

A Dual Model of Cooperative Value

Barry Feldman*

JUNE 15, 2002

ABSTRACT

An expanded model of value in cooperative games is presented in which value has either a linear or a proportional mode, and NTU value has either an input or an output basis. In TU games, the modes correspond to the Shapley (1953) and proportional (Feldman (1999) and Ortmann (2000)) values. In NTU games, the Nash (1950) bargaining solution and the Owen-Maschler (1989, 1992) value have a linear mode and an input basis. The egalitarian value (Kalai and Samet (1985)) has a linear mode and an output basis. The output-basis NTU proportional value (Feldman (1999)) and the input-basis variant, identified here, complete the model.

The TU proportional value is shown to have a random marginal contribution representation and to be in the core of a positive convex game. The output-basis NTU variant is shown to be the unique efficient Hart and Mas-Colell consistent NTU value based on equal proportional gain in two-player TU games. Both NTU proportional values are shown to be equilibrium payoffs in variations of the bargaining game of Hart and Mas-Colell (1996). In these variations, players' probabilities of participation at any point in the game are a function of their expected payoff at that time. Limit results determine conditions under which players with zero individual worth receive zero value. Further results show the distinctive nature of proportional allocations to players with small individual worths. In an example with a continuum of players bargaining with a monopolist, the monopolist obtains the entire surplus.

JEL Classification: C7.

Key words: cooperative game, value, mode, basis, bilateral cooperation, endogenous bargaining power, potential, equal proportional gain, consistency, noncooperative bargaining, zero players, monopoly.

* Ibbotson Associates, 225 North Michigan Avenue, Chicago, IL, 60601, USA. Email: bfeldman@ibbotson.com. Questions and comments are welcome. Thanks to participants at the 1999 Stony Brook International Conference on Game Theory, the First World Congress on Game Theory, the 2000 Econometric Society World Conference, and two anonymous referees for comments on earlier work. Thanks also to Ibbotson Associates for support of this work. ©2002, Barry Feldman.

1 Introduction

In cooperative games, players bargain over the distribution of the gains from cooperation. The Shapley (1953) value is an important model of how these gains might be distributed in TU (transferrable utility) games. Myerson (1977) characterizes the Shapley value in a model that assigns equal gains to players forming bilateral cooperation agreements. Hart and Mas-Colell (1989) show a related equal gain property of Myerson (1980) is intimately related to consistency and the existence of a difference potential in TU and NTU (non-transferrable utility) games. These results are used to characterize the Shapley and egalitarian (Kalai and Samet (1985)) values. Ortmann (2000) defines a ratio potential, derives the TU proportional value, and characterizes it through consistency with equal proportional gain bargaining in two-player games. In Feldman (1999), I derive similar and additional results for TU and NTU games.

This paper proposes a dual model of cooperative value. Equal and proportional division are generally considered to be the two fundamental models of fairness in allocation (see, e.g., Adams (1965), Homans (1977), Thompson (1998), or Young (1994)). The dual model identifies these methods with two *modes* of value allocation: linear and proportional. Substantive meaning is given to this duality by demonstrating properties of the proportional value and by providing TU and NTU characterizations of it and related values based on consistency, the Myerson (1977) bilateral cooperation model, and noncooperative implementation.

The principal characterizations are provided by a variation on the Hart and Mas-Colell (1996) noncooperative bargaining game. This variation uses a *proposed payoff-weighted participation mechanism*. Participation means making a proposal. Equilibrium payoffs in TU games correspond to the proportional value when the probability of participation at any point in the game is proportional to a player's average proposed payoff at that time. In the original game, players have equal probability of participation and payoffs correspond to the Shapely value in TU games, the Owen-Maschler (1989, 1992) value in NTU games, and the Nash (1950) bargaining solution in pure bargaining games.

The noncooperative characterization suggests that the proportional value provides a simple model of *endogenous bargaining power* in cooperative games. Bargaining power is endogenous in the sense that a player's degree of participation is a function of its individual endowment and the endowments of its coalitions. In contrast, the linear mode can be seen to reflect a model of *formal equality* in participation. Note, however, that all players in proposed payoff-weighted participation games have the same ability to reject proposals as players in the original Hart and Mas-Colell game.

In the dual model, NTU value also has a *basis*. The basis determines how allocations are compared. *Input-basis* allocation rules function as if players bargain over the resources allocated to them, rather than the utility they receive. The Nash bargaining solution and the Owen-Maschler value have an input basis. The coalitional resources necessary to implement a player's input-basis allocation are priced by the marginal rates of transformation of player's utilities. These are determined by the λ -weights generated

by the hyperplane supporting an efficient allocation. In contrast, *output-basis* allocation rules function as if players bargain directly over the utility they receive. The egalitarian value and the proportional value have an output basis. The logic of the dual model implies that there should also be a value with proportional mode and an input basis. This value is identified with the noncooperative game. In pure bargaining games, this value becomes a proportional analog of the Nash bargaining solution, completing the dual model. The egalitarian value is also implemented here.

Mode and basis are shown to be independent in NTU games. A player's probability of participation is independent of its average proposed payoff in the linear mode, but varies with it in the proportional mode. Similarly, participation is independent of a player's λ -weight with input-basis values, but varies with it in output-basis values. Thus, participation probabilities for implementing the output-basis proportional value are proportional to λ -weighted average proposed payoffs.

Several implications of proportional value allocation are studied. Players with zero individual worth, *zero players*, are shown to receive zero value if the coalition of all zero players has zero worth as well. Since the proportional value is not defined on games with zero coalitional worths, this result is in the limit as the worth every coalition of zero players goes to zero. The intuition from the bargaining game is that players with small individual worth have low probability of selection to propose in two-player stage games. Backward induction then determines the outcome. However, when the coalition of zero players has positive worth, the total allocation to zero players is positive. Results for zero players are generalized to games with *small* players. Conditions are developed under which these results, in turn, generalize to hierarchical classes of small players whose proportional value allocations are hierarchical as well.

An example demonstrating the relevance of the proportional mode to economic theory is provided by a game in which one large player and a variable number of small players engage in joint production. Under proportional value allocation, the share of profits obtained by the small players converges downward to their minimum core allocation as the number of small players grows, as should be expected. Under Shapley value allocation and constant returns to scale, and in the limit, the allocation to the large player is half its proportional value allocation. The total Shapley value allocation to small players in excess of their minimal core allocation is then equal to the larger player's excess allocation, implausibly implying that, in some sense, they have equal total bargaining power.

One objection to the dual model could be that only linear-mode input-basis value allocations are invariant to affine transformation of players' utility functions. In the proportional mode, translations lead to changes in allocations because they imply changes in the reference points for determining proportional gains. Rescalings of player's utility functions lead to changes in output-basis value allocations as well, because player's utilities are regarded as directly comparable. Invariance has no special status in the dual model. Instead, changes in real allocations resulting from affine transformations are understood to result from implied changes in interpersonal comparisons that arise naturally in the bargaining process. The conclusion expands on this perspective.

Section 2 of this paper defines the TU and output-basis NTU proportional values. Classical value properties are demonstrated, including random marginal contribution representation and inclusion in the core of a convex game. Section 3 characterizes the NTU proportional value through consistency. Section 4 develops a coalition formation result and presents the characterization based on the Myerson (1977) bilateral cooperation model. Section 5 presents the noncooperative implementations and identifies the input-basis proportional value. Section 6 considers games with zero and small players. Section 7 presents the production example. The conclusion and an appendix collecting several proofs follow.

2 The Proportional Value

2.1 Definition in TU and NTU Games

Let N be a set of n players in a TU cooperative game v in *characteristic function form*. Any coalition $S \subseteq N$ in such a game has a scalar *worth* $v(S)$. Let $s = |S|$ be the number of players in S . The worth of \emptyset , the *empty set*, is zero. Generally, TU games will be required to be *positive*, i.e., $v(S) > 0$ for $S \neq \emptyset$. Section 6 shows how the proportional value can be extended to games having coalitions with zero worth. The restriction of v to an $S \subseteq N$ is represented by the pair (S, v) . Thus, $(N, v) = v$ and $(T, v)(S) = v(S)$ for $T \supseteq S$. The operator “ \setminus ” is used to indicate set subtraction of a player or coalition. The *immediate subcoalitions* of S are the coalitions resulting from the removal of a single player. Small coalitions are usually identified with an overbar: i.e. $\bar{i} = \{i\}$ and $\bar{ij} = \{i, j\}$. Let $\mathcal{R}(S)$ be the set of all orders of the players in S and let $r = (r_1, r_2, \dots, r_s) \in \mathcal{R}(S)$ be any such order. Then define $R_i^r = \{r_j : j \leq i\}$ to be the coalition composed of the first i players in the order r .

The *ratio potential* of a positive TU cooperative game v is a function from v to the real numbers, $P : 2^N \rightarrow \mathbb{R}$, defined by the recursive relation

$$(2.1) \quad v(S) = \sum_{i \in S} \frac{P(S, v)}{P(S \setminus i, v)} \quad \text{or, equivalently,} \quad P(S, v) = v(S) \left(\sum_{i \in S} \frac{1}{P(S \setminus i, v)} \right)^{-1}$$

Given v and any $P(\emptyset, v) \neq 0$, potentials for all other $S \subseteq N$ are uniquely determined. The right-hand representation shows that the ratio potential of a coalition is the product of its worth and the harmonic mean of the potentials of its immediate subcoalitions. The *ordered worth product* of S according to any order of players $r \in \mathcal{R}(S)$ is $\prod_{i=1}^s v(R_i^r)$. The following lemma shows that the potential of a coalition is the harmonic mean of its ordered worth products when $P(\emptyset, v) = 1$.

Lemma 2.1 *For any positive TU game v , and for all $S \subset N$, $S \neq \emptyset$, the ratio potential*

of S , may be represented as:

$$P(S, v) = P(\emptyset, v) \left(\sum_{r \in \mathcal{R}(S)} \prod_{i=1}^s v(R_i^r)^{-1} \right)^{-1} = P(\emptyset, v) v(S) \left(\sum_{r \in \mathcal{R}(S)} \prod_{i=1}^{s-1} v(R_i^r)^{-1} \right)^{-1}.$$

Proof: The two formulations are equivalent because $v(S)$ is common to all of the ordered worth products on the left-hand side and factors out. If all potentials for coalitions of cardinality $s - 1$ satisfy the relation, then potentials of cardinality s must as well: Use the right-hand version of equation (2.1) and insert potentials of the immediate subcoalitions of the first form of the lemma and the right-hand version of the lemma results. To complete the proof, observe that it is clearly true for singleton coalitions: $P(\bar{i}, v) = P(\emptyset, v) v(\bar{i})$. \square

The *discrete derivative* of the ratio potential of a coalition S with respect to a player $i \in S$ in a game v is defined as the ratio of the potential of S to the potential of the immediate subcoalition of S without player i , coalition $S \setminus i$:

$$(2.2) \quad D^i P(S, v) = P(S, v) / P(S \setminus i, v).$$

The *proportional value* of a positive TU game assigns each player the discrete derivative of the potential of the coalition of all players with respect to that player:

$$(2.3) \quad \varphi_i(S, v) = D^i P(S, v) = v(S) \sum_{q \in \mathcal{R}(S \setminus i)} \prod_{j=1}^{n-1} v(R_j^q)^{-1} / \sum_{r \in \mathcal{R}(S)} \prod_{k=1}^{n-1} v(R_k^r)^{-1}.$$

Remark 2.1 *The proportional value of a player is a ratio of harmonic means. A harmonic mean may be considered a type of expectation. In this sense, the proportional value is a player's expected marginal proportional contribution.*

The proportional value for i in two-player games is then

$$(2.4) \quad \varphi_i(v) = \frac{v(\bar{i})}{v(\bar{i}) + v(\bar{j})} v(\bar{i}\bar{j}) = v(\bar{i}) + \frac{v(\bar{i})}{v(\bar{i}) + v(\bar{j})} (v(\bar{i}\bar{j}) - v(\bar{i}) - v(\bar{j})).$$

Clearly, the gain from cooperation is shared in proportion to each player's individual worth. The proportional value for a player in 3-player games has the following somewhat less intuitive form:

$$\varphi_1(v) = v(\overline{123}) \frac{v(\overline{1})v(\overline{12})v(\overline{13})(v(\overline{2})+v(\overline{3}))}{v(\overline{1})v(\overline{12})v(\overline{13})(v(\overline{2})+v(\overline{3})) + v(\overline{2})v(\overline{12})v(\overline{23})(v(\overline{1})+v(\overline{3})) + v(\overline{3})v(\overline{13})v(\overline{23})(v(\overline{1})+v(\overline{2}))}.$$

Many results in this paper are obtained for the more general NTU bargaining environment. In an NTU game V , the worth of a coalition S , $V(S)$, is represented by a set of feasible allocations. This set represents the utilities jointly achievable by the players

in S and is a subset of R^S , the s -dimensional Euclidean space indexed by the members of S . For every $S \subset N$, $S \neq \emptyset$, $V(S)$ must be comprehensive, closed, bounded, and positive. *Comprehensive* means that if $x \in V(S)$ and $y \leq x$, then $y \in V(S)$ as well. *Closed* means every convergent sequence in $V(S)$ converges to a point in $V(S)$. *Bounded* means there is some $y \in \mathbb{R}^S$ such that for any $x \in V(S)$, $x < y$. *Positive* means there is an $x \in V(S)$ such that $x \in R_{++}^S$. Section 6 shows how this constraint may be relaxed to allow coalitions with zero worth. Except for positivity, these are the same conditions used by Kalai and Samet (1985) in their characterization of the egalitarian value. An NTU game in characteristic function form is a complete collection of worths for all $S \subseteq N$.

