Implementation of the Levels Structure Value[*]

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Abstract

We implement the levels structure value (Winter, 1989) for cooperative transfer utility games with a levels structure. The mechanism is a generalization of the bidding mechanism by Pérez-Castrillo and Wettstein (2001).

Keywords: Levels structure value, implementation, TU games.

1 Introduction

A cooperative game describes a conflictive situation among a finite number of agents or players. Even though players are assumed to have independent interests, they can benefit from cooperation. When this cooperation is carried out, the question is how the benefit shall be distributed among the players.

This problem has been studied from different approaches. The aim is to define a *solution concept* which gives a "fair" (or at least "reasonable") allocation for each problem. This allotment must take into account the contribution of each player to the game.

Within cooperative games, transfer utility (TU) games have been deeply studied. In TU games, utility is freely transferable among members of a coalition. A widely studied solution concept for TU games is the Shapley value (presented by Shapley in 1953).

Once a solution concept has been established, the implementation for this solution aims to state a mechanism (or non-cooperative game) such that players, by behaving strategically, get as final outcome the one proposed by the solution concept.

In this context, we say that a mechanism implements the Shapley value (or any other) if two properties are satisfied. First, there must be some kind of equilibrium such that their final payoff is the Shapley value. Second, every equilibrium must have as final payoff the Shapley value. The first property is needed

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since, even if it is proved that the Shapley value arises in each equilibrium, it may occur that the non-cooperative game has no equilibria.

Implementation for the Shapley value in TU games has been studied by several authors. For example, Gul (1989), Hart and Moore (1990), Winter (1994), Dasgupta and Chiu (1998), Hart and Mas-Colell (1996) or Evans (1996). Recently, Pérez-Castrillo and Wettstein (2001), present a mechanism (the *bidding mechanism*) which has remarkable features.

In the bidding mechanism, one of the players (the *proposer*) should propose an allocation. If all the other players agree, this is the final payoff. If at least one of the other players does not accept the proposed allocation, the proposer leaves the game and the mechanism is repeated with the rest of the players.

A key feature in the bidding mechanism is the way the proposer is chosen. Since the final payoff depends on the identity of the proposer, in a first stage the players should bid for the right to be the proposer. The player who presents the highest net bid is chosen as proposer.

Pérez-Castrillo and Wettstein show that in equilibrium, all players have the same probability to be chosen as proposer. Furthermore, if the game is zero-monotonic, the equilibrium payoff is the Shapley value.

In equilibrium, the bidding mechanism may finish in one round (when no player drops out) or in more than one. However, the latter only happens when the game is not strictly zero-monotonic.

Frequently, we have more available information than those given by the characteristic function of the game. For example, let us consider the members of the European Union Parliament. Even though all of them have the same rights, they do not act independently, since they belong to different political parties. Furthermore, political parties are not completely independent from each other. On a higher level, parties of similar ideology may be formally associated, such like the Social-democratic or the Socialist Parties are, and so on.

We call this cooperation description of the players a *levels structure*. Solution concepts which take into account levels structures are the Owen value (presented by Owen in 1977) for a single level, and the levels structure value (suggested by Owen in 1977 and studied by Winter in 1989). The levels structure value is a generalization of the Owen value for more than one level. Furthermore, the Owen value is a generalization of the Shapley value.

In Vidal-Puga and Bergantiños (2001), the bidding mechanism by Pérez-Castrillo and Wettstein is generalized so that a single-level structure is taken into account. The resulting non-cooperative game implements the Owen value.

In this article, we move a step ahead. We modify the bidding mechanism so that a general levels structure is considered. To do so, we generalize the bidding mechanism to a new mechanism, called the *levels bidding mechanism*.

Given a levels structure with h levels, the levels bidding mechanism has h rounds. In Round 1, the members of the same coalition at this level play the bidding mechanism, trying to obtain the resources of the whole coalition. Eventually, we can find a player (called the *representative*) out of each coalition, who obtains the resources of his own coalition, or of a subcoalition of it if one or more players are removed. In the second round, the representatives who are in the same coalition at the second level repeat the process taking into account the resources obtained in the previous round. The process goes on until reaching the level h.

In Section 2 we present the notation and definitions. In Section 3 we define formally the coalitional bidding mechanism and prove that it implements the levels structure value.

2 The model

We consider a cooperative game in characteristic form (N, v), where $N = \{1, ..., n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is a *characteristic function* satisfying $v(\emptyset) = 0$. We denote by $TU(N)$ the set of cooperative games.

A *coalition* of (N, v) is a nonempty subset $S \subset N$. We say that $S \subset N$ *supports* v if $v(T) = v(S)$ for any $T \supset S$.

We say that (N, v) is *zero-monotonic* if $v(S \cup \{i\}) \geq v(S) + v(\{i\})$ for every $S \subset N \setminus \{i\}$.

We say that v is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for every $S, T \in N$ such that $S \cap T = \emptyset$.

Notice that superadditivity implies zero-monotonicity.

A *coalition structure* on N is a partition $\mathcal{C} = \{C_1, ..., C_m\}$ of N, i.e. $C_q \cap C_r = \emptyset$ when $C_q \neq C_r$ and $\bigcup_{C_q \in \mathcal{C}} C_q = N$.

Given $i \in C_q \in \mathcal{C}$, we denote by $\mathcal{C}_{-i}$ the coalition structure on $N \setminus \{i\}$ which equals $\mathcal{C}$ after removing player i, i.e. $\mathcal{C}_{-i} = \{C_1, ..., C_{q-1}, C_q \setminus \{i\}, C_{q+1} ..., C_m\}$.

Notice that this means that $\mathcal{C}_{-i}$ may have one less coalition than $\mathcal{C}$.

Given v characteristic function on N, and $S \subset N$, we define $(S, v_S) \in TU(S)$ as the game v restricted to the player set S, i.e. $v_S(T) = v(T)$ for all $T \subset S$.

In particular, we denote $v_{-i} = v_{N \setminus \{i\}}$ and $v_{-S} = v_{N \setminus S.}$

A *levels structure* on N is a sequence $\mathfrak{C} = (\mathcal{C}^0, \mathcal{C}^1, ..., \mathcal{C}^h)$, $h \geq 1$ with $\mathcal{C}^l$ $(0 \leq l \leq h)$ coalition structure on N such that:

1. $\mathcal{C}^0 = \{\{1\}, \{2\}, ..., \{n\}\}$.
2. $\mathcal{C}^h = \{N\}$.
3. If $C_q \in \mathcal{C}^l$ with $0 < l \leq h$ then $C_q = \bigcup_{S \in Q} S$ for some $Q \subset \mathcal{C}^{l-1}$.

