TECHNICAL REPORT SERIES

FUNCTIONAL FIXED POINTS

by

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Technical Report No. 33

THE ECONOMICS SERIES

STANFORD INSTITUTE FOR THEORETICAL ECONOMICS

FOURTH FLOOR, ENCINA HALL

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

1. Introduction

Many interesting and familiar problems in economics have a recursive structure; such problems are frequently solved by appealing to an appropriate fixed point construction (The classical problem of consumption and investment over an infinite horizon is a familiar example). There are some other important problems in economics, however, whose recursive structure seems to lead to an infinite regress, a regress that cannot be resolved by appealing to the traditional fixed point theories of analysis and topology. Examples include the regress that results from the formation of beliefs on other agents' beliefs, pioneered by Mertens and Zamir (1985); the regress that results from the process of deciding on how to decide what to do, pioneered by Lipman (1991); and the regress that results from the process of proposing rules to alter the rules of a given game, pioneered by Crawford (1985). The purpose of this paper is to introduce some notions from category theory, and use these notions to show that problems of infinite regress are generated when some operator (called a functor) is iterated; and that these problems are resolved when the functor satisfies a continuity condition that guarantees the existence of fixed points. There is, in fact, a theory of existence, uniqueness, approximation and continuous dependence on parameters of functorial fixed points that is presented here and which facilitates understanding the constructions of Mertens-Zamir and Lipman.

The paper is organized as follows: Section 2 introduces categories, functors and natural transformations, and illustrates them with simple examples. Section 3 introduces limits, continuous functors and greatest fixed points, and illustrates them with the construction of the universal beliefs space of Mertens and Zamir. Section 4 introduces colimits, cocontinuous functors and least fixed points, and illustrates them with a simplified version of Lipman's universal choice space. Section 5 concludes.

2. Categories, Functors, Natural Transformations

To motivate the definition of a category, we consider the collection of all sets; for any two sets A, B there are functions $f: A \to B$ and for any two functions $f: A \to B$ and $g: B \to C$ there is a composite function $gf: A \to C$; for each set A there is an identity function $id_A: A \to A$.

1. *Definition*: A category X consists of

    (i) a class of objects: A, B - - -

    (ii) for any two objects A, B a class of morphisms from A to B: a typical morphism is denoted by $f: A \to B$

    (iii) an operation on morphisms called composition: if $f: A \to B$ and $g: B \to C$ then there is a morphism $gf: A \to C$. Composition is assumed to be associative, i.e., $(fg)h = f(gh)$ for any morphisms f, g, h for which the compositions are defined.

    (iv) for each object A, an identity morphism $id_A: A \to A$ such that $fid_A = f$; $id_A g = g$ for any f and g for which the compositions are defined.

Examples of categories of sets: in all of theses examples, composition is ordinary composition of functions and identities are ordinary identity functions. Hence, we list only objects and morphisms.

Category	Objects	Morphisism
Set	all sets	all functions
Top	all topological spaces	all continuous functions
CompHaus	all compact Hausdorff sets	all continuous functions
Vect	all vector spaces	all linear functions

Examples of categories whose objects are not sets

Any partially ordered set (A, $\leq$) can be considered as a category: the objects are the elements of A, while there is a (unique) morphism from a to b if and only if a $\leq$ b. Any set A can be considered as a category with objects the elements of A and no morphisms except identities.

The set ω of natural numbers is a category with objects all natural numbers; there is a (unique) morphism m $\rightarrow$ n iff m $\leq$ n.

The category ω^{op} has as objects all natural numbers; there is a (unique) morphism m $\rightarrow$ n iff m $\geq$ n.

To motivate functors we begin with an example. For each compact set X let Δ(X) be the set of Borel regular probability measures on X endowed with the weak topology; it is well-known that Δ(X) is compact in this topology. For each continuous function f: X $\rightarrow$ Y, define a new function Δ(f): Δ(X) $\rightarrow$ Δ(Y) as follows: Δ(f) assigns to each measure μ in Δ(Y) a new measure $\Delta(f)(\mu)$ in Δ(Y) defined by $\Delta(f)(\mu)(E) = \mu(f^{-1}(E))$, E$\subseteq$Y Borel; Δ(f) is a continuous function and Δ preserves composition ($\Delta(fg) = \Delta(f)\,\Delta(g)$) and identity ($\Delta(id_x) = id_{\Delta(x)}$). Observe that the symbol Δ is being used like an ordinary function symbol: it assigns to each compact set X another compact set Δ(X) and to each continuous function f another continuous function Δ(f); in addition, it preserves composition and identify. We summarize this information by saying that $\Delta$is a functor from the category of compact sets to itself, and we write Δ: CompHaus $\rightarrow$ CompHaus.

Definition: A functor F from category X to category Y (we write F: X $\rightarrow$ Y) assigns to each object x in X an object F(x) in Y, to each morphism f: x $\rightarrow$ y in X a morphism F(f): F(x) $\rightarrow$ F(y) in Y, and preserves composition and identity

$$F(gf) = F(g)\,F(f)$$

$$F(id_x) = id_{F(x)}$$

Examples: The forgetful functor U: Vect $\rightarrow$ Set assigns to each vector space (X, τ) its underlying set X (where τ is the linear structure on X) and to each linear function f the function f itself.

The forgetful functor U: CompHaus $\rightarrow$ Set assigns to each compact Hausdorff space (X, τ) its underlying set X (where τ is the topology on X), and to each continuous function f the function f itself.

The functor "constant at x", K_x: C $\rightarrow$ C, where C is any category, assigns to each object y the object $K_x(y) = x$ and to each morphism f: y $\rightarrow$ z the identity morphism $K_x(f) = id_x$.

The identity functor I: X $\rightarrow$ X is defined by I(x) = x, I(f) = f.

Functors can be composed: if F: X $\rightarrow$ Y and G: Y $\rightarrow$ Z then G: X $\rightarrow$ Z is defined by GF (x) = G(F(x)), GF(f) = G(F(f)). Hence, we can form a new category, where the objects are categories and the morphisms are functors.

Natural transformations are essentially morphisms between functors that preserve the structure of the underlying categories. To introduce them, we present some elements of the Mertens-Zamir construction.