The NTU ratio potential is a straightforward generalization of the TU ratio potential. The TU condition (2.1) is modified to require that the discrete derivatives of the players in any coalition S identify an allocation on the efficient surface $\partial V(S)$ of $V(S)$. Given $(P(S \setminus i, V))_{i \in S}$, the *NTU ratio potential* $P(S, V)$ is the unique scalar that satisfies

$$(2.5) \quad \left(\frac{P(S, V)}{P(S \setminus i, V)} \right)_{i \in S} \in \partial V(S).$$

If V represents a TU game, then (2.5) clearly reduces to (2.1). The *output-basis NTU proportional value* for i in a positive NTU game V is

$$(2.6) \quad \varphi_i(V) = D^i P(N, V) = P(N, V) / P(N \setminus i, V).$$

The input-basis NTU proportional value is defined in Section 5.5.

2.2 Some Basic Properties of the Proportional Value

The proportional value has many properties classically associated with the Shapley value. In this connection, note that formula (2.3) for the TU proportional value gives equal weight to all possible orders of players. As is well known, the random arrival form of the Shapley value also has this important property. This subsection presents some additional properties of the proportional value. Reference to the NTU proportional value in this section should be understood to be the output-basis proportional value.

Lemma 2.2 *The proportional value is efficient, symmetric, and unique.*

Proof: TU efficiency follows directly from the definition of the ratio potential (2.1) and the definition of φ (2.2). Since $\varphi(S, V) \in \partial V(S)$, the NTU proportional value must be at least weakly Pareto efficient. When $V(S)$ is required to be nonlevel (i.e., allocations to all players change moving any direction in $\partial V(S)$), then $\varphi(S, V)$ must clearly be strongly Pareto efficient. Symmetry follows from the observation that any permutation of players' labels must lead to the same permutation of their value allocations. The TU proportional value is unique because it is the ratio of two potentials and Lemma 2.1 shows that potentials scale linearly with $P(\emptyset, v)$. It is easily determined that the same must be true for the NTU potential and value. \square

Let $c \in \mathbb{R}_{++}^N$, define x^* such that $x_i^* = c_i x_i$, and let $c \circ V$ be the game where $x^* \in (c \circ V)(S)$ if and only if $x \in V(S)$. A value is scale invariant if $\phi_i(c \circ V) = c_i \phi_i(V)$.

Lemma 2.3 *The proportional value is scale invariant in TU and NTU games.*

Proof: Guess that $P(S, c \circ V) = P(S, V) \prod_{j \in S} c_j$. This is clearly true for $n = 1$. Assume it is true for all $T \subsetneq S$. Then it must be true for S as well since $(c_i \varphi_i(S, V))_{i \in S} \in \partial(c \circ V)(S)$ and $c_i \varphi_i(S, V) = \{P(S, V) \prod_{j \in S} c_j\} / \{P(S \setminus i, V) \prod_{j \in S \setminus i} c_j\}$. \square

Define a value to be *monotonic* if an increase in the worth of a coalition never reduces the value allocated to any of its players. For TU games, ϕ is weakly monotonic if $\partial \phi_i(v) / \partial v(S) \geq 0$ for $S \ni i$, and strongly so if the inequality is strict. This definition is equivalent to that of Kalai and Samet (1985), but weaker than that of Young (1985). Ortmann (2000), Proposition 4.2, demonstrates monotonicity of the proportional value only with respect to $v(N)$ in TU games.

Lemma 2.4 *The proportional value is strongly monotonic in TU and weakly monotonic in NTU games.*

Proof: In the TU case, the potential is continuously differentiable with respect to the worths of coalitions in a positive game. Equation (2.1) implies $\partial P(S, v) / \partial v(S) > 0$ for any $S \subseteq N$. Further, (2.1) also implies $\partial P(R, v) / \partial P(S) > 0$ for any $R = S \cup \bar{j}$, $j \in N \setminus S$. Induction and the chain rule then imply $\partial P(T, v) / \partial v(S) > 0$ for any $T \supseteq S$. On the other hand, if $T \not\supseteq S$ then $P(T, v)$ is clearly independent of $v(S)$ and $\partial P(T, v) / \partial v(S) = 0$. Thus, for any $T \supseteq S$ and any $i \in T$, (2.2) implies that $\partial \varphi_i(T, v) / \partial v(S) > 0$.

In NTU games an “increase” in $V(S)$, an outward expansion of the efficient surface, may have no effect on $P(T, V)$, $S \subseteq T \subseteq N$. To see this, let $V' = V$ except that $V'(S) \supset V(S)$ is an enlargement of $V(S)$ such that there is no $y \in V'(S)$ such that for all $i \in S$, $y_i > \varphi_i(S, V)$. Then (2.5) requires that $P(S, V')$ be unchanged, and therefore, no potentials will change and the value allocation will not change. However, there is no way that an increase in $V(S)$ can lead to a decrease in $P(T, V)$ for $S \subseteq T \subseteq N$ or affect any $P(T, V)$ where $S \not\subseteq T$. \square

A value provides *equal proportional gain* to players if and only if the proportional change in value to a player j from a second player k joining a coalition is equal to that of k when j joins. This property is directly analogous to the “balanced contributions” property of Myerson (1980), which is called “preservation of differences” in Hart and Mas-Colell (1989). Ortmann defines a value with this property as “preserving ratios.” Ortmann (2000) Theorem 2.6 characterizes the TU proportional value with this property. This simpler proof applies also to NTU games.

Lemma 2.5 *The proportional value is the unique efficient value with the equal proportional gain property in TU and NTU games:*

$$\frac{\varphi_j(S, V)}{\varphi_j(S \setminus k, V)} = \frac{\varphi_k(S, V)}{\varphi_k(S \setminus j, V)}.$$

Proof: The proportional value has the property since both ratios reduce to $P(S, V) / P(S \setminus \bar{jk}, V)$. Assume a second efficient value ϕ also has the property. They must agree in two-player

games. Let S be a coalition of smallest cardinality where they do not agree. Since they agree for immediate subcoalitions, substitute potentials for values of immediate subcoalitions and rearrange to get $P(S \setminus j, V)\phi_j(S, V) = P(S \setminus k, V)\phi_k(S, V)$ for all $j, k \in S$. Since $\phi(S, V) \in \partial V(S)$, $P(S \setminus i)\phi_i(S, V) = P(S, V)$ for all $i \in S$ and $\phi(S, V) = \varphi(S, V)$ as well. \square

A *dummy player* adds exactly its individual worth to every coalition. Let i be a dummy player. In a TU game, for every $S \ni i$, $v(S) = v(S \setminus \bar{i}) + v(\bar{i})$. Similarly, in an NTU game, for every $S \ni i$, $V(S) = V(S \setminus \bar{i}) \times V(\bar{i})$. The following lemma is proved using a result for coalition structures developed in Section 4.

Lemma 2.6 *If i is a dummy player, then $\varphi_i(S, v) = v(\bar{i})$ in TU games and $\varphi_i(S, V) = \sup\{x \in V(\bar{i})\}$ in NTU games.*

Proof: Let $Q = \{N \setminus i, \bar{i}\}$ be a coalition structure. If i is a dummy player, then $V/Q = V$, where “/” is the refinement operator of (4.1) and (4.2). By Lemma 4.1, below, $P(S, V) = P(S \setminus i, V)P(\bar{i}, V)$. Thus, $\varphi_i(S, V) = P(\bar{i}, V) = \sup\{x \in V(\bar{i})\}$. The TU result is implied. \square

A value is *individually rational* if and only if it always allocates a player at least its individual worth. A game V is *superadditive* if, for any disjoint coalitions S and T , and for every $x \in V(S)$ and every $y \in V(T)$, the joint vector (x, y) is in the feasible set of the union: $(x, y) \in V(S \cup T)$.

Lemma 2.7 *The proportional value is individually rational in positive superadditive games: If V is superadditive, then $\varphi_i(S, V) \geq \sup\{x \in V(\bar{i})\}$.*

Proof: Take any game V and any $i \in N$. Create a game V^* such that $V^*(S) = V(S \setminus \bar{i}) \times V(\bar{i})$ if $i \in S$ and $V^*(S) = V(S)$ if $i \notin S$. By the superadditivity of V , $V^*(S) \subseteq V(S)$. By Lemma 2.6, $\varphi_i(S, V) = \sup\{x \in V(\bar{i})\}$ for every $S \ni i$, $S \subseteq N$. Now, by Lemma 2.4, the conclusion follows because V^* can be transformed into V by a sequence of games $V_1, V_2, V_3, \dots, V_k$, in which each game in the sequence differs from the immediately previous game by the enlargement of the worth (feasibility set) of a single coalition $S \ni i$. If $V_j(S)$ is enlarged by $V_{j+1}(S) = V(S)$, then $V_j(S) \subset V_{j+1}(S)$, and for every other $T \subseteq N$, $V_k(T) = V_{j+1}(T)$. Thus $\varphi_i(S, V_{j+1}) \geq \varphi_i(S, V_j)$ for every $j < k$ and every $S \subseteq N$. The TU result is implied. \square

2.3 Random Marginal Contribution Representation and the TU Core

The proportional value has a random marginal contribution representation. This implies the proportional value is in the core of a positive convex TU game. The following lemma is the basis for random marginal contribution representation. It is an obvious analog of a well-known representation of the Shapley value (see e.g., Pérez-Castrillo and Wettstein (2001)) and is closely related to the noncooperative implementation (compare with equations (5.2) and (5.3)). The lemma shows that the proportional value of a player i in a coalition S is the weighted sum of i 's marginal contribution to S and i 's value in the immediate subcoalitions of S . The weights for the subcoalitions are proportional to the

values of the excluded players. In contrast, equal weighting defines the Shapley value. To prove the lemma, represent values and $v(S \setminus i)$ in potential form on the right-hand-side and simplify.

Lemma 2.8 *For any positive TU game v and any $S \subset N$,*

$$\varphi_i(S, v) = v(S)^{-1} \left[\varphi_i(S, v) \left(v(S) - v(S \setminus i) \right) + \sum_{j \neq i} \varphi_j(S, v) \varphi_i(S \setminus j, v) \right].$$

The next lemma provides a representation of the proportional value as a weighted sum of marginal contributions. Recall coalition R_m^r is composed of the first m players in the order $r \in \mathcal{R}(S)$, and let $r(i)$ be i 's position in r .

Lemma 2.9
$$\varphi_i(S, v) = P(S, v) \sum_{r \in \mathcal{R}(S)} \prod_{m=1}^s v(R_m^r)^{-1} \left(v(R_{r(i)}^r) - v(R_{r(i)}^r \setminus i) \right).$$

Proof: Define $M_i(r) = v(R_{r(i)}^r) - v(R_{r(i)}^r \setminus i)$. Let the probability associated with any order $r_i \in \mathcal{R}(S)$ be $pr_{S,i} = \prod_{m=1}^s \varphi_{r_i(m)}(R_m^{r_i}, v) / v(R_m^{r_i})$. Assume $\varphi_k(S, v) = \sum_{r_i \in \mathcal{R}(S)} pr_{S,i} M_k(r_i)$. The result clearly holds for $s = 1$. Assume it holds for all $k \in T \subsetneq S$. Consider orders $r_j \in \mathcal{R}(S)$ such that $r_j = (r_i, q \in S)$, e.g., q is added to $r_i \in \mathcal{R}(T = S \setminus q)$. Then, by Lemma 2.8, $\varphi_k(S, v) = \sum_{r_j \in \mathcal{R}(S)} pr_{S,j} M_k(r_j)$ for all $k \in S$ as well. By induction, it holds for all S . Replace values in the probability formula by potentials and the result follows. \square

Observe that the weight associated with any order r is independent of i . The following reorganization of terms casts the proportional value in the standard Shapley value form.

Corollary 2.1
$$\varphi_i(v) = P(N, v) \sum_{\substack{S \subseteq N \\ S \ni i}} \sum_{\substack{r \in \mathcal{R}(N) \\ R_s^r = S \\ r^{(s)} = i}} \prod_{j=1}^n v(R_j^r)^{-1} \left(v(S) - v(S \setminus i) \right).$$

A TU game v is *convex* if and only if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$. Feldman (1999) shows only that the proportional value is in the core of a *log convex* game, i.e., when $\ln v$ is convex.

Theorem 2.1 *The proportional value is in the core of a positive convex TU game.*

Proof: The convexity of v implies $v(S \cup \bar{i}) - v(S) \geq v(S \cup \bar{i} \setminus j) - v(S \setminus j)$, the increasing marginal contributions of players. For any order $r \in \mathcal{R}(N)$, the marginal contributions when v is convex define a core allocation because the sum of the marginal contributions of any sequence of players must at least equal the worth of their coalition (the sum when they are first in the order). Lemma 2.9 shows the proportional value is a weighted sum of marginal contribution vectors. Since the core is a convex set, the result follows. \square

3 NTU Characterization Through Consistency

Consistency is a powerful property that means, roughly, a solution concept can be applied on a sequential basis and give the same result. Assume a first set of players in a game are given their values and “leave.” Consistency requires that the values of the remaining players in the resulting *reduced* game are equal to their value in the complete game. Young (1994) notes the consistent nature of many social allocation rules including rules for the apportionment of seats in legislative bodies. Hart and Mas-Colell (1989) discuss other applications of the consistency property in cooperative game theory. This section shows that the output-basis proportional value is the unique NTU value consistent with proportional bargaining in two-player TU games (i.e., equation (2.4)).