We call $\mathcal{C}^l$ the *l-th level* of $\mathfrak{C}$. We say that $\mathfrak{C}$ is a levels structure of *degree* h. Thus, the levels structure $\mathfrak{C}$ has $h + 1$ levels.

If $h = 1$, we say that $\mathfrak{C}$ is a *trivial* levels structure.

Given $i \in C_q \in \mathcal{C}^1$ with $n > 1$, we denote by $\mathfrak{C}_{-i}$ the levels structure on $N \setminus \{i\}$ which equals $\mathfrak{C}$ after removing player i. Namely, $\mathfrak{C}_{-i} = (\mathcal{C}^0_{-i}, \mathcal{C}^1_{-i}, ..., \mathcal{C}^h_{-i})$.

Given $S \in \mathcal{C}^l$, we denote by $\mathfrak{C}_{-S}$ the levels structure on $N \setminus S$ induced by $\mathfrak{C}$.

Assume $h \geq 2$. We define by $\mathfrak{C}/\mathcal{C}^1$ the levels structure induced by $\mathfrak{C}$ by dropping out the level $\mathcal{C}^0$ and considering the coalitions $C_q \in \mathcal{C}^1$ as players.

Whenever $C_q \in \mathcal{C}^1$ is considered as a player in $\mathfrak{C}/\mathcal{C}^1$, it is denoted by $[C_q]$. We also denote by $[\mathcal{C}^l]$ $(1 \leq l \leq h)$ the coalition structure which comes out from $\mathcal{C}^l$ by considering the coalitions of $\mathcal{C}^1$ as players.

Thus, we have $\mathfrak{C}/\mathcal{C}^1 = \left(\left[\mathcal{C}^1\right], \left[\mathcal{C}^2\right], ..., \left[\mathcal{C}^h\right]\right)$.

In particular for $l = 1$, if $\mathcal{C}^1 = \{C_1, ..., C_m\}$, we have $\left[\mathcal{C}^1\right] = \{\{[C_1]\}, ..., \{[C_m]\}\}$.

This new levels structure satisfies conditions 1, 2 and 3. Furthermore, $\mathfrak{C}/\mathcal{C}^1$ has degree $h - 1$.

Let $LTU(N)$ be the set of all $(N, v, \mathfrak{C})$ with $(N, v) \in TU(N)$ cooperative game and $\mathfrak{C}$ levels structure on N.

The *quotient game* $\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$ is the game $LTU\left(\mathcal{C}^1\right)$ defined on the coalition structure $\mathcal{C}^1$ with characteristic function:

$$\left(v/\mathcal{C}^1\right)(Q) = v\left(\bigcup_{[C_q]\in Q} C_q\right)$$

for all $Q \subset \mathcal{C}^1$.

A *solution concept* on $LTU(N)$ is a function $f : LTU(N) \to \mathbb{R}^N$ which assigns to each game $(N, v, \mathfrak{C}) \in LTU(N)$ a vector on $\mathbb{R}^N$, so that $f_i(N, v, \mathfrak{C})$ represents the payoff received by player $i \in N$.

In this article, we use two solution concepts for LTU. The *Shapley value* (Shapley, 1953) and the *levels structure value*, suggested by Owen (1977) and characterized by Winter (1989).

The Shapley value is given by next expression. Given $(N, v) \in TU(N)$ with $i \in N$:

$$\varphi_i(N, v) = \sum_{T \subset N \setminus \{i\}} \frac{|T|!(n - |T| - 1)!}{n!} \left[v\left(T \cup \{i\}\right) - v\left(T\right)\right].$$

The levels structure value is a generalization of the Shapley value to games with levels structure, i.e., when the levels structure is trivial, both solution concepts give the same payoff vector. In order to define it, we need some additional notation.

We denote by Π the set of all permutations on N. Given a levels structure $\mathfrak{C}$, we define by induction $\Pi_1(\mathfrak{C}) \subset \Pi_2(\mathfrak{C}) \subset ... \subset \Pi_h(\mathfrak{C})$ as follows

$$\Pi_h(\mathfrak{C}) = \Pi.$$

Given the sets $\Pi_{l+1}(\mathfrak{C}) \subset \Pi_{l+2}(\mathfrak{C}) \subset ... \subset \Pi_h(\mathfrak{C})$, we define:

$$\Pi_l(\mathfrak{C}) = \left\{\pi \in \Pi_{l+1}(\mathfrak{C}) : \forall j, k \in C_q \in \mathcal{C}^l, \forall i \in N, \pi(j) < \pi(i) < \pi(k) \Rightarrow i \in C_q\right\}.$$

In particular, permutations in $\Pi_1(\mathfrak{C})$ are those in which the players in the same coalition on any level appear always together.

Given $\pi \in \Pi$, $i \in N$, we denote by $P_i^\pi = \{j \in N : \pi(j) < \pi(i)\}$ the set of *predecessors* of i under π. We call *levels structure value* (Winter, 1989) to the solution concept $\Psi : LTU(N) \to \mathbb{R}^N$ given by

$$\Psi_i(N, v, \mathfrak{C}) = \frac{1}{|\Pi_1(\mathfrak{C})|} \sum_{\pi \in \Pi_1(\mathfrak{C})} [v(P_i^\pi \cup \{i\}) - v(P_i^\pi)]$$

for all $i \in N$.

This solution concept generalizes the Owen value (1977) for $h = 2$ with coalition structure $\mathcal{C}^1$ and the Shapley value for $h = 1$.

A simple and powerful characterization for the levels structure value is as follows (Calvo, Lasaga and Winter, 1996). The levels structure value is the only solution concept on $LTU(N)$ which satisfies efficiency and balanced contributions.

Efficiency. For any game $(N, v, \mathfrak{C}) \in LTU(N)$, we have $\sum_{i \in S} \Psi_i(N, v, \mathfrak{C}) = v(N)$.

