

Statistical limits to Equity

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abstract

We derive the most probable distribution of resources for a simple society.

We find that a probabilistic analysis forbids both too much and too less equity, and selects instead a minimally ordered state. We give the detailed calculations for a special model where the population and resources are fixed, and resources are owned only by individuals. We show that in general the equity is greater whenever the volume of the indifference manifold grows faster as a function of individual rent.

1 Introduction

Maybe the central task of the whole economical science is the clever administration of the resources of a society in order for everyone to fulfill their material needs. It is thus interesting the question of to what extent one could, in principle, distribute richness with equity. At first sight, it seems perfectly conceivable (although highly utopic) a society in which everyone access exactly the same 'amount of richness'. Is there some conceivable bound to get such a setting? Well, suppose we begin distributing goods to N people: there is just one 'configuration' in which everyone receives exactly the same amount of any good. If one decide being unfair by benefitting just someone, there are N ways to do so (one for every benefitted person). One can see that as we increase the 'unfairness', the number of possible configurations grow fastly. Thus, if we 'prepare the system' to be fair at the beginning, by exchange it will disorder with little probability of reordering. This reasoning puts a bound also on the unfairness: a few too wealthy people and the rest with nothing is again an ordered state.

Here we will give the detailed mathematics for a 'dilute' society, for which the following postulates holds:

1. The population N remains fixed
2. There is a fixed amount of richness, R

3. There are only individual patrimonias

Under these simplifications, we will find that we can find the probability distribution for individual wealthiness, giving a precise meaning to the above discussion. We will find the number of states for detailed distributions giving rise to the same global state (overall 'unfairness'). Those who know statistical mechanics will find a very close resemblance with ideal gases. Notice, however, that the following is not an analogy: everything is derived under the given and a few more assumptions. That much of the derivations are formally identical is quite amusing, but not that surprising once one realizes that both are statistics on the ways one can distribute something indestructible among a fixed number of 'objects'.

The conclusions drawn from this model are quite limited: richness is created and destroyed, populations change and in complex societies large amounts of resources belongs to 'composite objects' like associations, corporations or even the whole society ('large range interactions'). Even more seriously, actual societies are away from statistical equilibrium: there is a typical 'relaxation time' needed for the society to exchange goods until it gets to an equilibrium, but growth (especially sectorial growth) usually happens in shorter periods, so equilibrium is not reached. We know from other disciplines that systems far away from equilibrium can get extremely striking complexity (life being a paradigmatic thermodynamical example). Anyway, analyzing what happens at equilibrium is the mandatory beginning, and very important conclusions are available at this primitive stage.

2 Partition of resources: the distribution function

For clarity, we will suppose that each individual can have any patrimonium from a discrete set r_1, r_2, \dots, r_k . Every r_i could be composed of many combinations of different goods, say there are g_i such combinations for each r_i . We also suppose that any 'microscopical state' (any specific combination of goods) is accesible with equal probability for every individual ('fair rules'). We will call n_i to the number of people owning r_i each. Recall that the total population is N and the total wealthiness is R . We have the bounds

$$N = n_1 + n_2 + \dots + n_k \quad (1)$$

$$R = n_1 r_1 + n_2 r_2 + \dots + n_k r_k \quad (2)$$

We will count the total number of configurations and find which is the most probable distribution (that is, the $n(r)$ having the maximum amount of states). First, notice that we can fill the state r_1 with one person in $N g_i$ ways, then we have $N - 1$ remaining persons to accomodate. Once n_i persons have been placed into an r_i we must remember to divide by the $n_i!$ ways one does so, and we are left with

$$\Omega = \frac{N! g_1^{n_1} \dots g_k^{n_k}}{n_1! \dots n_k!} \quad (3)$$

We now suppose that there are lots of n_i 's at each r_i level, and we take them to be continuous. For each r_i we will find an n_i (we will call it $n(r_i)$) wich maximizes 3 subject to the bounds 1 and 2. In order to do so, we introduce the bounds as Lagrange multipliers, and extremize

$$S = \ln(\Omega) + \alpha(N - n_1 - n_2 - \dots - n_k) + \beta(R - n_1 r_1 - n_2 r_2 - \dots - n_k r_k) \quad (4)$$

