The NTU consistent coalitional value*

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Abstract

We introduce a new value for NTU games with coalition structure. This value coincides with the consistent value for trivial coalition structures, and with the Owen value for TU games with coalition structure. Furthermore, we present two characterizations: the first one using a consistency property and the second one using balanced contributions properties.

1. Introduction

One of the most important issues of cooperative game theory is to define "good" values, studying which interesting properties are satisfied by these values and obtaining axiomatic characterizations using some of these properties.

In cooperative games with transferable utility (TU games), Shapley (1953) introduced the Shapley value. He defines this value as the average of marginal contributions of players when all orders are equally likely. Moreover, he characterizes it as the only value satisfying efficiency, null player, symmetry, and additivity. Later, several authors obtain new characterizations of the Shapley value using other properties. For instance, Myerson (1980) using balanced contributions and Hart and Mas-Colell (1989) consistency.

There are several extensions of TU games. The most natural is to games without transferable utility (NTU games). Other extension is to TU games with

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a coalition structure. This model was introduced by Owen (1977) for studying situations where players are partitioned into several groups. Of course, a third extension is to NTU games with coalition structure. Since the Shapley value has a lot of interesting properties in TU games, many authors decided to propose, in these extended models, values which are generalizations of the Shapley value.

In NTU games the Harsanyi value (Harsanyi (1963)), and the Shapley NTU value (Aumann (1985)), are generalizations of the Shapley value. Later, Maschler and Owen (1989, 1992) define the consistent value for hyperplane games and NTU games respectively. The main idea behind this generalization is to maintain (as far as possible) the consistency property from the Shapley value. Maschler and Owen (1989) prove that, for hyperplane games, the consistent value can be obtained in a similar way that the Shapley value, *i.e.*, as the average of marginal contributions of players when all orders are equally likely. Later, Hart and Mas-Colell (1996) develop a bargaining mechanism which implements the consistent value and characterize it by means of balanced contributions.

Owen (1977) introduces a generalization of the Shapley value, called the Owen value, for TU games with coalition structure. He defines it as the average of marginal contributions of players assuming that: all orders in which players of the same element of the partition are together are equally likely; the rest of orders have probability 0. Moreover, he characterizes his value using similar axioms to those used by Shapley (1953). Later, Winter (1992) characterizes the Owen value using the consistency property and Calvo, Lasaga, and Winter (1996) using properties of balanced contributions.

NTU games with coalition structure are studied by Winter (1991), where he characterizes the Game Coalition Structure Value. This value is a generalization of the Harsanyi value for NTU games and the Owen value for TU games with coalition structure.

It was of our interest to know whether the consistent value and the Owen value could be generalized the same way to games with coalition structure. We know that the Shapley value, the consistent value, and the Owen value are obtained as an average of marginal contributions depending on equal-likely orders. Thus, it seems reasonable to generalize these values in the same way. We call random order coalitional value (Maschler and Owen (1992) also suggest the name random order value for the consistent value) to the value obtained in this way. Remarkably, this value misses most of the nice properties of the previous values (Shapley, Owen, and consistent); namely, it is not consistent, nor satisfies the balanced contributions properties.

Then, we introduce a new value, called the consistent coalitional value. This new value can be characterized in two ways: the first one using the consistency property and the second one using the balanced contributions properties. We must note that our characterizations generalize the results about consistency obtained by Maschler and Owen (1989) for the consistent value and Winter (1992) for the Owen value, and the results about balanced contributions obtained by Hart and Mas-Colell (1996) for the consistent value and Calvo *et al.* (1996) for the Owen value. We believe these characterizations make the consistent coalitional value a proper generalization of the consistent and the Owen value for NTU games with coalition structure.

The paper is organized as follows. In Section 2 we introduce the notation and some previous results. In Section 3 we define the consistent coalitional value and the random order coalitional value. In Section 4 we give a list of properties and study which are satisfied by both values. In Section 5 we present two axiomatic characterizations of the consistent coalitional value. Finally, in the Appendix, we present the proofs of the results obtained in the paper.

2. Definitions and Previous Results

Given a set A, $|A|$ denotes the cardinal of A. If $x, y \in \mathbb{R}^N$ we say $y \leq x$ when $y_i \leq x_i$ for each $i \in N$ and $x * y$ is the scalar product $\sum_{i \in N} x_i y_i$. We denote $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_i \geq 0, \forall i\}$ and $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N : x_i > 0, \forall i\}$. We say that $x \in \mathbb{R}^N$ is *normalized* if $\sum_{i \in N} |x_i| = 1$ (in this case $|x_i| = \max\{x_i, -x_i\}$). Let $\lambda \in \mathbb{R}^N$ be a vector orthogonal to some surface on $\mathbb{R}^N$, we say that λ is *orthonormal* if it is normalized.

A *game without transferable utility*, or simply an *NTU game*, is a pair (N, V) where $N = \{1, 2, ..., n\}$ is the set of players and V is a correspondence (*characteristic function*) which assigns to each coalition $S \subset N$ a subset $V(S) \subset \mathbb{R}^S$ which represents all the possible payoffs that members of S can obtain for themselves when play cooperatively. For $S \subset N$, when there is no ambiguity, we maintain the notation V when refer to the application V restricted to S as player set. We also denote $\overline{S} = N \backslash S$.

We impose the next conditions on the function V:

(A1) For each $S \subset N$, the set $V(S)$ is comprehensive (*i.e.*, if $x \in V(S)$ and $y \in \mathbb{R}^S$ with $y \leq x$, then $y \in V(S)$) and bounded above (*i.e.*, for each $x \in \mathbb{R}^S$, the set $\{y \in V(S) : y \geq x\}$ is compact).

(A2) For each $S \subset N$, the boundary of $V(S)$, which we denote by $\partial V(S)$, is smooth (on each point of the boundary there exists an unique outward orthonormal vector) and nonlevel (the outward vector on each point of $\partial V(S)$ has its coordinates positive). We denote these orthonormal vectors as $\lambda^S = \left(\lambda_i^S\right)_{i \in S}$.

(A3) These λ_i^S are continuous functions on $\partial V(S)$.

(A4) There exists a positive number δ, such that for each $S \subset N$ and $i \in S$, $\lambda_i^S > \delta$.

(A5) For each $S \subset N$, the origin $0_S = (0, ..., 0) \in \mathbb{R}^S$ belongs to $V(S)$.

Property (A5) is a normalization and does not affect our results.

We denote by $NTU(N)$ the set of NTU games over N and by NTU the set of all NTU games.

We now introduce two particular subclasses of NTU games studied in this paper.

We say that (N, V) is a *game with transferable utility* (or *TU game*) if it exists a function $v : 2^N \to \mathbb{R}$, called the characteristic function, satisfying that $v(\emptyset) = 0$ and for each $S \subset N$, $V(S) = \left\{ x \in \mathbb{R}^S : \sum\limits_{i \in S} x_i \leq v(S) \right\}$. Usually we represent a TU game as the pair (N, v). We denote by $TU(N)$ the set of TU games over N and by TU the set of all TU games.

We say that (N, V) is a *hyperplane game* if for all $S \subset N$ there exists $\lambda^S \in \mathbb{R}^S_{++}$ satisfying

$$V(S) = \left\{ x \in \mathbb{R}^S : \lambda^S * x \leq v(S) \right\} \tag{2.1}$$

for some $v : 2^N \to \mathbb{R}$.

Notice that each TU game is a hyperplane game (just take $\lambda_i^S = 1$ for each $S \subset N$ and $i \in S$).

A *coalition structure* C over N is a partition of the player set, *i.e.*, $C = \{C_1, C_2, ..., C_m\} \subset 2^N$ where $\bigcup\limits_{C_q \in C} C_q = N$ and $C_q \cap C_r = \emptyset$ when $q \neq r$. We denote by (N, V, C) an NTU game (N, V) with coalition structure C over N. We denote $CNTU(N)$ as the set of NTU games with coalition structure over N ($CTU(N)$ for TU games) and by $CNTU$ the set of all NTU games with a coalition structure (CTU for TU games).

Given $S \subset N$ we denote by C_S the structure C restricted to the players in S, *i.e.*, $C_S = \{C_q \cap S\}_{C_q \in C}$. Notice that this implies that C_S may have less or the same number of coalitions as C. By simplicity we use C_{-i} instead of $C_{N \setminus \{i\}}$.

A *payoff configuration* for (N,V) is a set of payoffs $x=\left(x^S\right)_{S\subset N}$ with $x^S\in V(S)$ for all $S\subset N$.

Given G a subset of $CNTU$ (or NTU), a *value* Γ on G is a correspondence which assigns to each $(N,V,C)\in G$ a subset $\Gamma(N,V,C)\subset V(N)$. We say that $\left(\Gamma^S\right)_{S\subset N}$ is a payoff configuration associated to Γ if $\Gamma^S\in\Gamma(S,V,C_S)$ for all $S\subset N$. When several NTU games or coalition structures are involved we write $\Gamma^S(V)$, $\Gamma^S(C)$, or $\Gamma^S(V,C)$ instead of Γ^S.

If $\Gamma(N,V,C)$ is a single point of $V(N)$ for all $(N,V,C)\in G$ we say that Γ is a *single value*. Of course each single value has an unique payoff configuration associated. Usually we write Γ^N instead of $\Gamma(N,V,C)$.

We denote by ϕ^N (or $\phi^N(v)$) the *Shapley value* (Shapley (1953)) of the TU game (N,v).

For TU games with coalition structure ϕ^N, or $\phi^N(v,C)$, denotes the *Owen value* (Owen (1977)), which is a generalization of the Shapley value (when $C=\{N\}$ or $C=\{\{1\},...,\{n\}\}$, the Owen value coincides with the Shapley value).

Let us mention two characterizations of the Owen value. Winter (1992) shows that the Owen value is the only value satisfying efficiency, individual symmetry, covariance, consistency, and $GBCP$ (Game Between Coalitions Property). Later, Calvo *et al.* (1996) show that the Owen value is the only value satisfying efficiency, balanced contributions among coalitions, and balanced contributions among players in the same coalition[1].

We say that a single value φ satisfies *balanced contributions among coalitions (BCAC)* if for each $C_q,C_r\in C$ with $q\neq r$,

$$\sum_{j\in C_q}\varphi_j^N-\sum_{j\in C_q}\varphi_j^{N\backslash C_r}=\sum_{j\in C_r}\varphi_j^N-\sum_{j\in C_r}\varphi_j^{N\backslash C_q}.$$

We say that a single value φ satisfies *balanced contributions among players in the same coalition (BCAP)* if for each $i,j\in C_q\in C$ with $i\neq j$,

$$\varphi_i^N-\varphi_i^{N\backslash\{j\}}=\varphi_j^N-\varphi_j^{N\backslash\{i\}}.$$

We now present the consistent value for NTU games following Maschler and Owen (1989,1992).

[1]Even though Calvo et al (1996) present these two balanced properties as only one, we think that for our paper is more intuitive the formulation as two properties.

