

# Risky versus Riskless Bargaining Procedures: The Aumann-Roth Controversy Revisited\*

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## Abstract

In a series of papers, Aumann and Roth discussed a game in which players can cooperate in pairs and two of them prefer to form a coalition with each other. Roth argued that the only rational outcome is that the players who prefer each other form a coalition; Aumann argued that all three coalitions are possible because the players have a problem of expectation coordination. A noncooperative analysis provides additional support for Aumann's arguments and shows that the difference between Aumann's and Roth's views can be traced back to a difference (risky versus riskless) in the bargaining procedure.

Keywords: coalition formation, noncooperative bargaining, NTU games, random proposers.

## 1 The Aumann-Roth Controversy

In a series of papers that appeared in *Econometrica* during the 80s, Aumann and Roth discussed the applicability of the Shapley (1969) NTU value or  $\lambda$ -transfer value. The controversy was centered around the following simple example:

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There are three players,  $N = \{1, 2, 3\}$ . If 1 and 2 form a coalition each of them gets  $\frac{1}{2}$ , whereas if any of them forms a coalition with player 3 the division is  $(p, 1-p)$  with  $0 \leq p < \frac{1}{2}$ . By choosing a pair of players at random, the grand coalition can achieve any convex combination of the payoff vectors  $(\frac{1}{2}, \frac{1}{2}, 0)$ ,  $(p, 0, 1-p)$  and  $(0, p, 1-p)$ .

Roth's (1980) point was that the only outcome consistent with rationality in this situation is coalition  $\{1, 2\}$ , associated with the payoff vector  $(\frac{1}{2}, \frac{1}{2}, 0)$ :

This is because, when  $p < \frac{1}{2}$ , the outcome  $(\frac{1}{2}, \frac{1}{2}, 0)$  is *strictly* preferred by *both* players 1 and 2 to every other feasible outcome, and because the rules of the game permit players 1 and 2 to achieve this outcome without the cooperation of player 3. So (...) there is really no conflict between players 1 and 2.

Solution concepts like the core and the von Neumann and Morgenstern solution make the same prediction as Roth. In contrast,  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  is the unique NTU value for  $p > 0$  (for  $p = 0$ ,  $(\frac{1}{2}, \frac{1}{2}, 0)$  is also an NTU value).

Roth's example can be interpreted in the context of government formation (see Aumann (1986)). Suppose there are three parties in Parliament and the distribution of seats is proportional to  $(\frac{p}{1+p}, \frac{p}{1+p}, \frac{1-p}{1+p})$ . 1 and 2 are small parties ( $p < \frac{1}{2}$ ) but any two parties have a majority ( $2p > \frac{1+p}{2}$ , or  $p > \frac{1}{3}$ ). If the parties in the government must split the payoff proportionally to their number of seats, this game is equivalent to Roth's example, with the additional restriction  $p > \frac{1}{3}$ . Roth's prediction would then be that the largest party is excluded from government. This phenomenon is known as "strength is weakness" and was early recognized by Caplow (1956) and Gamson (1961a) and observed in experiments (see e.g. Vinacke and Arkoff (1957), Gamson (1961b) and Murnighan (1978)).

Aumann (1985) argues that  $(\frac{1}{2}, \frac{1}{2}, 0)$  will not necessarily be the outcome, because players 1 and 2 may accept an offer from player 3 out of security considerations.

Suppose the players and the rules have just been announced on television. The amount 1 to be shared may be fairly large, so the players are rather excited. Suddenly the phone rings in 1's home; 3 is on the line with an offer. At first 1 is tempted to dismiss it. But then he realizes that if he does so, and if 3 manages to get

in touch with 2 before he (1) does, then he won't get anything at all out of the game, unless 2 also rejects 3's offer. "But wait a minute", 1 says now to himself; "2 will only reject 3's offer if he thinks that I will reject it. When he gets 3's phone call, he will go through the agonizing that I am going through now, and will realize that in this situation I would also agonize. (...) I'm beginning not to like this one bit".

