooperative game. *Naval Research Logistics Quarterly* 12, 223-259

[12] Hart, S., Kurz, M. (1983) Endogenous formation of coalitions. *Econometrica* 51, 1047-1064

[13] Hart, S., Kurz, M. (1984) Stable coalition structures. In: Holler, M. J. (ed.) *Coalitions and Collective Action*. Physica-Verlag, Würzburg

[14] Horowitz, A. D. (1973) The competitive bargaining set for cooperative n-person games. *Journal of Mathematical Psychology* 10, 265-289

[15] Horowitz, A. D., Rapoport, Am. (1974) Test of the kernel and two bargaining set models in four- and five-person games. In: Rapoport, An. (ed.) *Game Theory as a Theory of Conflict Resolution*. D. Reidel, Dordrecht.

[16] Komorita, S. S., Tumonis, T. M. (1980) Extensions and tests of some descriptive theories of coalition formation. *Journal of Personality and Social Psychology* 39, 256-268

[17] Maschler, M. (1992) The bargaining set, kernel, and nucleolus. In: Aumann, R.J., Hart, S. (eds.) *Handbook of game theory with economic applications*, *vol I*. North-Holland, Amsterdam

[18] Miller, C. E. (1980) Effects of payoffs and resources on coalition formation: a test of three theories. *Social Psychology Quarterly* 43, 154-164

[19] Montero, M. (2001) The nucleolus as a consistent power index in noncooperative majority games. CentER for Economic Research Discussion Paper No 2001-39

[20] Montero, M. (2002) Noncooperative bargaining in apex games and the kernel. *Games and Economic Behavior,* 309-321.

[21] Morelli, M. (1999) Demand Competition and Policy Compromise in Legislative Bargaining. *American Political Science Review* **93**, 809-20.

[22] Muthoo, A. (1999) *Bargaining Theory with Applications.* Cambridge University Press, Cambridge

[23] Okada, A. (1996) A noncooperative coalitional bargaining game with random proposers. *Games and Economic Behavior* 16, 97-108

[24] Osborne, M. J., Rubinstein, A. (1990) *Bargaining and Markets.* Academic Press

[25] Rapoport, Am., Kahan, J. P., Funk, S. G., Horowitz, A. D. (1979) *Coalition Formation by Sophisticated Players.* Springer-Verlag, Heidelberg

[26] Rapoport, Am., Kahan, J. P. and T. S. Wallstein. (1978) Sources of power in four-person apex games. In H. Sauermann (ed.) *Contributions to Experimental Economics, vol VIII: Coalition Forming Behavior.* J.C.B. Mohr, Tübingen.

[27] Schmeidler, D. (1969) The nucleolus of a characteristic function game *SIAM Journal on Applied Mathematics* 17, 1163-1170

[28] Selten, R. (1981) A noncooperative model of characteristic function bargaining. In: Böhm, V., Nachtkamp, H. H. *Essays in Game Theory and Mathematical Economics.* Bibliographisches Institut, Mannheim

[29] Selten, R., Schuster, K.G. (1968) Psychological variables and coalition-forming behavior. In: Borch, K., Mossin, J. *Risk and Uncertainty.* Macmillan, London

[30] Slikker, M. (2000) Decision making and cooperation restrictions. Ph.D. Thesis, Tilburg University.

[31] von Neumann, J. and O. Morgenstern (1944). *Theory of Games and Economic Behavior.* Princeton: Princeton University Press.

[32] Wallmeier, E. (1983) Der f-Nucleolus und ein dynamisches Verhandlungsmodell als Lösungskonzepte für kooperatieve n-Personenspiele. Inst. f. Math. Statistik, Münster