Let S be a compact Hausdorff space, and G: CompHaus $\rightarrow$ CompHaus the functor defined by

- $G(A) = \Delta(S \times A)$, on objects

- $G(f) = \Delta(id_s \times f)$, on morphisms

G assigns to each space A the beliefs on S x A; iterations of G generate the Mertens-Zamir beliefs hierarchy, defined by the functor $Q: \omega^{op} \to$ CompHaus as follows (1 below denotes any singleton set);

- $Q(o) = 1;$
- $Q(t+1) = G(Q(t)) = G^t(1), \quad t \geq o.$
- $Q(1 \to o) = M_o: \Delta(S \times 1) \to 1$ is the mapping that assigns everything to the single point in 1.
- $Q(t+1 \to t): Q(t+1) \to Q(t)$ is the mapping $G^t(M_o) = \underset{t\text{-times}}{\underline{Go\ Go\ \dots\ oG}}\ (M_o) = M_t$ that takes

  beliefs in the t+1 - level of the hierarchy and extracts from them the corresponding beliefs in the t-level of the hierarchy by "taking marginals."

A hierarchy of beliefs, then, is a point $q = (q(1), q(2), ..)$ in the product space $\prod_{t=1}^{\infty} Q(t)$. A hierarchy of beliefs q is consistent if, for all t, t, $M_{t+1}(q(t+1)) = q(t)$, i.e., if beliefs higher in the hierarchy have marginals consistent with beliefs lower in the hierarchy. Let $X \subset \prod_{t=1}^{\infty} Q(t)$ be a set of consistent belief hierarchies. For each level t in the hierarchy, there is a continuous function $\lambda_t: X \to Q(t)$, defined by $\lambda_t(q) = q(t)$, q in X. The consistency requirement, then, can be written as

$$M_{t+1}(\lambda_{t+1}(q)) = \lambda_t(q), \quad q \text{ in } X, \text{ all } t,$$

or, more concisely,

$$M_{t+1} \circ \lambda_{t+1} = \lambda_t, \quad \text{all } t.$$

Recalling that $M_t = Q(t+1 \to t)$, we have

$$Q(t+1 \to t) \circ \lambda_{t+1} = \lambda_t, \quad \text{all } t. \tag{1}$$

We say concisely that any collection of continuous maps $\langle \lambda_t : X \to Q(t) \rangle$ is a natural transformation $\lambda: X \to Q$ whenever the commutativity conditions (1) are satisfied. We generalize this to any category

Definition Let C be a category, x an object in C and $F: \omega^{op} \to C$ a functor. A natural transformation $\lambda: x \to F$ is a collection of morphisms in C, denoted by $\langle \lambda_t : x \to F(t) \rangle$, such that $F(t+1 \to t) \circ \lambda_{t+1} = \lambda_t, \quad \text{all } t \geq o.$

3. Limits, Greatest Fixed Points, Continuous Functors

In this section we present the Mertens-Zamir construction as a limit of a functor, and as the (greatest) fixed point of a functor. We explain why both notions are necessary, and why continuity of the relevant functor guarantees the two notions coincide, and constitute the universal beliefs space.

Recall from the previous section that a compact Hausdorff set X is a set of consistent beliefs if there is a natural transformation $\lambda: X \to Q$. Mertens and Zamir want to construct the set of all such consistent beliefs; naturally, they look at the "largest" set X for which a natural transformation $\lambda: X \to Q$ exists. This can be formalized as follows: X is a universal beliefs space if

- X is a consistent beliefs space, i.e., there is a natural transformation $\lambda: X \to Q$
- X is the largest consistent beliefs space: if Y is another consistent beliefs space, i.e., if there is a natural transformation $\mu: Y \to Q$, then there is a unique continuous function $f: Y \to X$ such that $\lambda_t o f = \mu_t$, all t

A (compact Hausdorff) X that satisfies these two properties is called a limit of the functor $Q: \omega^{op} \to$ **CompHaus**. It is instructive to see X constructed explicitly. Let $\pi_t: \prod_{t=1}^{\infty} Q(t) \to Q(t)$ be projections on each of the levels of the belief hierarchy. Then, define

- $X = \{q \in \prod_{t=1}^{\infty} Q(t): M_{t+1}(\pi_{t+1}(q)) = \pi_t(q)\}$.
- $\lambda_t: X \to Q(t), \quad \lambda_t(q) = \pi_t(q)$.
- The topology of X as the smallest topology that makes all λ_t continuous.

Clearly, by the construction of X, λ is a natural transformation. To see that X is the greatest consistent beliefs space, let $\mu: Y \to Q$ be another natural transformation. Then, there exists exactly one function $f: Y \to X$ such that $\lambda_t o f = \mu_t$, and it is defined by $f(y) = (\mu_1(y), \mu_2(y), ...)$; f is continuous because each μ_t is, and f maps into X because μ is *a* natural transformation.

We now generalize the preceding discussion

Definition: Let C be a category, $F: \omega^{op} \to C$ a functor, x an object in C, and $\lambda: x \to F$ a natural transformation. Then, (x, λ) is a limit of F if for any other natural transformation $\mu: y \to F$ there is a unique morphism $f: y \to x$ in C such that $\lambda_t o f = \mu_t$, for all $t \geq 1$.

Limits do not exist in all categories, or for all functors. We have provided a sketch of a constructive existence proof for the Mertens-Zamir functor Q; a theorem in MacLane ((1971), p. 121) shows that all functors that map into CompHaus have limits. When limits do exist, though, they are unique up to isomorphism.

Definition: Let C be a category and $f: x \to y$ a morphism in C; f is an isomorphism if there exists a morphism $g: y \to x$ in C such that $f \circ g = id_y$; $g \circ f = id_x$.

Theorem: Let $F: \omega^{op} \to C$ be a functor, and let $\langle x, \lambda \rangle$ $\langle x', \lambda' \rangle$ be limits of F. Then there exists a (unique) isomorphism $f: x \to x'$; in other words, limits are unique up to isomorphism.

Proof: $\langle x, \lambda \rangle$ is a limit of F and $\lambda': x'. \to F$ is another natural transformation. By the definition of a limit, there is a unique morphism $f: x' \to x$ in C such that

$$\lambda_t \circ f = \lambda'_t, \quad \text{all } t \tag{1}$$

At the same time, $\langle x', \lambda' \rangle$ is a limit of F and $\lambda: x \to F$ is another natural transformation. By the definition of a limit, there is a unique continuous function $g: x \to x'$ such that

$$\lambda'_t \circ g = \lambda_t \quad , \quad \text{all } t \tag{2}$$

By (1) and (2)

$$\lambda_t o f o g = \lambda'_t o g = \lambda_t = \lambda_t o id_x \tag{3}$$

$$\lambda'_t o g o f = \lambda_t o f = \lambda'_t = \lambda'_t \; o id_x \tag{4}$$

Consider now the natural transformation $\mu: x \to F$, defined by $\mu_t = \lambda_t o f o g$. Given that $\langle x, \lambda \rangle$ is a limit of F, there is a unique $h: x' \to x$ such that

$$\lambda_t o h = \mu_t = \lambda_t o f o g \tag{5}$$

By (3)(5) and the uniqueness of h, $h = fog$ and $h = id_x$, i.e.,

$$fog = id_x \tag{6}$$

Similarly, repeating the procedure below (4) for $\langle x', \lambda' \rangle$:

$$gof = id_{x'} \tag{7}$$

By (6) and (7), f is an isomorphism.