Let Π be the set of all orders over N. Given $\pi \in \Pi$ we define the set of *predecessors* of i under π as

$$P(\pi, i) = \{j \in N : \pi(j) < \pi(i)\}.$$

We call the *marginal contribution* of player $i \in N$ to the game V in the order π to

$$d_i(\pi) = \max\left\{x_i : \left(\left(d_j(\pi)\right)_{j \in P(\pi,i)}, x_i\right) \in V\left(P(\pi, i) \cup \{i\}\right)\right\}.$$

So, $d_i(\pi)$ is the maximum that player i can obtain in $V(S)$ after his predecessors obtain their respective $d_j(\pi)$'s. We denote $d(\pi) = \left(d_i\left(\pi\right)\right)_{i \in N}$.

It is straightforward to prove that if (N, V) is a hyperplane game,

$$d_i(\pi) = \frac{v(P(\pi, i) \cup \{i\}) - \sum_{j \in P(\pi,i)} \lambda_j^{P(\pi,i)\cup\{i\}} d_j(\pi)}{\lambda_i^{P(\pi,i)\cup\{i\}}}.$$

Given a hyperplane game (N, V), the *consistent value* Ψ^N (or $\Psi^N(V)$), Maschler and Owen (1989), is the vector of expected marginal contributions, where each $\pi \in \Pi$ is equally likely, *i.e.*

$$\Psi^N = \frac{1}{n!} \sum_{\pi \in \Pi} d(\pi).$$

Notice that each $d(\pi)$ is an efficient vector (it belongs to the boundary of $V(N)$). Since we are dealing with hyperplane games, this boundary is flat and the consistent value is also an efficient value.

Maschler and Owen (1989) prove that, given $i \in N$,

$$\Psi_i^N = \frac{1}{|N|\lambda_i^N}\left(\sum_{j \in N\setminus\{i\}} \lambda_i^N \Psi_i^{N\setminus\{j\}} + v\left(N\right) - \sum_{j \in N\setminus\{i\}} \lambda_j^N \Psi_j^{N\setminus\{i\}}\right). \tag{2.2}$$

One way to extend a hyperplane solution to the general class of NTU games with convex $V\left(S\right)$'s is to pass arbitrary hyperplanes to the various sets $V\left(S\right)$. These hyperplanes determine a hyperplane game for which we know the solution. If this solution belongs to $V\left(N\right)$ we say that this is a solution of the NTU game (N, V). This is the way adopted by Maschler and Owen (1992) for extending the consistent value to the class of NTU games.

Formally, given an NTU game (N,V) we say that (N,V') is a supporting hyperplane game for (N,V) if for each $S \subset N$,

$$V'(S) = \left\{x \in \mathbb{R}^S : \lambda^S * x \leq v(S)\right\}$$

where λ^S is orthonormal to the boundary of $V(S)$ and $v(S) = \max\left\{\lambda^S * x : x \in V(S)\right\}$. Notice that $V(S) \subset V'(S)$.

Given an NTU game (N,V) a payoff configuration x is a *consistent value* for (N,V) if there exists a supporting hyperplane game for (N,V) such that $x^S = \Psi^S(V')$ for all $S \subset N$.

It is remarkable that Maschler and Owen (1992) even suggest the name random order value instead of consistent value.

3. The Consistent Coalitional Value

In this section we define two NTU values for NTU games with coalition structure, which generalize the consistent NTU value and the Owen value. The random order coalitional value generalizes the definition of Ψ as the average of marginal contributions. The consistent coalitional value generalizes the expression (2.2) of Ψ.

We first introduce the random order coalitional value for hyperplane games. Let (N,V,C) be an NTU game with coalition structure. We say that an order $\pi \in \Pi$ is *admissible* with respect to C if given $i,j \in C_q \in C$ and $k \in N$ such that $\pi(i) < \pi(k) < \pi(j)$ then $k \in C_q$. We denote by Π^C the set of all orders over N admissible with respect to C.

Given a hyperplane game (N,V,C), the *random order coalitional value* Φ^N (or $\Phi^N(V,C)$) is defined as the expected marginal contributions when all the admissible orders with respect to C are equally likely, *i.e.*

$$\Phi^N = \frac{1}{|\Pi^C|} \sum_{\pi \in \Pi^C} d(\pi).$$

It is trivial to see that if (N,V) is a TU game then Φ coincides with the Owen value. Moreover if $C = \{N\}$ or $C = \{\{1\},...,\{n\}\}$ then Φ coincides with the consistent value.

Notice that Φ is a single value. Then, there is only one payoff configuration $\Phi = \left(\Phi^S\right)_{S \subset N}$ associated to Φ, which satisfies that $\Phi^S = \Phi^S(V, C_S) \in \partial V(S)$ for all $S \subset N$.

We now define the consistent coalitional value for hyperplane games.

Given a hyperplane game (N, V, C), the *consistent coalitional value* Υ^N (or $\Upsilon^N(V, C)$) is the only point satisfying the following two conditions:

For all $C_q \in C$,

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N =$$

$$\frac{1}{|C|}\left[\sum_{C_r \in C \setminus C_q}\left(\sum_{j \in C_q} \lambda_j^N \Upsilon_j^{N \setminus C_r}\right) + v(N) - \sum_{C_r \in C \setminus C_q}\left(\sum_{j \in C_r} \lambda_j^N \Upsilon_j^{N \setminus C_q}\right)\right]. \quad (3.1)$$

For all $i \in C_q \in C$,

$$\Upsilon_i^N =$$

$$\frac{1}{|C_q|\lambda_i^N}\left(\sum_{j \in C_q \setminus \{i\}} \lambda_i^N \Upsilon_i^{N \setminus \{j\}} + \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N - \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \Upsilon_j^{N \setminus \{i\}}\right). \quad (3.2)$$

Remark 1. It is straightforward to prove that Υ is well defined and $\sum_{j \in N} \lambda_j^N \Upsilon_j^N = v(N)$.

Since Υ is a single value, there is only one consistent coalitional payoff configuration $\Upsilon = \left(\Upsilon^S\right)_{S \subset N}$, which satisfies that $\Upsilon^S = \Upsilon^S(V, C_S) \in \partial V(S)$ for all $S \subset N$.

We must admit that the definition of the consistent coalitional value is not so intuitive as the definition of Φ, which is the natural extension to hyperplane games of the expression of the Owen value in terms of expected marginal contributions. Nevertheless, we believe that Υ is a more suitable value for hyperplane games (and NTU games) than Υ. The reason is that, as we will prove later, Υ satisfies more interesting properties. Moreover, Υ can be characterized generalizing axiomatic characterizations of the Owen value and the consistent value.

The generalization of Υ to NTU games is done analogously to the consistent value. For an NTU game with coalition structure (N, V, C), we take for each coalition $S \subset N$ a orthonormal vector λ^S to the boundary of $V(S)$. Let (N, V', C) be the resulting hyperplane game and $\Upsilon = \left(\Upsilon^S\right)_{S \subset N}$ the consistent coalitional payoff configuration associated to (N, V', C). If Υ is feasible in (N, V, C) then we say that Υ is a *consistent coalitional payoff configuration.*

We can extend the random order coalitional Φ to NTU games in a similar way.

It is straightforward to prove that if $C=\{N\}$ or $C=\left\{\{i\}_{i\in N}\right\}$ then $\Upsilon^N=\Psi^N$. Then, the consistent coalitional value is a generalization of the consistent value for NTU with coalition structure. Moreover, for TU games with coalition structure the consistent coalitional value coincides with the Owen value (we will see it later in Corollary 1).

The random order coalitional value also generalizes the consistent NTU value and the Owen value.

We now compute Φ and Υ in the following example:

Example 1. (Owen (1972)). Let (N,V,C) be the hyperplane game such that $N=\{1,2,3\}$ and

$$\begin{aligned}
V(\{i\}) &= \{x_i\in\mathbb{R}^{\{i\}}: x_i\le 0\},\ \forall i\in N,\\
V(\{1,2\}) &= \{(x_1,x_2)\in\mathbb{R}^{\{1,2\}}: x_1+4x_2\le 1, x_1\le 1, x_2\le \frac{1}{4}\},\\
V(\{1,3\}) &= \{(x_1,x_3)\in\mathbb{R}^{\{1,3\}}: x_1\le 0, x_3\le 0\},\\
V(\{2,3\}) &= \{(x_2,x_3)\in\mathbb{R}^{\{2,3\}}: x_2\le 0, x_3\le 0\},
\end{aligned}$$

and

$$V(N)=\{x\in\mathbb{R}^N: \sum_{i\in N}x_i\le 1; x_i\le 1\ \forall i\in N; x_i+x_j\le 1\ \forall i,j\in N\}.$$

If $C=\{\{1,2\},\{3\}\}$, making some computations we obtain that

$$\Phi^N=\left(\tfrac{8}{16},\tfrac{5}{16},\tfrac{3}{16}\right) \text{ and } \Upsilon^N=\left(\tfrac{13}{32},\tfrac{13}{32},\tfrac{6}{32}\right).$$

However, for $C=\{\{1\},\{2,3\}\}$ both values coincide because

$$\Upsilon^N=\Phi^N=\left(\frac{1}{2},\frac{5}{16},\frac{3}{16}\right).$$

In the following lemma we prove that the random order coalitional value also satisfies (3.1).

Lemma 1. Given a hyperplane game (N,V,C), for all $C_q\in C$,

$$\sum_{j\in C_q}\lambda_j^N\Phi_j^N=\frac{1}{|C|}\left[\sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_q}\lambda_j^N\Phi_j^{N\backslash C_r}\right)+v(N)-\sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N\Phi_j^{N\backslash C_q}\right)\right].$$

Proof. See the Appendix.

Since Φ and Υ are different (Example 1) we conclude that Φ does not satisfy (3.2).

In next theorem we prove the existence of consistent coalitional payoff configurations.

Theorem 1: Every NTU game has a consistent coalitional payoff configuration.

Proof. See the Appendix.

Using arguments similar to those used in the proof of Theorem 1 we can conclude that every NTU game has a random order coalitional payoff configuration.

4. Properties

In this section we present several desirable properties and study which of them are satisfied by the consistent coalitional value and the random order coalitional value.

We now define some properties of NTU values. Some of them are well known in the literature of NTU games. Others are introduced in this paper generalizing properties of TU games. We present the definitions for single values. The definition for payoff configurations associated to general values is straightforward.

We say that a value Γ satisfies *efficiency (EF)* if for each $(N, V, C) \in CNTU$, $\Gamma^N \in \partial V(N)$.

Remark 2. Since V satisfies A2 we have that if Γ satisfies efficiency then for each $(N, V, C) \in CNTU$ and $S \subset N$, there exists $\lambda^S \in \mathbb{R}^S_{++}$ satisfying $\lambda^S * \Gamma^S = v(S)$ where $v(S) = \max\left\{\lambda^S * x : x \in V(S)\right\}$. Of course the reciprocal is also true.

Given an $CNTU$ game (N, V, C) we say that two players $i, j \in N$ are *symmetrics* if : For each $S \subset N \backslash \{i, j\}$ if $x \in V(S \cup \{i\})$, $y_j = x_i$, and $y_k = x_k$ for each $k \in S$ then, $y \in V(S \cup \{j\})$. For each $S \supset \{i, j\}$ if $x \in V(S)$, $y_i = x_j$, $y_j = x_i$, and $x_k = y_k$ for each $k \in S \backslash \{i, j\}$ then, $y \in V(S)$.