Hart and Mas-Colell (1989) define a new reduced game and show that the Shapley value is consistent with it. The worth of a coalition S in this reduced game is what remains of the worth of their union with all reduced players $i \in T^C$, after the reduced players are given their value in the game based on this union. Consistency in this paper should be understood to be with respect to the Hart and Mas-Colell reduced game:

$$(3.1) \quad V_T^\phi(S) = \left\{ x \in R^S : \left(x, (\phi_i(S \cup T^C, V))_{i \in T^C} \right) \in V(S \cup T^C) \right\}.$$

Formally, consistency requires $\phi_i(S, V_T^\phi) = \phi_i(S \cup T^C, V)$, for all $S \subseteq T$ and all $T \subset N$.

Theorem 3.1 *The output-basis proportional value is consistent in NTU games.*

Proof: When i is reduced from a two-player game we see from (3.1) that $V_{-i}^\varphi(\bar{j}) = \{x \leq \varphi_j(\bar{i}\bar{j}, V)\}$. Therefore $\varphi_j(\bar{j}, V_{-i}^\varphi) = P(\bar{j}, v_{-i}^\varphi) = P(\bar{i}\bar{j}, V)/P(\bar{i}, V) = \varphi_j(\bar{i}\bar{j}, V)$. Assume φ is consistent in games of $m-1$ players or less and create V_{-i}^φ by reducing i from V . Then for any $S \not\ni i$ with $s < m$, and any $j \in S$, $\varphi_j(S, V_{-i}^\varphi) = \varphi_j(S \cup \bar{i}, V)$. Consider the ratio potential for V_{-i}^φ . For any $S \not\ni i$ with $s < m$, and any $j \in S$, $\varphi_j(S, V_{-i}^\varphi) = P(S, V_{-i}^\varphi)/P(S \setminus j, V_{-i}^\varphi) = P(S \cup \bar{i}, V)/P((S \cup \bar{i}) \setminus j, V)$. Thus, there must be a scalar constant c such that $P(S, V_{-i}^\varphi) = c P(S \cup \bar{i}, V)$ for the potentials of all these coalitions. Now consider a coalition $R \not\ni i$ with m players. The efficiency of the proportional value and the definition of the reduced game requires that values of the players of R in the reduced game together with $\varphi_i(R \cup \bar{i}, V)$ must lie in $\partial V(R \cup \bar{i})$:

$$\left(\left(\frac{P(R, V_{-i}^\varphi)}{cP((R \cup \bar{i}) \setminus j, V)} \right)_{j \in S}, \varphi_i(R \cup \bar{i}) \right) \in V(R \cup \bar{i}).$$

Thus $P(R, V_{-i}^\varphi) = cP(R \cup \bar{i}, V)$ and $\varphi_j(R, V_{-i}^\varphi) = \varphi_j(R \cup \bar{i}, V)$ for all $j \in R$. Since additional players can be sequentially reduced from V_{-i}^φ with the same result, φ must be consistent in m -player games as well. The conclusion then follows by induction. \square

Theorem 3.2 *The output-basis proportional value is the unique consistent NTU allocation rule that, in two-player NTU games, is efficient and gives players equal proportional gains.*

Proof: Assume a second NTU allocation rule ϕ that is also consistent and, in two-player NTU games, has efficient equal proportional gain outcomes. Thus, φ and ϕ agree in two-player games. Their singleton reduced games must agree. Consistency requires that both values in the singleton reduced games are equal to the values in the two-player games. Thus, φ and ϕ must agree for one-player games as well.

Assume that φ and ϕ agree for games of $s - 1$ players or less, but not in s -player games. Choose a coalition S with s players and any $i, j \in S$. Construct the reduced games consisting of this pair of players, V_{ij}^φ and V_{ij}^ϕ , one for each value, according to (3.1). The players $S \setminus \overline{ij}$ are reduced from the game. Since φ and ϕ agree for games of $s - 1$ players the individual worths of i and j must be the same in both reduced games.

Since both rules give equal proportional gain outcomes in 2-player games and the individual worths in both reduced games are the same, each player will gain in the same proportion in both games. Thus both players' allocations will be equal or larger according to one of the values than the other. By consistency, these values are equal to player values in the game (S, V) . This outcome applies to any pair of players in S . Thus, all players' allocations according to one of the rules must be at least as great as according to the other. The (output-basis) proportional value is efficient by definition. If ϕ is efficient in s -player games it must be equal to φ in s -player games, a contradiction. But ϕ must be efficient in s -player games by (3.1) since it must be efficient in $s - 1$ -player games and is consistent in s -player games. \square

The following theorem is similar to Theorem E of Hart and Mas-Colell (1989). It shows that the output-basis NTU proportional value is determined by consistency and principles that are independent of the NTU environment. The proof may be found in the appendix. In Feldman (2002), I provide one characterization of TU proportional bargaining in two-player games.

Theorem 3.3 *The output-basis proportional value is the unique consistent NTU value that, in two-player TU games, is efficient and gives players equal proportional gains.*

The relative importance of consistency in cooperative game theory has been unclear since the Hart and Mas-Colell (1996) conclusion that the Maschler-Owen value (1989, 1992) should be considered the proper coalitional NTU extension of the Shapley value and Nash bargaining solutions. The dual model and the consistency of the proportional value affirm the continuing relevance of consistency to NTU cooperative value theory. The output basis is the consistent basis.

4 Characterization Through Bilateral Cooperation

Myerson (1977) characterizes the Shapley value in a model of bilateral cooperation in which pairs of cooperating players share the benefits of cooperation equally and all pairs of players cooperate. Parallel results are demonstrated here for the proportional value in TU and NTU games. This section also demonstrates that the ratio potential for games with coalition structures has a structure closely related to the difference potential for these games. This implies that the coalition formation results of work such as Winter (1992), Qin (1996), and Slikker (2001) also have proportional analogs.

4.1 Coalition Structures and the Ratio Potential

Let $Q = \{Q_1, Q_2, \dots, Q_k\}$ be a coalition structure, a partition of the players of N into k coalitions. For any $S \subseteq N$, let the *refinement* of S by Q be the partition of S into subcoalitions such that every such coalition in S/Q is the intersection of S with a component of Q :

$$S/Q = \{S_1, S_2, \dots, S_k\} : \cup_{S_i \in S/Q} S_i = S \text{ and } \forall S_i \in S/Q, \exists Q_j \in Q \text{ s.t. } S_i = S \cap Q_j.$$

In TU games, define the worth $v(S/Q)$ to be equal to the sum of worths of the subcoalitions of S relative to the coalition structure Q :

$$(4.1) \quad v(S/Q) = \sum_{S_i \in S/Q} v(S_i).$$

Similarly, for NTU games, define the worth $V(S/Q)$ to be equal to the product space of the worths of the subcoalitions S relative to the coalition structure Q :

$$(4.2) \quad V(S/Q) = \times_{S_i \in S/Q} V(S_i).$$

Lemma 4.1 *The ratio potential of a coalition S in a game V^* refined by a coalition structure Q , $V^*(S) = V(S/Q)$ is the product of the potentials of its refined subcoalitions:*

$$P(S, V^*) = \prod_{S_j \in S/Q} P(S_j, V).$$

The proof may be found in the appendix. The Shapley and egalitarian values for games with coalition structures is described by the difference potential relationship $P^d(S, V^*) = \sum_{S_j \in S/Q} P^d(S_j, V)$.

4.2 Bilateral Cooperation Games

In Myerson (1977) the complete worth of a coalition is only realized if all players cooperate at least indirectly through a chain of bilateral cooperation agreements between players in the coalition. The set of bilateral agreements in a game is its *cooperation structure* and is represented by a graph g . The *nodes* of g are the players $i \in N$. A bilateral agreement between i and j is represented by the *edge* $i:j \in g$. A coalition S is *connected* in a coalition structure g if and only if for every $i, j \in S$, there is a set of links in g such that $i:r_1, r_1:r_2, \dots, r_k:j$ and $r_1, \dots, r_k \in S$. The notation S/g is used to represent the partition of S into its connected subcoalitions. The worth of any coalition S given a cooperation structure g is defined in TU games as

$$v(S/g) = \sum_{S_i \in S/g} v(S_i), \quad \text{and, in NTU games, as } V(S/g) = \times_{S_i \in S/g} V(S_i).$$

Lemma 4.2 *The TU and output-basis NTU proportional values assign equal proportional gains to players forming a bilateral cooperation agreement.*

Proof: Consider a game V with a cooperation graph g such that two players, $i, j \in S$ do not cooperate directly and the graph $g^* = g \cup \overline{i:j}$ where they do. By Lemma 2.5

$$\frac{\varphi_i(S, V/g)}{\varphi_i(S \setminus j, V/g)} = \frac{\varphi_j(S, V/g)}{\varphi_j(S \setminus i, V/g)}$$

and the same relationship obtains when g^* is substituted for g . For any restricted game (S, V) in which either i or j are missing, $\varphi_i(R, V/g) = \varphi_i(R, V/g^*)$ and similarly for j because the loss of this agreement has no effect on any of the coalitions in the restricted game. Thus $\varphi_i(S, V/g^*)/\varphi_i(S, V/g) = \varphi_j(S, V/g^*)/\varphi_j(S, V/g)$ and the result follows. \square

Lemma 4.3 *The TU and output-basis NTU proportional values of a graph-restricted game completely distribute the worth of a connected coalition to its members. Further, if $i \in S_j \in T/g$, $T \subseteq N$, then $\varphi_i(T, V) = \varphi_i(S_j, V)$.*

Proof: This follows directly from Lemma 4.1. Since the potential of a coalition is the product of the potentials of its connected subcoalitions, we have for any $i \in S_j \in T/g$, that $\varphi_i(T, V) = P(T, V)/P(T \setminus \overline{i}, V) = P(S_j, V)P(T \setminus S_j, V)/[P(S_j \setminus \overline{i}, V)P(T \setminus S_j, V)] = P(S_j, V)/P(S_j \setminus \overline{i}, V) = \varphi_i(S_j, V)$. By (2.1), for TU games, $\sum_{i \in S_j} \varphi_i(S_j, v) = v(S_j)$; and by (2.5), for NTU games, $(\varphi_i(S_j, V))_{i \in S} \in \partial V(S_j)$. \square

Theorem 4.1 *The TU and output-basis NTU proportional values are the unique TU and NTU values that give equal proportional gains to bilateral cooperators and completely distribute the worth of a connected coalition to its members.*

Proof: Assume there are two values which both satisfy these properties. For any game V , choose any minimal graph g_0 such that these values produce different allocations, and choose a component $S \in N/g_0$. Since bilateral cooperators gain in equal proportion under both values, for any two players $i, j \in S$ with $i:j \in g_0$:

$$\frac{Y_i(V/g_0)}{Y_i(V/(g_0 \setminus \overline{i:j}))} = \frac{Y_j(V/g_0)}{Y_j(V/(g_0 \setminus \overline{i:j}))} \text{ and } \frac{Z_i(V/g_0)}{Z_i(V/(g_0 \setminus \overline{i:j}))} = \frac{Z_j(V/g_0)}{Z_j(V/(g_0 \setminus \overline{i:j}))}.$$

Then because both values must agree when any link is removed, $Y_i(V/(g_0 \setminus \overline{i:j})) = Z_i(V/(g_0 \setminus \overline{i:j}))$ and $Y_j(V/(g_0 \setminus \overline{i:j})) = Z_j(V/(g_0 \setminus \overline{i:j}))$. Thus, division gives $Y_i(V/g_0)/Z_i(V/g_0) = Y_j(V/g_0)/Z_j(V/g_0)$. Since this must be true for every such pair $i, j \in S$, there must be a constant $c > 0$ such that $Y_i(V/g_0)/Z_i(V/g_0) = c$ for every $i \in S$. If $c = 1$ then the values are identical, but $c \neq 1$ contradicts the assumption of efficiency for both values. \square

Remark 4.1 *It is easily shown that the egalitarian value results when players receive equal gain in NTU games.*

The Myerson (1977) bilateral cooperation game has been much cited and has been remarkably influential in the development of cooperative game theory. In fact, while the characterization of value by consistency and by the Hart and Mas-Colell (1996) noncooperative game may appear to be fundamentally different approaches; it will be seen that, from the bilateral cooperation perspective, this noncooperative game is merely a mechanism that properly guides the generalization of the equal gain relationship in a more complex environment. The fact that the bilateral cooperation model naturally generates equal proportional gain relationships is a necessary demonstration of the validity of the dual model of value.

5 Noncooperative Characterization

Hart and Mas-Colell (1996) introduce an infinite horizon noncooperative bargaining game. The game is modified here by introducing a mechanism to allow participation probabilities to be a function of currently proposed payoffs. This game is used here primarily to implement the proportional value in TU and NTU games. The egalitarian value and an input-basis proportional value are characterized as well.

5.1 The Hart and Mas-Colell (1996) Bargaining Game

In the Hart and Mas-Colell game a player is selected in every round (*stage game*) to propose a division of the worth of the coalition of the current players. If no player rejects the proposal, the division is effected and the game ends. If any player rejects, the proposal is rejected. The game continues directly with probability ρ . Otherwise a negotiation breakdown occurs. The proposer is removed from the game and receives zero final payoff, and then the next round begins. A player is assumed to always accept a proposal when indifferent between acceptance and rejection. There is no time discounting.