This concludes the discussion of the universal beliefs space seen as a limit of a functor. We now come to the second view of this space, namely that of the greatest fixed point of a (different but related) functor. Recall that $G(A) = \Delta(S \times A)$ and that $Q(1) = \Delta(S)$, $Q(t+1) = G(Q(t))$, i.e., that *Q is the functor obtained by iterating* G. The universal beliefs space X was obtained as a limit of these iterations, but X would not have a claim to be a *universal* beliefs space unless X was isomorphic to $G(X)$: for otherwise $G(X)$ would contain some beliefs not represented by points in X, and X would not represent *all* the possible beliefs in the situation described by S. Note here that isomorphism is all we can hope for, since equality of X and $G(X)$ is impossible, given that $G(X)$ is a space of probabilities on $S \times X$. This motivates

Definition: Let C be a category and $F: C \to C$ a functor. An object x of C is a fixed point of F if there is an isomorphism $f: x \to F(x)$ in C; we write $x \cong F(x)$ to denote a fixed point of F.

Hence, we are now looking for a fixed point of the beliefs functor G: *CompHaus* $\to$ *CompHaus*, and we want this fixed point to be the universal beliefs space X previously constructed as a limit. To achieve this, we will show that G preserves limits.

Definition: A functor $F: C \to C$ is continuous if it preserves limits, i.e., if whenever $\langle x, \lambda \rangle$ is a limit of the functor $T: \omega^{op} \to C$, then $\langle F(x), F(\lambda) \rangle$ is a limit of the composite functor $F \circ T: \omega^{op} \to G$. To denote continuity of F, we write $F(LimT) \cong Lim(F \circ T)$.

A simple example might clarify the notion of a continuous functor. Let C be a poset: recall that $x \to y$ *iff* $x \leq y$. Then, a functor $F: C \to C$ is simply an order-preserving function: $x \leq y$ implies $Fx \leq Fy$. A functor $T: \omega^{op} \to C$ is simply a descending chain $T(1) \geq T(2) \geq \ldots$ in the poset C. The limit of T is simply the greatest lower bound of the descending chain. Continuity of F, then, means that F preserves greatest lower bounds of descending chains.

We now take up again the beliefs functor G, where $G(A) = \Delta(S \times A)$, and its relation with the functor Q; recall that 1 is any singleton set, that M_o: $G(1) \to 1$ is the only possible function between these two sets, and that

- $Q(t+1) = G^{t+1}(1)$, $t \geq o$
- $Q(t+1 \to t) = G^{t}(M_o)$, $t \geq o$

Q is the functor of iterations of G on the function M_o. We generalize this to arbitrary functors.

Definition: Let $F: C \to C$ be a functor and $f: F(x_o) \to x_o$ a morphism in C. The functor $I_f: \omega^{op} \to C$ of iterations of F on f is defined by:

- $I_f(o) = x_o, \quad I_f(t+1) = F^{t+1}(x_o), \quad t \geq o.$
- $I_f(t+1 \to t) = F^t(f_o) \quad , \quad t \geq o.$

Referring back to the Mertens-Zamir construction, $Q = I_{M_o}$. Recall, however, that the universal beliefs space X is the limit of Q, and that we want X to be a fixed point of G. This is achieved by the following theorem.

Basic Fixed Point Theorem I: Let $F: C \to C$ be a functor, and $f: F(x_o) \to x_o$ a morphism in C. Suppose that

- $I_f: \omega^{op} \to C$ has a limit $\langle x, \lambda \rangle$
- F preserves this limit.

Then, x is a fixed point of F, called the greatest fixed point of F below x_o.

Sketch of a proof:

$$\begin{aligned} F(x) &= F(Lim I_f) \simeq Lim(F o I_f) = \\ &= \underset{t}{Lim} F(F^t(x_o)) = \underset{t}{Lim} F^{t+1}(x_o) = \\ &= \underset{t}{Lim} F^t(x_o) = x. \end{aligned}$$

The first equality follows by the definition of $x = Lim I_f = \underset{t}{Lim} F^t(x_o)$. The second equality (in fact, isomorphism), follows from the assumption that F preserves $Lim I_f$. The third equality is a symbolic way to remember the definition if I_f. The rest is obvious. A complete proof can be found in Manes and Arbib ((1986), Theorem 14, p. 276).

The terminology "greatest fixed point of F below x_o" originates in the case C is a poset. Then, as it was argued before, x is the greatest lower bound of the chain $x_o \geq Fx_o \geq F^2x_o \geq \ldots$ and by the Fixed Point Theorem it is a fixed point of F below x_o. If $y \leq x_o$ is any other fixed point of F below x_o, we have, recalling that F preserves order, $y = Fy \leq Fx_o, y = F^2y \leq F^2x_o$, etc, i.e., $y \leq F^tx_o$ for all t, and therefore $y \leq glbF^t(x_o) = x$.

Back to the Mertens-Zamir case. By the Fixed Point Theorem, all we need to guarantee that the universal beliefs space X is a fixed point of G is to show that G preserves the limit of $Q = I_{M_o}$. This was shown by Mertens and Zamir (1985) themselves (not in this terminology), and by Vassilakis ((1989, pp. 12-17); the latter paper establishes that G in fact preserves the limits of all functors $T: \omega^{op} \to CompHaus$.

The Fixed Point Theorem guarantees existence, and provides a method for approximating, the greatest fixed point of a functor below an object. Recalling that this fixed point is a limit, and that limits are unique up to isomorphism, we have a

Uniqueness Theorem: If $F: C \to C$ is a functor and $f: Fx_o \to x_o$ a morphism in C, the greatest fixed point of F below x_o is unique up to isomorphism.