We say that a value Γ satisfies *individual symmetry (IS)* if for each pair of symmetric players $i, j \in C_q \in C$,

$$\Gamma_i^N = \Gamma_j^N.$$

We now generalize the property of covariance to hyperplane games following Maschler and Owen (1989). Let (N, V, C) and $(N, \widetilde{V}, C)$ be two hyperplane games such that for each $S \subset N$,

$$V(S) = \left\{x \in \mathbb{R}^S : \lambda^S * x \leq v(S)\right\} \text{ and } \widetilde{V}(S) = \left\{x \in \mathbb{R}^S : \widetilde{\lambda}^S * x \leq \widetilde{v}(S)\right\}.$$

We say that (N, V, C) and $(N, \widetilde{V}, C)$ are *equivalent under a linear transformation of player i's utility* if there exist two constants $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that for all $S \subset N$: $\widetilde{\lambda}_i^S = \dfrac{\lambda_i^S}{a}$, $\widetilde{\lambda}_j^S = \lambda_j^S$ if $j \neq i$, $\widetilde{v}(S) = v(S) + \dfrac{b\lambda_i^S}{a}$ if $i \in S$, and $\widetilde{v}(S) = v(S)$ if $i \notin S$. Notice that if (N, V, C) and $\left(N, \widetilde{V}, C\right)$ are equivalent under a linear transformation of player i's utility then, $\widetilde{x} \in V(S)$ if and only if there exists $x \in V(S)$ satisfying: $\widetilde{x}_i = ax_i + b$ and $\widetilde{x}_j = x_j$ if $j \in S \setminus \{i\}$.

We say that a value Γ satisfies *covariance (COV)* if, given two hyperplane games (N, V, C) and $(N, \widetilde{V}, C)$, equivalent under a linear transformation of some player i's utility,

$$\begin{aligned} \Gamma_i\left(N, \widetilde{V}, C\right) &= a\Gamma_i(N, V, C) + b \text{ and} \\ \Gamma_j\left(N, \widetilde{V}, C\right) &= \Gamma_j(N, V, C) \text{ if } j \in N \setminus \{i\}. \end{aligned}$$

Thus, covariance just states that, if we linearly change player i's utility function, his final payoff change the same way, while other players' payoffs remain constant.

Hart and Mas-Colell (1989) characterize the Shapley value as the only value on TU games satisfying consistency and other properties. Later, Winter (1992) extends the definition of consistency to TU games with coalition structure.

Maschler and Owen (1989) show that if we define the property of consistency, of Hart and Mas-Colell (1989) in NTU games as in the TU case, there is no value satisfying consistency and other "basic" properties (for instance, efficiency). Then they provide a weaker definition of consistency for hyperplane games called bilateral consistency.

We now present a generalization of the property of bilateral consistency to hyperplane games with coalition structure. Our bilateral consistency generalizes the bilateral consistency of Maschler and Owen (1989) in the same way that the consistency of Winter (1992) generalizes the consistency of Hart and Mas-Colell (1989).

Given a value Γ, a hyperplane game (N, V, C), and $S \subset C_q \in C$, the *reduced game* $(S, V_S, \{S\})$ is defined, for each $T \subset S$, as follows:

$$V_S(T) = \left\{ x \in \mathbb{R}^T : \left(x, \left(\Gamma_i^{T \cup \overline{S}} \right)_{i \in \overline{S}} \right) \in V\left(T \cup \overline{S}\right) \right\}.$$

It is straightforward to prove that V_S is the hyperplane game given, for each $T \subset S$, by

$$V_S(T) = \left\{ x \in \mathbb{R}^T : \left(\lambda_i^{T \cup \overline{S}} \right)_{i \in T} * x \leq v\left(T \cup \overline{S}\right) - \sum_{i \in \overline{S}} \lambda_i^{T \cup \overline{S}} \Gamma_i^{T \cup \overline{S}} \right\}.$$

We say that a value Γ satisfies *l-consistency* if for each hyperplane game (N, V, C), $C_q \in C$ with $l \leq |C_q|$, and $i \in C_q$,

$$\sum_{S \subset C_q, i \in S, |S| = l} \Gamma_i^S(V_S) = \binom{|C_q| - 1}{l - 1} \Gamma_i^N(V).$$

By simplicity we will take $\Gamma_i^S(V_S) = \Gamma_i^S(V_S, \{S\})$ and $\Gamma_i^N(V) = \Gamma_i^N(V, C)$.
We call *bilateral consistency (BCONS)* to 2-consistency.

Myerson (1980) characterizes the Shapley value using efficiency and balanced contributions (BC). Hart and Mas-Colell (1996) introduce the following generalization of BC for NTU games.

We say that a value Γ satisfies *average balanced contributions (ABC)* if for each NTU game (N, V), $S \subset N$, and $i \in S$, there exists $\lambda^S \in \mathbb{R}_{++}^S$ such that

$$\sum_{j \in S \setminus \{i\}} \lambda_i^S \left(\Gamma_i^S - \Gamma_i^{S \setminus \{j\}} \right) = \sum_{j \in S \setminus \{i\}} \lambda_j^S \left(\Gamma_j^S - \Gamma_j^{S \setminus \{i\}} \right).$$

Later, Calvo *et al.* (1996) generalize the property of balanced contributions for TU games with a coalition structure obtaining two properties: $BCAC$ and $BCAP$.

We now introduce the properties of average balanced contributions among coalitions and average balanced contributions among players in the same coalition for NTU games with coalition structure. Our average balanced properties generalize the balanced properties of Calvo *et al.* (1996) in the same way that the average balanced property of Hart and Mas-Colell (1996) generalizes the balanced property of Myerson (1980).

We say that a value Γ satisfies *average balanced contributions among coalitions (ABCAC)* if for each NTU game (N,V,C), $S\subset N$, and $C'_q = C_q\cap S\in C_S$, there exists $\lambda^S\in\mathbb{R}^S_{++}$ such that

$$\sum_{C'_r\in C_S\backslash C'_q}\left[\sum_{j\in C'_q}\lambda^S_j\left(\Gamma^S_j-\Gamma^{S\backslash C'_r}_j\right)\right]=\sum_{C'_r\in C_S\backslash C'_q}\left[\sum_{j\in C'_r}\lambda^S_j\left(\Gamma^S_j-\Gamma^{S\backslash C'_q}_j\right)\right].$$

We say that a value Γ satisfies *average balanced contributions among players in the same coalition (ABCAP)* if for each NTU game (N,V,C), $S\subset N$, $C'_q = C_q\cap S\in C_S$, and $i\in C'_q$, there exists $\lambda^S\in\mathbb{R}^S_{++}$ such that

$$\sum_{j\in C'_q\backslash\{i\}}\lambda^S_i\left(\Gamma^S_i-\Gamma^{S\backslash\{j\}}_i\right)=\sum_{j\in C'_q\backslash\{i\}}\lambda^S_j\left(\Gamma^S_j-\Gamma^{S\backslash\{i\}}_j\right).$$

Before studying the properties satisfied by the consistent coalitional value we need a previous result.

Lemma 2. Given a hyperplane game (N,V,C) and $i\in S\subset C_q\in C$,

$$(S\backslash\{i\},V_S,\{S\backslash\{i\}\})=\left(S\backslash\{i\},V_{S\backslash\{i\}},\{S\backslash\{i\}\}\right).$$

Proof. This result is due to Maschler and Owen (1989). ■

Notice that Lemma 2 says that if we pass to the reduced game V_S and then remove a player (i) we obtain the same game as if we remove the player first $(N\backslash\{i\})$ and then pass to the reduced game $V_{S\backslash\{i\}}$.

Proposition 1. The consistent coalitional value satisfies l-consistency for each l with $1\le l\le n$.

Proof. See the Appendix.

In next theorem we study which of these properties are satisfied by the consistent coalitional value.

Theorem 2. The consistent coalitional value satisfies EF, IS, COV, $BCONS$, $ABCAC$, and $ABCAP$.

Proof. See the Appendix.

Remark 3. The random order coalitional value satisfies EF, IS, COV, and $ABCAC$.

It is trivial to see that Φ satisfies EF and IS.

Maschler and Owen (1989) show that, for any permutation π, the vector $d(\pi)$ satisfies COV. Since Φ is the mean of some of these $d(\pi)$'s, we conclude that Φ also satisfies COV.

By Lemma 1, Φ satisfies (3.1). Now using arguments similar to those used in the proof of Theorem 2 for Υ we can conclude that Φ also satisfies $ABCAC$.

Later, we will obtain, as a consequence of theorems 3 and 4, that Φ does not satisfy neither $BCONS$ nor $ABCAP$.

By Theorem 2 we know that Υ satisfies, in NTU games or hyperplane games, all the interesting properties that the Owen value satisfies in TU games. Although, by Remark 3, Φ does not.

5. Axiomatic characterizations

In this section we present two axiomatic characterizations of the consistent coalitional value. The first one on the set of hyperplane games using consistency. The second one on the set of NTU games using balanced contributions.

Hart and Mas-Colell (1989) characterize the Shapley value on the class of TU games as the only single value satisfying EF, SYM (if i and j are symmetric players then must receive the same), COV, and $CONS$. Later, Maschler and Owen (1989) and Winter (1992) extend this result in two different ways.

Maschler and Owen (1989) extend this result to the class of hyperplane games. They prove that the consistent value is the only single value satisfying EF, SYM, COV, and $CONS$.

Winter (1992) extends it to the class of TU games with coalition structure. He proves that the Owen value is the only single value satisfying EF, IS, COV, $CONS$, and $GBCP$ (Game Between Coalitions Property).

We say that a single value φ satisfies $GBCP$ if for each TU game (N, v, C) and $C_q \in C$,

$$\sum_{i \in C_q} \varphi_i(N, v, C) = \varphi_{C_q}\left(M, v^{[C]}, \{M\}\right)$$

where $M = \{C_1, ..., C_m\}$ and $v^{[C]}(S) = v\left(\bigcup_{C_r \in S} C_r\right)$ for each $S \subset M$. This property says that the amount received by a coalition in the game played by the coalitions (all coalitions act as a single player) coincides with the sum of the amounts received by the members of this coalition in the original game.

This property can not be exported to hyperplane games.

It is easy to check that the proof of Winter's result about the characterization of the Owen value is also valid if we replace $GBCP$ by $BCAC$. Then, the Owen value is the only single value satisfying EF, IS, COV, $CONS$, and $BCAC$.

In Theorem 3 below we generalize the results of Hart an Mas-Colell (1989), Maschler and Owen (1989), and Winter (1992) to hyperplane games with coalition structure.

Theorem 3: The consistent coalitional value is the only single value on the class of hyperplane games satisfying EF, IS, $COV, BCONS$, and $ABCAC$.

Proof. See the Appendix.

Remark 4. The properties used in this theorem are independent (see the Appendix).

