A Consumer-Based Model of Competitive Diffusion of Two Goods: The Effects of Network Externalities and Local Interactions

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ABSTRACT

The diffusion of two competitive, interchangeable, and durable goods is studied under the framework of a spatial game where consumers are distributed on a two-dimensional square lattice and play 3×3 symmetric coordination-like games with their nearest neighbors. There are three strategies, either consuming a product A or B, or a strategy C of not consuming either A or B. The payoff matrix of the game contains the positive effects of network externality, that is, the payoffs are increasing functions of the number of agents adopting the strategies A or B. Both simulations and mean-field approximation show that the existence of the positive effects of the network externality amplifies any slight initial difference in the number of agents who adopt either A or B and eventually promotes the superior product to take over the entire market. On the other hand, without effects of the network externality the slight initial difference is not enlarged and both superior and inferior products are observed to coexist by forming clusters in the market. Moreover, the effects of innovation factors that help an inferior product to retake the market are studied. It is shown that both the timing and size of the innovation factor matter for an inferior product in order to retake the market.

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I. INTRODUCTION

Diffusion of durable goods is often believed to be explainable by the logistic model, which originates from Verhulst (1838) who applied the model to the studies of demography [3,23]. There exist numerous studies on diffusion of durable goods with the logistic model [12], and these studies showed that the logistic model could replicate the data of full diffusion of various durable goods. A discrete version of the logistic model is given by

$$p_X(t+1) - p_X(t) = \lambda \; p_X(t) \; (1 - p_X(t)) \tag{1}$$

where $p_X(t)$ represents the fraction of the people who already have a product X at a unit time t and λ is a constant parameter, which controls the initial slope of the diffusion curve. In the logistic model, the diffusion rate $p_X(t+1) - p_X(t)$ is given as the multiplication of the parameter λ, $p_X(t)$, and $1 - p_X(t)$, that is, the rate is a function of the fraction of both people who already have adopted the product by the time t and who have not yet.

Besides the studies on diffusion phenomena with the logistic model, large numbers of diffusion phenomena, such as expansion of forest fires, disease and so on, have been studied by utilizing the framework of cellular automata [24]. A study of diffusion of interchangeable and durable goods in a market based on the framework of the two-dimensional cellular automata is thought to be meaningful because it is commonly observed that people try to imitate the most successful strategy from their local neighbors. This imitation dynamics is known as "copy cat." The spatiality such as local interactions within neighborhood is one of the most important features of cellular automata. A number of studies show that the effects of physical or abstract spatiality are critical for people's behavior [2,7,11,13-19,21]. Under the framework of the cellular automata based on the two-dimensional square lattice, each consumer on each cell only interacts with his nearest neighbors. If the field is large enough and if the initial seed $p_X(1)$ is much smaller than the entire population in the whole system, then the diffusion progresses only gradually.

In such case as a spatial game, the number of neighbors is fixed as eight (the Moore neighborhood) in this paper and the updating probability, which Huberman and Glance (1993) introduced, is thought to act like the parameter λ in Eq. (1) [8]. In the game field, consumers who do not have the product X are assumed to be able to adopt the product X only when there is at least one neighbor who already has the product X, that is, only such a neighbor can be a source of copy of the strategy that is to adopt the product X. When there is at least one neighbor who already has the product X in a consumer i's neighborhood, he is assumed to copy his neighbor's strategy with the following updating probability $\mu(t)$ that is given as an increasing function of the fraction of the people who has adopted the product X:

$$\mu(t) = \nu(p_X(t-1))^{\xi} \tag{2}$$

where t is greater than or equal to two, and ν $(0 < \nu \leq 1)$ and ξ are parameters. This particular updating probability function is one of the arbitrary increasing functions of the fraction of the people who has adopted the product X, and it is introduced here since this function with a proper set of parameters ν and ξ gives the time evolution of $p_X(t)$, which is obtained from a simulation, a good fit to the logistic curve with a given λ in Eq.(1).