Players 1 and 2 have to solve an expectations coordination problem: if 2 would refuse the offer from 3, it is optimal for 1 to refuse it too; if 2 would accept the offer from 3, it may be optimal for player 1 to accept, since there is a risk that he will get nothing otherwise.

Commenting on Roth's paper, Harsanyi (1980) proposes "to define the solutions for cooperative games by means of suitable bargaining models, having the nature of noncooperative games"; this approach is in line with the Nash (1953) program. Aumann (1985) studied two noncooperative models and showed that coalition  $\{1, 2\}$  is not the only outcome consistent with rationality. In the first model a player  $i$  is picked at random to be the proposer. This player chooses another player  $j$  and makes him an offer. If  $j$  rejects,  $i$  makes an offer to  $k$ , but  $k$  does not know of the previous offer to  $j$ . If  $k$  also rejects, coalition  $\{j, k\}$  forms. In the second model, the three pairs of players are ordered at random and given the opportunity to agree; the first pair that does so forms a coalition, and if no pair agrees they all get zero. Players are only informed of proposals involving them.

There are two equilibria of the first model, differing on what happens when player 3 is selected to be the proposer: in one of them both 1 and 2 accept 3's offer and in the other both reject. The second model also admits two equilibria if  $p > \frac{1}{4}$ : one in which  $\{1, 2\}$  always forms and other in which the first selected pair forms.

This paper provides additional support for Aumann's arguments. We make the following points:

1. Aumann (1985, footnote 18) traces the difference between his and Roth's predictions to a difference between perfect and imperfect information: "If we demand that all bargaining takes place in public, and that the extensive form be finite, then all perfect equilibrium points

do lead to  $(\frac{1}{2}, \frac{1}{2}, 0)$ ". He seems to implicitly assume that the opportunity to form each of the coalitions arises exactly once. By dispensing with this assumption we show that, keeping the assumptions of perfect information and a finite extensive form, there is a natural bargaining procedure in the line of Rubinstein (1982) that has two types of equilibria if  $p$  is large enough.

2. We try to select one of the two types of equilibria by using refinements and looking at perturbed games. If  $p$  is large enough, *no* subgame perfect equilibria surviving simple refinements supports Roth's arguments; moreover, if we perturb the game slightly *no* subgame perfect equilibrium supports Roth's arguments.
3. Our bargaining procedure selects proposers randomly in each period. Comparing it with a deterministic procedure of the same family, we show that the difference between Roth's and Aumann's arguments can be traced back to a difference between deterministic and random bargaining procedures. We will argue that the random procedure is more realistic.

## 2 The Bargaining Procedure

### 2.1 Preliminaries

This procedure is based on Rubinstein (1982) and has been studied by Okada (1996) in the context of TU games. It uses a probability vector  $\theta \in \mathbb{R}^3$  with  $\theta_i > 0$  for all  $i$  and  $\sum_{i \in N} \theta_i = 1$ ; we will refer to  $\theta$  as the *protocol*.

Let  $(N, V)$  be the three-person game described in the introduction. Time is discrete,  $t : 1, \dots, T$ . Bargaining proceeds as follows:

A player  $i$  is selected to be the proposer according to  $\theta$ . Player  $i$  chooses another player  $j$ .<sup>1</sup> If  $j$  accepts,  $\{i, j\}$  is formed and the game ends. If  $i$  rejects and  $t < T$ , the game proceeds to the next period and a new proposer is randomly selected according to  $\theta$ .

If no coalition is formed at time  $T$ , all players get zero.

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<sup>1</sup>The game can be extended to allow players to form singletons or the grand coalition without substantially affecting the results.

We do not introduce discounting in the model, but assume that players break ties in favor of an early agreement.

There are two obvious choices for  $\theta$ : the *egalitarian protocol*, with  $\theta_i = \frac{1}{3}$  for  $i = 1, 2, 3$ , and the *proportional protocol*, with  $\theta = (\frac{p}{1+p}, \frac{p}{1+p}, \frac{1-p}{1+p})$  (consistent with the interpretation of the players as parties in parliament).