Balanced contributions. For any $(N, v, \mathfrak{C}) \in LTU(N)$ and any $S, T \in \mathcal{C}^l$ with $0 \leq l < h$ such that $S, T \subset R \in \mathcal{C}^{l+1}$, $S \neq T$, we have

$$\Psi_S(N, v, \mathfrak{C}) - \Psi_S(N \backslash T, v_{-T}, \mathfrak{C}_{-T}) = \Psi_T(N, v, \mathfrak{C}) - \Psi_T(N \backslash S, v_{-S}, \mathfrak{C}_{-S}).$$

Furthermore, the levels structure value also satisfies additivity and quotient game property (Winter, 1989).

Additivity. For any $(N, v, \mathfrak{C})$, $(N, w, \mathfrak{C}) \in LTU(N)$, we have

$$\Psi(N, v + w, \mathfrak{C}) = \Psi(N, v, \mathfrak{C}) + \Psi(N, w, \mathfrak{C})$$

with $(N, v + w)$ the TU game defined on N by $(v + w)(S) = v(S) + w(S)$ for all $S \subset N$.

Quotient game property. For any $(N, v, \mathfrak{C}) \in LTU(N)$, we have

$$\sum_{i \in C_q} \Psi_i(N, v, \mathfrak{C}) = \Psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right).$$

3 The levels bidding mechanism

Given a cooperative game (N, v), Pérez-Castrillo and Wettstein (2001) design a non-cooperative game, called the *bidding mechanism*. In the bidding mechanism, players bid for the right to propose a payoff, which should be accepted by all the other players. Otherwise the prososer leaves the game. Pérez-Castrillo and Wettstein prove that the payoff of any subgame perfect Nash equilibrium (SPNE from now) of this mechanism always coincides with the Shapley value of the cooperative game (N, v). Thus, this mechanism implements the Shapley value in SPNE.

Our mechanism is played in several rounds. In each round, coalitions in each coalition structure play a bidding mechanism in order to obtain the resources of their own coalition. Namely, they bid for the right to propose a payoff. If this offer is not accepted by the other members of the coalition, the proposer leaves the game. If the offer is accepted by all the members of the coalition, and this happens in every coalition, the proposers become representatives of their coalitions and they move to the next round.

We now present the *levels bidding mechanism (LBM)* formally. We proceed by double induction on h (degree of $\mathfrak{C}$) and n (number of players).

For $h = 1$, the players play a single round. This round comprises the bidding mechanism (Pérez-Castrillo and Wettstein, 2001) associated with the game (N, v).

Assume that we know the rules of the LBM when the levels structure has degree $h - 1$, and it comprises $h - 1$ rounds.

If there is only one player i, he obtains $v(\{i\})$. Assume now that we know the rules of the LBM when played by $n - 1$ players. Then, for a set of players $N = \{1, ..., n\}$ and a levels structure $\mathfrak{C} = (\mathcal{C}^0, \mathcal{C}^1, ..., \mathcal{C}^h)$ with $\mathcal{C}^1 = \{C_1, ..., C_m\}$, the LBM proceeds as follows,

Round 1. The players of any coalition $C_q \in \mathcal{C}^1$ play the bidding mechanism trying to obtain the resources of C_q. Formally, if there is only one player i, then this player has his resources. Assume now that we know the rules when played by $|C_q| - 1$ players. For $|C_q| > 1$ it proceeds as follows

Stage 1. Each player $i \in C_q$ makes bids $b_j^i \in \mathbb{R}$ for every $j \in C_q \setminus \{i\}$. For each $i \in C_q$, we take $B^i = \sum_{j \in C_q \setminus \{i\}} b_j^i - \sum_{j \in C_q \setminus \{i\}} b_i^j$. Assume that $\alpha_q = argmax_i\{B^i\}$. In the case of a non-unique maximizer, α_q is randomly chosen among the maximizing indices.

Stage 2. Player α_q, called the *proposer*, makes an offer $y_i^{\alpha_q}$ to every player $i \in C_q \setminus \{\alpha_q\}$.

Stage 3. The players of $C_q \setminus \{\alpha_q\}$, sequentially, either accept or reject the offer. If a rejection is encountered, we say the offer is rejected. Otherwise, we say the offer is accepted.

The coalitions of $\mathcal{C}^1$ play sequentially in the order $C_1, ..., C_m$ until either we find $C_{q_0} \in \mathcal{C}^1$ and $\alpha_{q_0} \in C_{q_0}$ such that the offer of α_{q_0} is rejected, or for any $C_q \in \mathcal{C}^1$ the offer of α_q is accepted.

In the first case, player α_{q_0} pays $b_i^{\alpha_q}$ to every player $i \in C_q \setminus \{\alpha_q\}$ and leaves the non-cooperative game obtaining $v(\{\alpha_{q_0}\}) - \sum\limits_{i \in C_{q_0} \setminus \{\alpha_{q_0}\}} b_i^{\alpha_{q_0}}$. All players other than α_{q_0} proceed to play the LBM with $(N', v', \mathfrak{C}')$ where $N' = N \setminus \{\alpha_{q_0}\}$, $v' = v_{-\alpha_{q_0}}$, and $\mathfrak{C}' = \mathfrak{C}_{-\alpha_{q_0}}$. Any player $i \in C_{q_0} \setminus \{\alpha_{q_0}\}$ obtains as final payoff the sum of the bids received, $b_i^{\alpha_{q_0}}$, and the payoff outcome of the mechanism corresponding to (N', C', v'). Any player $i \in N \setminus C_{q_0}$ obtains as final payoff the payoff outcome of the mechanism corresponding to (N', C', v').

In the second case, for any $C_q \in \mathcal{C}^1$, player α_q pays $b_i^{\alpha_q} + y_i^{\alpha_q}$ to every $i \in C_q \setminus \{\alpha_q\}$ and becomes the *representative* of coalition C_q. This means that player α_q goes to Round 2 with all the resources of C_q. Moreover, the payoff obtained by this player in this round is $p^1_{\alpha_q} = -\sum\limits_{i \in C_q \setminus \{\alpha_q\}} \left(b_i^{\alpha_q} + y_i^{\alpha_q}\right)$. Any other player $i \in C_q \setminus \{\alpha_q\}$ leaves the non-cooperative game obtaining a final payoff of $b_i^{\alpha_q} + y_i^{\alpha_q}$.

After finishing Round 1, for any $C_q \in \mathcal{C}^1$ we can find the representative (denoted by r_q) of this coalition.

Rounds 2 throught h. The representatives play the LBM associated with the quotient game $(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1)$, where each r_q plays the role of $[C_q]$. These rounds are well defined by induction on h. For any representative r_q, we denote by $p^2_{r_q}$ the payoff obtained by r_q (or $[C_q]$) in these rounds.