A *payoff configuration* is a set of payoff vectors, $a = \{a_S\}_{S \subseteq N}$, one for each $S \subseteq N$. Each a_S specifies payoffs for all $i \in S$ with player i receiving a_S^i when the players are S . The proposed payoff $a_{S,k}^j$ is to j by k when the remaining players are S . Player k 's complete proposal to the members of S is $a_{S,k}$. Boldface indicates a profile of proposal strategies, so that \mathbf{a}_S represents a complete proposal profile for the players in S and $\mathbf{a} = \{\mathbf{a}_S\}_{S \subseteq N}$ is a complete strategy profile. Proposals are required to be nonnegative and feasible in order to simplify the participation mechanism.

Section 2.1 requires NTU games to be comprehensive, closed, bounded, and positive. In order to guarantee the existence of equilibria in general NTU games, feasible sets are also required to be smooth, nonlevel, and convex. *Smoothness* means that the tangent hyperplane at any point $x \in \partial V(S)$ is well-defined. A surface is *nonlevel* if the outward normal vector at any point in the surface is positive in all directions. *Convexity* requires that for any $x, y \in V(S)$, and any $\alpha : 0 \leq \alpha \leq 1$, $\alpha x + (1 - \alpha)y \in V(S)$ as well. Further, games must be *weakly monotonic*. A TU game is weakly monotonic if $S \supset T$

implies $v(S) \geq v(T)$. An NTU game V is weakly monotonic if $S \supset T$ implies $V(S) \supseteq V(T) \times \{0^{S \setminus T}\}$. Any feasible allocation for the players of T , $x \in V(T)$ must also be feasible in $V(S)$. These are the Hart and Mas-Colell (1996) assumptions.

Hart and Mas-Colell (1996) Proposition 1 develops the basic conditions of the SSP equilibrium for both TU and NTU games and extend these conditions to games with unequal participation probabilities. This result depends on the monotonicity condition. Let Δ^S be an s -simplex, and let $p_S \in \Delta^S$ be a vector of participation probabilities for the players of S . Let $a_S = \sum_{j \in S} p_S^j a_{S,j}$ be the expected payoff vector for S given proposal profile \mathbf{a}_S , where p_S^j is the probability of j being selected to propose when the players are S . Then all proposals are positive and efficient and

$$(5.1) \quad a_{S,k}^j = \rho a_S^j + (1 - \rho) a_{S \setminus k}^j,$$

for all $j, k \in S$, $j \neq k$, and all $S \subseteq N$. Each player is offered the weighted average of its continuation value in the current stage game and the continuation value conditional on ejection of the proposer. The weights are the probabilities of continuation and breakdown.

Hart and Mas-Colell's Proposition 9 implies that continuation values in TU games satisfy the following recursive relation:

$$(5.2) \quad a_S^j = \sum_{k \in S \setminus j} p_S^k a_{S \setminus k}^j + p_S^j (v(S) - v(S \setminus j)).$$

A *hyperplane* game is an NTU game where the efficient surfaces of coalitional feasibility sets are hyperplanes. The marginal rates of substitution between player allocations are defined by a fixed set of λ -transfer weights. Thus $V(S) = \{x \in R^S : x \cdot \lambda_S \leq V_S\}$, where $\lambda_S \in R_{++}^S$ and $V_S > 0$. Hart and Mas-Colell show equation (5.2) generalizes naturally to hyperplane games by λ -weighting marginal contributions:

$$(5.3) \quad a_S^j = \sum_{k \in S \setminus j} p_S^k a_{S \setminus k}^j + p_S^j \frac{1}{\lambda_S^j} \left(\sum_{k \in S} \lambda_S^k a_S^k - \sum_{k \in S \setminus j} \lambda_S^k a_{S \setminus j}^k \right).$$

In both (5.2) and (5.3), a player's continuation value is the sum of its expected payoffs contingent on the departure of each of the other players, each weighted by the participation probability of the other player, plus its marginal contribution weighted by its own probability of participation. Note the similarity of these conditions to Lemma 2.8.

Multiple equilibria may exist in general NTU games. Let $\Gamma_S : \Delta^S \rightrightarrows (V(S) \cap \mathbb{R}_+^S)^S$ be the correspondence identifying the equilibrium proposals in a stage game with the players in S as a function of participation probabilities, expected payoffs in games based on the immediate subcoalitions, and ρ . The following lemma demonstrates technical properties of Γ_S needed later. The proof may be found in the appendix.

Lemma 5.1 *For every $0 \leq \rho < 1$ the stage game equilibrium correspondence Γ_S is continuous and single valued in TU and hyperplane games. It is (i) upper hemi-continuous, (ii) locally unique, and (iii) lower hemi-continuous in general NTU games.*

5.2 Proposed Payoff-Weighted Participation Games

A *proposed payoff-weighted participation mechanism* is a function $\Xi : (V(S) \cap \mathbb{R}_+^S)^S \rightarrow \Delta^S$ that determines participation probabilities on the basis of players' proposed payoffs. A bargaining game incorporating this mechanism is a *proposed payoff-weighted participation game*. As an extension to the Hart and Mas-Colell (1996) bargaining game, the mechanism works as follows: In every round, all current players submit proposals. The mechanism then determines participation probabilities and selects a player's proposal accordingly. The mechanism allows inefficient bids, as in the original game.

In TU games, let $\hat{a}_{S,k} = \sum_j a_{S,k}^j$ and define $a_{S,k}^0 = \frac{1}{s}(v(S) - \hat{a}_{S,k})$. In NTU games, define $a_{S,k}^0$ as the solution to the equation $(a_{S,k}^j + a_{S,k}^0)_{j \in S} \in \partial V(S)$. Then define player j 's *adjusted proposed payoff* by k as $a_{S,k}^j + a_{S,k}^0$ and player j 's *adjusted average proposed payoff* as $\bar{a}_S^j = \frac{1}{s} \sum_{k \in S} (a_{S,k}^j + a_{S,k}^0)$. The mechanism Ξ_0 sets

$$(5.4) \quad p_S^i(\mathbf{a}_S) = \Xi_0(\mathbf{a}_S) = \bar{a}_S^i / \sum_{j \in S} \bar{a}_S^j.$$

Player's participation probabilities under Ξ_0 are proportional to their average adjusted proposed payoffs. Proposed payoffs in an inefficient proposal are adjusted to make the adjusted proposal efficient. Adjustment applies only to the determination of participation probabilities. Adjustment guarantees the continuity of p_S as a function of \mathbf{a}_S .

In NTU games proposals may be λ -weighted. In general NTU games the average of players' proposals will not usually be efficient. Therefore an appropriate point in $\partial V(S)$ is selected in order to generate λ weights. Let $a_S^* = \kappa(a_S)$ be the intersection of the ray passing from the origin through the average proposal \bar{a}_S and $\partial V(S)$, and let $\lambda(a_S^*)$ be the weights associated with the hyperplane tangent to $V(S)$ at a_S^* . Now define player j 's *λ -weighted adjusted proposed payoff* by k as $\lambda_j(a_S^*)(a_{S,k}^j + a_{S,k}^0)$ and j 's *λ -weighted adjusted average proposed payoff* as: $\bar{b}_S^j = \lambda_j(a_S^*) \bar{a}_S^j$. The *λ -weighted proposed payoff-weighted participation mechanism* Ξ_λ is then defined as

$$(5.5) \quad p_S^i(\mathbf{a}_S) = \Xi_\lambda(\mathbf{a}_S) = \bar{b}_S^i / \sum_{j \in S} \bar{b}_S^j.$$

Lemma 5.2 *The Hart and Mas-Colell bargaining game modified by incorporating a continuous participation mechanism Ξ has an SSP equilibrium in TU, hyperplane, and general NTU games for every $0 \leq \rho < 1$. In every such equilibrium all proposals are efficient and the first proposal is accepted by all players.*

Proof: Equilibria trivially exist for the singleton coalitions. Assume that an equilibrium strategy profile of the game including a continuous participation mechanism exists for all $R \subset S$. Let $\mathbf{c}_S = \{c_{S,j}\}_{j \in S}$ be the proposals submitted to the mechanism when the current players are S and $\mathbf{e}_S = \{e_{S,j}\}_{j \in S}$ be player's equilibrium proposals conditional on $p_S = \Xi(\mathbf{c}_S)$ and (5.4) or (5.5). For any equilibrium proposal profile \mathbf{e}_S , $e_{S,k}^j \geq 0$, for every $j, k \in S$ and for every

$k \in S$, $e_{S,k}$ must be efficient, by Hart and Mas-Colell Proposition 1 (minimally modified to allow unequal proposal probabilities) and the positivity and monotonicity of V . Further, conditional on reaching any stage game, Hart and Mas-Colell Proposition 1 guarantees that an equilibrium proposal will be immediately accepted by all players.

In hyperplane NTU games, $c_{S,k}$ and $e_{S,k}$ must lie within the compact, closed, convex set $A_S^k = \{x \in \mathbb{R}_+^S : \lambda \cdot x \leq v_S\}$. (In TU games $\lambda = \mathbf{1}$.) Let $A_S^S = \times_{k \in S} A_S^k$. The composition of any continuous participation mechanism Ξ with (5.2) in TU games or (5.3) in hyperplane games and then (5.1) clearly generates a continuous function $A_S^S \rightarrow A_S^S$. In general NTU games, \mathbf{c}_S and \mathbf{e}_S are elements of $(V(S) \cap \mathbb{R}_+^S)^S$. Lemma 5.1 shows the correspondence $\Gamma_S : \Delta^S \rightrightarrows (V(S) \cap \mathbb{R}_+^S)^S$ is upper- and lower hemi-continuous and locally unique. This implies Γ_S is a set of non-intersecting continuous functions. Select one of these functions, $f^*(p_S)$. The composition $f^* \cdot \Xi : (V(S) \cap \mathbb{R}_+^S)^S \rightarrow (V(S) \cap \mathbb{R}_+^S)^S$ is clearly continuous as well. In all cases then, a fixed point must exist by the Brouwer fixed point theorem. Further, all equilibrium proposals in the fixed point mapping must be efficient since all proposals $e_{S,j} \in \mathbf{e}_S$ are efficient. The conclusion follows by induction. \square

5.3 Implementation of the Proportional Value

Theorem 5.1 *When the underlying cooperative game is TU, and in the limit as $\rho \rightarrow 1$, the (TU) proportional value defines payoffs in the unique equilibrium of the modified Hart and Mas-Colell bargaining game incorporating participation mechanism Ξ_0 .*

Proof: By Lemma 5.2 a proposed payoff-weighted participation equilibrium exists for every $0 \leq \rho < 1$. In the limit as $\rho \rightarrow 1$, $a_{S,j} \rightarrow a_S$ for every $j \in S$ and all $S \subseteq N$. Thus $\bar{a}_{S,j} \rightarrow a_S$. The proportional value is the equilibrium proposal profile for singleton coalitions. Assume this is true for all coalitions $R \subset S$. If $a_S^i = \varphi_i(S, v)$, then p_S^i can be written $p_S^i = 1/v(S) \times (P(S, v)/P(S \setminus i, v))$. Clearly $\sum_{i \in S} p_S^i = 1$. Lemma 2.8 shows that the proportional value is a solution to equation (5.2). Thus, the proportional value is an equilibrium payoff configuration by induction. Uniqueness follows from Lemma 5.3. \square

Lemma 5.3 *When the outcomes of cooperation are defined by an NTU hyperplane game, and in the limit as $\rho \rightarrow 1$, the output-basis proportional value defines payoffs in the unique equilibrium of the modified Hart and Mas-Colell bargaining game incorporating participation mechanism Ξ_λ .*

Proof: Lemma 5.2 guarantees that there is an equilibrium for every ρ . Proceed by induction. The result is trivially true for singleton coalitions. Assume that $a_R^i = \varphi_i(R, V)$ for all $i \in R \subset S$. In equilibrium, and in the limit as $\rho \rightarrow 1$, $\bar{b}_S^j \rightarrow \lambda_S^j a_S^j$. Substitute into equation (5.3) to obtain

$$(5.6) \quad a_S^i = \sum_{j \in S \setminus i} \frac{\lambda_S^j}{V_S} a_S^j \varphi_i(S \setminus j, V) + \frac{a_S^i}{V_S} \left(V_S - \sum_{k \in S \setminus i} \lambda_S^k \varphi_k(S \setminus i, V) \right),$$

where $V_S = \sum_{j \in S} \lambda_S^j a_S^j$. Now replace φ for players in the immediate subcoalitions by the potential representation, rearrange, and simplify to get

$$(5.7) \quad a_S^i P(S \setminus i, V) = \frac{1}{\sum_{j \in S \setminus i} \beta_S^j} \sum_{j \in S \setminus i} \beta_S^j a_S^j P(S \setminus j, V),$$

where $\beta_S^j = \lambda_S^j / P(S \setminus ij, V)$, for all $j \in S, j \neq i$. All products $a_S^j P(S \setminus j, V)$ must have the same value since each is equal to the weighted average of all the others. Thus $a_S^i = \varphi_i(S, V)$ for all $i \in S$ and the product is the potential $P(S, V)$. The conclusion follows by induction. \square

Theorem 5.2 *When the underlying cooperative game is NTU, and in the limit as $\rho \rightarrow 1$, the output-basis proportional value defines payoffs in the unique equilibrium of the modified Hart and Mas-Colell bargaining game incorporating participation mechanism Ξ_λ .*

Proof: The existence of a limit equilibrium in the stage game using participation mechanism Ξ_λ exactly parallels the existence argument in Hart and Mas-Colell (1996) Proposition 8, which requires the smoothness of $\partial V(S)$. The proportional value defines a limit equilibrium payoff configuration because Lemma 5.3 shows it defines the unique equilibrium payoff configuration for the hyperplane game generated by the supporting tangent hyperplane. It defines the unique equilibrium payoff configuration. Assume otherwise. Then there is another limit equilibrium with a distinct supporting tangent hyperplane for at least one coalition S . All equilibria must trivially agree for singleton coalitions. Let S be a coalition with a limit equilibrium not corresponding to the proportional value such that all immediate subcoalitions have unique limit equilibria corresponding to the proportional value. Then a ratio potential defines payoffs for all immediate subcoalitions. Lemma 5.3 then shows equilibrium payoffs must satisfy equation (5.7), thus payoffs for coalition S must be described by the proportional value as well, contradicting the assumption of a second equilibrium payoff configuration. \square

5.4 Implementation of the Egalitarian Value

The egalitarian value is implemented in the Hart and Mas-Colell game by making the probability of participation proportional to a player's λ weight. Let $\lambda_S^* = \sum_{i \in S} \lambda_i(a_S)$.