Back to the Mertens-Zamir case. The universal beliefs space X depends on the original space of uncertainty S. If S itself is obtained as a limit of some functor $R\colon \omega^{op} \to \mathit{CompHaus}$ and X(t) is the universal beliefs space generated by R(t), can we conclude that $X = \mathit{Lim}_t X(t)$? This is, of course, a continuous dependence question, and is answered in the affirmative by a theorem of Lehmann and Smyth ((1981), p. 119), theorem 4.1). We will use the continuous dependence theorem in a substantive way in the next section.

4. Colimits, Least Fixed Points, Cocontinuous Functors

4.1 *Lipman's Regress*

In this section we present a version of Lipman's (1991) construction as a colimit of a functor, and as the least fixed point of a functor. We explain why both notions are necessary, and why cocontinuity of the relevant functor guarantees that the two notions coincide, and constitute the universal choice space.

An agent, as described by Lipman (1991), can perform computations (to discover the consequences of his actions). The outcome of those computations, though, is not known to the agent in advance. Lipman introduces a set L, the language, and a set S, the states of nature, to model computations. Each member of L (a word) is an assertion; if computations were deterministic, then each computation σ would produce an assertion upon termination ($x = 3$, *or* $\forall x\, \forall y (x + y = y + x)$ are examples of such assertions). To model the uncertainty of computations, Lipman takes them to be functions $\sigma\colon S \to L$ from the set of states of nature to the set of words: $\sigma(s)$ is the outcome of computation σ in state of nature s. For our purposes, and without loss of generality, we suppress S and take computations to be subsets of L, i.e., we identify the function σ with its range. We impose on additional restriction (different from that in p. 1113 of Lipman (1991)):

(A1) A computation σ is a *finite* subset of L.

Lipman now introduces the set A of feasible actions of the agent, and the set of decision procedures on A. A decision procedure is a tree whose nodes are labelled with computations or with actions: if a node is labelled with action a, the node is terminal and we say that the decision procedure might recommend action a. If a node is labelled with a decision procedure σ, and σ has n elements, then this node has n immediate successors: each successor node correspond to a possible outcome of computation σ. Depending on the decision procedure, each such successor node can be labelled either with an action or with another computation, thus allowing the agent to choose to perform computations or actions contingent on the outcome of other computations.

We now formalize this discussion. Let Σ be the set of all computations (recall that Σ is a set of finite subsets of L). The elements of Σ are ranked by a function $\nu: \Sigma \to \omega$ as follows: $\nu(\sigma)$ = cardinality of σ. Decision procedures on A are then trees whose nodes are labelled with elements of $\Sigma \cup A$, called Σ-trees on A.

Definition 1: Σ-trees on A are defined as follows:

- Each action in A is a Σ-tree on A
- If $\nu(\sigma) = m$ and $t_1, \ldots, t_m$ are Σ-trees on A, then $\sigma t_1 \ldots t_m$ is a Σ-tree on A.
- nothing else is a Σ-tree on A.

The set of all Σ-trees on A will be denoted by $T_\Sigma(A)$: it is also the set of all decision procedures on A. Note that the inclusion function inc: $A \to T_\Sigma(A)$ embeds A into $T_\Sigma(A)$, where inc $(a) = a$.

Lipman's regress can now be visualized as generated by the iteration of T_Σ on A: an agent can choose an element of A without deliberation, or he can choose a decision procedure on A (i.e., an element of $T_\Sigma(A)$); or he can choose a decision procedure on $T_\Sigma(A)$ (i.e., an element of $T_\Sigma^2(A)$), and so on. The hierarchy generated by the iteration can be written as $A \to T_\Sigma(A) \to T_\Sigma^2(A) \to \ldots$; note that the direction of the arrows is the opposite from that in the Mertens-Zamir hierarchy $1 \leftarrow G(1) \leftarrow G^2(1) \leftarrow \ldots$; while higher-order beliefs can be projected back on lower order beliefs, higher-order decision procedures can not; on the contrary, lower-level decision procedures can be included in the sets of higher-order procedures. It is this difference in the direction of the arrows that necessitates a separate (although completely dual) treatment of *co* limits, *least* fixed points and *co* continuous functions; for what Lipman seeks to resolve his regress is the least fixed point U_A *of the functor* T_Σ that is above A (i.e., includes A), and at the same time the colimit U_A of the chain $A \to T_\Sigma(A) \to T_\Sigma^2(A) \to \ldots$. The set U_A is the universal choice space generated by A, and captures all the choice opportunities open to the players in the context defined by $<A, L>$.

Our objective in the rest of this section is to establish the existence of U_A for each A. We show that (1) for each A, $T_\Sigma(A)$ is the least fixed point of some functor R_A; (2) T_Σ can be defined on functions as well, and T_Σ: $SET \to SET$ is then a cocontinuous functor; and (3) the colimit U_A *of* $A \to T_\Sigma(A) \to T_\Sigma^2(A) \to \ldots$ is also the least fixed point of T_Σ above A.

4.2 $T_\Sigma(A)$ as the least fixed point of a functor R_A.

The key in discovering the definition of R_A is in the definition of Σ-trees: if $\nu(\sigma) = k$ and $t_1 \_ t_k$ are Σ-trees in a set B, then $(\sigma, t_1 \_ t_k)$ is also a Σ-tree. Note that $(\sigma, t_1 \_ t_k)$ belongs to the set $\{\sigma\}xB^{\nu(\sigma)}$. Hence, if B is a *set* of Σ-trees, then the union $U(\{\sigma\}xB^{\nu(\sigma)}: \sigma\epsilon\Sigma)$ is also a set of Σ-trees.

We fix the name of such unions in the following

Definition 2: Let $<A_i : i\epsilon I>$ be a family of sets. Their coproduct is the set $\coprod_{i\epsilon I} A_i = \bigcup_{i\epsilon I} (\{i\}xA_i)$. If $I = \{1, 2\}$ we write $A_1 + A_2$ instead of $\coprod_{i\epsilon I} A_i$.

The functor R_A is now easily given by

Definition 3: R_A: *SET* $\rightarrow$ *SET* is defined

- on objects X, by $R_A(X) = A + \coprod_{\sigma\epsilon\Sigma} X^{\nu(\sigma)}$ with typical element either $a \epsilon A$ or $(\sigma, x_1 \_ x_K)$, $K = \nu(\sigma)$, $\sigma\epsilon\Sigma$, $x_i \epsilon X$.