Myerson (1980) characterizes the Shapley value on the class of TU games as the only single value satisfying EF and BC. Later, Calvo *et al.* (1996) and Hart and Mas-Colell (1996) extends this result in two different ways.

Calvo *et al.* (1996) extend it to the class of TU games with coalition structure. They prove that the Owen value is the only single value satisfying EF, $BCAP$, and $BCAC$. Hart and Mas-Colell (1996) extends Myerson's result to the class of NTU games proving that the consistent value is the only value satisfying EF and ABC.

In Theorem 4 below we generalize the results of Myerson (1980), Calvo *et al.* (1996), and Hart and Mas-Colell (1996) to NTU games with coalition structure.

Theorem 4. The consistent coalitional value is the only value on the class of NTU games with coalition structure satisfying EF, $ABCAC$, and $ABCAP$.

Proof. See the Appendix.

Remark 5. The properties used in this theorem are independent (see the Appendix).

We now prove that the consistent coalitional value generalizes the Owen value.

Corollary 1: For each TU game (N, v, C) the Owen value is the only consistent coalitional value.

Proof. See the Appendix.

The results obtained in this section about the consistent coalitional value and the relation with other values can be summarized in the following table.

About consistency			
Without coalition structure		With coalition structure	
TU	Hyperplane	TU	Hyperplane
Shapley	**Consistent**	**Owen**	**Consistent Coalitional**
EF SYM COV $CONS$	EF SYM COV $BCONS$	EF IS COV $CONS$ $BCAC$	EF IS COV $BCONS$ $ABCAC$
About balanced contributions			
Without coalition structure		With coalition structure	
TU	NTU	TU	NTU
Shapley	**Consistent**	**Owen**	**Consistent Coalitional**
EF BC	EF ABC	EF $BCAC$ $BCAP$	EF $ABCAC$ $ABCAP$

Then, the consistent coalitional value is the right generalization of the Owen value and the consistent value to NTU games with coalition structure if we focus in the properties of consistency and balanced contributions of both values.

6. Appendix

Proof of Lemma 1: Let $\Phi = \left(\Phi^S\right)_{S\subset N}$ be the random order coalitional payoff configuration for (N, V, C). By definition, Φ_j^N is the expected marginal contribution of player j over all the $|\Pi^C|$ admissible orders of players with respect to C. We classify these orders in $|C|$ groups according the last coalition C_r in such orders.

Let $\Pi^C(C_r)$ be the set of admissible orders with respect to C in which players of coalition C_r are in the last position. Notice that $|\Pi^C| = |C||\Pi^C(C_r)|$ for each

$C_r \in C$.

If $C_r \neq C_q$, then the expected marginal contribution for each player $j \in C_q$ in the orders of $\Pi^C(C_r)$ coincides with the expected marginal contribution of player j in the game $(N\backslash C_r, V, C\backslash C_r)$, which is $\Phi_j^{N\backslash C_r}$, *i.e.*

$$\frac{1}{|\Pi^C(C_r)|}\sum_{\pi\in\Pi^C(C_r)} d_j(\pi) = \frac{1}{|\Pi^{C\backslash C_r}|}\sum_{\pi\in\Pi^{C\backslash C_r}} d_j(\pi) = \Phi_j^{N\backslash C_r}. \tag{6.1}$$

Moreover, for each $\pi \in \Pi^C(C_q)$,

$$\sum_{j\in C_q}\lambda_j^N d_j(\pi) = v(N) - \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N d_j(\pi)\right).$$

Then, for each $C_q \in C$,

$$\begin{aligned}
&\frac{1}{|\Pi^C(C_q)|}\sum_{\pi\in\Pi^C(C_q)}\left[\sum_{j\in C_q}\lambda_j^N d_j(\pi)\right] \\
= & \frac{1}{|\Pi^C(C_q)|}\sum_{\pi\in\Pi^C(C_q)}\left(v(N) - \sum_{C_r\in C\backslash C_q}\sum_{j\in C_r}\lambda_j^N d_j(\pi)\right) \\
= & \frac{1}{|\Pi^C(C_q)|}\sum_{\pi\in\Pi^C(C_q)} v(N) - \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\frac{1}{|\Pi^C(C_q)|}\sum_{\pi\in\Pi^C(C_q)} d_j(\pi)\right)\right) \\
= & v(N) - \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N\Phi_j^{N\backslash C_q}\right).
\end{aligned} \tag{6.2}$$

We have then:

$$\begin{aligned}
\sum_{j\in C_q}\lambda_j^N\Phi_j^N &= \sum_{j\in C_q}\lambda_j^N\frac{1}{|\Pi^C|}\sum_{\pi\in\Pi^C} d_j(\pi) \\
&= \sum_{j\in C_q}\lambda_j^N\left(\sum_{C_r\in C}\frac{1}{|\Pi^C|}\sum_{\pi\in\Pi^C(C_r)} d_j(\pi)\right)
\end{aligned}$$

since $\left|\Pi^C\right| = |C|\left|\Pi^C(C_r)\right|$, the last expression can be rewritten as

$$\sum_{j\in C_q}\lambda_j^N\frac{1}{|C|}\sum_{C_r\in C}\frac{1}{|\Pi^C(C_r)|}\sum_{\pi\in\Pi^C(C_r)}d_j(\pi)=$$

$$\frac{1}{|C|}\left[\sum_{j\in C_q}\lambda_j^N\sum_{C_r\in C\setminus C_q}\underbrace{\frac{1}{|\Pi^C(C_r)|}\sum_{\pi\in\Pi^C(C_r)}d_j(\pi)}+\underbrace{\frac{1}{|\Pi^C(C_q)|}\sum_{\pi\in\Pi^C(C_q)}\left(\sum_{j\in C_q}\lambda_j^N d_j(\pi)\right)}\right]$$

the terms above brackets are those given in (6.1) and (6.2), so:

$$\begin{aligned}
&= \frac{1}{|C|}\left[\sum_{j\in C_q}\lambda_j^N\sum_{C_r\in C\setminus C_q}\Phi_j^{N\setminus C_r}+v(N)-\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\Phi_j^{N\setminus C_q}\right)\right]\\
&= \frac{1}{|C|}\left[\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\Phi_j^{N\setminus C_r}\right)+v(N)-\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\Phi_j^{N\setminus C_q}\right)\right]
\end{aligned}$$

which is precisely the statement of this lemma. ■

Proof of Theorem 1. The structure of the proof is analogous to the proof of Theorem 3.3 in Maschler and Owen (1992), where they prove the existence of the consistent value for general NTU games.

We make use of induction to prove the following claim:

> *Given $\left(x^T\right)_{T\subsetneq N}$ with $x^T\in\mathbb{R}^T$ such that, for any $S\subsetneq N$, the collection $\left(x^T\right)_{T\subset S}$ is a consistent coalitional payoff configuration of the game (S,V,C_S), there exists $x^N\in\partial V(N)$ such that $\left(x^T\right)_{T\subset N}$ is a consistent coalitional payoff configuration of (N,V,C).*

For $n=1$ the claim is trivially true, being the collection the empty set $\emptyset$.

Assume now the claim holds for less than n players. Thus, there exists a collection $\left(x^T\right)_{T\subsetneq N}$ such that, for any $S\subsetneq N$, $\left(x^T\right)_{T\subset S}$ is a consistent coalitional payoff configuration of the game (S,V,C_S).

Assume that $z\in\partial V(N)$. For each $T\subsetneq N$, let $\lambda^T=\left(\lambda_i^T\right)_{i\in T}$ be the orthonormal vector outwards x^T. Moreover, $\left(\lambda_i^N\right)_{i\in N}$ is the orthonormal vector outwards z.

Let $\left(\Upsilon^S(z)\right)_{S\subset N}$ be the (unique) consistent coalitional payoff configuration for the hyperplane game (N,V^z,C) where for each $S\subset N$,

$$V^z(S)=\left\{y\in\mathbb{R}^S:\lambda^S*y\le v\left(S\right)\right\}.$$

By definition of V^z, $\Upsilon^S(z) = x^S$ for all $S \subsetneq N$, independently of the chosen z.

We want to show that there exists a point $x^N \in \partial V(N)$ such that the collection $\left(x^T\right)_{T\subset N}$ is a consistent coalitional payoff configuration for (N, V, C). Notice that it is enough to prove that $\Upsilon^N\left(x^N\right) = x^N$. We make use of a fixed point theorem. Since Υ satisfies (3.1) and (3.2) and the λ_i^S's are strictly positive and continuous functions, $\Upsilon^N(z)$ is also a continuous function of z.

We define $M = \max\left\{\frac{\left|x_i^T\right|}{\delta} : i \in T \subsetneq N\right\}$, where δ is given by (A4).

Given $C_q \in C$, by (3.1),

$$|C| \sum_{j\in C_q} \lambda_j^N \Upsilon_j^N(z) = \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_q}\lambda_j^N x_j^{N\backslash C_r}\right) + v(N) - \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N x_j^{N\backslash C_q}\right).$$

By (A5), $v(N) \geq 0$, and since the λ_j^N's are positive,

$$\begin{aligned}
&\geq \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_q}\lambda_j^N(-M\delta)\right) - \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N(M\delta)\right)\\
&= -(|C|-1)M\delta\sum_{j\in C_q}\lambda_j^N - M\delta\sum_{j\in N\backslash C_r}\lambda_j^N\\
&\geq -(|C|-1)M\delta - M\delta = -|C|M\delta
\end{aligned}$$

where the last inequality comes because λ^N is normalized.

So, $\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N(z) \geq -M\delta$ for each $C_q \in C$.

Given $i \in C_q \in C$, by (3.2),

$$\begin{aligned}
|C_q|\,\Upsilon_i^N(z) &= \sum_{j\in C_q\backslash\{i\}} x_i^{N\backslash\{j\}} + \frac{\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N(z) - \sum_{j\in C_q\backslash\{i\}}\lambda_j^N x_j^{N\backslash\{i\}}}{\lambda_i^N}\\
&\geq \sum_{j\in C_q\backslash\{i\}}(-M\delta) + \frac{-M\delta - \sum_{j\in C_q\backslash\{i\}}\lambda_j^N M\delta}{\lambda_i^N}\\
&= -(|C_q|-1)M\delta - \frac{M\delta}{\lambda_i^N} - \frac{\sum_{j\in C_q\backslash\{i\}}\lambda_j^N M\delta}{\lambda_i^N}
\end{aligned}$$

since $\lambda_i^N > \delta$, λ^N is normalized, and $\delta < 1$,

$$\begin{aligned}
&> -\left(|C_q| - 1\right) M\delta - M - M \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \\
&> -\left(|C_q| - 1\right) M\delta - M - M \\
&> -\left(|C_q| - 1\right) M - 2M \\
&\geq -2|C_q|M.
\end{aligned}$$

So, $\Upsilon_i^N(z) > -2M$.

The rest of the proof is analogous to Maschler and Owen's (1992) and we just give a geometric description for the case $n = 2$.