Theorem 5.3 *Let the probability of participation of player i be $p_S^i = \lambda_i(a_S) / \lambda_S^*$ in the stage game of the Hart and Mas-Colell bargaining game with players S . Then, when the underlying cooperative game is NTU, and in the limit as $\rho \rightarrow 1$, the egalitarian value defines payoffs of the unique equilibrium of the game.*

Proof: The argument parallels that of Lemma 5.3 and Theorem 5.2. All that must be proved here is that the egalitarian value defines equilibrium payoffs in the λ -weighted hyperplane game. The egalitarian value satisfies (5.3) for singleton coalitions. Assume that the egalitarian value is the continuation value for players in all coalitions $R \subset S$. The egalitarian value has the difference potential representation $E_i(S, V) = P^d(S, V) - P^d(S \setminus i, V)$. Replace values in immediate subcoalitions of (5.3) by their potential form, simplify to obtain

$$a_S^i = \sum_{k \in S} \frac{\lambda_S^k}{\lambda_S^*} a_S^k + \sum_{k \in S \setminus i} \frac{\lambda_S^k}{\lambda_S^*} \left(P^d(S \setminus k, V) - P^d(S \setminus i, V) \right),$$

and rearrange by collecting all a_S^i and $P(S \setminus i)$ terms and dividing by their weight to get

$$a_S^i + P^d(S \setminus i, V) = \frac{1}{\sum_{k \in S \setminus i} \lambda_S^k} \sum_{k \in S \setminus i} \lambda_S^k \left(a_S^k + P^d(S \setminus k, V) \right).$$

A solution is $a_S^i = E_i(S, V) = P^d(S, V) - P^d(S \setminus i, V)$, for all $i \in S$. Since this is one equation in a non-degenerate s -equation linear system, the solution must be unique. The hyperplane result follows by induction. The proof for general NTU games is as for Theorem 5.2. \square

5.5 The Input-Basis NTU Proportional Value

Since the Hart and Mas-Colell bargaining framework can be used to implement linear values with either input or output basis, the principle of duality of properties between linear and proportional values suggests that a proportional value with input basis exists and can be implemented with a suitable participation mechanism. In fact, using the participation mechanism Ξ_0 instead of Ξ_λ leads to equilibrium payoffs that can be interpreted as a proportional input-basis value.

Definition 5.1 *The input-basis proportional value φ^λ satisfies the following conditions for every $S \subseteq N$, where (iii) is for all $i \in S$ and the dependence on V is understood:*

- (i) $\varphi^\lambda(S, V) \in \partial V(S)$,
- (ii) $\lambda_S = \lambda(\varphi^\lambda(S, V))$ is the supporting hyperplane at $\varphi^\lambda(S, V)$, and
- (iii)
$$\sum_{j \in S \setminus i} \varphi_j^\lambda(S) \lambda_S^i (\varphi_i^\lambda(S) - \varphi_i^\lambda(S \setminus j)) = \sum_{j \in S \setminus i} \varphi_i^\lambda(S) \lambda_S^j (\varphi_j^\lambda(S) - \varphi_j^\lambda(S \setminus i)).$$

The first condition requires that payoffs are efficient. The second condition requires that λ -transfer weights λ_S are generated by the tangent hyperplane supporting $V(S)$ at $\varphi^\lambda(S, V)$. The final condition is a generalized weighted proportional gain condition.

Theorem 5.4 *When the underlying cooperative game is NTU, and in the limit as $\rho \rightarrow 1$, the input-basis proportional value φ^λ defines equilibrium payoffs in all equilibria of the modified Hart and Mas-Colell bargaining game incorporating participation mechanism Ξ_0 .*

Proof: Let $p_S^i(\mathbf{a}_S) = \Xi_0(\mathbf{a}_S) = \bar{a}_S^i / \sum_{k \in S} \bar{a}_S^k$. Existence of the limit equilibrium is guaranteed by an argument parallel to the existence argument of in Theorems 5.2 and 5.3. In the limit as $\rho \rightarrow 1$, $p_S^i = \Xi_0(\mathbf{a}_S) = a_S^i / \sum_{k \in S} a_S^k$. To see that a_S is the input-basis proportional value, substitute into (5.3), collect terms by moving the first summation and $p_S^j / \lambda_S^j (\lambda_S^j a_S^j)$ to the left-hand side, multiply both sides by λ_S^j , and then resolve $(1 - a_S^j) a_S^j$ into $\sum_{k \neq j} a_S^k a_S^j$. \square

Remark 5.1 *Definition 5.1 is the direct proportional analog to Hart and Mas-Colell (1996) Proposition 4. The conditions are identical except that condition (iii) adds multiplication by $\varphi_j(S, V)$ on the LHS and multiplication by $\varphi_i(S, V)$ on the RHS. This definition is a generalization of a weighted equal proportional gain condition in the sense*

that if the following condition, equation (5.8), is satisfied for all players then condition (iii) will hold as well. The converse, however, need not be true. Note that when $\lambda_S^j = \lambda_S^k$, equation (5.8) reduces to the equal proportional gain condition of Lemma 2.5.

$$(5.8) \quad \lambda_S^j \frac{\varphi_j(S, V) - \varphi_j(S \setminus k, V)}{\varphi_j(S, V)} = \lambda_S^k \frac{\varphi_k(S, V) - \varphi_k(S \setminus j, V)}{\varphi_k(S, V)}.$$

Remark 5.2 *Theorem 5.4 leads to the following characterization of the λ -weighted proportional value φ^λ in two-player games:*

$$(5.9) \quad \lambda_1 \frac{\varphi_1^\lambda(\overline{12}, V) - \varphi_1^\lambda(\overline{1}, V)}{\varphi_1^\lambda(\overline{12}, V)} = \lambda_2 \frac{\varphi_2^\lambda(\overline{12}, V) - \varphi_2^\lambda(\overline{2}, V)}{\varphi_2^\lambda(\overline{12}, V)},$$

where $(\lambda_1, \lambda_2) = \lambda(\varphi^\lambda(\overline{12}, V))$, with $\varphi^\lambda(\overline{12}, V) \in \partial V(\overline{12})$.

5.6 Interpretation of the Noncooperative Results

Hart and Mas-Colell (1989) distill the essential characteristics of equal gain relationships into the difference potential and the consistency property. The Hart and Mas-Colell (1996) bargaining game jettisons consistency and the potential but reflects a generalized weighted equal gain property characterized by their Proposition 4 and inherent in the structure of the game. Weighted equal gain relationships for players in any coalition hold “on average.” One way of understanding this generalization is to consider *parallel* hyperplane games, games where $\lambda_S = \{\lambda_N^i\}_{i \in S}$. The Owen-Maschler value of these games can be found to be equal to the weighted Egalitarian value with weights $\omega^i = 1/\lambda_N^i$. Thus, the Owen-Maschler value is a generalization of the weighted egalitarian value where each coalition has weights equal to the inverse of the λ -transfer weights supporting the value restricted to that coalition. A further evident implication is that if participation is set such that $p_S^i \propto \omega_S^i \lambda_S^i$, a generalized egalitarian value with independent weights for each coalition results. (If $p_S^i \propto \omega_S^i$, a weighted Owen-Maschler value results.)

Similar reasoning implies that setting participation such that $p_S^i \propto \omega^i \varphi_i(S, V)$ identifies weighted TU and input-basis NTU proportional value. When this restriction is relaxed to allow independent weights ω_S for each coalition, generalized weighted proportional value results. Output-basis weighted proportional value results when $p_S^i \propto \omega^i \lambda_S^i \varphi_i(S, V)$. Weighted values can be recursively determined by applying the conditions of Definition 5.1, appropriately modifying condition (iii). These observations correct an error in my earlier work. In Feldman (1999), Section 3.1.6, I proposed a weighted proportional value defined by $V(S) = \sum_i \omega_i P^\omega(S, V) / P^\omega(S \setminus i, V)$. However, the weights have no net effect. This potential defines the *unweighted* proportional value.

A deeper implication of the noncooperative results is that equity-based relationships are more fundamental to the characterization of cooperative value than either consistency or potential. Further, these relationships are also more fundamental than the classical

axiomatic approach. These inferences are already implicit in Hart and Mas-Colell's conclusion that the Owen-Maschler value is the proper representation of linear NTU value. Hart and Mas-Colell base their conclusion on the authority of the noncooperative implementation. The results developed here support this conclusion. The generalized weighted equal and proportional gain relationships emerge directly from the equilibrium conditions of the game. (In the terminology of the proof of Lemma 5.1, $a_s = A_S^{-1}B_S$.) All forms of TU and NTU value are identifiable by equity-based gain conditions.

6 Games with Zero and Small Players

Zero players are players with zero individual worth. Results for games with zero players raise the issue of how similar results would be for players with *almost* zero wealth. *Small* players are players with individual worths that are small relative to other players. This section derives some basic results for games with zero and small players.

6.1 Zero Players

Define Z as the coalition of all zero players. *Zero coalitions* are coalitions with zero worth. In monotonic games only zero players can be in zero coalitions. Zero players may not be *null* players, players that never add worth to a coalition. However, there will be circumstances where all zero coalitions are *null coalitions* as well and do not add worth in combination with coalitions $S \subset N \setminus Z$. If $S \subset N \setminus Z$ and R is a null coalition, then, in a TU game, $v(S \cup R) = v(S)$. If R is a null coalition in an NTU game then require $\partial V(S) \times \mathbf{0}^R \subset \partial V(S \cup R)$. When the players of R receive no allocation, the set of feasible allocations for the players of S is equal to $V(S)$. Assign the worth of a coalition S in a game V to a coalition $T \subset S$ in the game W . If V and W are TU, then $w(T) = v(S)$. In an NTU game, the same assignment is represented by *restriction* of the worth $V(S)$ to the players in T and denoted $W(T) = V(S)|_T$. Thus, $W(T) = \{x \in R^T : (x, \mathbf{0}^{S \setminus T}) \in V(S)\}$.

The ratio potential is not defined for zero coalitions because at least one ordered worth product will be zero. Conditions are identified here under which values for at least some players in such games are nonetheless well-defined. These results are based on limit sequences of games. In each such sequence, the worth of non-zero coalitions is fixed and the worth of all zero coalitions goes to zero in the limit. In a TU game sequence $\{v_p\}_{p=1}^\infty$, $0 < v_{p+1}(S) \leq v_p(S)$ and $\lim_{p \rightarrow \infty} v_p(S) = 0$ for all zero coalitions. In an NTU game sequence $\{V_p\}_{p=1}^\infty$, $\mathbf{0} \subset V_{p+1}(S) \subset V_p(S)$, and $\lim_{p \rightarrow \infty} V_p(S) = \mathbf{0}$ for all zero coalitions. Let $\{V_p^1\}_{p=1}^\infty$ and $\{V_p^2\}_{p=1}^\infty$ be two limit sequences based on V . The value of $\varphi(S, V)$ is *strongly defined* if $\lim_{p \rightarrow \infty} \varphi(S, V_p^1) = \lim_{p \rightarrow \infty} \varphi(S, V_p^2)$ for any two such sequences. Then $\varphi(S, V) = \lim_{p \rightarrow \infty} \varphi(S, V_p)$.

Theorem 6.1 *Let V be monotonic with $\sup\{x \in V(Z)\} = \mathbf{0}$. Then $\varphi(S, V)$ is strongly defined for all $S \subset N$ and $\varphi_i(V, S) = 0$ for all $S \ni i \in Z$, and $\varphi_j(V, S) > 0$ for all $S \ni j \notin Z$.*

Proof: Consider the TU case and sequences $\{v_p\}_{p=1}^\infty$ of the form $v_p(R) = \epsilon_p v_0(R)$ for $R \subseteq Z$ and $v_p(R) = v_0(R)$ otherwise, defined by a v_0 and a sequence $\epsilon_p \rightarrow 0$. These are clearly sufficient to define sequences that approximate a more general sequence in the limit. Then for any coalition S containing m zero players and for ϵ_p sufficiently small,

$$(6.1) \quad P(S, v_p) \approx \epsilon_p^m \left(\sum_{\substack{r \in \mathcal{R}(S) \\ R_m^r \subseteq Z}} \prod_{i=1}^s v_p(R_i^r)^{-1} \right)^{-1},$$

where the summation is over all orders where all zero players in S are first in the order. This is because the harmonic weight of ordered worth products with weight larger than ϵ_p^m , that is, with fewer than m zero players at the front of the order, goes to zero. The value of a zero player i will be the ratio of a potential with m zero players to one with $m - 1$ players, hence $\varphi_i(S, v_p) = O(\epsilon)$. The value of a nonzero player $j \in S$ is the ratio of two potentials with m zero players, hence $\varphi_j(S, V) \gg 0$ and is well-defined.