## 2.2 Subgame perfect equilibria of the game

Given a strategy combination, we will denote player  $i$ 's *continuation value* - that is, his expected payoff at a node where he receives a proposal and rejects it - by  $z_i$ . The strategy combination and node associated to  $z_i$  will usually be clear from the context; sometimes we will want to emphasize the round number and denote the continuation value by  $z_i^t$ . In equilibrium, the continuation values must be consistent with the strategies played and the strategies must be optimal given the continuation values.

We now proceed to solve the game starting at time  $T$ .

**Lemma 1** *At time  $T$ , players 1 and 2 propose coalition  $\{1, 2\}$  and player 3 proposes either  $\{1, 3\}$  or  $\{2, 3\}$ ; all proposals are accepted.*

Since player 3 is indifferent between proposing  $\{1, 3\}$  and  $\{2, 3\}$ , there is a continuum of subgame perfect equilibria at time  $T$ , parametrized by the probability that 3 proposes to 1. We will denote the probability that 3 proposes to 1 at time  $t$  by  $\pi_1^t$ .

What will happen at  $t < T$ ? In a subgame perfect equilibrium, players 1 and 2 will always propose to each other and accept each other's proposals. But will they manage to coordinate on rejecting 3's proposals? If so, we will talk about a *Roth-equilibrium*; if not we will talk about an *Aumann-equilibrium*. We will see that a Roth-equilibrium always exists, and an Aumann-equilibrium exists if  $p$  is large enough.

**Proposition 2** *For  $p < \frac{1-\theta_3}{2}$ , there are only Roth-equilibria. For  $p \geq \frac{1-\theta_3}{2}$ , both types of equilibria exist.*

**Proof.** 1) Let  $p < \frac{1-\theta_3}{2}$ . Since 1 and 2 propose  $\{1, 2\}$  and it is accepted,  $z_i^t \geq \frac{1-\theta_3}{2} > p$  for  $i = 1, 2$  and  $t < T$  and any proposal by 3 will be rejected.

2a) Let  $p \geq \frac{1-\theta_3}{2}$ . Consider the following strategy of player 3 for time  $T$ : 3 proposes to 1 with probability  $\pi_1 \in [0, 1]$  unless 3 has been selected as

a proposer in period  $T - 1$ , in which case he proposes *to the same player he has proposed to at period  $T - 1$* . This strategy is part of a subgame perfect equilibrium at time  $T$ . It has the peculiarity that at time  $T - 1$  the continuation value of 1 and 2 given that they reject a proposal by 3 is  $(1 - \theta_3)\frac{1}{2} + \theta_3 p > p$ , and 3 cannot make acceptable proposals at time  $T - 1$ . In general, suppose at time  $t$  player 3 proposes to player 1 with probability  $\pi_1 \in [0, 1]$  unless he has been selected to be proposer at time  $t - 1$ , in which case he proposes to the same player he chose at time  $t - 1$ . This strategy for player 3 together with 1 and 2 proposing to each other and all players accepting offers that give them at least their continuation value constitutes a Roth-equilibrium.

2b) Let  $p \geq \frac{1-\theta_3}{2}$  and suppose that  $\pi_1^T = 1$ . Then  $z_2^{T-1} = \frac{1-\theta_3}{2}$  and it is a best response for 3 to propose  $\{2, 3\}$  at time  $T - 1$ . Then  $z_1^{T-2} = \frac{1-\theta_3}{2}$  and it is optimal for 3 to propose  $\{1, 3\}$  at time  $T - 2$ ... Iterating this reasoning we conclude that there is a subgame perfect equilibrium in which 3's proposals are always accepted: an Aumann-equilibrium. ■

### 2.3 Equilibrium refinements and perturbations of the game

The tie-breaking rule used by 3 played an important role in the previous analysis. One may use different criteria to choose some tie-breaking rules over others. One criterion is simplicity: strategies that do not depend on what players did in previous periods do not require any memory on the part of the players. Other criterion is based on credible threats: player 3 can solve his indifferences at time  $T$  in any way he wants, thus it seems reasonable to assume that he will choose the rule that, if anticipated, will lead to the best outcome for him.<sup>2</sup>