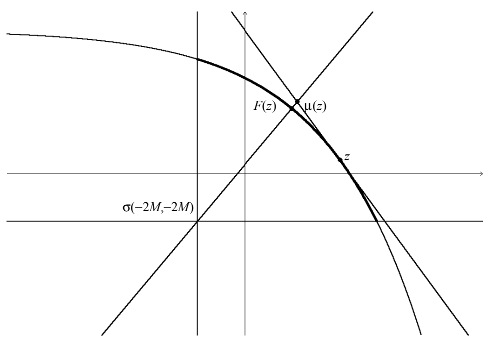

figure 1

We define $D = \left\{x \in \mathbb{R}^N : x_i \geq -2M \text{ for all } i \in N\right\}$. Given a vector z on $\partial V(N) \cap D$ (which is the thick line in figure 1) we have proved that $\Upsilon^N(z) \in D$; and so the point $F(z)$ obtained by applying a projection centered at $\sigma = (-2M, ..., -2M) \in \mathbb{R}^N$, also belongs to $\partial V(N) \cap D$ (see figure 1). By applying a standard fixed point theorem over the (continuous) function F, we find the desired x^N. ■

Proof of Proposition 1. We proceed by induction on l. The theorem is trivially true for $l = 1$. Assume it is true for at most $l - 1$.

If we apply the induction hypothesis to the game $(N \backslash \{j\}, V, C_{-j})$ with $j \in C_q \backslash \{i\}$ (if $C_q = \{i\}$, the result is trivially true for C_q) then,

$$\sum_{T \subset C_q \backslash \{j\} : i \in T, |T| = l-1} \Upsilon_i^T (V_T) = \tbinom{|C_q|-2}{l-2} \Upsilon_i^{N \backslash \{j\}} (V) . \tag{6.3}$$

We wish to prove that for each $C_q \in C$ and $i \in C_q$,

$$l \lambda_i^N \sum_{S \subset C_q : i \in S, |S| = l} \Gamma_i^S (V_S) = l \lambda_i^N \tbinom{|C_q|-1}{l-1} \Gamma_i^N (V) . \tag{6.4}$$

To do so, we analyze the left side of this expression. Assume that $i \in S \subset C_q$ and $|S| = l$. Applying (3.2) to the game $(S, V_S, \{S\})$,

$$l \lambda_i^N \Upsilon_i^S (V_S) = \sum_{j \in S \backslash \{i\}} \lambda_i^N \Upsilon_i^{S \backslash \{j\}} (V_S) + \sum_{j \in S} \lambda_j^N \Upsilon_j^S (V_S) - \sum_{j \in S \backslash \{i\}} \lambda_j^N \Upsilon_j^{S \backslash \{i\}} (V_S) .$$

If we compute Υ in the game V_S we obtain that

$$\sum_{j \in S} \lambda_j^N \Upsilon_j^S (V_S) = v(N) - \sum_{j \in \overline{S}} \lambda_j^N \Upsilon_j^N (V) ,$$

hence,

$$l \lambda_i^N \Upsilon_i^S (V_S) = \sum_{j \in S \backslash \{i\}} \lambda_i^N \Upsilon_i^{S \backslash \{j\}} (V_S) + v(N) - \sum_{j \in \overline{S}} \lambda_j^N \Upsilon_j^N (V) - \sum_{j \in S \backslash \{i\}} \lambda_j^N \Upsilon_j^{S \backslash \{i\}} (V_S) .$$

Since there are $\tbinom{|C_q|-1}{l-1}$ possible sets $S \subset C_q$ with $i \in S$ and $|S| = l$,

$$l \lambda_i^N \sum_{S \subset C_q : i \in S, |S| = l} \Upsilon_i^S (V_S) =$$

$$\sum_{S \subset C_q : i \in S, |S| = l} \left(\sum_{j \in S \backslash \{i\}} \lambda_i^N \Upsilon_i^{S \backslash \{j\}} (V_S) \right) + \binom{|C_q| - 1}{l - 1} v(N)$$
$$- \sum_{S \subset C_q : i \in S, |S| = l} \left(\sum_{j \in \overline{S}} \lambda_j^N \Upsilon_j^N (V) \right) - \sum_{S \subset C_q : i \in S, |S| = l} \left(\sum_{j \in S \backslash \{i\}} \lambda_j^N \Upsilon_j^{S \backslash \{i\}} (V_S) \right) .$$

rearranging the order of summation, we have:

$$\sum_{j\in C_q\setminus\{i\}}\left(\sum_{S\subset C_q:i,j\in S,|S|=l}\lambda_i^N\Upsilon_i^{S\setminus\{j\}}(V_S)\right)+\binom{|C_q|-1}{l-1}v(N)$$
$$-\sum_{j\in N\setminus\{i\}}\left(\sum_{S\subset C_q:i\in S,j\notin S,|S|=l}\lambda_j^N\Upsilon_j^N(V)\right)-\sum_{j\in C_q\setminus\{i\}}\left(\sum_{S\subset C_q:i,j\in S,|S|=l}\lambda_j^N\Upsilon_j^{S\setminus\{i\}}(V_S)\right).$$

We now analyze the four terms separately:

1. First term is equal, by Lemma 2, to

$$\sum_{j\in C_q\setminus\{i\}}\lambda_i^N\left(\sum_{T\subset C_q\setminus\{j\}:i\in T,|T|=l-1}\Upsilon_i^T(V_T)\right)$$

   which coincides, by (6.3), with

$$\binom{|C_q|-2}{l-2}\sum_{j\in C_q\setminus\{i\}}\lambda_i^N\Upsilon_i^{N\setminus\{j\}}(V).$$

2. Since $v(N)=\lambda_i^N\Upsilon_i^N(V)+\sum_{j\in C_q\setminus\{i\}}\lambda_j^N\Upsilon_j^N(V)+\sum_{j\in N\setminus C_q}\lambda_j^N\Upsilon_j^N(V)$, second term is equal to

$$\binom{|C_q|-1}{l-1}\lambda_i^N\Upsilon_i^N(V)+\binom{|C_q|-1}{l-1}\sum_{j\in C_q\setminus\{i\}}\lambda_j^N\Upsilon_j^N(V)$$
$$+\binom{|C_q|-1}{l-1}\sum_{j\in N\setminus C_q}\lambda_j^N\Upsilon_j^N(V).$$

3. Third term is equal to

$$-\sum_{j\in C_q\setminus\{i\}}\left(\sum_{S\subset C_q:i\in S,j\notin S,|S|=l}\lambda_j^N\Upsilon_j^N(V)\right)-\sum_{j\in N\setminus C_q}\left(\sum_{S\subset C_q:i\in S,j\notin S,|S|=l}\lambda_j^N\Upsilon_j^N(V)\right)$$

since: for each $j \in C_q \backslash \{i\}$, there are $\binom{|C_q|-2}{l-1}$ possible sets S such that $S \subset C_q$, $i \in S$, $j \notin S$, and $|S| = l$, and for each $j \in N \backslash C_q$, there are $\binom{|C_q|-1}{l-1}$ possible sets S such that $S \subset C_q$, $i \in S$, $j \notin S$, and $|S| = l$, last expression coincides with

$$-\binom{|C_q|-2}{l-1} \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \Upsilon_j^N (V) - \binom{|C_q|-1}{l-1} \sum_{j \in N \backslash C_q} \lambda_j^N \Upsilon_j^N (V).$$

4. Fourth term is equal, by Lemma 2, to

$$-\sum_{j \in C_q \backslash \{i\}} \lambda_j^N \left(\sum_{T \subset C_q \backslash \{i\} : j \in T, |T| = l-1} \Upsilon_j^T (V_T) \right)$$

which coincides, by (6.3), with

$$-\binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \Upsilon_j^{N \backslash \{i\}} (V).$$

Since $\binom{|C_q|-1}{l-1} = \binom{|C_q|-2}{l-1} + \binom{|C_q|-2}{l-2}$, adding terms 2 and 3 we obtain

$$\binom{|C_q|-1}{l-1} \lambda_i^N \Upsilon_i^N (V) + \binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \Upsilon_j^N (V).$$

Then,

$$l \lambda_i^N \sum_{S \subset C_q : i \in S, |S| = l} \Upsilon_i^S (V_S) =$$

$$\binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_i^N \Upsilon_i^{N \backslash \{j\}} (V) + \binom{|C_q|-1}{l-1} \lambda_i^N \Upsilon_i^N (V)$$
$$+ \binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \left(\Upsilon_j^N (V) - \Upsilon_j^{N \backslash \{i\}} (V) \right)$$

We will prove in Theorem 2, without using this proposition, that Υ satisfies $ABCAP$ and hence,

$$= \binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_i^N \Upsilon_i^{N \backslash \{j\}} (V) + \binom{|C_q|-1}{l-1} \lambda_i^N \Upsilon_i^N (V)$$
$$+ \binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_i^N \left(\Upsilon_i^N (V) - \Upsilon_i^{N \backslash \{j\}} (V) \right)$$

$$= \binom{|C_q|-1}{l-1} \lambda_i^N \Upsilon_i^N(V) + \binom{|C_q|-2}{l-2} \left(\sum_{j \in C_q \setminus \{i\}} \lambda_i^N \Upsilon_i^N(V) \right)$$

Since $\binom{|C_q|-1}{l-1} + \binom{|C_q|-2}{l-2}(|C_q|-1) = l\binom{|C_q|-1}{l-1}$ the last expression coincides with

$$l \binom{|C_q|-1}{l-1} \lambda_i^N \Upsilon_i^N(V)$$

which is precisely the right side of (6.4). ■

Proof of Theorem 2. It is straightforward to prove that Υ satisfies EF and IS. By Proposition 1 we know that Υ satisfies $BCONS$.

We now prove that Υ satisfies $ABCAC$. In order to simplify the notation we assume that $S = N$. By EF, $v(N) = \sum_{C_r \in C} \left(\sum_{j \in C_r} \lambda_j^N \Upsilon_j^N \right)$. Applying this to (3.1) we obtain that for all $C_q \in C$,

$$|C| \sum_{j \in C_q} \lambda_j \Upsilon_j^N =$$

$$= \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \Upsilon_j^{N \setminus C_r} \right) + \sum_{C_r \in C} \left(\sum_{j \in C_r} \lambda_j^N \Upsilon_j^N \right) - \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N \Upsilon_j^{N \setminus C_q} \right)$$

$$= \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \Upsilon_j^{N \setminus C_r} \right) + \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N + \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N \left(\Upsilon_j^N - \Upsilon_j^{N \setminus C_q} \right) \right)$$

If we subtract $\sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N \right) = (|C|-1) \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N$ in both sides,

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N =$$

$$\sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \left(\Upsilon_j^{N \setminus C_r} - \Upsilon_j^N \right) \right) + \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N + \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N \left(\Upsilon_j^N - \Upsilon_j^{N \setminus C_q} \right) \right)$$

Then,

$$0=\sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_q}\lambda_j^N\left(\Upsilon_j^{N\backslash C_r}-\Upsilon_j^N\right)\right)+\sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\Upsilon_j^N-\Upsilon_j^{N\backslash C_q}\right)\right)$$

which means that Υ satisfies $ABCAC$.