The final payoff obtained by any representative r_q is the sum of the payoffs obtained in all the rounds, *i.e.* $p^1_{r_q} + p^2_{r_q}$.

We must note that the LBM terminates in a finite number of moves.

Remark 1 *Assume that in Round 1 the offer from player α_q is accepted for any $q < q_0$, but the offer of α_{q_0} is rejected. Then a new subgame begins, which coincides with the LBM associated to $\left(N \setminus \{\alpha_{q_0}\}, v_{-\alpha_{q_0}}, \mathfrak{C}_{-\alpha_{q_0}}\right)$. Moreover, when all the offers are accepted in Round 1, another subgame begins, which is equivalent to the LBM associated to $\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$.*

Before the characterization of the SPNE outcomes of the levels bidding mechanism we need the following result.

Proposition 2 *Given a triple $(N, v, \mathfrak{C}) \in LTU(N)$ such that (N, v) is zero-monotonic, $j \in C_q \in \mathcal{C}^1 \in \mathfrak{C}$ and $\{j\} \subsetneq C_q$ then*

$$\sum_{i \in C_q} \Psi_i(N, v, \mathfrak{C}) \geq \sum_{i \in C_q \setminus \{j\}} \Psi_i(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}) + v(\{j\}).$$

Proof. We take $\mathfrak{C}/\mathcal{C}^1 = \left(\left[\mathcal{C}^1\right], ..., \left[\mathcal{C}^h\right]\right)$ levels coalition structure. Assume $\mathcal{C}^1 = \{C_1, ..., C_m\}$ and $M = \{1, 2, ..., m\}$. Let $\mathfrak{Q} = \left(Q^1, ..., Q^h\right)$ be the levels structure on M which equals $\mathfrak{C}/\mathcal{C}^1$ except for the name of the players, i.e.

$\{q_1, q_2, ..., q_k\} \in Q^l :\Leftrightarrow \{[C_{q_1}], [C_{q_2}], ..., [C_{q_k}]\} \in [\mathcal{C}^l] \qquad 1 \leq l \leq h.$

We define the following games on M. For all $R \subset M$,

$$\begin{aligned}
u(R) &= v\left(\bigcup_{r\in R} C_r\right) \\
w_1(R) &= \left\{ \begin{array}{ll} v\left(\bigcup\limits_{r\in R} C_r \backslash \{j\}\right) & \text{if } q \in R \\ v\left(\bigcup\limits_{r\in R} C_r\right) & \text{if } q \notin R \end{array} \right\} \\
w_2(R) &= \left\{ \begin{array}{ll} v\left(\{j\}\right) & \text{if } q \in R \\ 0 & \text{if } q \notin R \end{array} \right\} \\
w &= w_1 + w_2.
\end{aligned}$$

Notice that the game u on M equals the quotient game $v/\mathcal{C}^1$ on $\mathcal{C}^1$. Thus, their levels structure values are the same

$$\Psi_q(M, u, \mathfrak{Q}) = \Psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right).$$

Given $C_q \in \mathcal{C}^1$, by the quotient game property, $\sum\limits_{i\in C_q} \Psi_i(N, v, \mathfrak{C}) = \Psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$. Thus,

$$\Psi_q(M, u, \mathfrak{Q}) = \sum_{i\in C_q} \Psi_i\left(N, v, \mathfrak{C}\right).$$

Analogously, the game w_1 on M equals the quotient game $v_{-j}/\mathcal{C}^1_{-j}$ on $\mathcal{C}^1_{-j}$. Thus:

$$\Psi_q(M, w_1, \mathfrak{Q}) = \Psi_{[C_q\backslash\{j\}]}\left(\mathcal{C}^1_{-j}, v_{-j}/\mathcal{C}^1_{-j}, \mathfrak{C}_{-j}/\mathcal{C}^1_{-j}\right) = \sum_{i\in C_q\backslash\{j\}} \Psi_i\left(N\backslash\{j\}, v_{-j}, \mathfrak{C}_{-j}\right).$$

Finally, the levels structure value of q for the game w_2 is:

$$\Psi_q(M, w_2, \mathfrak{Q}) = v(\{j\}).$$

By applying the zero-monotonicity of v, we get $\Psi_q(M, u, \mathfrak{Q}) \geq \Psi_q(M, w, \mathfrak{Q})$. By additivity of the levels structure value

$$\begin{aligned}
\sum_{i\in C_q} \Psi_i(N, v, \mathfrak{C}) &= \Psi_q(M, u, \mathfrak{Q}) \geq \Psi_q(M, w, \mathfrak{Q}) = \Psi_q(M, w_1 + w_2, \mathfrak{Q}) \\
&= \Psi_q(M, w_1, \mathfrak{Q}) + \Psi_q(M, w_2, \mathfrak{Q}) \\
&= \sum_{i\in C_q\backslash\{j\}} \Psi_i\left(N\backslash\{j\}, v_{-\alpha}, \mathfrak{C}_{-j}\right) + v(\{j\}).
\end{aligned}$$

■

In order to cope with technical problems of ties, we need an additional assumption on the SPNE's. These problems appear when players are indifferent between two or more strategies yielding the same payoff. In the last section we study an example of a game such that the associated LBMs have SPNE outcomes whose payoff is different from the levels structure value.

Vidal-Puga and Bergantiños (2002) make a modification to their mechanism, so that the player who rejects an offer, and the proposer whose offer is rejected, must pay a small penalty $\varepsilon > 0$.

In this article, we will not move in that direction. Moldovanu and Winter (1994) assume that players prefer agreements which involves large coalitions better than smaller ones (provided his final payoff is the same in both agreements). Hart and Mas-Colell (1996) assume that players "break ties in favor of quick termination of the game"[1]. In this paper we make both assumptions.

As a consequence of our assumptions, we can define a tie-breaking rule satisfying:

- If a player is indifferent between accepting or rejecting an offer from a proposer, he always accepts the offer.
- If a proposer $\alpha \in C_q$ is indifferent between offering b^α or $\widetilde{b}^\alpha$ being b^α due to be rejected by some player $i \in C_q \setminus \{\alpha\}$, and $\widetilde{b}^\alpha$ being accepted by every player in $C_q \setminus \{\alpha\}$, he always offers $\widetilde{b}^\alpha$.

In the rest of the section, by SPNE we mean SPNE satisfying this tie-breaking rule.