In $\ln(\Omega)$ we use Stirling formula. We get then

$$\frac{\partial S}{\partial n_i} = \ln(g_i) - \ln(n_i) - \alpha - \beta r_i = 0 \quad (5)$$

while the conditions $\frac{\partial S}{\partial \alpha}$ and $\frac{\partial S}{\partial \beta}$ just imposes the bounds. Thus we have got a Boltzman-like distribution for richness

$$n(r) = g_i e^{-\alpha} e^{-\beta r} \quad (6)$$

with α and β determined by the conditions

$$N = e^{-\alpha} \sum_r e^{-\beta r}$$

$$R = e^{-\alpha} \sum_r r e^{-\beta r}$$

3 Particular models

Lets consider some concrete distributions in order to see what do we have here. The above distribution will be considered continuous, that is, $n(r)$ means that there are $n(r)dr$ individuals with patrimonia between r and $r+dr$. Then, since the probability is exponentially damped, we will forget to bound r and we will take it to range between zero and infinity (in fact, it couldnt exceed R , but the probability of being R while one expect it to be about R/N is about e^{-N} , that is, astronomically small). We do this because it is much easier to calculate. The distribution is thus

$$n(r) = g(r)e^{-\alpha}e^{-\beta r}$$

In order to give a $g(r)$ we will consider that we have m goods whose unitary price is much cheaper that the mean wealthy (we will clarify this somewhat dark condition below) so we can think that we have a continuum of goods. For a given r we have an indifference hipersurface such that $r = r_1 + \dots + r_m$ (r_i being the price of the ammount of the i -th good owned by the individual). So, $g(r)$ is proportional to the $(m-1)$ -dimmmensional volume of the hypersurface: $S(r) = \sqrt{m}r^{m-1}$. That surface is an hyperplane bounded by the walls defined by $r_i = 0$, and its 'volume' is of the form Cr^{m-1} . So we take as our distribution

$$n(r) = Cr^{m-1}e^{-\alpha}e^{-\beta r} \quad (7)$$

Let us first solve for the parameters α and β . We have

$$N = Ce^{-\alpha} \int_0^{\infty} r^{m-1} e^{-\beta r} dr$$

$$R = Ce^{-\alpha} \int_0^{\infty} r^m e^{-\beta r} dr$$

This gives the result

$$n(r) = \frac{N}{(m-1)!} \left(\frac{m}{\bar{r}}\right)^m r^{m-1} e^{-mNr/R} \quad (8)$$

where $\bar{r} = R/N$ is the mean rent per capita. From 8 we can tell what is the condition for low unitary prices: we mean that every price should be much

lower than $\frac{R}{mN}$. If one has many prices independent of R , as R grows we should include more and more goods, so m depends indeed on R . We see that the more goods available, the more peaky will the distribution be. As m grows, the position of the peak approach from below the rent per capita R/N . The variance for the individual rent (that is roughly the dispersion of income) is

$$\Delta r = \frac{1}{\sqrt{m}} \frac{R}{N} \quad (9)$$

4 General Discussion

For computational ease, we solved for an artificially simple example in wich any good could be obtained from any combination summing the same price. In general, however, one should have an 'indifference manifold' whose volume will in general depend on the income. What is important for the calculations above is the 'dispersion relation' connecting the individual rent and the volume of the indifference manifold. If it follows a power law $V = Cr^m$, all the above applies including the calculation for the dispersion. It shows that equity will grow if the elasticity m grows.

We have shown that a closed economy at equilibrium cannot be perfectly fair. Anyway, a complex society with large individual rent and large diversity can reach a state quite close to equity. On the other hand, subsistence economies in wich only a few goods are exchanged are mostly composed of poor individuals.

An important remark on fluctuations is in order here: the 'occupation density' $d = n(r)/N$ will indeed fluctuate around the distribution found, and one should wonder how often it will differ appreciably from that distribution. If the answer is 'very often', then all the analysis done is nearly useless. The answer is found by calculating the standard deviation Δd , which is found to be d/\sqrt{N} . One could be tempted to take $N = 6000000000$, but it must be remembered that larger populations have larger time relaxations, depending on the intensity of the trade exchange. That time for an entire continent would probably be many years, so normal suceses impeded reaching equilibrium. Where interchange is strong enough in order to have a somewhat small relaxation time is inside a country, a city, etc where population is typically a few million people, so fluctuations in d can be estimated in 10^{-3}

of d , which is quite small. However, economies are continuously changing, and are surely much away from equilibrium.