We now prove that Υ satisfies $ABCAP$. In order to simplify the notation we assume that $S=N$. Given $i\in C_q\in C$, by (3.2),

$$\begin{aligned}
|C_q|\lambda_i^N\Upsilon_i^N &= \sum_{j\in C_q\backslash\{i\}}\lambda_i^N\Upsilon_i^{N\backslash\{j\}}+\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N-\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\Upsilon_j^{N\backslash\{i\}}\\
&= \sum_{j\in C_q\backslash\{i\}}\lambda_i^N\Upsilon_i^{N\backslash\{j\}}+\lambda_i^N\Upsilon_i^N+\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\left(\Upsilon_j^N-\Upsilon_j^{N\backslash\{i\}}\right)\\
&= \sum_{j\in C_q\backslash\{i\}}\lambda_i^N\left(\Upsilon_i^{N\backslash\{j\}}-\Upsilon_i^N\right)+|C_q|\lambda_i^N\Upsilon_i^N+\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\left(\Upsilon_j^N-\Upsilon_j^{N\backslash\{i\}}\right).
\end{aligned}$$

Then,

$$0=\sum_{j\in C_q\backslash\{i\}}\lambda_i^N\left(\Upsilon_i^{N\backslash\{j\}}-\Upsilon_i^N\right)+\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\left(\Upsilon_j^N-\Upsilon_j^{N\backslash\{i\}}\right)$$

which means that Υ satisfies $ABCAP$.

We now prove that Υ satisfies COV. Given $i\in C_q\in C$, let $\left(N,\widetilde{V},C\right)$ be obtained from (N,V,C) by a change in player i's utility. Let a and b be the corresponding constants. We proceed by induction over the number of coalitions of C.

If C has only one coalition ($C=\{N\}$) then, $\Upsilon_i^N\left(\widetilde{V}\right)=\Psi_i^N\left(\widetilde{V}\right)=a\Psi_i^N(V)+b=a\Upsilon_i^N(V)+b$ and $\Upsilon_j^N\left(\widetilde{V}\right)=\Psi_j^N\left(\widetilde{V}\right)=\Psi_j^N(V)=\Upsilon_j^N(V)$ for each $j\in N\backslash\{i\}$ because Ψ satisfies COV.

Assume the result holds when $|C|$ has at most $m-1$ coalitions. We prove it when $|C|=m$.

By (3.1)

$$|C|\sum_{j\in C_q}\tilde{\lambda}_j^N\Upsilon_j^N\left(\widetilde{V}\right)=$$

$$\sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \tilde{\lambda}_j^N \Upsilon_j^{N \setminus C_r} \left(\widetilde{V} \right) \right) + \tilde{v}(N) - \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_r} \tilde{\lambda}_j^N \Upsilon_j^{N \setminus C_q} \left(\widetilde{V} \right) \right).$$

By induction hypothesis $\Upsilon_i^{N \setminus C_r} \left(\widetilde{V} \right) = a \Upsilon_i^{N \setminus C_r}(V) + b$ when $C_r \neq C_q$ and $\Upsilon_j^{N \setminus C_r} \left(\widetilde{V} \right) = \Upsilon_j^{N \setminus C_r}(V)$ when $j \neq i$. Then,

$$\begin{aligned} |C| \sum_{j \in C_q} \tilde{\lambda}_j^N \Upsilon_j^N \left(\widetilde{V} \right) &= \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q \setminus \{i\}} \lambda_j^N \Upsilon_j^{N \setminus C_r}(V) + \lambda_i^N \Upsilon_i^{N \setminus C_r}(V) + \frac{b\lambda_i^N}{a} \right) \\ &\quad + v(N) + \frac{b\lambda_i^N}{a} - \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N \Upsilon_j^{N \setminus C_q}(V) \right) \\ &= \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \Upsilon_j^{N \setminus C_r}(V) \right) + v(N) \\ &\quad - \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N \Upsilon_j^{N \setminus C_q}(V) \right) + |C| \frac{b\lambda_i^N}{a} \\ &= |C| \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N(V) + |C| \frac{b\lambda_i^N}{a}, \end{aligned}$$

where the last equality comes because Υ satisfies (3.1).

Given $k \in C_q$, by (3.2),

$$|C_q| \tilde{\lambda}_k^N \Upsilon_k^N \left(\widetilde{V} \right) = \sum_{j \in C_q \setminus \{k\}} \tilde{\lambda}_k^N \Upsilon_k^{N \setminus \{j\}} \left(\widetilde{V} \right) + \sum_{j \in C_q} \tilde{\lambda}_j^N \Upsilon_j^N \left(\widetilde{V} \right) - \sum_{j \in C_q \setminus \{k\}} \tilde{\lambda}_j^N \Upsilon_j^{N \setminus \{k\}} \left(\widetilde{V} \right).$$

If $k = i$, by the induction hypothesis and the previous result,

$$\begin{aligned} |C_q| \tilde{\lambda}_i^N \Upsilon_i^N \left(\widetilde{V} \right) &= \sum_{j \in C_q \setminus \{i\}} \left(\lambda_i^N \Upsilon_i^{N \setminus \{j\}}(V) + \frac{b\lambda_i^N}{a} \right) \\ &\quad + \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N(V) + \frac{b\lambda_i^N}{a} - \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \Upsilon_j^{N \setminus \{i\}}(V) \\ &= \sum_{j \in C_q \setminus \{i\}} \lambda_i^N \Upsilon_i^{N \setminus \{j\}}(V) + \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N(V) \end{aligned}$$

$$
\begin{aligned}
& - \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \Upsilon_j^{N \backslash \{i\}} (V) + \frac{b \lambda_i^N}{a} |C_q| \\
= \; & |C_q| \lambda_i^N \Upsilon_i^N (V) + \frac{b \lambda_i^N}{a} |C_q|,
\end{aligned}
$$

where the last equality comes because Υ satisfies (3.2).

Then,

$$\Upsilon_i^N \left(\widetilde{V} \right) = a \Upsilon_i^N (V) + b.$$

If $k \neq i$, by the induction hypothesis and the previous result

$$
\begin{aligned}
|C_q| \lambda_k^N \Upsilon_k^N \left(\widetilde{V} \right) &= \sum_{j \in C_q \backslash \{k\}} \lambda_k^N \Upsilon_k^{N \backslash \{j\}} (V) + \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N (V) + \frac{b \lambda_i^N}{a} \\
& - \sum_{j \in C_q \backslash \{k,i\}} \lambda_j^N \Upsilon_j^{N \backslash \{k\}} (V) - \lambda_i^N \Upsilon_i^{N \backslash \{k\}} (V) - \frac{b \lambda_i^N}{a} \\
&= \sum_{j \in C_q \backslash \{k\}} \lambda_k^N \Upsilon_k^{N \backslash \{j\}} (V) + \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N (V) - \sum_{j \in C_q \backslash \{k\}} \lambda_j^N \Upsilon_j^{N \backslash \{k\}} (V) \\
&= |C_q| \lambda_k^N \Upsilon_k^N (V).
\end{aligned}
$$

Then, $\Upsilon_k^N \left(\widetilde{V} \right) = \Upsilon_k^N (V)$.

Given $C_r \in C \backslash C_q$ using arguments similar to those used for C_q we can conclude that

$$\sum_{j \in C_r} \widetilde{\lambda}_j^N \Upsilon_j^N \left(\widetilde{V} \right) = \sum_{j \in C_r} \lambda_j^N \Upsilon_j^N (V).$$

Now using (3.2) it is easy to conclude that for each $j \in C_r$, $\Upsilon_j^N \left(\widetilde{V} \right) = \Upsilon_j^N (V)$.

Then, Υ satisfies COV. ■

Proof of Theorem 3. In Theorem 2 we proved that the consistent coalitional value satisfies these five properties in the class of hyperplane games.

We now prove the reciprocal. Let $\widetilde{\Upsilon}$ be a single value satisfying these five properties. We will show that $\widetilde{\Upsilon} = \Upsilon$. We proceed by induction on the number of players. If there is only one player then, by EF, $\widetilde{\Upsilon} = \max \{x : x \in V(\{i\})\} = \Upsilon$.

Assume that $|N| = 2$. We can assume without loss of generality that $\lambda_i^{\{i\}} = \lambda_j^{\{j\}} = 1$. There are two possible coalition structure, $C^1 = \{i, j\}$ or $C^2 = \{\{i\}, \{j\}\}$.

Given $a \in \mathbb{R}$, let (N, v^a) be the TU game given by $v^a(\{i\}) = v^a(\{j\}) = a$ and $v^a(N) = 1$.

Since $\widetilde{\Upsilon}$ satisfies EF and IS we conclude that

$$\widetilde{\Upsilon}_i^N\left(v^a, C^1\right) = \widetilde{\Upsilon}_j^N\left(v^a, C^1\right) = \frac{1}{2}.$$

Since $\widetilde{\Upsilon}$ satisfies EF and $ABCAC$ we conclude that

$$\widetilde{\Upsilon}_i^N\left(v^a, C^2\right) = \widetilde{\Upsilon}_j^N\left(v^a, C^2\right) = \frac{1}{2}.$$

A similar result can be obtained for Υ.

As any hyperplane game with two players (N, V, C) can be obtained from v^a (for some a) by linear transformation of utilities of players, and Υ and $\widetilde{\Upsilon}$ satisfy COV it is straightforward to prove that for each $i \in N$,

$$\widetilde{\Upsilon}_i^N = \frac{v(N) + \lambda_i^N v(\{i\}) - \lambda_j^N v(\{j\})}{2\lambda_i^N} = \Upsilon_i^N.$$

Moreover,

$$\lambda_i^N \Upsilon_i^N - \lambda_j^N \Upsilon_j^N = \lambda_i^N \widetilde{\Upsilon}_i^N - \lambda_j^N \widetilde{\Upsilon}_j^N = \lambda_i^N v(\{i\}) - \lambda_j^N v(\{j\}). \tag{6.5}$$

Assume that $\widetilde{\Upsilon} = \Upsilon$ for hyperplane games with at most $n-1$ players with $n \geq 3$. We will prove it when (N, V, C) is a hyperplane game with n players.