Figure 1 shows numerically solved $p_X(t)$ (square dots) in Eq. (1) with the initial value $p_X(1)$=20/101^2 and the parameter λ=0.32 and the value $p_X(t)$ (black circles) that is obtained as a result of the cellular-automata based simulation mentioned above with the initial value $p_X(1)$=20/101^2 and the parameters ν=1 and $\xi = 0.5$. In the simulation consumers are placed in the field of 101×101 cells of a two-dimensional square lattice, therefore the size of the whole population $|N|$ is given as 101^2. It is observed that two plots in Fig. 1 are almost identical, and it suggests that the $p_X(t)$ obtained from the simulation could be described by the logistic equation in Eq. (1). This fact will be utilized later in Sec. V where a mean-field theory is conducted to recover the results of the simulations in Secs. III and IV.

In this paper the diffusion phenomena of two competitive, interchangeable, and durable goods are studied based on the framework of spatial game where consumers are distributed on two-dimensional square lattice and play 3×3 symmetric coordination like games with their nearest neighbors [20,25]. The detailed rules of the game are explained in the next section. The payoff matrix of the game contains the positive effects of the network externality, that is, the payoff elements are increasing functions of the number of agents adopting each strategy. The network externalities play an important role in diffusion of interchangeable goods [1,4-6,9,10]. In Sec. III, the results of simulations based on the rules of the game in Sec. II are shown to clarify the payoff and initial-condition-dependent behavior of the system and the positive effects of the network externality. The effects of innovation factors that help an inferior product to retake the market are illustrated in Sec. IV. In Sec. V, a mean-field theory is formulated to approximate the results of the simulations in Secs. III and IV. Discussions are given in the last section.

II. THE MODEL

Consumers are placed in a two-dimensional square lattice. Each consumer has one of three possible strategies, consuming either a product A or B, or a strategy C of consuming neither A nor B. Those who have the strategy C are the potential consumers of either the product A or B in the future. In the following, adopting either the product A, B, or C is expressed by $+1$, -1, or 0, respectively, that is, at time t the consumer in the i-th cell ($1 \leq i \leq |N|$) takes a strategy $\sigma_i(t)$ that is either $+1$, -1, or 0. The consumer i plays one-shot 3×3 symmetric coordination like games with his eight immediate neighbors, denoted as $\tilde{n}(i)$, under the payoff matrix given in Table I that describes the payoff matrix for a row player.

In Table I, $R_i(\pm 1)$ denotes a consumer i's payoff derived from a product itself and is given as

$$R_i(\pm 1) = r(\pm 1) \pm w\theta_i \tag{3}$$

where the parameter $r(\pm 1)$ is assumed to be greater than zero. The random value θ_i is uniformly generated between -0.5 and 0.5, and w is assumed to be a positive small number compared to $r(\pm 1)$, that is, $0 < w \ll r(\pm 1)$. Introducing such $w\theta_i$ enables $R_i(\pm 1)$ to contain a small amount of fluctuation reflecting the fact that the utility obtained from a product is not exactly the same but slightly different for each other.

Additionally, introducing such $R_i(\pm 1)$ allows the model to have consumers with slightly biased preferences [5].

The enhanced payoffs by sharing the same kind of an interchangeable product with neighbors, $S(\pm 1, t)$, are defined as

$$\begin{aligned} S(+1,t) &= s(+1)p_A(t-1) \text{ and} & (4)\\ S(-1,t) &= s(-1)p_B(t-1), & (5) \end{aligned}$$

respectively, where the parameter $s(\pm 1)$ is assumed to be greater than or equal to zero and $p_A(t)$ and $p_B(t)$ represent the fraction of consumers who are adopting the product A or B in the whole population, respectively, at the time t. Both $p_A(0)$ and $p_B(0)$ are assumed to be zero, therefore the initial value, $S(\pm 1, 1)$, is also assumed to be zero. The $S(\pm 1, t)$, which is a function of $p_A(t-1)$ and $p_B(t-1)$, respectively, reflects the effects of the network externalities from the whole population [1]. Because of the assumption that having a product A or B strictly dominates having neither of these products, the third row in Table I is all filled by zeros. This assumption corresponds to the scenario that all agents sooner or later adopt the product A or B in the diffusion process.