For the NTU case, consider the recursive definition of the potential and NTU sequences as described for the TU case. For any $S \subset Z$, clearly $P(S, V_p) = \epsilon_p^m P(S, V)$ and when $S \cap Z = \emptyset$, $P(S, V_p) = P(S, V)$. When $m = |S \cap Z| > 0$, the potentials of immediate subcoalitions with m zero players have the highest harmonic weight and must dominate as $p \rightarrow \infty$ so $P(S, V_p) = O(\epsilon^m)$. Thus for a zero player i , $\varphi_i(S, V_p) = O(\epsilon)$ and the value for non-zero players is positive and well-defined. \square

Corollary 6.1 *Let V be monotonic with $\sup\{x \in V(Z)\} = \mathbf{0}$. Let W be a game based on the players $N \setminus Z$. Define $W(S) = V(S \cup Z)|_S$. Then $\varphi_i(V) = \varphi_i(W)$.*

Proof: In the TU case, for any order $r_{S \setminus Z}$ of the players in $S \setminus Z$ there is an order r in the summation of equation (6.1) for every order $r_{S \cap Z}$ of players in $S \cap Z$ such that $r = (r_{S \cap Z}, r_{S \setminus Z})$. Thus $P(S, V_p) \approx \epsilon_p^m c P(S \setminus Z, W)$, where $c > 0$ depends only on $S \cap Z$. In the NTU case, $\varphi_S(S \cup Z, V) \in \partial W(S)$ for all $S \subseteq N \setminus Z$ since $\varphi_k(S \cup Z, V) = 0$ for all $j \in Z$ and φ is efficient. Since $\varphi_S(S \cup Z, V)$ has the equal proportional gain property for players $i, j \in S$, it must be equal to $\varphi_S(S, W)$ as well, both assertions due to Lemma 2.5. \square

Theorem 6.2 *Let V be monotonic with (i) $\sup\{x \in V(Z)\} > \mathbf{0}$, (ii) $\sup\{x \in V(S)\} = \mathbf{0}$ for $S \subsetneq Z$, and (iii) all $S \subsetneq Z$ are null coalitions. Then (1) $\varphi_j(S, V) > 0$ is well-defined for all $S \ni j \notin Z$, (2) when $S \supset Z$, $\sum_{i \in Z} \varphi_i(S, V) > 0$ is well-defined, and (3) when $S \not\supset Z$, $\varphi_i(S, V) = 0$ for $i \in Z$.*

Proof: Consider the TU case. For any S such that $Z \not\subseteq S$, Theorem 6.1 applies. When $S \supseteq Z$, for any $i \notin Z$ and $j \in Z$ the following relation may be obtained by summing over proportionality relationships (2.5) between i and the players $j \in Z$:

$$(6.2) \quad \varphi_i(Z \cup \bar{i}, v_p) = \frac{1}{v_p(Z)} \sum_{j \in Z} \varphi_i(Z \cup \bar{i} \setminus j, v_p) \varphi_j(Z \cup \bar{i}, v_p).$$

Theorem 6.1 requires $\lim_{p \rightarrow \infty} \varphi_i(Z \cup \bar{i} \setminus j, v_p) = v(Z \cup \bar{i} \setminus j)$ and because v is coalitionally null, $\lim_{p \rightarrow \infty} v_p(Z \cup \bar{i} \setminus j) = v(\bar{i})$. Equation (6.2) then simplifies and can be solved for $\varphi_i(v_p, Z \cup \bar{i})$:

$$(6.3) \quad \lim_{p \rightarrow \infty} \varphi_i(Z \cup \bar{i}, v_p) = \frac{v(\bar{i})}{v(Z) + v(\bar{i})} v(Z \cup \bar{i}).$$

For any $S \supseteq Z$, and in the limit as $p \rightarrow \infty$, the players of Z may be treated as a merged player by the following argument. For $i, j \in Z \cap S$ and $k \in S \setminus Z$ we have $\varphi_i(S, v)/\varphi_i(S \setminus k, v) = \varphi_k(S, v)/\varphi_k(S \setminus i, v)$ and $\varphi_j(S, v)/\varphi_j(S \setminus k, v) = \varphi_k(S, v)/\varphi_k(S \setminus j, v)$ by Lemma 2.5. But since all $R \subsetneq Z$ are coalitionally null, $\varphi_k(S \setminus i, v) = \varphi_k(S \setminus j, v)$ by Theorem 6.1. Therefore $\varphi_i(S, v)/\varphi_i(S \setminus k, v) = \varphi_j(S, v)/\varphi_j(S \setminus k, v)$. Let $\varphi_Z(S, v)$ represent the collective allocation to the players of Z . Note equation (6.3) implies $\varphi_Z(R, v) > 0$ when $m = z + 1$ and $v(Z) > 0$.

For any $S \supset Z$, value relationships satisfy Lemma 2.5 when the players in Z are considered as a merged player. Given $\varphi_Z(Z \cup j, V)$ and $\varphi_j(Z \cup j, V)$ are well-defined for any $j \notin Z$, equal proportional gain relationships must define the values for Z and all other players in S as well and the conclusion follows. The NTU case is identical in structure. (Worths of coalitions must be replaced by the sum of values of players, e.g., $v_p(S)$ is replaced by $\sum_{i \in S} \varphi_i(S, V_p)$). \square

Corollary 6.2 *Let v be a TU game satisfying conditions (i) – (iii) of Theorem 6.2. Let w be an $m - z + 1$ player game with player set $(N \setminus Z) \cup k$, where k “represents” the zero coalition. Define $w(S) = v((S \cup Z) \setminus k)$ for all $S \ni k$, and $w(S) = v(S)$ otherwise. Then w is strongly defined, $\varphi_i(v) = \varphi_i(w)$ for $i \in N \setminus Z$, and $\sum_{j \in Z} \varphi_j(v) = \varphi_k(w)$.*

Remark 6.1 *Construct a statistical cooperative game (Feldman (2002)) associated with a statistical model by making a one-to-one correspondence between players and independent variables and defining the worth of a coalition to be the marginal contribution to explained variance of its associated independent variables. If a subset of independent variables Z are all identical, the game meets conditions (i) – (iii) of Theorem 6.2. More complex forms of multicollinearity result in variations on this basic game.*

6.2 Small Players

What are the implications of value allocation to zero players for *small players*, players that are small, but have positive individual worth? Let v be a positive, symmetric TU game. Then, for any coalition S there is a constant K_S such that for any order $r \in \mathcal{R}(S)$, $K_S = \prod_{i=1}^s v(R_i^r)$, where, again, R_i^r is the coalition of the first i players in the order r . Let Z be the coalition of all small players, with $m = |Z|$. Let $z_S = |S \cap Z|$. Define the *small player game* v_η for an $0 < \eta < 1$ as

$$(6.4) \quad v_\eta(S) = \begin{cases} v(S) & \text{if } S \not\subseteq Z, \text{ and} \\ \eta v(S) & \text{otherwise.} \end{cases}$$

Lemma 6.1 *Let v_η be a game with small players derived from a symmetric game v using (6.4). Then the potential of any coalition $S \subseteq N$ with z_S small players is*

$$P(S, v_\eta) = \eta^k K_s \left((s - z_S) z_S! \sum_{k=0}^{z_S} \frac{(s - k - 1)!}{(z_S - k)!} \right)^{-1}.$$

Proof: The value of any ordered worth product is a function of the number of coalitions $R \subseteq Z$ in the order. If there are k such coalitions in an order r , then $\prod_{i=1}^s v(R_i^r) = \eta^k K_S$. Calculating the potential is then a counting exercise. For $0 < k \leq z_S$, there are $z_S! / (z_S - k)! (s - z_S) (s - k - 1)!$ orders where k small players are the first k players in an order, and a regular player follows. Factoring out common terms and rearranging leads to the result. \square

Corollary 6.3 *Let v_η be a game with small players based on a positive, symmetric, TU game v , let $i \in Z \cap S$, and $j \in S \setminus Z$. Then*

$$\frac{\varphi_i(S, v_\eta)}{\varphi_j(S, v_\eta)} = \frac{P(S \setminus j, v_\eta)}{P(S \setminus i, v_\eta)} \approx \eta \frac{s - z_S}{z_S}.$$

This result generalizes easily to games with a hierarchy of player classes. Again start with a positive, symmetric, TU game v . Define a partition of S , $\mathcal{P} = \{S_1, S_2, \dots, S_k\}$ and a vector $\eta^* = (\eta_1^*, \eta_2^*, \dots, \eta_k^*)$, with $0 < \eta_i^* \ll \eta_j^*$ for $i < j$. Also define $Z_j = \cup_{i=1}^j S_i$. Then define the *hierarchical* game v_{η^*} as follows:

$$(6.5) \quad v_{\eta^*}(S) = \eta_j^* v(S) \text{ for } j : S \subseteq Z_j \text{ and } S \cap S_j \neq \emptyset.$$

The element of the η^* vector applied to $v(S)$ is determined by the highest size class represented in S . For a coalition S , let $z_S = (z_{S_1}, z_{S_2}, \dots, z_{S_k})$ be a vector representing the number of players of each size class, i.e., $s = \sum_{i=1}^k z_{S_i}$. The potential function for v_{η^*} is more complex than for (6.4), but from Lemma 6.1 it can be seen that the leading term of the summation should determine the potential when the differences between consecutive elements of η^* are sufficiently large. Therefore, consider only orders where the players of size class S_i all are before the players of class S_j , for all $i < j$. There are $z_{S_1}! z_{S_2}! \dots z_{S_k}!$ such orders. For each such $r \in \mathcal{R}(S)$, $\prod_{i=1}^s v(R_i^r) = K_S \eta_1^{*z_{S_1}} \eta_2^{*z_{S_2}} \dots \eta_k^{*z_{S_k}}$. Thus,

$$(6.6) \quad P(S, v_{\eta^*}) \approx K_s \frac{\eta_1^{*z_{S_1}} \eta_2^{*z_{S_2}} \dots \eta_k^{*z_{S_k}}}{z_{S_1}! z_{S_2}! \dots z_{S_k}!}.$$

Theorem 6.3 *Let v_{η^*} be a game with hierarchical small players based on a positive symmetric TU game v , hierarchical partition $\mathcal{P} = \{S_1, S_2, \dots, S_k\}$, and $\eta^* > 0$. Then for any $j \in S_j$ and for an η^* where the differences between consecutive elements of η^* are sufficiently large,*

$$\varphi_i(S, v_{\eta^*}) \approx \frac{1}{z_{S_i}} \frac{\eta_i^*}{\sum_{j=1}^k \eta_j^*} v_{\eta^*}(S).$$

Proof: Start with (6.6) and use $P(S \setminus i, v_{\eta^*})/P(S \setminus j, v_{\eta^*}) = \varphi_i(S, v_{\eta^*})/\varphi_j(S, v_{\eta^*})$ to show that $\eta_j^* z_{S_i} \varphi_i(S, v_{\eta^*}) \approx \eta_i^* z_{S_j} \varphi_j(S, v_{\eta^*})$. Sum over j on both sides and solve for $\varphi_i(S, v_{\eta^*})$. \square

Computational experiments suggest that an order of magnitude difference between consecutive elements of η^* is usually more than sufficient to obtain close approximation to the true value even in games with relatively small numbers of players.

Theorem 6.3 shows that proportional value allocation in these hierarchical games is approximately pure proportional bargaining between the player *classes*. Class i 's allocation is proportional to η_i^* and is independent of the number of its members z_{S_i} .

Remark 6.2 *The primary purpose of these games is to estimate an upper bound on the expected allocations to small players using generally liberal assumptions regarding their potential contributions to general coalitions. It is necessary, however, to place an upper limit on the maximum possible contributions of small players. Here this limit is the marginal contribution of the largest players. This is the reason for the underlying symmetric game. If a small player's contributions to coalitions including large players is large enough (they must be proportionally larger than the individual worths are small), the impact of small individual worth can be overcome.*

A general interpretation of these games consistent with their structure could be as *information* games. All players are equal except for the information they possess. All players of the same class possess the same information. The information of any class includes the information of all classes below it. The information possessed by a coalition is union of all information held by its players.

7 Monopoly in a Production Game

Consider a game with a single large player B that controls a first factor of production x and $n - 1$ small players with equal shares of a second fixed factor of production y . Profits from production are described by a generalized Cobb-Douglas function. Coalition S controls y_S of the second factor. Let v_n represent the profits from production by different coalitions. Then

$$(7.1) \quad v_n(S) = \begin{cases} \xi x^\alpha & \text{when } S = \{B\} \\ y_S^\beta & = \left[\frac{s}{n-1} y\right]^\beta \text{ when } B \notin S \\ x^\alpha y_S^\beta & = x^\alpha \left[\frac{s-1}{n-1} y\right]^\beta \text{ otherwise,} \end{cases}$$

where $0 < \xi \leq v_n(\bar{i})$ for $i \neq B$, $x > 1$ and $y = y_N > 1$ are fixed, and $\alpha, \beta > 0$. When the first factor is missing production takes place with value y_S^β . The normalization factor ξ determines the net value of production when the second factor is not present. If $\xi > v_n(\bar{i})$, $i \neq B$, then $v_n(\bar{B}) > v_n(\bar{B}i)$ and v_n is not superadditive. If $v_n(\bar{i}) > 1$ for $i \neq B$, it is convenient to consider $\xi = 1$. Non-transferrable utility, costs, and prices are not modelled in order to allow an analytic solution. If $\beta > 1$ it can be assumed that production is

taking place on an increasing returns portion of the production function since the total quantities of productive factors are fixed.