We first prove that for each $C_q \in C$,

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N(V) = \sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N(V). \tag{6.6}$$

By induction hypothesis we know that $\widetilde{\Upsilon}^S(V) = \Upsilon^S(V)$ for each $S \subsetneq N$. Given $C_q \in C$, by (3.1),

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N(V) =$$

$$= \frac{1}{|C|}\left[\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\Upsilon_j^{N\setminus C_r}(V)\right)+v(N)-\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\Upsilon_j^{N\setminus C_q}(V)\right)\right]$$

$$= \frac{1}{|C|}\left[\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\widetilde{\Upsilon}_j^{N\setminus C_r}(V)\right)+v(N)-\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\widetilde{\Upsilon}_j^{N\setminus C_q}(V)\right)\right]$$

Since $\widetilde{\Upsilon}$ satisfies EF, $v(N)=\sum_{C_r\in C}\left(\sum_{j\in C_r}\lambda_j^N\widetilde{\Upsilon}_j^N(V)\right)$. Then,

$$\begin{aligned}|C|\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N(V) &= \sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\widetilde{\Upsilon}_j^{N\setminus C_r}(V)\right)+\sum_{j\in C_q}\lambda_j^N\widetilde{\Upsilon}_j^N(V)\\ &\quad+\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\widetilde{\Upsilon}_j^N(V)-\widetilde{\Upsilon}_j^{N\setminus C_q}(V)\right)\right)\end{aligned}$$

We add and subtract $\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\widetilde{\Upsilon}_j^N(V)\right)=(|C|-1)\sum_{j\in C_q}\lambda_j^N\widetilde{\Upsilon}_j^N(V)$,

$$\begin{aligned}&= \sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\left(\widetilde{\Upsilon}_j^{N\setminus C_r}(V)-\widetilde{\Upsilon}_j^N(V)\right)\right)+|C|\sum_{j\in C_q}\lambda_j^N\widetilde{\Upsilon}_j^N(V)\\ &\quad+\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\widetilde{\Upsilon}_j^N(V)-\widetilde{\Upsilon}_j^{N\setminus C_q}(V)\right)\right)\end{aligned}$$

So,

$$|C|\left(\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N(V)-\sum_{j\in C_q}\lambda_j^N\widetilde{\Upsilon}_j^N(V)\right)=$$

$$\begin{aligned}&= \sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\widetilde{\Upsilon}_j^N(V)-\widetilde{\Upsilon}_j^{N\setminus C_q}(V)\right)\right)\\ &\quad-\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\left(\widetilde{\Upsilon}_j^N(V)-\widetilde{\Upsilon}_j^{N\setminus C_r}(V)\right)\right).\end{aligned}$$

Since $\widetilde{\Upsilon}$ satisfies $ABCAC$ we conclude that the last expression is equal to 0. Then,

$$\sum_{j\in C_q} \lambda_j^N \Upsilon_j^N (V) = \sum_{j\in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N (V).$$

We now prove that $\widetilde{\Upsilon}_i^N = \Upsilon_i^N$ for each $i \in C_q \subset N$. We denote by V_S and $\widetilde{V}_S$ the reduced games associated to Υ and $\widetilde{\Upsilon}$ respectively.

If $C_q = \{i\}$, by (6.6) we conclude that $\widetilde{\Upsilon}_i^N = \Upsilon_i^N$.

Assume that $C_q \neq \{i\}$. For each $j \in C_q \setminus \{i\}$ we consider $S = \{i, j\}$. We know that V_S and $\widetilde{V}_S$ are hyperplane games. Then, we denote by v_S and $\tilde{v}_S$ the associated functions to V_S and $\widetilde{V}_S$. By the definition of reduced game and the induction hypothesis,

$$\widetilde{V}_S (\{i\}) = V_S (\{i\}) \text{ and } \widetilde{V}_S (\{j\}) = V_S (\{j\}).$$

Hence, $v_S (\{i\}) = \tilde{v}_S (\{i\})$ and $v_S (\{j\}) = \tilde{v}_S (\{j\})$.

Since $\widetilde{\Upsilon}$ satisfies EF we conclude that $v(N) = \sum_{k\in N} \lambda_k^N \widetilde{\Upsilon}_k^N (V)$. Then,

$$\widetilde{V}_S (S) = \left\{(x_i, x_j) \in \mathbb{R}^{\{i,j\}} : \lambda_i^N x_i + \lambda_j^N x_j \leq \lambda_i^N \widetilde{\Upsilon}_i^N (V) + \lambda_j^N \widetilde{\Upsilon}_j^N (V)\right\}.$$

By the efficiency of $\widetilde{\Upsilon}$ and (6.5),

$$\lambda_i^N \widetilde{\Upsilon}_i^S \left(\tilde{V}_S\right) + \lambda_j^N \widetilde{\Upsilon}_j^S \left(\tilde{V}_S\right) = \lambda_i^N \widetilde{\Upsilon}_i^N (V) + \lambda_j^N \widetilde{\Upsilon}_j^N (V)$$

$$\lambda_i^N \widetilde{\Upsilon}_i^S \left(\tilde{V}_S\right) - \lambda_j^N \widetilde{\Upsilon}_j^S \left(\tilde{V}_S\right) = \lambda_i^N \tilde{v}_S (\{i\}) - \lambda_j^N \tilde{v}_S (\{j\}).$$

If we sum on $C_q \setminus \{i\}$ both expressions

$$\lambda_i^N \sum_{j\in C_q\setminus\{i\}} \widetilde{\Upsilon}_i^S \left(\tilde{V}_S\right) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S \left(\tilde{V}_S\right) = \lambda_i^N (|C_q| - 1) \widetilde{\Upsilon}_i^N (V) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^N (V)$$

$$\lambda_i^N \sum_{j\in C_q\setminus\{i\}} \widetilde{\Upsilon}_i^S \left(\tilde{V}_S\right) - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S \left(\tilde{V}_S\right) = \lambda_i^N (|C_q| - 1) \tilde{v}_S (\{i\}) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \tilde{v}_S (\{j\}).$$

Since $\widetilde{\Upsilon}$ satisfies $BCONS$, $\sum_{j\in C_q\setminus\{i\}} \widetilde{\Upsilon}_i^S (V_S) = (|C_q| - 1) \widetilde{\Upsilon}_i^N (V)$, and hence

$$\lambda_i^N \widetilde{\Upsilon}_i^N (V) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S \left(\tilde{V}_S\right) = \sum_{j\in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N (V)$$

$$(|C_q|-1)\,\lambda_i^N \widetilde{\Upsilon}_i^N(V) - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S\left(\tilde{V}_S\right) = \lambda_i^N(|C_q|-1)\,\tilde{v}_S(\{i\}) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \tilde{v}_S(\{j\}).$$

A similar analysis for Υ yields,

$$\lambda_i^N \Upsilon_i^N(V) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Upsilon_j^S(V_S) = \sum_{j\in C_q} \lambda_j^N \Upsilon_j^N(V)$$

$$(|C_q|-1)\,\lambda_i^N \Upsilon_i^N(V) - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Upsilon_j^S(V_S) = \lambda_i^N(|C_q|-1)\,v_S(\{i\}) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N v_S(\{j\}).$$

By (6.6)

$$\begin{aligned} &\lambda_i^N \widetilde{\Upsilon}_i^N(V) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S\left(\tilde{V}_S\right) \\ &= \lambda_i^N \Upsilon_i^N(V) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Upsilon_j^S(V_S). \end{aligned} \tag{6.7}$$

Since $\tilde{v}_S(\{i\}) = v_S(\{i\})$ and $\tilde{v}_S(\{j\}) = v_S(\{j\})$,

$$\begin{aligned} &(|C_q|-1)\,\lambda_i^N \widetilde{\Upsilon}_i^N(V) - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S\left(\tilde{V}_S\right) \\ &= (|C_q|-1)\,\lambda_i^N \Upsilon_i^N(V) - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Upsilon_j^S(V_S). \end{aligned} \tag{6.8}$$

Adding (6.7) and (6.8),

$$|C_q|\lambda_i^N \Upsilon_i^N(V) = |C_q|\lambda_i^N \widetilde{\Upsilon}_i^N(V)$$

which means that $\widetilde{\Upsilon}_i^N(V) = \Upsilon_i^N(V)$. ■

Proof of Remark 4. $ABCAC$ is independent of the rest of properties because the consistent value satisfies EF, IS, COV, and $BCONS$ but not $ABCAC$.

Using arguments similar to those used by Winter (1992) we can conclude that the rest of properties are independent. ■

Proof of Theorem 4. By Theorem 2 we know that Υ satisfies these properties.

We now prove the reciprocal. We proceed by induction on the number of players. The result is trivially true for $n = 1$. Assume the result holds for each $S \subsetneq N$.

Assume now $\left(\tilde{\Upsilon}^S\right)_{S\subset N}$ is a payoff configuration associated to a value $\tilde{\Upsilon}$ satisfying these properties. Since $\tilde{\Upsilon}$ satisfies EF, by Remark 2, for each $S\subset N$ there exists $\lambda^S\in\mathbb{R}^S_{++}$ satisfying $\lambda^S * \tilde{\Upsilon}^S = v\left(S\right)$ where $v(S)=\max\left\{\lambda^S * x : x\in V(S)\right\}.$ Let (N,V',C) be the corresponding hyperplane game, *i.e.* for each $S\subset N$,

$$V'(S)=\left\{y\in\mathbb{R}^S : \lambda^S * y\leq v(S)\right\}.$$

By induction hypothesis, for each $S\subsetneq N$, $\tilde{\Upsilon}^S=\Upsilon^S\left(V'\right).$ We will show that $\tilde{\Upsilon}^N=\Upsilon^N\left(V'\right).$ By simplicity we take $\Upsilon^N=\Upsilon^N\left(V'\right).$ Assume that $i\in C_q\in C.$

Since $\tilde{\Upsilon}$ satisfies EF and $ABCAC$, using arguments similar to those used in the proof of Theorem 3 we can conclude that for each $C_q\in C$,

$$\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N=\sum_{j\in C_q}\lambda_j^N\tilde{\Upsilon}_j^N.$$

By (3.2),

$$|C_q|\lambda_i^N\Upsilon_i^N=\sum_{j\in C_q\backslash\{i\}}\lambda_i^N\Upsilon_i^{N\backslash\{j\}}+\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N-\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\Upsilon_j^{N\backslash\{i\}}.$$

Since $\sum\limits_{j\in C_q}\lambda_j^N\Upsilon_j^N=\sum\limits_{j\in C_q}\lambda_j^N\tilde{\Upsilon}_j^N$ and the induction hypothesis,

$$\begin{aligned}|C_q|\lambda_i^N\Upsilon_i^N &= \sum_{j\in C_q\backslash\{i\}}\lambda_i^N\tilde{\Upsilon}_i^{N\backslash\{j\}}+\sum_{j\in C_q}\lambda_j^N\tilde{\Upsilon}_j^N-\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\tilde{\Upsilon}_j^{N\backslash\{i\}}\\ &= \sum_{j\in C_q\backslash\{i\}}\lambda_i^N\tilde{\Upsilon}_i^{N\backslash\{j\}}+\lambda_i^N\tilde{\Upsilon}_i^N+\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\left(\tilde{\Upsilon}_j^N-\tilde{\Upsilon}_j^{N\backslash\{i\}}\right)\end{aligned}$$

if we add and substract $\sum\limits_{j\in C_q\backslash\{i\}}\lambda_i^N\tilde{\Upsilon}_i^N=\left(|C_q|-1\right)\lambda_i^N\tilde{\Upsilon}_i^N$ we obtain:

$$=\sum_{j\in C_q\backslash\{i\}}\lambda_i^N\left(\tilde{\Upsilon}_i^{N\backslash\{j\}}-\tilde{\Upsilon}_i^N\right)+|C_q|\lambda_i^N\tilde{\Upsilon}_i^N+\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\left(\tilde{\Upsilon}_j^N-\tilde{\Upsilon}_j^{N\backslash\{i\}}\right).$$

Then,

$$|C_q|\lambda_i^N\left(\Upsilon_i^N-\tilde{\Upsilon}_i^N\right)=\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\left(\tilde{\Upsilon}_j^N-\tilde{\Upsilon}_j^{N\backslash\{i\}}\right)-\sum_{j\in C_q\backslash\{i\}}\lambda_i^N\left(\tilde{\Upsilon}_i^N-\tilde{\Upsilon}_i^{N\backslash\{j\}}\right).$$

Since $\tilde{\Upsilon}$ satisfies $ABCAP$ we conclude that the last expression is equal to 0. Then, $\tilde{\Upsilon}_i^N = \Upsilon_i^N$. ■

Proof of Remark 5. EF is independent of the rest of properties. The value $\Gamma_i^N = 0$ for each NTU game (N, V, C) and $i \in N$ satisfies $ABCAC$ and $ABCAP$ but not EF.