One may also wonder if Roth-equilibria are robust to small perturbations in the game. Consider the following variant of Roth's example:  $N = \{1, 2, 3\}$ ,  $V(1, 2) = (\frac{1}{2}, \frac{1}{2})$ ,  $V(1, 3) = (p_1, 1 - p_1)$ ,  $V(2, 3) = (p_2, 1 - p_2)$ , with  $\frac{1}{2} > \max(p_1, p_2)$ , and  $p_1 = p_2 - \varepsilon$  with  $\varepsilon > 0$ .<sup>3</sup> We will think of  $\varepsilon$  as being small. Roth's and Aumann's arguments apply to this perturbation of the game: players 1 and 2 strictly prefer coalition  $\{1, 2\}$  to any other

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<sup>2</sup>The idea that players can solve their indifferences strategically is behind the credibility refinement of Bennett and van Damme (1991).

<sup>3</sup>One can also introduce an asymmetric payoff division for coalition  $\{1, 2\}$  (keeping it preferred to any other outcome) without affecting the results.

outcome of the game, but they may want to form a coalition with 3 out of security considerations.

**Proposition 3** (1) If  $\pi_1^t$  is independent of the history of the game, there are only Aumann-equilibria for  $p \geq \frac{1-\theta_3}{2-\theta_3}$ . (2) If players are allowed to make credible threats, there are only Aumann-equilibria for  $p > \frac{1-\theta_3}{2}$ . (3) If we perturb the game, there are only Aumann-equilibria for  $p_1 > \frac{1-\theta_3}{2}$ .

**Proof.** (1) If  $\pi_1^T$  is independent of the history of the game it holds that  $\min(z_1^{T-1}, z_2^{T-1}) \leq (1 - \theta_3)\frac{1}{2} + \theta_3\frac{1}{2}p$ . If  $p \geq (1 - \theta_3)\frac{1}{2} + \theta_3\frac{1}{2}p$ , (that is,  $p \geq \frac{1-\theta_3}{2-\theta_3}$ ) an offer by 3 to the player with the smallest continuation value will be accepted. We can iterate this reasoning and conclude that player 3's proposals are accepted at all periods.

(2) Suppose the players can announce their tie-breaking rules before the game is played. Then player 3 can announce the following rule: he will propose to each player with probability  $\frac{1}{2}$  whenever he is indifferent unless he has been selected to be the proposer at time  $t - 1$ , in which case he proposes to the other player. This tie-breaking rule ensures that 3 can make acceptable proposals at  $T - 1$  since  $p > \frac{1-\theta_3}{2}$ ; iterating the argument we see that 3's proposals are accepted in all periods. 1 and 2 are never indifferent between accepting and rejecting a proposal on the equilibrium path, thus any tie-breaking rules by 1 and 2 play no role.<sup>4</sup>

(3) At time  $T$ , player 3 proposes to player 1. At time  $T - 1$  we have the following continuation values

$$\begin{aligned} z_1^{T-1} &= (1 - \theta_3)\frac{1}{2} + \theta_3 p_1 > p_1 \\ z_2^{T-1} &= (1 - \theta_3)\frac{1}{2} \end{aligned}$$

If player 3 proposes coalition  $\{1, 3\}$ , this proposal will be rejected. If he proposes coalition  $\{1, 2\}$  it will be accepted since  $p_2 > p_1 > (1 - \theta_3)\frac{1}{2}$ . Player 3 will propose  $\{2, 3\}$  if getting  $1 - p_2$  at time  $T - 1$  is preferable to making a proposal that will be rejected; in the latter case, player 3's expected payoff is  $\theta_3(1 - p_1)$ . If  $p_1$  is close to  $p_2$  we have  $1 - p_2 > \theta_3(1 - p_1)$ . We can iterate this reasoning and conclude that 3 will propose to 1 in odd periods and to 2 in even periods. ■

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<sup>4</sup>To be sure, in the non-generic case  $p = \frac{1-\theta_3}{2-\theta_3}$ , players 1 and 2 profit from announcing that they will reject a proposal if indifferent, and the refinement eliminates the Aumann-equilibrium.