A similar approach by means of tie-breaking rule for SPNE's can be found in Navarro and Perea (2001). In their model, a player must choose prices, propose offers and accept or reject offers[2]. If a player is indifferent between accepting or rejecting an offer, he is supposed to accept. If, under certain circumstances, a player is indifferent between proposing Δ or $\widetilde{\Delta}$ with $\Delta < \widetilde{\Delta}$, he is supposed to propose $\widetilde{\Delta}$. If a player is indifferent between choosing price p or $\widetilde{p}$ with $p < \widetilde{p}$, he is supposed to choose price p.

Theorem 3 *The LBM implements the levels structure value in SPNE for superadditive games.*

Proof. The structure of this proof is similar to that of the main result by Vidal-Puga and Bergantiños (2002). However, the computations are different.

We proceed by double induction on h and n. For $h = 1$, the mechanism coincide with those by Pérez-Castrillo and Wettstein (2001). Thus, we assume the players play according to a strategy profile described in Pérez-Castrillo and Wettstein (2001) when they construct, for any zero-monotonic game, an SPNE that yields the Shapley value of this game as a payoff outcome. It is easy to check

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[1] However, tie-breaking rules are not needed in Hart and Mas-Colell's model.

[2] These offers are differences in payoffs to be received at the end of the mechanism.

that this SPNE satisfies the tie-breaking rule. So, the mechanism implements the levels structure value.

Assume the result is true for levels structures of degree at most $h-1$.

We now prove the result when the degree is h. If there is only a player it is trivial. Assume that if there are at most $n-1$ players the LBM implements the levels structure value in SPNE and, moreover, all the offers of Round 1 are accepted in equilibrium. We now prove that the same holds when there are n players.

We first prove that the levels structure value is indeed an equilibrium outcome. We explicitly construct an SPNE which yields the levels structure value as an SPNE outcome.

We consider the following strategies.

Round 1. First, we define the strategies in the LBM associated to any $C_q \in \mathcal{C}^1$.

Stage 1. For any $i \in C_q$, $b_j^i = \Psi_j\left(N, v, \mathfrak{C}\right) - \Psi_j\left(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i}\right)$ for any $j \in C_q \setminus \{i\}$.

Stage 2. Player α_q, the proposer, offers $y_j^{\alpha_q} = \Psi_j\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right)$ to every $j \in C_q \setminus \{\alpha_q\}$.

Stage 3. Any player $i \in C_q \setminus \{\alpha_q\}$ accepts the offer of α_q if and only if $y_j^{\alpha_q} \geq \Psi_j\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right)$ for every $j \in C_q \setminus \{\alpha_q\}$.

If some offer is rejected, for instance, the offer of α_{q_0}, we go to the subgame where all players other than α_{q_0} play this mechanism in $\left(N \setminus \{\alpha_{q_0}\}, v_{-\alpha_{q_0}}, \mathfrak{C}_{-\alpha_{q_0}}\right)$. We assume that players in $N \setminus \{\alpha_{q_0}\}$ play according to the strategies profiles of some SPNE with payoff associated $\Psi\left(N \setminus \{\alpha_{l_0}\}, v_{-\alpha_{l_0}}, \mathfrak{C}_{-\alpha_{l_0}}\right)$ (by induction hypothesis on n we can find such SPNE).

Rounds 2 through h. We assume that the representatives play according to the strategies of some SPNE with associated payoff $\Psi\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$. Again, by induction hypothesis on h, we can find such SPNE.

It is straightforward to prove that these strategies satisfy the tie-breaking rule.

First, we prove that according to these strategies any player $i \in N$ receives as payoff the levels structure value $\Psi_i(N, C, v)$. We must note that for any $C_q \in \mathcal{C}^1$ the offer from α_q is accepted. Then player α_q goes to Round 2 as the representative of C_q.

Given $C_q \in \mathcal{C}^1$ and $i \in C_q \setminus \{\alpha_q\}$, the payoff obtained by player i is $b_i^{\alpha_q} + y_i^{\alpha_q} =$

$$\Psi_i(N, v, \mathfrak{C}) - \Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right) + \Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right) = \Psi_i(N, v, \mathfrak{C}).$$

We now compute the payoff of any representative r_q. As v is superadditive we have that $v/\mathcal{C}^1$ is also superadditive. By induction hypothesis on h, we know that the payoff obtained by r_q in Rounds 2 through h ($p^2_{r_q}$) coincides with the levels structure value of $\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$. Then the final payoff obtained by r_q is

$$\begin{aligned} p^1_{r_q} + p^2_{r_q} &= -\sum_{i \in C_q \setminus \{r_q\}} b_i^{r_q} - \sum_{i \in C_q \setminus \{r_q\}} y_i^{r_q} + \Psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right) \\ &= -\sum_{i \in C_q \setminus \{r_q\}} \left[\Psi_i(N, v, \mathfrak{C}) - \Psi_i\left(N \setminus \{r_q\}, v_{-r_q}, \mathfrak{C}_{-r_q}\right)\right] \\ &\quad - \sum_{i \in C_q \setminus \{r_q\}} \Psi_i\left(N \setminus \{r_q\}, v_{-r_q}, \mathfrak{C}_{-r_q}\right) + \Psi_{[C_q]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right) \end{aligned}$$

by rearranging and applying the quotient game property,

$$= -\sum_{i \in C_q \setminus \{r_q\}} \Psi_i(N, v, \mathfrak{C}) + \sum_{i \in C_q} \Psi_i(N, v, \mathfrak{C}) = \Psi_{r_q}(N, v, \mathfrak{C}).$$

We now prove that these strategies are an SPNE. By induction hypothesis on h, we conclude that in the subgames obtained after Round 2 these strategies induce an SPNE.

By induction hypothesis on n, in all the subgames obtained after the rejection of the offer of some proposer α_q, these strategies induce an SPNE.

We only have to prove that these strategies induce an SPNE in the bidding mechanism associated to any coalition C_q (Round 1).

Stage 3. Assume that player i rejects the offer of α_q. Then the LBM mechanism of $\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right)$ is played and, by induction hypothesis on n, after the rejection player i can obtain at most $\Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, C_{-\alpha_q}\right)$. Hence, if player i rejects the offer of α_q, he obtains, at most,

$$b_i^{\alpha_q} + \Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right) = \Psi_i(N, v, \mathfrak{C}).$$

This means that player i does not improve his payoff.