The payoff function for the consumer i in a game with a consumer j can be denoted as $f_i(\sigma_i(t), \sigma_j(t))$, that is, $f_i(+1,+1) = R_i(+1) + S(+1,t)$, $f_i(+1,-1) = f_i(+1,0) = R_i(+1)$, $f_i(-1,-1) = R_i(-1) + S(-1,t)$, $f_i(-1,0) = f_i(-1,+1) = R_i(-1)$, and $f_i(0,+1) = f_i(0,-1) = f_i(0,0) = 0$. The utility at time t, $u_i(\sigma_i(t))$, of the consumer i with the strategy $\sigma_i(t)$ is defined as the sum of the resultant payoffs obtained by playing the games with the consumer i's eight immediate neighbors:

$$\begin{aligned} u_i(\sigma_i(t)) &= \sum_{j \in \tilde{n}(i)} f_i(\sigma_i(t), \sigma_j(t)) \\ &= (R_i(+1) + S(+1,t)p_i(\sigma_i(t),t))\frac{n_0}{2}\sigma_i(t)(\sigma_i(t)+1) \\ &+ (R_i(-1) + S(-1,t)p_i(\sigma_i(t),t))\frac{n_0}{2}\sigma_i(t)(\sigma_i(t)-1) \qquad (6) \end{aligned}$$

where n_0 stands for the number of neighbors and $p_i(\sigma_i(t), t)$ is the fraction of the agents with the strategy

$\sigma_i(t)$ among the immediate neighbors and given by

$$\begin{aligned} p_i(\sigma_i(t), t) &= \frac{1}{n_0} \sum_{j \in \tilde{n}(i)} \{ \frac{1}{4} \sigma_i(t)(\sigma_i(t)+1)\sigma_j(t)(\sigma_j(t)+1) \\ &\quad + \frac{1}{4}\sigma_i(t)(\sigma_i(t)-1)\sigma_j(t)(\sigma_j(t)-1) \\ &\quad + (\sigma_i(t)+1)(\sigma_i(t)-1)(\sigma_j(t)+1)(\sigma_j(t)-1)\} \\ &= \frac{1}{2n_0} \sum_{j \in \tilde{n}(i)} \{(3\sigma_i^2(t)-2)\sigma_j^2(t) + \sigma_i(t)\sigma_j(t) - 2(\sigma_i^2(t)-1)\} \end{aligned} \tag{7}$$

The updating rule adopted in this paper is so-called "copy cat," that is, the consumer i's strategy at time $t+1$ is defined as follows:

$$\sigma_i(t+1) = \{\sigma_j(t) \mid u_j(t) = \max_{j \in n(i)} u_j(t)\} \tag{8}$$

where $n(i)$ represents i's neighborhood, which contains both $\tilde{n}(i)$ and i itself. In the copy cat, each consumer imitates the most successful strategy in his neighborhood in terms of consumers' utilities. The copy cat is adopted in this paper because it is commonly observed that people try to imitate a strategy of their most successful neighbor [2,22]. Note that switching costs that consumers pay when they switch from the product A to B or B to A are assumed to be negligible in this paper and set as zero for simplicity. If such costs are highly significant, then consumers ' incentive to switch their products dramatically decreases, therefore it is expected that results of the simulations will be highly dependent on the initial configurations.

III. NUMERICAL RESULTS OF THE MODEL

Simulations are conducted based on the rules of the game explained in the previous section. The results of the extensive and intensive parameter running are shown in this section. The agents with the three distinct strategies are homogeneously and randomly distributed at time $t = 1$ in the game field of two-dimensional square lattice. Fifty realizations with fifty different initial random configurations are examined in order to obtain the frequencies of three possible equilibria denoted as A*, B*, and P*. The symbol A* denotes the equilibrium where the product A takes over the whole market, and B* for the product B. The symbol P* stands for a polymorphic equilibrium where the product A and B coexist. The small fluctuation term in payoff, w, is fixed as 0.001 in this paper.