## 2.4 Some Alternative Bargaining Procedures

### 2.4.1 The infinite-horizon variant

If we look at subgame perfect equilibria, there is no difference between the finite horizon and the infinite horizon game: there are always subgame perfect equilibria in which 3's proposals are always rejected, but for  $p$  large enough ( $p \geq (1 - \theta_3)\frac{1}{2}$ ) there are also subgame perfect equilibria in which 3's proposals are always accepted. The difference between the finite and infinite horizon game is that the refinements and perturbations considered in the previous section have no bite in the generic case: their effect was based on what happened at time  $T$ . Moreover, the refinement of stationarity of equilibria works in favor of Roth's arguments: only for  $p \geq \frac{1-\theta_3}{2-\theta_3}$  there are stationary perfect equilibria in which 3's proposals are always accepted.

**Proposition 4** *In the infinite horizon game with random proposals a Roth-equilibrium exists for all values of  $p$ . An Aumann-equilibrium exists for  $p \geq \frac{1-\theta_3}{2}$ , and a stationary Aumann-equilibrium exists for  $p \geq \frac{1-\theta_3}{2-\theta_3}$ .*

**Proof.** If 1 and 2 always propose to each other and reject 3's proposals, their continuation value is  $\frac{1}{2} > p$ , thus a Roth-equilibrium exists for all values of  $p$ . An Aumann-equilibrium can be constructed by making 3 alternate between proposing to 1 and 2. A stationary Aumann-equilibrium can be constructed in which 3 proposes to each player with probability  $\frac{1}{2}$ . ■

### 2.4.2 The deterministic variant

This variant was studied by Selten (1981), Chatterjee et al (1993) and Moldovanu and Winter (1995). The crucial difference with the random proposers game is that when a player rejects a proposal he automatically becomes the next proposer.

**Proposition 5** *In any subgame perfect equilibrium of the deterministic bargaining procedure coalition  $\{1, 2\}$  always forms.*

**Proof.** If either player 1 or 2 is selected to be the proposer, he will propose coalition  $\{1, 2\}$  and this proposal will be accepted, since none of the two players can possibly get a higher payoff from the game and players

solve ties in favor of immediate agreement. This implies that each player's continuation value equals  $\frac{1}{2}$ .

Suppose that player 3 is selected to be proposer. If he proposes to, say, player 1, this proposal is rejected because  $p < \frac{1}{2}$ . Player 1 can confidently reject the proposal and then propose to player 2. This reasoning does not depend on whether the game has a finite or infinite horizon. ■

The deterministic bargaining procedure basically assumes away the problem of expectations coordination. If player 1 rejects 3's proposal he does not have to worry that player 3 may get to 2 before him: player 3 is simply not allowed to do so.

### 3 Conclusion

We may say that, staying in the framework of perfect information, the difference between Aumann's and Roth's predictions can be traced to a difference between deterministic and random bargaining procedures. The safe bargaining procedure supports Roth's arguments, whereas the risky bargaining procedure supports Aumann's arguments, especially strongly if the game has a finite horizon. The safe bargaining procedure assumes away the problem of expectation coordination. We believe that this problem is important in practice. The random proposer model captures an important aspect of bargaining in real situations, namely that if players reject proposals from others they cannot be sure to be the first to make a counterproposal, and thus face a risk of being excluded from a coalition. The noncooperative approach is often criticized because the equilibrium is too sensitive to the details of the bargaining procedures. We believe that this difference between the two bargaining procedures is fundamental and should not be regarded as an unimportant procedural detail.

Our paper gives additional support to the view that dominance is not the only criterion to take into account in cooperative game theory. However, it does not lend support to the Shapley NTU value as such.

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