Stage 2. If player α_q offers to some player $i \in C_q$ less than $\Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, C_{-\alpha_q}\right)$, the offer is rejected and, therefore, player α_q obtains a final payoff of

$$v(\alpha_q) - \sum_{i \in C_q \setminus \{\alpha_q\}} \left[\Psi_i(N, v, \mathfrak{C}) - \Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_l}, \mathfrak{C}_{-\alpha_q}\right)\right].$$

By Proposition 2, this payoff is not larger than $\Psi_{\alpha_q}(N, v, \mathfrak{C})$, which means that player α_q does not improve his payoff.

If player α_q offers to any player $i \in C_q \setminus \{\alpha_q\}$ at least $\Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right)$, the offer is accepted. It is easy to prove that player α_q obtains at most $\Psi_{\alpha_q}(N, v, \mathfrak{C})$.

Stage 1. First, we prove that for any $i \in C_q \in \mathcal{C}^1$, $B^i = 0$.

$$\begin{aligned} B^i &= \sum_{j \in C_q \setminus \{i\}} b_j^i - \sum_{j \in C_q \setminus \{i\}} b_i^j \\ &= \sum_{j \in C_q \setminus \{i\}} \left(\Psi_j(N, v, \mathfrak{C}) - \Psi_j\left(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i}\right)\right) \\ &\quad - \sum_{j \in C_l \setminus \{i\}} \left[\Psi_i(N, v, \mathfrak{C}) - \Psi_i\left(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}\right)\right]. \end{aligned}$$

As the levels structure value satisfies balanced contributions, we have that for any $j \in C_q \setminus \{i\}$,

$$\Psi_i\left(N, v, \mathfrak{C}\right) - \Psi_i\left(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}\right) = \Psi_j\left(N, v, \mathfrak{C}\right) - \Psi_j\left(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i}\right)$$

and hence $B^i = 0$.

Assume that player $i \in C_q$ makes a different bid b^*. If $B^{*i} < 0$, the proposer will be another player of C_q. Then player i can not increase his payoff.

If $B^{*i} > 0$, he becomes the proposer but he must pay $\sum\limits_{j \in C_q \setminus \{i\}} b_j^{*i}$ to the other players of $C_q \setminus \{i\}$. It is straightforward to prove that player i can obtain, at most, a final payoff of

$$\Psi_i(N, v, \mathfrak{C}) - \sum_{j \in C_q \setminus \{i\}} b_j^{*i} + \sum_{j \in C_q \setminus \{i\}} b_j^i$$

which is smaller than $\Psi_i(N, v, \mathfrak{C})$.

If $B^{*i} = 0$ and player i is not the proposer, using similar arguments to those used when $B^{*i} < 0$, we can conclude that player i does not increase his payoff. If $B^{*i} = 0$ and player i is the proposer, using similar arguments to those used when $B^{*i} > 0$ we can conclude that player i does not increase his payoff.

We now prove that the payoff in all SPNE outcomes coincides with the levels structure value. We do it in several steps.

Step A. At every SPNE outcome, and for every $C_q \in \mathcal{C}^1$, the offer from the proposer α_q to each player $i \in C_q \setminus \{\alpha_q\}$ is $y_i^{\alpha_q} = \Psi_i\left(N \setminus \{\alpha_q\}, v_{-\alpha_q}, \mathfrak{C}_{-\alpha_q}\right)$ and every $i \in C_q \setminus \{\alpha_q\}$ accepts this offer.

Assume that in each coalition $C_q \in \{C_1, ..., C_{m-1}\}$, the offer from a proposer $\alpha_q \in C_q$ is accepted, and consider the subgame starting with the last coalition C_m. Let $\alpha_m \in C_m$ be the proposer in C_m. Let y^{α_m} be an offer from α_m. Let the order of reply of the players in $C_m \setminus \{\alpha_m\}$ be $i_1, ..., i_k$.

Claim 1: At every SPNE, the strategies of the players in $C_m \setminus \{\alpha_m\}$ must satisfy the following statements:

(i) If $y_i^{\alpha_m} \geq \Psi_i\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$ for every $i \in C_m \setminus \{\alpha_m\}$, then every $i \in C_m \setminus \{\alpha_m\}$ accepts y^{α_m}.

(ii) If $y_j^{\alpha_m} < \Psi_j\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$ for some $j \in C_m \setminus \{\alpha_m\}$, then some player in $C_m \setminus \{\alpha_m\}$ rejects y^{α_m}.

(i) Consider the strategy of the last player i_k. Assuming that his decision node is reached, if he accepts the offer y^{α_m}, then he receives $b_{i_k}^{\alpha_m} + y_{i_k}^{\alpha_m}$, whereas if he rejects y^{α_m}, then by the induction hypothesis he obtains $b_{i_k}^{\alpha_m} + \Psi_{i_k}\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$. Hence, at any SPNE,

- if $y_{i_k}^{\alpha_m} > \Psi_{i_k}\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$, then i_k accepts the offer because it is optimal;
- if $y_{i_k}^{\alpha_m} = \Psi_{i_k}\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$, then i_k accepts the offer because of the tie-breaking rule.

Repeating the same argument backwards, we can show that players $i_{k-1}, ..., i_1$ accept the offer.

(ii) Suppose, to the contrary, that there exists $j \in C_m \setminus \{\alpha_m\}$ with $y_j^{\alpha_m} < \Psi_j\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$, but all the players in $C_m \setminus \{\alpha_m\}$ accept the offer y^{α_m}. Then, player j receives $b_j^{\alpha_m} + y_j^{\alpha_m}$. However, if player j deviates and rejects the offer, then he obtains $b_j^{\alpha_m} + \Psi_j\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$, which is larger than $b_j^{\alpha_m} + y_j^{\alpha_m}$. Hence, the strategies of the players in $C_m \setminus \{\alpha_m\}$ cannot constitute an SPNE.

Claim 2: At every SPNE outcome, every $i \in C_m \setminus \{\alpha_m\}$ accepts the offer from the proposer α_m.