Extensive parameter running is performed with regard to the combinations of three conditions, (ic), (r), and (s), each of which stands for an initial fraction of agents who have the product A and who have the product B, $r(\pm 1)$, and $s(\pm 1)$, respectively. The parameter sets are constructed as the combinations of these three conditions. The condition (ic) has two categories that are (ic-1) $\{p_A(1), p_B(1), p_C(1)\} = \{10/101^2, 10/101^2, 1-20/101^2\}$ and (ic-2) $\{p_A(1), p_B(1), p_C(1)\} = \{9/101^2, 11/101^2, 1-20/101^2\}$. In the case of (ic-1) there is symmetry in the initial number of those who adopt the product A and those who adopt the product B. On the other hand, in the case of (ic-2), the product B is designed to have slight superiority in number at initial point. The condition on the parameters $r(\pm 1)$ has three categories that are (r-1) $\{r(+1), r(-1)\} = \{1, 1\}$, (r-2) $\{r(+1), r(-1)\} = \{1, 2\}$, and (r-3) $\{r(+1), r(-1)\} = \{2, 1\}$. The condition on the parameters $s(\pm 1)$ has four categories that are (s-1) $\{s(+1), s(-1)\} = \{0, 0\}$, (s-2) $\{s(+1), s(-1)\} = \{1, 1\}$, (s-3) $\{s(+1), s(-1)\} = \{1, 2\}$, and (s-4) $\{s(+1), s(-1)\} = \{2, 1\}$. All the combinations of the above three conditions count 24. However, only 19 cases excluding one side of symmetric cases are examined.

Table II shows the frequencies of the equilibria A*, B*, and P* for the 19 cases. The last column in Table II will be explained in Sec. V. The time after which the system reaches an equilibrium depends on the parameters ν and ξ in Eq. (2), and it is observed that t=150 (=t^*) is long enough for the system to reach an equilibrium when $\nu = 1$ and $\xi = 0.5$ are used.

In the cases 1 and 8, there are no effects of the network externalities because both the parameters $s(\pm 1)$ are set as zero. In such cases, it is observed that the system has stable polymorphic equilibria. Note that even though there is symmetry in the initial number, the configuration of the consumers with each strategy is random. Due to this randomness the equilibria in the case 1 is not exactly as $\{p_A(t^*), p_B(t^*), p_C(t^*)\} = \{0.5, 0.5, 0\}$ but most likely close to these values. On the other hand, the polymorphic equilibria in the case 8 move slightly toward B* because of the product B's slight superiority in number at initial point. In the cases

2 and 9, now there exist positive and symmetric effects of the network externalities on the product A and B since the parameters $s(\pm 1)$ are set as the same non-zero value. In such cases, polymorphic equilibria tend to bifurcate into either A^* or B^* and the system becomes sensitive to its initial configuration and fraction. In the case 2, it now can have three equilibria, A*, B*, or P*. The frequencies of those three equilibria are distributed nearly equally because of the symmetry in number in the condition (ic-1). In the case 9, however, the system reaches B* more frequently due to the product B's initial slight superiority in number as in the condition (ic-2). These results in the cases of 1, 2, 8, and 9 suggest that the existence of the positive effects of the network externality makes the system inherent three stable equilibria, A*, B*, and P*, and if there is a difference in initial fraction between agents who adopt A and who does B, the difference is eventually amplified and decides which equilibrium the system attains. This corresponds to the effects of increasing returns, which is discussed by Arthur (1989) [1]. In contrast, without effects of the network externality the slight initial difference is not enlarged and both superior and inferior products are observed to coexist by forming local clusters in the market (Figure of the game field that contains local clusters is not shown).

In the cases of 3, 4, 5, 6, 10, 12, 13, and 14 the product B is designed to has a strict superiority or superiorities in its payoff. As a result, in all those cases, the product B always takes over the market. The same type of argument applies to the cases 11, 16, 17, and 19, but in these cases, the product A is the one that takes over the market.