$ABCAP$ is independent of the rest of properties. The random order coalitional value satisfies EF and $ABCAC$ but not $ABCAP$.

$ABCAC$ is independent of the rest of properties.

Given a hyperplane game (N, V, C) we define, for each $i \in N$,

$$\Omega_i^N = \frac{v(N)}{|N| \lambda_i^N}.$$

Let $\pi \in \Pi_q$ be an order of players in C_q. We consider $f(\pi) \in \mathbb{R}^{C_q}$ such that for each $i \in C_q$,

$$f_i(\pi) = \max\left\{x_i : \left(\left(\Omega_j^S\right)_{j \in \overline{C_q}}, \left(f_j(\pi)\right)_{j \in P(\pi,i)}, x_i\right) \in V(S)\right\}$$

where $S = \overline{C_q} \cup P(\pi, i) \cup \{i\}$.

It is straightforward to prove that

$$f_i(\pi) = \frac{v(S) - \sum\limits_{j \in \overline{C_q}} \lambda_j^S \Omega_j^S - \sum\limits_{j \in P(\pi,i)} \lambda_j^S f_j(\pi)}{\lambda_i^S}.$$

Then, given $i \in C_q \in C$, we define Γ as follows:

$$\Gamma_i^N = \frac{1}{|\Pi_q|} \sum_{\pi \in \Pi_q} f_i(\pi).$$

Since Ω satisfies EF, for each $C_q \in C$ and $\pi \in \Pi_q$, $\sum\limits_{j \in C_q} \lambda_j^N \Omega_j^N = \sum\limits_{j \in C_q} \lambda_j^N f_j(\pi)$ and hence, $\sum\limits_{j \in C_q} \lambda_j^N \Omega_j^N = \sum\limits_{j \in C_q} \lambda_j^N \Gamma_j^N$. Then, it is trivial to see that Γ satisfies EF in the class of hyperplane games.

We now prove that Γ satisfies $ABCAP$.

For each $j \in C_q$ we denote by $\Pi_q(j)$ the set of orders of Π_q where j is the last player. If $j \neq i$, then player i's expected marginal contribution conditioned to j being last, is the same as in the game $(N \backslash \{j\}, V, C_{-j})$, which is $\Gamma_i^{N\backslash\{j\}}$, *i.e.*

$$\frac{1}{|\Pi_q(j)|}\sum_{\pi\in\Pi_q(j)} f_i(\pi) = \frac{1}{|\Pi_q^{C_{-j}}|}\sum_{\pi\in\Pi_q^{C_{-j}}} f_i(\pi) = \Gamma_i^{N\backslash\{j\}}.$$

Given $\pi \in \Pi_q(i)$,

$$\begin{aligned} f_i(\pi) &= \frac{v(N) - \sum_{j\in\overline{C_q}} \lambda_j^N \Omega_j^N - \sum_{j\in C_q\backslash\{i\}} \lambda_j^N f_j(\pi)}{\lambda_i^N} \\ &= \frac{\sum_{j\in C_q} \lambda_j^N \Omega_j^N - \sum_{j\in C_q\backslash\{i\}} \lambda_j^N f_j(\pi)}{\lambda_i^N} \\ &= \frac{\sum_{j\in C_q} \lambda_j^N \Gamma_j^N - \sum_{j\in C_q\backslash\{i\}} \lambda_j^N f_j(\pi)}{\lambda_i^N}. \end{aligned}$$

Then,

$$\begin{aligned} \frac{1}{|\Pi_q(i)|}\sum_{\pi\in\Pi_q(i)} f_i(\pi) &= \frac{\sum_{j\in C_q} \lambda_j^N \Gamma_j^N}{\lambda_i^N} - \frac{1}{|\Pi_q(i)|}\sum_{\pi\in\Pi_q(i)} \frac{\sum_{j\in C_q\backslash\{i\}} \lambda_j^N f_j(\pi)}{\lambda_i^N} \\ &= \frac{\sum_{j\in C_q} \lambda_j^N \Gamma_j^N}{\lambda_i^N} - \frac{1}{\lambda_i^N}\sum_{j\in C_q\backslash\{i\}} \lambda_j^N \frac{1}{|\Pi_q(i)|}\sum_{\pi\in\Pi_q(i)} f_j(\pi) \\ &= \frac{\sum_{j\in C_q} \lambda_j^N \Gamma_j^N}{\lambda_i^N} - \frac{1}{\lambda_i^N}\sum_{j\in C_q\backslash\{i\}} \lambda_j^N \Gamma_j^{N\backslash\{i\}} \\ &= \frac{\sum_{j\in C_q} \lambda_j^N \Gamma_j^N - \sum_{j\in C_q\backslash\{i\}} \lambda_j^N \Gamma_j^{N\backslash\{i\}}}{\lambda_i^N} \end{aligned}$$

Thus, for each $i \in C_q \in C$,

$$\Gamma_i^N = \frac{1}{|\Pi_q|}\sum_{\pi\in\Pi_q} f_i(\pi)$$

$$= \frac{1}{|\Pi_q|}\sum_{j\in C_q\backslash\{i\}}\sum_{\pi\in\Pi_q(j)} f_i(\pi)+\frac{1}{|\Pi_q|}\sum_{\pi\in\Pi_q(i)} f_i(\pi)$$

since $|\Pi_q| = |C_q|\,|\Pi_q\,(j)|$ for each $j\in C_q$, the last expression can be rewritten as

$$\frac{1}{|C_q|}\left[\sum_{j\in C_q\backslash\{i\}}\frac{1}{|\Pi_q\,(j)\,|}\sum_{\pi\in\Pi_q(j)} f_i(\pi)+\frac{1}{|\Pi_q\,(i)\,|}\sum_{\pi\in\Pi_q(i)} f_i(\pi)\right]$$

$$= \frac{1}{|C_q|}\left[\sum_{j\in C_q\backslash\{i\}}\Gamma_i^{N\backslash\{j\}}+\frac{\sum\limits_{j\in C_q}\lambda_j^N\Gamma_j^N-\sum\limits_{j\in C_q\backslash\{i\}}\lambda_j^N\Gamma_j^{N\backslash\{i\}}}{\lambda_i^N}\right].$$

Then,

$$|C_q|\lambda_i^N\Gamma_i^N=\sum_{j\in C_q\backslash\{i\}}\lambda_i^N\Gamma_i^{N\backslash\{j\}}+\sum_{j\in C_q}\lambda_j^N\Gamma_j^N-\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\Gamma_j^{N\backslash\{i\}}.$$

Since $|C_q|\lambda_i^N\Gamma_i^N=\sum\limits_{j\in C_q}\lambda_i^N\Gamma_i^N$ we conclude that Γ satisfies $ABCAP$.

If we proceed with Γ in the same way that we did with Υ we can extend Γ to the set of NTU games and prove that Γ also satisfies EF and $ABCAP$ in the class of NTU games. ■

Proof of Corollary 1: Since each TU game is a hyperplane game we conclude that the consistent coalitional value is a single value. Repeating the same arguments that in the proof of Theorem 4 for TU games we can obtain that there is at most a value (on the set of TU games) satisfying EF, $ABCAC$, and $ABCAP$. Then, we only need to prove that the Owen value ϕ satisfies these properties.

We know that ϕ satisfies EF. We now prove that ϕ satisfies $ABCAC$ and $ABCAP$. By simplicity we assume that $S = N$.

Since ϕ satisfies $BCAC$, for each $C_q, C_r \in C$

$$\sum_{j\in C_q}\left(\phi_j^N-\phi_j^{N\backslash C_r}\right)=\sum_{j\in C_r}\left(\phi_j^N-\phi_j^{N\backslash C_q}\right).$$

Then,

$$\sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_q}\left(\phi_j^N-\phi_j^{N\backslash C_r}\right)\right)=\sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\left(\phi_j^N-\phi_j^{N\backslash C_q}\right)\right)$$

which means that ϕ satisfies $ABCAC$ in TU games.

Since ϕ satisfies $BCAP$, for each $C_q \in C$ and $i, j \in C_q$

$$\phi_i^N - \phi_i^{N\setminus\{j\}} = \phi_j^N - \phi_j^{N\setminus\{i\}}.$$

Then,

$$\sum_{j\in C_q\setminus\{i\}} \left(\phi_i^N - \phi_i^{N\setminus\{j\}}\right) = \sum_{j\in C_q\setminus\{i\}} \left(\phi_j^N - \phi_j^{N\setminus\{i\}}\right)$$

which means that ϕ satisfies $ABCAP$ in TU games. ■

7. References

Aumann R (1985) *An axiomatization of the non-transferable utility value.* Econometrica 53**:** 599-612.

Calvo, Lasaga and Winter (1996). *The Principle of Balanced Contributions and Hierarchies of Cooperation.* Mathematical Social Sciences. No. 31, 171-182.

Harsanyi J (1963) A simplified bargaining model for the n-person cooperative game. International Economic Review 4**:** 194-220.

Hart, S. and Mas-Colell, A. (1989) *Potential, value, and consistency.* Econometrica 57: 589-614.

Hart, S. and Mas-Colell, A. (1996). *Bargaining and Value.* Econometrica. Vol. 64. No. 2, 357-380.

Maschler and Owen (1989). *The consistent Shapley Value for Hyperplane Games.* International Journal of Game Theory, 18, 389-407.

Maschler and Owen (1992). *The Consistent Shapley Value for Games without Side Payments.* Rational Interaction. Ed. by R. Selten. New York. Springer-Verlag, 5-12.

Myerson R.B. (1980) *Conference structures and fair allocation rules.* International Journal of Game Theory 9: 169-182.

Owen G (1972) A value for non-transferable utility games. International Journal of Game Theory 1: 95-109.

Owen G. (1977) *Values of games with a priori unions.* In: Henn R., Moeschlin O. (eds) Essays in Mathematical Economics and Game Theory, Springer-Verlag, Berlin Heidelberg New York: 76-88.

Shapley (1953). *A Value for n–Person Games.* Contributions to the Theory of Games II. Annals of Mathematics Studies, 28. Ed. by H. W. Huhn and A. W. Tucker. Princeton. Princeton University Press, 307-317.
Winter E. (1991) *On Non-Transferable Utility Games with Coalition Structure.* International Journal or Game Theory 20: 53-63.
Winter, E. (1992) *The consistency and potential for values of games with coalition structure.* Games and Economic Behavior 4: 132-144.