- on morphisms $f: X \rightarrow Y$, by $R_A(f): R_A(X) \rightarrow R_A(Y)$ where

$$R_A(f)(a) = a \quad \text{if } a\epsilon A$$

$$R_A(f)(\sigma, x_1 \_ x_k) = (\sigma, f(x_1),\ldots, f(x_k)), \text{ if } \sigma\epsilon\Sigma,\ k = \nu(\sigma),\ x_i\epsilon X.$$

It is easy to verify that R_A preserves composition and identity, and to guess that the set of Σ-trees on A, namely $T_\Sigma(A)$, is generated by iterating R_A on the empty set ϕ: $A = R_A(\phi)$ consists of trees of depth at most one, $R_A^2(\phi)$ consists of trees of depth at most two, and so on. Recall that there is a unique function $!: \phi \to A$, namely the function with empty graph. Then, iterating R_A on ϕ we obtain a right chain

$$\phi \xrightarrow{!} R_A(\phi) \xrightarrow{R_A(!)} R_A^2(\phi) \xrightarrow{R_A^2(!)} \dots$$

Imitating the Mertens-Zamir case, we define the functor $J^A: \omega \to SET$, of iterations of R_A on $!$ by

- $J^A(\phi) = \phi$
- $J^A(n+1) = R_A(J^A(n)) = R_A^{n+1}(\phi)$
- $J^A(o \to 1) = !$
- $J^A(n+1 \to n+2) = R_A(J^A(n \to n+1)) = R_A^{n+1}(!)$

We will write occasionally J_n^A to denote $J^A(n \to n+1)$ and J_{nm}^A, to denote the composite map $J_{m-1}^A \; o \; J_{m-2}^A \; o \; .. \; oJ_n^A$, *for* $m \geq n+1$

We now show, motivated again by the analogy with Mertens-Zamir, that there is a natural transformation $\lambda^A: J^A \to T_\Sigma(A)$. First, we define $\lambda_o^A: J^A(o) \to T_\Sigma(A)$ in the only possible way, i.e., $\lambda_o^A = !$

Then, we define $\lambda_{n+1}^A: J^A(n+1) \to T_\Sigma(A)$ inductively:

- $\lambda_{n+1}^A(a) = a$ *if* $a \in A$
- $\lambda_{n+1}^A(\sigma, t_1, t_2, \dots, t_m) = (\sigma, \lambda_n^A(t_1), \lambda_n^A(t_2), \dots, \lambda_n^A(t_m))$, *if* $t_1, t_2, \dots, t_m \in J^A(n)$, and $m = \nu(\sigma)$.

Fact 1: $\lambda^A: J^A \to T_\Sigma(A)$ is a natural transformation, i.e., $\lambda_n^A = \lambda_{n+1}^A \; o \; J_n^A$, *all* $n \geq o$.

Proof: By induction on n; if n = o, then $\lambda_1^A \circ J_o^A$ maps $J^A(o) = \phi$ into $T_\Sigma(A)$, hence $\lambda_1^A \circ J_o^A = ! = \lambda_o^A$.

For the induction step, suppose $\lambda_n^A = \lambda_{n+1}^A \circ J_n^A$ for all $n \leq k$. We have to show, then, that for all $x \in J^A(k+1)$

$$\lambda_{k+1}^A(x) = \lambda_{k+2}^A(J_{k+1}^A(x)) \tag{1}$$

Recall that $J^A(k+1) = A + \coprod_{\sigma\in\Sigma} (J^A(k))^{v(\sigma)}$. Hence either $x \in A$ or $x = (\sigma, t_1, \ldots t_m)$ for some $\sigma \in \Sigma$ and some $t_i \in J^A(k)$.

If $x \in A$, we show that (1) holds:

$$\lambda_{k+2}^A(J_{k+1}^A(x)) = \lambda_{k+2}^A(x) = x = \lambda_{k+1}^A(x)$$

If $x = (\sigma, t_1, \ldots, t_m)$, then we show that (1) holds:

$$\begin{aligned}
&\lambda_{k+2}^A(J_{k+1}^A(x)) = \lambda_{k+2}^A(R_A(J_k^A)(x)) = \\
&= \lambda_{k+2}^A((\sigma, J_k^A(t_1), \ldots, J_k^A(t_m)) = \\
&= (\sigma, (\lambda_{k+1}^A \circ J_k^A)(t_1), \ldots, (\lambda_{k+1}^A \circ J_k^A)(t_m)) = \\
&= (\sigma, \lambda_k^A(t_1), \ldots, \lambda_k^A(t_m)) = \\
&= \lambda_{k+1}^A((\sigma, t_1, \ldots t_m)) = \\
&\lambda_{k+1}^A(x).
\end{aligned}$$

The first equality follows from the definition of J^A, the second from the definition of R_A, the third from the definition of λ^A, the fourth from the induction hypothesis, the fifth from the definition of λ^A, and the sixth from the definition of x.

Motivated by this example, we define natural transformations from a functor $J: \omega \to C$ to an object x.

Definition 4: Let C be a category, x an object in C, and $J: \omega \to c$ a functor. A natural transformation $\lambda: J \to \bar{x}$ is a collection $\langle \lambda_n: J(n) \to x \rangle$ of morphisms in C that satisfy for all n the commutativity conditions $\lambda_{n+1} \circ J(n \to n+1) = \lambda_n$.

The meaning of the commutativity conditions in our example becomes clear if we recall that $J_n^A: J^A(n) \to J^A(n+1)$ embeds trees of depth of most n into the set of trees of depth at most $n+1$. Hence, if a tree t belongs to $J^A(n)$. it also belongs (with the name $J_{n,m}^A(t)$) to all $J^A(m)$, $m \geq n+1$. The union $\bigcup_{m \to \omega} J^A(m)$, therefore, contains all Σ-trees, but it contains many different names for each tree (it contains "junk"). $T_\Sigma(A)$ is a subset of this union, resulting from identifying the different names of the same tree. The commutativity conditions $\lambda_{n+1}^A \circ J_n^A = \lambda_n^A$ impose this identification of tree names. For if $x \in J^A(n)$ is a tree, it is embedded into $T_\Sigma(A)$ as $\lambda_n^A(x)$. At the same time $J_n^A(x)$ (the name of x when embedded in $T(n+1)$) is embedded into $T_\Sigma(A)$ as $\lambda_{n+1}^A(J_n^A(x))$. Hence, naturality here is a "no junk" condition, as opposed to the consistency condition it was in the Mertens-Zamir case.