Suppose, to the contrary, that at some SPNE outcome, there exists $i \in C_m \setminus \{\alpha_m\}$ who rejects the offer y^{α_m}. Then, the proposer obtains

$$e = v\left(\{\alpha_m\}\right) - \sum_{i \in C_m \setminus \{\alpha_m\}} b_i^{\alpha_m}.$$

Suppose that the proposer α_m proposes $z_i^{\alpha_m} = \Psi_i\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$ to every $i \in C_m \setminus \{\alpha_m\}$. By Claim 1 (i), every $i \in C_m \setminus \{\alpha_m\}$ accepts z^{α_m}. Hence, player α_m is the representative of coalition C_m in Round 2. Now, in Rounds 2 through h, there are m players $\{\alpha_1, ..., \alpha_m\}$, where, for any coalition $C_q \in \mathcal{C}^1$, α_q is the representative of coalition C_q. As the representatives are playing an SPNE of the LBM associated to $\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$, by induction hypothesis on h we know that the payoff obtained by player α_m in Rounds 2 through h is $\Psi_{[C_m]}\left(\mathcal{C}^1, v/\mathcal{C}^1, \mathfrak{C}/\mathcal{C}^1\right)$, which, by the quotient game property, equals $\sum_{i \in C_m} \Psi_i\left(N, v, \mathfrak{C}\right)$. Then, the final payoff of player α_m is

$$\widetilde{e} = \sum_{i \in C_m} \Psi_i(N, v, \mathfrak{C}) - \sum_{i \in C_m \setminus \{\alpha_m\}} \Psi_i\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right) - \sum_{i \in C_m \setminus \{\alpha_m\}} b_i^{\alpha_m}.$$

By Proposition 2, we know that

$$\sum_{i \in C_m} \Psi_i(N, v, \mathfrak{C}) - \sum_{i \in C_m \setminus \{\alpha_m\}} \Psi_i\left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right) \geq v\left(\{\alpha_m\}\right).$$

Thus, $e \leq \widetilde{e}$.

- If $e < \widetilde{e}$, to offer y^{α_m} cannot be an SPNE strategy of the proposer α_m, which is a contradiction.
- If $e = \widetilde{e}$, then α_m is indifferent between offering y^{α_m} or z^{α_m}. By Claim 1 (i), offer z^{α_m} is accepted by every $i \in C_m \backslash \{\alpha_m\}$. By the tie-breaking rule α_m must propose z^{α_m} better than y^{α_m}, which is a contradiction.

Claim 3: At every SPNE, and for every $i \in C_m \setminus \{\alpha_m\}$, we have $y_i^{\alpha_m} = \Psi_i \left(N \setminus \{\alpha_m\}, C_{-\alpha_m}, v_{-\alpha_m}\right)$.

Let y^{α_m} be the offer from α_m at an SPNE. By Claim 2, y^{α_m} must be accepted by every $i \in C_m \setminus \{\alpha_m\}$. Then, it follows from Claim 1 (ii) that for every $i \in C_m \setminus \{\alpha_m\}$, $y_i^{\alpha_m} \geq \Psi_i \left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$. Suppose that for some $j \in C_m \setminus \{\alpha_m\}$, $y_j^{\alpha_m} > \Psi_j \left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$. For each $i \in C_m \setminus \{\alpha_m\}$, define $w_i^{\alpha_m} = \Psi_i \left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right)$. Suppose that the proposer α_m deviates and offers w^{α_m}. Then, by Claim 1 (i), every $i \in C_m \setminus \{\alpha_m\}$ accepts w^{α_m}. Moreover, since

$$\sum_{i \in C_m \setminus \{\alpha_m\}} w_i^{\alpha_m} = \sum_{i \in C_m \setminus \{\alpha_m\}} \Psi_i \left(N \setminus \{\alpha_m\}, v_{-\alpha_m}, \mathfrak{C}_{-\alpha_m}\right) < \sum_{i \in C_m \setminus \{\alpha_m\}} y_i^{\alpha_m},$$

the proposer α_m obtains a greater payoff by offering w^{α_m} than by offering y^{α_m}. Hence, to offer y^{α_m} cannot be an SPNE strategy, which is a contradiction.

Repeating the same arguments for coalitions $C_{m-1}, ..., C_1$, we prove Step A.

Step B. Assume that we are in Stage 1 of Round 1 of the LBM associated to $C_q \in \mathcal{C}^1$. Then in any SPNE, $B^i = 0$ for any $i \in C_q$.

It is easy to prove that $\sum\limits_{i \in C_q} B^i = 0$. We take $X = \left\{ i \in C_q : B^i = \max\limits_{j \in C_q} B^j \right\}$. If $X = C_q$, the result holds because $\sum\limits_{i \in C_q} B^i = 0$.

If $X \neq C_q$, we get a contradiction by proving that player $i \in X$ has a deviation which improves his final payoff. We take $j \in C_q \backslash X$ such that $B^j \geq B^k$ for any $k \in C_q \setminus X$. Assume that player i makes a new bid b'^i, where $b'^i_k = b^i_k + \delta$ if $k \in X \setminus \{i\}$, $b'^i_j = b^i_j - |X|\delta$, and $b'^i_k = b^i_k$ if $k \in C_q \setminus (X \cup \{j\})$.

For any $k \in C_q$, we compute B'^k assuming that $b'^k = b^k$ for any $k \in C_q \setminus \{i\}$. Then $B'^k = B^k - \delta$ if $k \in X$, $B'^j = B^j + |X|\delta$, and $B'^k = B^k$ if $k \in C_l \backslash (X \cup \{j\})$.

Since $B^j < B^i$, we can find $\delta > 0$ satisfying $B^j + |X|\delta < B^i - \delta$. Moreover, $X' = \left\{ k \in C_q : B'^k = \max\limits_{h \in C_q} B'^h \right\} = X$. This means that any player of X is the proposer with the same probability under b^i and b'^i. When player i is not the proposer, which happens with probability $\frac{|X|-1}{|X|}$, he obtains, by Step A, the same making a bid b^i or b'^i. But if player i is the proposer, which happens with probability $\frac{1}{|X|}$, he obtains, by Step A, δ units more with b'^i than with b^i.

Step C. Assume that we are in Stage 1 of Round 1 of the LBM associated to $C_l \in C$. Then, at every SPNE, the payoff of any player $i \in C_q$ is the same regardless of who is chosen as the proposer.

By Step B, we know that $B^i = 0$ for any $i \in C_l$.

Assume that some player i strictly prefers to be (not to be) the proposer. Then player i can improve his payoff by slightly increasing (decreasing) one of his bids b_j^i. But this is impossible in an SPNE.

Step D. In any SPNE outcome of LBM any player $i \in N$ obtains as final payoff his levels structure value.