Interestingly enough, in the cases of 7and 15, the product B still always takes over the market even though the product A has a superiority on its network externality parameter $s(+1)$ over $s(-1)$ as seen in the condition (s-4). This is because the product B has its superiority on the parameter $r(-1)$ over $r(+1)$ in addition to its initial number superiority for the case 15, and the level of the superiority in network externality for the product A could not overwhelm the product B's superiority. It is expected that A* could be seen if $s(+1)$ is set higher, and for example in the case of 7, it is observed that $s(+1) = 6$ is large enough for the product A to most likely take over the market as is shown in Table III. The explanation on the last column in Table III will be given in Sec. V. The case 18 is symmetric against the case 15 except that not only the condition (s-3) but also the condition (ic-2) works against the product A, but the frequency of A* is still unity due to the condition (r-3).

IV. INTRODUCING AN INNOVATION FACTOR

Now an innovation factor is introduced into the model. Table IV shows frequencies of the three equilibria, A*, B*, and P* when an innovation factor is introduced in the case 5 in Table II. The last column in Table IV will be explained in Sec. V. For the cases 5(a), 5(b), and 5(c) in Table IV, all the parameters as well as initial conditions are the same as the case 5 up to $t = 24$, and after $t = 25$ the parameter $s(+1)$, now denoted as $s(+1, t \geq 25)$, is increased to 5(a) 2, 5(b) 4, and 5(c) 6, that is, an innovation is introduced to the product A's side at $t = 25$. One can see that the innovation factor 5(c) $s(+1, t \geq 25) = 6$ is large enough for the product A to retake the market while 5(a) $s(+1, t \geq 25) = 2$ and 5(b) $s(+1, t \geq 25) = 4$ are too small. Figure 2(a) represents the trajectories of $\{p_A(t), p_B(t)\}$ on the $p_A(t)$-$p_B(t)$ plane with (triangle dots) and without (black circles) the innovation factor $s(+1, t \geq 25) = 6$ in the case 5(c) in Table IV, respectively. It is observed that the product A regains its market share after $t = 25$ and eventually takes over the whole market. Here, note that introducing an innovation factor could make the system finally arrive a polymorphic equilibrium as one can see in Table IV. On the other hand, in the case 5(d) in Table IV, the same size of innovation factor as the case 5(c) is introduced but at time $t = 35$. In this case, the product B still always takes over the market, that is, the time $t = 35$ is too late for the product A with the innovation factor given as 6 or the innovation factor $s(+1, t \geq 35) = 6$ is too small for introducing at $t = 35$ to retake the market. These results suggest that both the timing and size of the innovation factor matter for an inferior product in order to retake the market.

Figure 2(b) illustrates the overall mean utilities corresponding to the Fig. 2(a) where the product A successfully retakes the market. The overall mean utilities denoted as $U_A(t)$ and $U_B(t)$ are defined as follows:

$$
\begin{aligned}
U_A(t) &= \frac{1}{|N_A(t)|}\sum_{i\in N_A(t)} u_i(+1,t) \\
&= \frac{1}{|N_A(t)|}\sum_{i\in N_A(t)} n_0\{R_i(+1)+S(+1,t)p_i(+1,t)\} \\
&= n_0\{r(+1)+\frac{\sum_{i\in N_A(t)} p_i(+1,t)}{|N_A(t)|}s(+1)p_A(t-1)\} \qquad (9)\\
U_B(t) &= \frac{1}{|N_B(t)|}\sum_{i\in N_B(t)} u_i(-1,t) \\
&= \frac{1}{|N_B(t)|}\sum_{i\in N_B(t)} n_0\{R_i(-1)+S(-1,t)p_i(-1,t)\} \\
&= n_0\{r(-1)+\frac{\sum_{i\in N_B(t)} p_i(-1,t)}{|N_B(t)|}s(-1)p_B(t-1)\} \qquad (10)
\end{aligned}
$$