In the preceding paragraph, we have talked informally about "embedding" $J^A(n)$ into $T_\Sigma(A)$ via λ_n^A, and $J^A(n)$ into $J^A(n+1)$ via $J^A(n)$. We make this formal in

Fact 2: Each λ_n^A and each J_n^A *is a one-to-one mapping.*

Proof: We show that $\lambda_n^A(x) = \lambda_n^A(y)$ implies $x = y$ by induction on n. It is obviously true when $n=o$. Suppose it is true for all $n \leq k$. Then, if $\lambda_{k+1}^A(x) = \lambda_{k+1}^A(y)$, we have several cases.

Case 1: $x, y \in A$. Then $\lambda^A_{k+1}(x) = x$, $\lambda^A_{k+1}(y) = y$ and, obviously, $x = y$.

Case 2: $x \notin A$; $y \notin A$. Then $x = (\sigma, t_1, \ldots t_m)$ $y = (\psi, u_1, \ldots u_p)$, σ, ψ in $\sum$, and

$$\lambda^A_{k+1}(x) = (\sigma, \lambda^A_k(t_1), \ldots, \lambda^A_k(t_m))$$
$$\lambda^A_{k+1}(y) = (\psi, \lambda^A_k(u_1), \ldots, \lambda^A_k(u_p))$$

Since the left-hand sides are equal, we have $\sigma = \psi$, $m = p$, $\lambda^A_k(t_i) = \lambda^A_k(u_i)$; by the induction hypothesis, $t_i = u_i$ for all i. Hence, $x=y$.

The other two cases, namely $x \in A$, $y \notin A$, and $x \notin A$, $y \in A$, are incompatible with the assumption $\lambda^A_{k+1}(x) = \lambda^A_{k+1}(y)$.

The proof that each J^A_n is one-to-one is similar.

We now continue our discussion of the set $T_\Sigma(A)$ of Σ-trees on A. It is clear from its definition that $T_\Sigma(A)$ has a minimality property, in the sense that it is the "smallest" set that contains all $J^A(n)$. To capture this precisely we prove

Fact 3: Let E be any set and μ: $J^A \to E$ any natural transformation. Then there exists a unique function f: $T_\Sigma(A) \to E$ such that $f \circ \lambda^A_n = \mu_n$, all n.

Proof: The Naturality of λ^A and μ implies

$$\lambda^A_{n+1} o J^A_n = \lambda^A_n \tag{2}$$

$$\mu_{n+1} o J^A_n = \mu_n \tag{3}$$

To define f on $T_\Sigma(A)$ recall that $T_\Sigma(A)$ is a set of trees of finite depth. Hence for each t in $T_\Sigma(A)$ there is some n and some $x \in J^A(n)$ such that $\lambda^A_n(x) = t$ (recall that $J^A(n)$ contains all trees of length at most n). Hence, if f exists, it must satisfy

$$f(t) = f(\lambda^A_n(x)) = \mu_n(x) \tag{4}$$

Equation (4) defines f unambiguously, because if for some $m \geq n$ and some $y \in J^A(m)$, we have $\lambda^A_m(y) = t$, then $\mu_n(x) = \mu_m(y)$. To see this note that naturality of λ^A implies

$$(\lambda^A_m o J^A_{n,m})\ (x) = \lambda^A_n(x) \qquad i.e.,$$
$$\lambda^A_m(J^A_{n,m}(x)) = t \qquad i.e.,$$
$$\lambda^A_m(J^A_{n,m}(x)) = \lambda^A_m(y) \qquad i.e., \text{ by fact 2}$$

$$J^A_{n,m}(x) = y. \tag{5}$$

But then by naturality of μ and (5):

$$\mu_m(y) = \mu_m(J^A_{n,m}(x)) = \mu_n(x)$$ Q.E.D.

Facts (1) and (3) describe the crucial properties of the set of trees $T_\Sigma(A)$; they define it as the colimit of the functor J^A. We provide the formal definition,

Definition 5: Let C be a category, J: $\omega \rightarrow C$ a functor, x an object in C. We say that $\langle x, \lambda \rangle$ is a colimit of J if

- λ:$J \rightarrow x$ is a natural transformation

- If $\mu: J \to y$ is another natural transformation, then there is a unique morphism $f: x \to y$ in C such that $f \circ \lambda_n = \mu_n$, all n.

Colimits are unique up to isomorphism; we will sometimes talk about *the* colimit of a functor to emphasize this.

Fact 4: The set of Σ-trees on A, $T_\Sigma(A)$, is the colimit of $J^A: \omega \to SET$, where J^A is the functor of iterations of R_A on!: $\phi \to R_A(\phi)$

Proof: Facts 1 and 3. We write $T_\Sigma(A) = \mathit{Colimit}\ J^A = \mathit{Colimit}\ R_A^k(\phi)$.

The existence of colimit J^A established by facts 1 and 3 is no accident. We have the

Colimit Existence Theorem: *SET* has all colimits, i.e., if $J: \omega \to SET$ is a functor, then colimit J exists.

Proof: Manes and Arbib ((1986), example 7, p. 260).

We have now the same problem as in the Mertens-Zamir case: $T_\Sigma(A)$ cannot be the set of *all* decision procedures unless it is isomorphic to $R_A(T_\Sigma(A))$. The problem is resolved exactly as in that case: we show that R_A preserves colimits (i.e., that it is cocontinuous). The key to that is to observe that the definition of R_A involves only coproducts and products of sets: such functors are called *polynomial* by Manes and Arbib ((1986), p. 243).

Definition 6: A functor $F: C \to C$ is cocontinuous if it preserves colimits of functors $J: \omega \to C$,, i.e., if whenever $<x, \lambda>$ is a colimit of J, $<F(x), F(\lambda)>$ is a colimit of the composite functor $F \cdot J$. We write $F\ (Colimit J) = \text{Colimit}\ ((F o J)$

Definition 7: A functor $F: SET \to SET$ is polynomial if it is obtained from constant and identity functors by finitely many applications of the operations of product, coproduct and composition.

Fact 5: Polynomial functors are cocontinuous

Proof: Manes and Arbib ((1986), corollary 12, p. 270)

Fact 6: R_A: $SET \to SET$ is cocontinuous provided Σ is finite.

Proof: Fact 5 and the fact that R_A is polynomial.

Another, simpler, example of a cocontinuous functor is obtained when C is a poset and $F: C \to C$ an order preserving map. Then, a functor $J: \omega \to C$ is just an ascending chain $J(1) \leq J(2) \leq \ldots$ in C; colimit $J = \underset{n}{LUB} J(n)$; and cocontinuty of F means that F preserves least upper bounds of ascending chains.