Assume that players are playing according to some SPNE. Given $i \in C_q \in \mathcal{C}^1$, we denote by p_i the final payoff obtained by player i in this SPNE.

By Step B, we know that any player of C_q is the proposer with probability $\frac{1}{|C_q|}$.

If player i is the proposer, we know, by Step A, that his final payoff is

$$\sum_{j \in C_q} \Psi_j\left(N, v, \mathfrak{C}\right) - \sum_{j \in C_q \setminus \{i\}} \Psi_j\left(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i}\right) - \sum_{j \in C_q \setminus \{i\}} b_j^i .$$

If $j \in C_q \setminus \{i\}$ is the proposer then the final payoff of player i is, by Step A,

$$b_i^j + \Psi_i\left(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}\right).$$

By Step C, we know that

$$\begin{aligned} |C_q| p_i &= \sum_{j \in C_q} \Psi_j\left(N, v, \mathfrak{C}\right) - \sum_{j \in C_q \setminus \{i\}} \Psi_j\left(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i}\right) - \sum_{j \in C_q \setminus \{i\}} b_j^i \\ &\quad + \sum_{j \in C_q \setminus \{i\}} \left[b_i^j + \Psi_i\left(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}\right) \right]. \end{aligned}$$

By Step B, we know that $-\sum_{j \in C_q \setminus \{i\}} b_j^i + \sum_{j \in C_q \setminus \{i\}} b_i^j = -B^i = 0$.

Hence,

$$|C_q| p_i = \sum_{j \in C_q \setminus \{i\}} \left[\Psi_i\left(N \setminus \{j\}, v_{-j}, \mathfrak{C}_{-j}\right) - \Psi_j\left(N \setminus \{i\}, v_{-i}, \mathfrak{C}_{-i}\right) \right] + \sum_{j \in C_q} \Psi_j\left(N, v, \mathfrak{C}\right).$$

Since the levels structure value satisfies the property of balanced contributions, we have that

$$\begin{aligned} |C_q| p_i &= \sum_{j \in C_q \setminus \{i\}} \left[\Psi_i\left(N, v, \mathfrak{C}\right) - \Psi_j\left(N, v, \mathfrak{C}\right) \right] + \sum_{j \in C_q} \Psi_j\left(N, v, \mathfrak{C}\right) \\ &= \left(|C_q| - 1\right) \Psi_i\left(N, v, \mathfrak{C}\right) - \sum_{j \in C_q \setminus \{i\}} \Psi_j\left(N, v, \mathfrak{C}\right) + \sum_{j \in C_q} \Psi_j\left(N, v, \mathfrak{C}\right) \\ &= |C_q| \Psi_i\left(N, v, \mathfrak{C}\right). \end{aligned}$$

Then $p_i = \Psi_i\left(N, v, \mathfrak{C}\right)$. ■

4 Conclusion

In this paper we have developed a bidding mechanism which implements the levels structure value of every superadditive game with a levels structure of cooperation. The mechanism is a generalization of the bidding model presented by Pérez-Castrillo and Wettstein (2001).

In equilibrium, we have imposed that players prefer large coalitions to small ones. Next example shows that this condition is needed. Notice that players from coalition $\{1,2\}$ are indifferent between leaving the game or not (they obtain 0 anyway). However, players in $\{3,4\}$ are sensitive to player 1 or player 2 leaving the game.

Consider $(N,v,\mathfrak{C})$ with $h=2$, where $N=\{1,2,3,4\}$, $\mathfrak{C}=\{\mathcal{C}^0,\mathcal{C}^1,\mathcal{C}^2\}$, $\mathcal{C}^1=\{\{1,2\},\{3,4\}\}$. Moreover, v is the characteristic function associated to the weighted majority game where the quota is 3 and the weights are 1, 1, 1, and 2 respectively. This means that $v(S)=1$ if and only if S contains some of the following subsets: $\{1,2,3\}$, $\{1,4\}$, $\{2,4\}$, or $\{3,4\}$.

It is straightforward to prove that

$$\begin{aligned}
\Psi\left(N,v,\mathfrak{C}\right) &= \left(0,0,\tfrac{1}{2},\tfrac{1}{2}\right)\\
\Psi\left(N\setminus\{1\},v_{-1},\mathfrak{C}_{-1}\right) &= \left(-,0,\tfrac{1}{4},\tfrac{3}{4}\right)\\
\Psi\left(N\setminus\{2\},v_{-2},\mathfrak{C}_{-2}\right) &= \left(0,-,\tfrac{1}{4},\tfrac{3}{4}\right)\\
\Psi\left(N\setminus\{3\},v_{-3},\mathfrak{C}_{-3}\right) &= \left(\tfrac{1}{4},\tfrac{1}{4},-,\tfrac{1}{2}\right)\\
\Psi\left(N\setminus\{4\},v_{-4},\mathfrak{C}_{-4}\right) &= \left(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{2},-\right).
\end{aligned}$$

We now define an SPNE whose payoff outcome is $\left(0,0,\frac{1}{4},\frac{3}{4}\right)$.

Round 1. First, we describe the strategies of players 1 and 2. The bids are $b_2^1=b_1^2=0$. Then, the proposer α is randomly chosen between 1 and 2. Moreover, $y_j^\alpha=0$ and player j accepts the offer of α if and only if α offers him something strictly positive.

We now describe the strategies of players 3 and 4. In the subgame obtained after the offer of α is accepted, the strategies of players 3 and 4 coincide with the strategies whose payoff outcome is the levels structure value. We know that these strategies exist by Theorem 3. In the subgame obtained after the offer of α is rejected, the strategies of players 3 and 4 coincide with the strategies whose payoff outcome is the levels structure value of $(N\setminus\{\alpha\},v_{-\alpha},\mathfrak{C}_{-\alpha})$.

Round 2. We assume the representatives play according to the strategies described in Pérez-Castrillo and Wettstein (2001), which implement the levels structure value.

It is not difficult to check that these strategies are an SPNE. However, they do not satisfy the tie-breaking rule.

According to these strategies, the offer of player α is rejected, which means that player α obtains a final payoff of $v\left(\{\alpha\}\right)=0$. Then players of $N\setminus\{\alpha\}$ obtain as final payoff $\Psi\left(N\setminus\{\alpha\},v_{-\alpha},\mathfrak{C}_{-\alpha}\right)$. This means that the final payoff induced by these strategies is $\left(0,0,\frac{1}{4},\frac{3}{4}\right)$.

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