where $u_i(\pm1,t) = u_i(\sigma_i(t) = \pm1)$, $N_A(t)$ and $N_B(t)$ are the sets of agents who consume the product A and B in N, respectively, $|N_A(t)|$ and $|N_B(t)|$ are the size of the sets $N_A(t)$ and $N_B(t)$, respectively. The ex and square dots represent $U_A(t)$ and $U_B(t)$, respectively, in Fig 2(a). Equations (9) and (10) are the key factors to control the equilibrium of the system since utility functions realize all the parameters in payoffs, which decide the payoff matrix on which the system mainly is dependent. As in Fig. 2(b), when the inferior product A comes from behind to retake the market due to the innovation factor, there exists crossover of $U_A(t)$ and $U_B(t)$. This fact is utilized when a mean-field theory is constructed to recover the results of the simulations.

V. MEAN-FIELD THEORY

In this section, a mean-field approximation is conducted to recover the results of the simulations in the previous section. Here we introduce the local densities of i's neighbors who are adopting either the strategy A, B, or C at time t that are given as $\ell_i(+1,t)$, $\ell_i(-1,t)$, and $\ell_i(0,t)$, respectively, as follows:

$$\ell_i(\sigma,t) = \frac{1}{2(n_0+1)}\sum_{j\in n(i)}\{(3\sigma^2-2)\sigma_j^2(t)+\sigma\sigma_j(t)-2(\sigma^2-1)\} \qquad (11)$$

where $\sigma = +1, -1$, or 0. Now, if we let $\sigma^M_{k\in n(i)}(t)$ symbolize the $\sigma_i(t+1)$ that satisfies the right hand side of Eq. (8), then from Eq. (11) the time evolutions of the local densities are given as

$$\begin{aligned}
\ell_i(+1,t+1)-\ell_i(+1,t) &= \frac{1}{2(n_0+1)}\sum_{j\in\tilde{n}(i)}\{(\sigma^M_{k\in n(i)}(t))^2-\sigma_j^2(t)+(\sigma^M_{k\in n(i)}(t)-\sigma_j(t))\} \\
&= \frac{1}{2(n_0+1)}\{\sum_{j\in\tilde{n}(i)}\sigma_j(t)(\sigma_j(t)-1)\frac{\sigma^M_{k\in n(j)}(t)(\sigma^M_{k\in n(j)}(t)+1)}{2} \\
&\quad -\sum_{j\in\tilde{n}(i)}\sigma_j(t)(\sigma_j(t)+1)\frac{\sigma^M_{k\in n(j)}(t)(\sigma^M_{k\in n(j)}(t)-1)}{2} \\
&\quad +\sum_{j\in\tilde{n}(i)}2(1+\sigma_j(t))(1-\sigma_j(t))\frac{\sigma^M_{k\in n(j)}(t)(\sigma^M_{k\in n(j)}(t)+1)}{2}\}, \qquad (12)
\end{aligned}$$

$$\begin{aligned}
\ell_i(-1,t+1)-\ell_i(-1,t) &= \frac{1}{2(n_0+1)}\sum_{j\in\tilde{n}(i)}\{(\sigma^M_{k\in n(i)}(t))^2-\sigma_j^2(t)-(\sigma^M_{k\in n(i)}(t)-\sigma_j(t))\} \\
&= \frac{1}{2(n_0+1)}\{-\sum_{j\in\tilde{n}(i)}\sigma_j(t)(\sigma_j(t)-1)\frac{\sigma^M_{k\in n(j)}(t)(\sigma^M_{k\in n(j)}(t)+1)}{2} \\
&\quad +\sum_{j\in\tilde{n}(i)}\sigma_j(t)(\sigma_j(t)+1)\frac{\sigma^M_{k\in n(j)}(t)(\sigma^M_{k\in n(j)}(t)-1)}{2} \\
&\quad +\sum_{j\in\tilde{n}(i)}2(1+\sigma_j(t))(1-\sigma_j(t))\frac{\sigma^M_{k\in n(j)}(t)(\sigma^M_{k\in n(j)}(t)-1)}{2}\}, \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
\ell_i(0,t+1)-\ell_i(0,t) &= \frac{1}{2(n_0+1)}\sum_{j\in\tilde{n}(i)}(-2)\{(\sigma^M_{k\in n(i)}(t))^2-\sigma_j^2(t)\} \\
&= \frac{1}{2(n_0+1)}\{-\sum_{j\in\tilde{n}(i)}2(1+\sigma_j(t))(1-\sigma_j(t))(\sigma^M_{k\in n(j)}(t))^2\}. \qquad (14)
\end{aligned}$$