Finally, we establish that the set of Σ-trees on A is a least fixed point of R_A, in addition to being a colimit of the functor of iterations of R_A on $!: \phi \to R_A(\phi)$.

Definition 8: Let $F: C \to C$ be a functor and $f: a \to Fa$ a morphism in C. The functor $J^f: \omega \to C$ of iterations of F on f is defined by

- $J^f(o) = a$
- $J^f(n+1) = F(J^f(n)) = F^{n+1}(a)$

- $J^f(o \to 1) = f$
- $J^f(n+1 \to n+2) = F(J^f(n \to n+1)) = F^{n+1}(f)$.

Basic Fixed Point Theorem 2: Let $F: C \to C$ be a functor, $f: a \to Fa$ a morphism in C, and $J^f: \omega \to C$ the functor of iterations of F on f. Suppose that

- J^f has a colimit $<x, \lambda>$
- F preserves this colimit

Then, x is a fixed point of F, called the least fixed point of F above a.

Sketch of a proof:

$$
\begin{aligned}
Fx &= F\,(\text{colimit } J^f) = \text{colimit } (FoJ^f) = \\
&= \text{colimit } (FoF^i(a)) = \textit{colimit } F^{i+1}(a) = \\
&= \textit{colimit } F^i(a) = x.
\end{aligned}
$$

where the first and last equality follow from the definition of x as $\text{colimit } J^f = \text{colimit } F^i(a)$, the second equality (in fact, isomorphism) follows from the fact that F preserves this colimit, and the rest is obvious. A complete proof can be found in Manes and Arbib ((1985), p. 270, theorem 13).

The terminology "least fixed point of F above a" originates in the case C is a poset. Then, as it was argued before, x is the least upper bound of the chain $a \leq Fa \leq F^2a \leq$... and, by the fixed point theorem, it is a fixed point of F above a. If $y \geq a$ is any other fixed point of F

above a then we have, recalling that F preserves order, $y = Fy \geq Fa$, $y = F^2y \geq F^2a$, etc.; in other words $y \geq F^t(a)$ for all t, and therefore $y \geq LUB\,F^t(a) = x$.

We now conclude the discussion in this section with

Fact 7: The set $T_\Sigma(A)$ of Σ-trees on A is the least fixed point of the functor $R_A: SET \to SET$.

Proof: It follows from facts 4, 6 and the basic fixed point theorem with $C = SET$, $F = R_A$, $a = \phi$, $f = !$.

4.3 $T_\Sigma: SET \to SET$ *is a Cocontinuous Functor*

Obviously, we can think of T_Σ as a map that assigns to each set A the set $T_\Sigma(A)$ of Σ-trees on A. It is possible, and necessary for what we are after, to define T_Σ on functions as well and turn it into a functor. The way this is done illustrates why it is useful to think of $T_\Sigma(A)$ as a fixed point of R_A. Let $h: A \to B$ be any function; we want to construct $T_\Sigma(h): T_\Sigma(A) \to T_\Sigma(B)$ in such a way that T_Σ is a functor. Let $<\lambda_n^A: J^A \to T_\Sigma(A)>$ and $<\lambda_n^B: J^B \to T_\Sigma(B)>$ be the colimit natural transformations associated with $T_\Sigma(A)$, $T_\Sigma(B)$ respectively. Our objective is to construct a natural transformation $<\mu_n; J^A \to T_\Sigma(B)>$; Then, the fact that $T_\Sigma(A) = Colimit\,J^A$ will imply the existence of a unique function $g: T_\Sigma(A) \to T_\Sigma(B)$ such that $\mu_n = g \circ \lambda_n^A$ for all n. We will then define $T_\Sigma(h) = g$.

Step 1: Define a family of functions $f_n: J^A(n) \to J^B(n)$ by $f_0 = !$, $f_1 = h$, $f_{n+1}(a) = a$ *if* $a \in A$, and $f_{n+1}((\sigma, t_1, \ldots, t_m)) = (\sigma, f_n(t_1), \ldots, f_n(t_m))$, for $t_1, \ldots, t_m$ in $J^A(n)$ and for $m = v(\sigma)$.

Note that the definition of f_{n+1} is complete because $J^A(n+1) = R_A(J^A(n)) = A + \coprod_{\sigma\in\Sigma}(J^A(n))^{\nu(\sigma)}$. Hence, the typical element of $J^A(n+1)$ is either $a\in A$ or $(\sigma, t_1, \_ t_m)$ for some $\sigma\in\Sigma$ and some $t_i\in J^A(n)$.

Step 2: The family $\langle f_n \rangle$ satisfies the following commutativity conditions for all n

$$f_{n+1} o J_n^A = J_n^B o f_n \tag{1}$$

Proof: Recall that J_n^A: $J^A(n) \to J^A(n+1)$, and similarly for J_n^B. We proceed by induction on n. If $n=o$, then (1) is true because both sides of equality (1) map the empty set into J^B (1), and there is only one such map, namely !. Suppose now that (1) holds for some $n \geq o$, and that $x\in J^A(n+1)$. We have two cases

Case 1: $x\in A$. Then, $(f_{n+2} o J_{n+1}^A)(x) = x = (J_{n+1}^B o f_{n+1})(x)$ by the fact that on A all the functions involved are identities.

Case 2: $x = (\sigma, t_1, \_ t_m)$ for some $\sigma\in\Sigma$ and some $t_i\in J^A(n)$. Then,

$f_{n+2}(J_{n+1}^A(x)) = f_{n+2}((\sigma, J_n^A(t_1), ..., J_n^A(t_m))) =$
$(\sigma,(f_{n+1} o J_n^A)(t_1), ..., (f_{n+1} o J_n^A)(t_m)) =$
$(\sigma,(J_n^B o f_n)(t_1), ..., (J_n^B o f_n)(t_m)) =$
$J_{n+1}^B((\sigma, f_n(t_1), ..., f_n(t_m))) =$
$(J_{n+1}^B o f_{n+1})((\sigma, t_1, \_ t_m)) =$
$(J_{n+1}^B o f_{n+1})(x).$

where the first equality follows from the definition of J^A_{n+1}, the second from the definition of f_{n+2}, the third from the induction hypothesis, the fourth from the definition of J^B_{n+1}, the fifth from the definition of f_{n+1}, and the sixth from the definition of x.