The game could represent either a monopolist seller of an input needed by small producers or a monopsonist purchaser of the small producers' product. Intuitively, the small players are at an increasing bargaining disadvantage as their numbers grow large. In the limit, they should be price-takers.

The proportional value of player B in any coalition S may be determined by computing the potentials of $v_n(S)$ and $v_n(S \setminus B)$. Every ordered worth product contributing to a potential $P(S, v_n)$ with $B \notin S$ is the same: $[s!y_S^s/s^s]^\beta$. Then $P(S, v_n) = s!^{\beta-1} s^{-s\beta} y_S^{s\beta}$ for $B \notin S$. When $S = \overline{B}$, $P(S, v_n) = \xi x^\alpha$. When $\overline{B} \subsetneq S$, B will show up at any position in an order with equal probability. The resulting coalition, and all following will have worth increased by the factor x^α over the worth without B . If B enters at position k then the magnitude of the ordered worth product increases by $x^{\alpha(s-k+1)}$. Also, when B enters at k , there is an additional correction because the coalitions R_k^r and R_{k-1}^r both have $(k-1)/(s-1)y_S$ units of the second productive factor. There are $(s-1)!$ orders associated with B entering at any position. Thus for a coalition $S \ni B$,

$$(7.2) \quad P(S, v_n) = (s-1)!^{\beta-1} (s-1)^{-(s-1)\beta} y_S^{(s-1)\beta} \left(\xi^{-1} x^{-\alpha s} + y_S^{-\beta} \sum_{i=1}^{s-1} x^{-i\alpha} \left[\frac{s-1}{s-i} \right]^\beta \right)^{-1}.$$

Luck has it that the first three terms of $P(S, v)$ are equal to $P(S \setminus B, v)$. Therefore,

$$(7.3) \quad \varphi_B(S, v_n) = \left(\xi^{-1} x^{-s\alpha} + y_S^{-\beta} \sum_{i=1}^{s-1} x^{-i\alpha} \left[\frac{s-1}{s-i} \right]^\beta \right)^{-1}.$$

Consider $\varphi_B(v_n)$ as $n \rightarrow \infty$. The first term goes to zero. The summation may be broken into two components. The first component is $y_S^{-\beta} \sum_{i=1}^n [x^\alpha]^{-i}$. In the limit as $n \rightarrow \infty$, $\sum_{i=1}^\infty [x^\alpha]^{-i} = (x^\alpha - 1)^{-1}$. The second component with individual terms $x^{-i\alpha} y_S^{-\beta} \{[(s-1)/(s-i)]^\beta - 1\}$ clearly goes to zero. Thus

$$(7.4) \quad \lim_{n \rightarrow \infty} \varphi_B(v_n) = (x^\alpha - 1) y^\beta.$$

Core allocations in this game require a collective allocation to the small players of at least y^β when $\beta \geq 1$. (When $\beta < 1$ the core does not exist in the limit since collective individually rational allocations to small players become infinite: $\lim_{n \rightarrow \infty} (n-1)[1/(n-1)y]^\beta = \infty$.) As the number of small players grows their total value shrinks to their minimum core allocation when $\beta \geq 1$.

Remark 7.1 *Since v_n can be interpreted as being derived from a symmetric game where $v(S) = (s/n y)^\beta$ by rule (6.4), the results of Theorem 6.3 apply when $x^\alpha \gg 1$. Set $\eta^* = (1, x^\alpha)$, which is approximately correct in the limit (it is not true for $v(\overline{B})$), and Theorem 6.3 implies $\lim_{n \rightarrow \infty} \varphi_B(v_n) = x^{2\alpha} y^\beta / (x^\alpha + 1)$. Thus if $x^\alpha = 3$, the approximation implies $\varphi_B(v_n) = 2.25 y^\beta$, whereas the exact result is $2y^\beta$.*

The Shapley value for this game may also be determined. The marginal contribution of player B is ξx^β when no other players are in S , and is $(x^\alpha - 1)y_S^\beta$ otherwise. Then for any coalition $S \ni B$,

$$(7.5) \quad Sh_B(S, v_n) = \frac{1}{s} \left(\xi x^\alpha + (x^\alpha - 1) \sum_{i=1}^{s-1} \left[\frac{i}{s-1} y_S \right]^\beta \right).$$

In the limit as $n \rightarrow \infty$, the average of the summation converges to a simple integral, thus

$$(7.6) \quad \lim_{n \rightarrow \infty} Sh_B(v_n) = (x^\alpha - 1)y^\beta \int_0^1 i^\beta di = \frac{1}{\beta + 1} (x^\alpha - 1)y^\beta.$$

In the limit B 's Shapley value is half its proportional value when the second factor exhibits constant returns to scale. For every $\beta > 0$, $Sh_B < \varphi_B$. Observe that $Sh_B(v_n)$ decreases as n grows. To see this, consider the interval $[(i-1)/(n-1), i/(n-1)]$. For any x in this interval except the right boundary, $x^\beta < (i/(n-1))^\beta$. Therefore $1/s \sum_{i=1}^{s-1} (i/(s-1))^\beta < \int_0^1 i^\beta di$.

The Shapley value of B should be expected to decrease as the number of small players increases primarily because their collective probability of selection to propose is increasing as well. As this probability increases, the weight on B 's marginal contribution in the computation of the continuation value by equation (5.2) (with equal participation probabilities) decreases. In the case of the proportional value, first, since v is convex for $\beta \geq 1$, $\varphi(v_n) \in \text{core}(v_n)$ by Theorem 2.1. This places a lower limit on the payoffs to small players. Allocations to small players are driven down to this point by B 's high probability of selection to propose under proportional value allocation combined with the steep drop in their payoffs when B is removed from the game.

Remark 7.2 Let (a_1, a_2) , be an allocation where the first allocation is to B and the second is the sum of allocations to all small players. Since all solutions to be considered are efficient, $a_1 + a_2 = x^\alpha y^\beta$. When $\beta \geq 1$ the core is defined by $a_2 \in [y^\beta, x^\alpha y^\beta]$. The nucleolus and kernel both give $a_1 = \frac{1}{2}(x^\alpha - 1)y^\beta$, the Shapley value solution when $\beta = 1$. Thus for $\beta \geq 1$,

$$(7.7) \quad \lim_{n \rightarrow \infty} \varphi_B(v_n) \gg \lim_{n \rightarrow \infty} Nuc_B(v_n) = \lim_{n \rightarrow \infty} Ker_B(v_n) \geq \lim_{n \rightarrow \infty} Sh_B(v_n).$$

These results are consistent with the literature on nonatomic games with large players. For example, Einy, Moreno, and Shitovitz (1999) show the asymptotic nucleolus is equal or larger than the Shapley value in nonatomic exchange economies.

8 Discussion and Conclusion

This paper presents a model of cooperative value as the extension of the traditional fairness or equity-based methods of sharing to coalitional cooperative games. The basic

forms of cooperative value are shown to be concisely represented in this framework. Section 5.6 makes the case that this framework, derivative of Myerson (1977 and 1980), should be considered the fundamental characterization of cooperative value.

The Hart and Mas-Colell (1996) bargaining game has been used here, and by Hart and Mas-Colell, primarily as an instrument. This game has some undesirable features which, however, should be of little consequence here. There are no alternative noncooperative models of NTU value. The TU and output-basis equilibria agree with the equity, consistency, and potential models of value. The input-basis equilibria define generalized equity relationships. These generalizations, described by Hart and Mas-Colell Proposition 4 and by Definition 5.1 here, have independent significance. They can be seen to be the natural result of extending the input-basis bargaining model to coalitional games by allowing for the variation in marginal rates of efficient substitution between players' utilities characteristic of general NTU games (see Section 5.6).

Implementation of output-basis values requires λ -weighting of participation probabilities. This may appear awkward. A functional institutional perspective is that practices embodying such characteristics will evolve if there is a preference for the outcomes they produce. Output-basis bargaining is similar to bargaining over after-tax dollars or services received instead of the cost of provision. Clearly, there are situations where bargaining appears to take place on this basis. Regardless, this paper establishes that the proportional mode is not only a creature of the output basis in NTU games.

The dual model does not determine which type of value should be observed in any particular circumstance. The properties, characterizations, and examples presented here, however, should be a useful starting point for the formation of hypotheses. In particular, the production game of Section 7 clearly implies that the proportional mode may be useful in the study of bargaining power in imperfectly competitive economic environments.

Returning to an issue raised in the introduction, the invariance of physical allocations to affine transformations of players' utility functions is a cornerstone of economic and game theory. Myerson (1992: 18) advises suspicion of "any theory of economic behavior" without this property. Hart and Mas-Colell cross the path that leads to these results but explicitly reject the study of proportionality for just this reason (1989: 595).

Invariance implies the impossibility of interpersonal comparison of utility. Aversion to interpersonal comparisons of utility in general economic theory and in social choice theory is not surprising since either impersonal market forces or a "central planner" would have to make these comparisons. In bargaining, however, these assessments must be understood to be made by the bargainers themselves. Whether or not such comparisons can actually be properly made, there is little question that agreement on such comparisons is regarded as an important component of successful bargaining outcomes (see, e.g., Thompson (1998)). Proportional mode and input basis values provide a theoretical structure for the representation of these comparisons. In this respect, my argument bears some resemblance to the proposition that individual utility functions should allow reference points or frames (e.g., the prospect theory of Kahneman and Tversky (1979)).

In fact, assuming players to have expected utility functions specifically enables in-

terpersonal comparison of utility. Proportional allocation requires agreement on appropriate reference points to base computation of proportional gains. The translations of bargainers' nominal utility functions implied by these reference points are guaranteed to still properly represent their preferences under expected utility. Similarly, in egalitarian allocation bargainers must implicitly or explicitly agree on rescalings of their utility functions so that units represent equal amounts of utility. These rescalings will still properly represent the preferences of players with expected utility functions. To be sure, our interpretation of players' utility functions must then be that are no longer fully invariant precisely because they incorporate the relevant interpersonal comparisons.

The dual model may also provide a productive framework for the study of problems where invariance is not remotely an issue. For example, in a *statistical cooperative game* the worth of a coalition represents the explanatory power of an associated set of independent variables in a statistical model. Feldman (2002) shows that two existing variance decomposition methods can be understood as the Shapley value of statistical cooperative games; and that consistency, non-negativity of decompositions, and a weakened form of linearity in two-player games uniquely identifies proportional variance decomposition. Cooperative methods may be applicable to other such joint-effects problems.

It appears that proportional bargaining has never been tested for, either experimentally or otherwise. Much might be learned simply from the reanalysis of studies of two-player bargaining situations. Most bargaining experiments set individual worths to zero, effectively eliminating the possibility of proportional bargaining. However, empirical studies such as Svejnar's (1986) study of union bargaining and the now extensive studies of bargaining effects in intrahousehold resource allocation following Manser and Brown (1980) represent a wealth of data. Particularly in the case of intrahousehold resource allocation, it appears that the formulation of a test of a proportional bargaining hypothesis should not be too difficult. Nash bargaining implies that the effect of an extra dollar of maternal income should be independent of the level of maternal income.

The experimental record with respect to the basis of NTU bargaining outcomes in linear bargaining environments is very different. Since Nydegger and Owen (1974), tests of Nash bargaining have revealed the strong tendency of subjects to reach equal sharing agreements. (See, particularly, Weg, Rapoport, and Felsenthal (1990). See Roth (1995) for a survey.) Measures such as anonymity and denying "weaker" bargainers knowledge of other player's payoffs appear necessary to generate a high frequency of Nash-like results. Surprisingly, no experimental study or commentary appears to consider the fact that such outcomes have a theoretical basis represented by the egalitarian value.

9 Appendix

9.1 Proof of Theorem 3.3

Given Theorem 3.2 all that must be proved is that consistency with equal proportional gain bargaining in TU two-player games implies it in NTU two-player games as well. The proof

strategy is the same as in Hart and Mas-Colell (1989) Lemma 6.9: A two-player NTU game $(\bar{i}\bar{j}, V)$ is embedded in a three-player game $(\widehat{ijk}, \widehat{V})$ whose other coalitional worths are all transferable. Because $\widehat{V}(\widehat{ijk})$ is equivalent to a transferable worth, all games reduced by one player are equivalent to TU games. Consistency identifies the values of the original game.