Step 3: Define $\mu_n: J^A(n) \to T_\Sigma(B)$ by $\mu_n = \lambda^B_n o f_n$. Then $\mu: J^A \to T_\Sigma(B)$ is a natural transformation, i.e., for all n

$$\mu_{n+1} o J^A_n = \mu_n.$$

Proof: $\mu_{n+1} o J^A_n = \lambda^B_{n+1} o f_{n+1} o J^A_n = \lambda^B_{n+1} o J^B_n o f_n = \lambda^B_n o f_n = \mu_n$, where the first equality follows from the definition of μ_{n+1}, the second from Step 2, the third from the naturality of λ^B and the fourth from the definition of μ_n.

Step 4: There is a unique function $g: T_\Sigma(A) \to T_\Sigma(B)$ such that $g o \lambda^A_n = \mu_n$ for all n.

Proof: $\lambda^A: J^A \to T_\Sigma(A)$ is a colimit natural transformation and $\mu: J^A \to T_\Sigma(B)$ is another natural transformation with the same source.

Step 5: Define $T_\Sigma(h) = g$

Fact 8: $T_\Sigma: SET \to SET$ is a functor.

Proof: in the appendix. This result is also stated in Arbib and Manes ((1976), observation 7, p. 159). We are now ready to sketch a proof of

Fact 9: T_Σ: $SET \to SET$ is cocontinuous.

We want to show that

$T_\Sigma\ (colimit\ A_n) = Colimit\ T_\Sigma(A_n)$ where $A_0 \to A_1 \to A_2 \to \ldots$ is a right chain of sets. Recall that

$$T_\Sigma(A) = Y(R_A) = Colimit\ R^k(\phi) \tag{1}$$

where $Y(R_A)$ is the least fixed point of R_A above ϕ. Note that Y can be thought of as a map that assigns to each cocontinuous functor F: $SET \to SET$ $Y(F)$, its least fixed point above ϕ. Denote the category of these cocontinuous functors by $[SET \to SET]$. Lehmann and Smyth ((1981), p. 199, Th. 4.1) have shown the

Continuous dependence theorem: Y: $[SET \to SET] \to SET$ is a cocontinuous functor, i.e., for any right chain $J_0 \to J_1 \to \ldots$ of cocontinuous functors, we have:

$$Y\ (Colimit\ J_n) = Colimit\ (Y(J_n)) \tag{2}$$

In our case $J_n = R_{An}$, i.e., $J_n(X) = A_n + \coprod_{\sigma \in \Sigma} X^{\nu(\sigma)}$. Note that R_{A_n} depends on A_n only through a coproduct operation. Our previous result on the cocontinuity of polynomial functors yields, therefore

$$Colimit\ R_{A_n} = R_{Colimit\ A_n} \tag{3}$$

We have, by (1), (2) and (3), that

$T_\Sigma\ (Colimit A_\omega) = Y(R_{Colimit\ A_\omega}) =$

$Y\ (Colimit R_{A_\omega}) = Colimit\ (Y(R_{A_\omega})) =$

$Colimit\ T_\Sigma(A_\omega)$, *Q.E.D.*

Finally, we obtain

Lipman's Theorem: For any set A, there is a set U_A, called the universal choice space generated by A, that satisfies

- U_A is the least fixed point of T_Σ above A.
- $U_A = Colimit\ T_\Sigma^k(A)$

Proof: Obvious from fact 9 and the basic fixed point theorem 2, with $C = SET,\ F = T_\Sigma,\ a = A,\ f = \lambda_1^A$.

5. Conclusions

This paper has argued for functorial fixed points as the right tool in resolving problems of infinite regress. We have sketched the constructions of Mertens-Zamir (1985) and Lipman (1991) to support this argument. The same tools have been used to resolve Crawford's (1985) regress in Vassilakis (1991, 1992).

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Appendix

Proof of Step 6: :Let h^1: $A_1 \to A_2$ and h^2: $A_2 \to A_3$ be two functions, and h: $A_1 \to A_3$ their composite. We show that $T_\Sigma(h) = T_\Sigma(h^2) o T_\Sigma(h^1)$. To achieve this, we will construct the natural transformations μ^1, μ^2 and μ that correspond to h^1, h^2 and h, respectively. By Step 3, we have

$$\mu_n^1 = \lambda_n^{A_2} o f_n^1 \tag{1}$$

$$\mu_n^2 = \lambda_n^{A_3} o f_n^2 \tag{2}$$

$$\mu_n = \lambda_n^{A_3} o f_n \tag{3}$$

while by Step 1

$$f_o^i = ! = f_o, \quad i=1.2 \tag{4}$$

$$f_1^i = h^i, f_1 = h \tag{5}$$

$$f_{n+1}^i(a) = a = f_{n+1}(a), \quad a \in A \tag{6}$$

$$f_{n+1}^i(\sigma, t_{1, \ldots,} t_m) = (\sigma, f_n^i(t_1), \ldots, f_n^i(t_m)) \tag{7}$$

$$f_{n+1}(\sigma, t_{1, \ldots,} t_m) = (\sigma, f_n(t_1), \ldots, f_n(t_m)) \tag{8}$$

Finally, by Steps 4 and 5, $T_\Sigma(h^i)$, $T^\Sigma(h)$ are the unique functions that satisfy:

$$T_\Sigma(h^i) o \lambda_n^{A_i} = \mu_n^i \quad , \quad i=1.2 \tag{9}$$

$$T_\Sigma(h) o \lambda_n^{A_1} = \mu_n \tag{10}$$

We will show that $T_\Sigma(h_2) o T_\Sigma(h_1) o \lambda_n^{A_1} = \mu_n$ and that, therefore, $T_\Sigma(h) = T_\Sigma(h_2) o T_\Sigma(h_1)$.

$$T_\Sigma(h_2) o T_\Sigma(h_1) o \lambda_n^{A_1} = \qquad \text{by (9)}$$

$$T_\Sigma(h_2) o \mu_n^1 = \qquad \text{by (1)}$$

$$T_\Sigma(h_2) o \lambda_n^{A_2} o f_n^1 = \qquad \text{by (9)}$$

$$\mu_n^2 o f_n^1 = \qquad \text{by (2)}$$

$$\lambda_n^{A_3} o f_n^2 o f_n^1 =$$

$$\lambda_n^{A_3} o f_n = \qquad \text{by (3)}$$

$$\mu_n.$$

(The equality $f_n^2 o f_n^1 = f_n$ follows easily by induction on n from (4), (5), (6), (7) and (8)).