Let $\alpha_i = \sup\{x_i \in V(\bar{i})\}$, define α_j similarly, and set $\alpha_k > 0$. Define \widehat{V} as follows:

$$\widehat{V}(S) = \begin{cases} V(S) & S = \bar{i}, \bar{j}, \text{ or } \bar{i}\bar{j}, \\ \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq \sum_{i \in S} \alpha_i\} & \text{otherwise.} \end{cases}$$

Let ϕ satisfy the conditions of the theorem and let $(y_i, y_2, y_3) = \phi(\widehat{ijk}, \widehat{V})$. Define TU versions of the two-player reduced games: $\widehat{v}_{-i}^\phi, \widehat{v}_{-j}^\phi$, and \widehat{v}_{-k}^ϕ . In \widehat{v}_{-i}^ϕ and \widehat{v}_{-j}^ϕ the individual worths of i and j must be their values under $\phi(\bar{i}\bar{j}, V)$, which is what is left over after the other player is given his value under $\phi(\bar{i}\bar{j}, V)$, by efficiency and the definition of the reduced game. All other two-player games are additive so players get their individual worth:

$$\begin{aligned} \widehat{v}_{-i}^\phi(\bar{j}) &= \phi_j(\bar{i}\bar{j}, V), & \widehat{v}_{-i}^\phi(\bar{k}) &= \alpha_k, & \widehat{v}_{-i}^\phi(\bar{i}\bar{k}) &= y_j + y_k, \\ \widehat{v}_{-j}^\phi(\bar{i}) &= \phi_i(\bar{i}\bar{j}, V), & \widehat{v}_{-j}^\phi(\bar{k}) &= \alpha_k, & \widehat{v}_{-j}^\phi(\bar{j}\bar{k}) &= y_i + y_k, \\ \widehat{v}_{-k}^\phi(\bar{i}) &= \alpha_i, & \widehat{v}_{-k}^\phi(\bar{j}) &= \alpha_j, & \widehat{v}_{-k}^\phi(\bar{i}\bar{j}) &= y_i + y_j. \end{aligned}$$

The TU values for these games are easily calculated from (2.4). Since ϕ is consistent, these games generate a system of equations equating allocations in two of the reduced games, one for each player. Dividing $\phi_i(\widehat{v}_{-j}^\phi)$ by $\phi_j(\widehat{v}_{-i}^\phi)$ and substituting on the left-hand side gives

$$\frac{\phi_i(\widehat{ijk}, \widehat{V})}{\phi_j(\widehat{ijk}, \widehat{V})} = \frac{\widehat{v}_{-j}^\phi(\bar{i})}{\widehat{v}_{-i}^\phi(\bar{j})} \frac{\widehat{v}_{-i}^\phi(\bar{j}) + \alpha_k}{\widehat{v}_{-j}^\phi(\bar{i}) + \alpha_k} \frac{y_i + y_k}{y_j + y_k}.$$

The game \widehat{v}_{-k}^ϕ implies that $\phi_i(\widehat{ijk}, \widehat{V})/\phi_j(\widehat{ijk}, \widehat{V}) = \alpha_i/\alpha_j$. Equating $\phi_k(\widehat{v}_{-i}^\phi)$ and $\phi_k(\widehat{v}_{-j}^\phi)$ in the first two games implies $(\widehat{v}_{-j}^\phi(\bar{i}) + \alpha_k)/(\widehat{v}_{-i}^\phi(\bar{j}) + \alpha_k) = (y_i + y_k)/(y_j + y_k)$. Using these results to simplify the principal equation and substituting $\phi_i(\bar{i}\bar{j}, V) = \widehat{v}_{-j}^\phi(\bar{i})$ and $\phi_j(\bar{i}\bar{j}, V) = \widehat{v}_{-i}^\phi(\bar{j})$ shows that $\phi_i(\bar{i}\bar{j}, V)/\alpha_i = \phi_j(\bar{i}\bar{j}, V)/\alpha_j$. By Theorem 3.2, $\phi = \varphi$. \square

9.2 Proof of Lemma 4.1

Assume the lemma is true for all immediate subcoalitions of S and consider the potential of S . For an $i \in S_i$,

$$(9.1) \quad P(S \setminus i, V^*) = P(S_i \setminus i, V) \prod_{\substack{S_j \in S/Q \\ i \notin S_j}} P(S_j, V) = \frac{P(S_i \setminus i, V)}{P(S_i, V)} \prod_{S_j \in S/Q} P(S_j, V),$$

where the second step results from multiplying the numerator and denominator of the right-hand terms by $P(S_i, V)$. Rearranging leads to

$$(9.2) \quad \varphi_i(S_i, V) = \frac{P(S_i, V)}{P(S_i \setminus i, V)} = \frac{\prod_{S_j \in S/Q} P(S_j, V)}{P(S \setminus i, V^*)}.$$

Since this relation is true for every i , the product must be the potential. For $n = 2$ the only nontrivial coalition structure is $Q = \{\bar{1}, \bar{2}\}$. Then $V^*(\bar{i}\bar{j}) = V(\bar{i}\bar{j}/Q) = \{x : x_i \leq V(\bar{i}), x_j \leq V(\bar{j})\}$. In this case (9.2) reduces to $P(\bar{i}, V) = P(\bar{i}, V) = \varphi_i(\bar{i}, V) = \sup\{x : x \in V(\bar{i})\}$. The conclusion follows by induction. \square

9.3 Proof of Lemma 5.1

By Hart and Mas-Colell Proposition 6 an efficient SSP equilibrium exists for every $0 \leq \rho < 1$. Let the weights $\lambda_{S,j} = (\lambda_{S,j}^k)_{k \in S}$ and the constant $V_S^j > 0$ define the tangent hyperplane supporting player j 's equilibrium proposal $a_{S,j} \in \partial V(S)$ for all $k \in S$. Substitute into Hart and Mas-Colell Proposition 1 (equation (5.1) here), for j 's proposal to any player $k \neq j$, $a_{S,j}^k = \rho \sum_i p_S^i a_{S,i}^k + (1 - \rho) a_{S \setminus j}^k$, where $a_{S,j}^k = a_S^k$ for $k \neq j$ and $a_{S,j}^j = 1/\lambda_{S,j}^j (V_S^j - \sum_{i \neq j} \lambda_{S,j}^i a_{S,i}^j)$. Define the $s \times s$ matrix A_S such that for $j \neq i$, $A_S^{ij} = \rho p_S^i \lambda_{S,i}^j / \lambda_{S,i}^i$, and $A_S^{ii} = 1 - \rho + \rho p_S^i$. Define the $s \times 1$ column vector B_S with $B_S^i = V_S^i / \lambda_{S,i}^i + (1 - \rho) \sum_{j \neq i} p_S^j a_{S \setminus i}^j$. Then $A_S a_S = B_S$. Both A_S and B_S are continuous in p_S , λ_S , and $\{V_S^j\}_{j \in S}$.

The inverse A_S^{-1} exists if and only if the matrix Λ_S , generated by multiplying each row by $\lambda_{S,i}^i / (\rho p_S^i)$, has an inverse. For $i \neq j$ then, $\Lambda_S^{ij} = \lambda_{S,i}^j$. For $i = j$,

$$(9.3) \quad \Lambda_S^{ii} = \lambda_{S,i}^i \left(1 + \frac{1 - \rho}{\rho p_S^i} \right).$$

Λ_S can be interpreted as defining the polyhedral cone associated with the convex cone generated by the intersection of the half-spaces defined by the supporting tangent hyperplanes defined by λ_S and $\{V_S^j\}_{j \in S}$. Equation (9.3) shows that the effective weight defining a player's self allocation increases with decreases in ρ . An increasing λ -transfer weight implies that this player's utility is more "costly" to produce. Then if $\rho_1 < \rho_2$, the cone generated by ρ_1 is clearly contained in the cone generated by ρ_2 . Thus, for $\rho < 1$, equation (9.3) shows the cone must have s generators, even in TU and hyperplane games. (In the case of $p_S^i = 0$ for at least one player i , modify Λ by multiplying row i by p_S^i for all such players. Then $\Lambda_S^{ij} = 0$ for $j \neq i$ and $\Lambda_S^{ii} = \lambda_{S,i}^i (1 - \rho)$. Λ is still clearly nonsingular since $\Lambda_S x = \mathbf{0}$ implies $x_i = 0$ for every i such that $p_S^i = 0$. The submatrix defined by the rows and columns of players with $p_S^i > 0$ is immediately seen to be nonsingular.) Thus, $\Lambda_S x = \mathbf{0}$ implies $x = \mathbf{0}$ and A_S^{-1} exists for every equilibrium proposal profile generated by any $p_S \in \Delta^S$.

For TU and hyperplane NTU games, and for a fixed ρ , A_S and B_S are functions only of p_S . Therefore $\Gamma_S(p_S) = A_S^{-1}(p_S) B_S(p_S)$. Clearly, Γ_S is single-valued and continuous in this case. Evidently, this result characterizes both (5.2) (Hart and Mas-Colell Proposition 9) and (5.3).

Now consider the general NTU case. Assume a fixed ρ , sequence $p_{S^i} \rightarrow p_S$ and sequences $\lambda_S(p_{S^i}) \rightarrow \lambda_S$, $A_S(p_{S^i}) \rightarrow A_S^*$, $a_S(p_{S^i}) \rightarrow a_S^*$, and $B_S(p_{S^i}) \rightarrow B_S^*$ such that $A_S^*(p_{S^i}) a_S^*(p_{S^i}) = B_S^*(p_{S^i})$ for every p_{S^i} . Then $A_S^* a_S^* = B_S^*$ and Γ_S is upper hemi-continuous.

Local uniqueness and lower-hemicontinuity follow from application of the implicit function theorem. Let $F(p_S, c_S) = A_S(p_S, \lambda(p_S)) c_S - B_S(p_S, \lambda(p_S))$, for c_S in a neighborhood $U \in \mathbb{R}^S$ of a_S where $A_S(p_S, \lambda(p_S)) a_S = B_S(p_S, \lambda(p_S))$. Clearly, $F(p_S, a_S) = 0$. Further, $\partial F / \partial p_S(c_S)$ is continuous due to (5.1) and the smoothness of $\partial V(S)$ ($\partial V(S)$ must be at least C^2). With these conditions, together with the existence of $A^{-1}(p_S, \lambda(p_S))$, the implicit function theorem guarantees the local uniqueness of Γ_S and its continuity. \square

10 References

- ADAMS, S. (1965): "Inequity in Social Exchange," in *Advances in Experimental Social Psychology*, ed. by L. Berkowitz. New York: Academic Press.
- EINY, EZRA, DIEGO MORENO, AND BENYAMIN SHITOVITZ (1999): "The Asymptotic Nucleolus of Large Monopolistic Market Games," *Journal of Economic Theory*, 89, 186-206.
- FELDMAN, BARRY (1999): *The Proportional Value of a Cooperative Game*. Manuscript. Chicago: Scudder Kemper Investments.
- (2002): *Statistical Cooperative Games*. Manuscript. Chicago: Ibbotson Associates.
- GUL, FARUK (1989): "Bargaining Foundations of the Shapley Value," *Econometrica* 57, 81-95.
- HART, SERGIU, AND ANDREU MAS-COLELL (1989): "Potential, Value and Consistency," *Econometrica*, 57, 589-614.
- AND ——— (1996): "Bargaining and Value," *Econometrica*, 64, 357-380.
- HOMANS, GEORGE S. (1961,1977): *Social Behavior: Its Elementary Forms*. New York: Harcourt, Brace and Jovanovich.
- KAHNEMAN, DANIEL AND AMOS TVERSKY: (1979) "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica*, 47, 263-292.
- KALAI, EHUD AND DOV SAMET (1985): "Monotonic Solutions to General Cooperative Games." *Econometrica*, 53, 307-327.
- MANSER, MARILYN AND MURRAY BROWN (1980): "Marriage and Household Decision Making: A Bargaining Analysis," *International Economic Review*, 21, 31-44.
- MASCHLER, MICHAEL, AND GUILLERMO OWEN (1989): "The Consistent Shapley Value for Hyperplane Games," *International Journal of Game Theory* 18, 389-407.
- , AND ——— (1992): "The Consistent Shapley Value for Games Without Side Payments," in *Rational Interaction*, ed. by R. Selten. New York: Springer-Verlag, 5-12.
- MYERSON, ROGER (1977): "Cooperation on Graphs," *Mathematics of Operations Research*, 2, 225-229.
- (1980): "Conference Structures and Fair Allocation Rules," *International Journal of Game Theory*, 9, 169-82.
- (1991): *Game Theory: Analysis of Conflict*. Cambridge: Harvard University Press.
- NASH, JOHN F., JR. (1950): "The Bargaining Problem," *Econometrica*, 18, 155-162.
- NYDEGGER, RUDY V., AND GUILLERMO OWEN (1974): "Two-Person Bargaining: An Experimental Test of the Nash Axioms," *International Journal of Game Theory*, 3, 239-49.
- ORTMANN, K. MICHAEL (2000): "The Proportional Value for Positive Cooperative Games," *Mathematical Methods of Operations Research* 51, 235-248.
- PÉREZ CASTRILLO, DAVID AND DAVID WETTSTEIN (2001): "Bidding for the Surplus: A Non-cooperative Approach to the Shapley Value," *Journal of Economic Theory*, 100, 274-94.

- QIN, CHENG-ZHONG (1996) "Endogenous Formation of Coalition Structures," *Journal of Economic Theory* 69, 218-226.
- ROTH, ALVIN E. (1995): "Introduction to Experimental Economics," in *The Handbook of Experimental Economics*, John H. Kagel and Alvin E. Roth, eds., Princeton: Princeton University Press, pp. 3-109.
- SHAPLEY, LLOYD S. (1953): "Additive and Non-Additive Set Functions," Ph.D. Thesis. Princeton: Princeton University.
- SLIKKER, MARCO (2001): "Coalition Formation and Potential Games," *Games and Economic Behavior*, 37, 436-448.
- SVEJNAR, JAN (1986): "Bargaining Power, Fear of Disagreement, and Wage Settlements: Theory and Evidence From U.S. Industry," *Econometrica*, 54, 1055-78.
- THOMPSON, LEIGH (1998): *The Mind and Heart of the Negotiator*. Upper Saddle River, New Jersey: Prentice Hall.
- WEG, EYTHAN, AMNON RAPOPORT, AND DAN S. FELSENTHAL (1990): "Two-Person Bargaining Behavior in Fixed Discounting Factors Games with Infinite Horizon," *Games and Economic Behavior*, 2, 76-95.
- WINTER, EYAL (1992): "The Consistency and Potential for Values of Games with Coalition Structure," *Games and Economic Behavior*, 4, 132-44.
- YOUNG, H. PEYTON (1985): "Monotonic Solutions of Cooperative Games," *International Journal of Game Theory* 14, 65-72.
- (1994): *Equity: In Theory and Practice*. Princeton: Princeton University Press.