Here the global densities of the strategy A, B, and C consumers are introduced as

$$p_A(t) = \frac{1}{|N|}\sum_{i\in N}\ell_i(+1,t), \qquad (15)$$

$$p_B(t) = \frac{1}{|N|}\sum_{i\in N}\ell_i(-1,t), \text{ and} \qquad (16)$$

$$p_C(t) = \frac{1}{|N|}\sum_{i\in N}\ell_i(0,t), \qquad (17)$$

respectively, where N is the set of the whole customers. Certainly it holds

$$p_A(t)+p_B(t)+p_C(t)=1. \qquad (18)$$

Now the local densities $\ell_i(+1,t)$, $\ell_i(-1,t)$, and $\ell_i(0,t)$ are replaced by the global density $p_A(t)$, $p_B(t)$, and $p_C(t)$, respectively, and the following equations are obtained:

$$\begin{aligned}
&\frac{1}{|N|}\sum_{i\in N}[\ \ell_i(\pm1,t+1)-\ell_i(\pm1,t)\ ]\\
=\ &\pm Pr(\sigma^M_{k\in n(j\in n(i))}(t)=+1\mid\forall i\in B)\ \frac{1}{|N|}\sum_{i\in N}\frac{1}{2(n_0+1)}\sum_{j\in\tilde{n}(i)}\sigma_j(t)(\sigma_j(t)-1)\\
&\mp Pr(\sigma^M_{k\in n(j\in n(i))}(t)=-1\mid\forall i\in A)\ \frac{1}{|N|}\sum_{i\in N}\frac{1}{2(n_0+1)}\sum_{j\in\tilde{n}(i)}\sigma_j(t)(\sigma_j(t)+1)\\
&+Pr(\sigma^M_{k\in n(j\in n(i))}(t)=\pm1\mid\forall i\in C)\ \frac{1}{|N|}\sum_{i\in N}\frac{1}{2(n_0+1)}\sum_{j\in\tilde{n}(i)}2(1+\sigma_j(t))(1-\sigma_j(t))
\end{aligned}\tag{19}$$

and

$$\begin{aligned}
&\frac{1}{|N|}\sum_{i\in N}[\ \ell_i(0,t+1)-\ell_i(0,t)\ ]\\
=\ &Pr(\sigma^M_{k\in n(j\in n(i))}(t)=+1\cup-1\mid\forall i\in C)\ \frac{1}{|N|}\sum_{i\in N}\frac{1}{2(n_0+1)}\sum_{j\in\tilde{n}(i)}2(1+\sigma_j(t))(1-\sigma_j(t))
\end{aligned}\tag{20}$$

that lead to

$$p_A(t+1)-p_A(t) = \alpha(t)\ p_B(t)-\beta(t)\ p_A(t)+\gamma(t)\ p_C(t), \tag{21}$$

$$p_B(t+1)-p_B(t) = -\alpha(t)\ p_B(t)+\beta(t)\ p_A(t)+\delta(t)\ p_C(t),\ \text{and} \tag{22}$$

$$p_C(t+1)-p_C(t) = -\epsilon(t)\ p_C(t) \tag{23}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$, and $\epsilon(t)$ are transition probabilities and therefore approximated as

$$\begin{aligned}
\alpha(t) &= Pr(\sigma^M_{k\in n(j\in n(i))}(t)=+1\mid\forall i\in B)\\
&\simeq \begin{cases}\frac{a}{2}\{1+\text{sign}[\hat{U}_A(t)-\hat{U}_B(t)]\} & \text{if } \mathrm{p_A}(1)\neq 0\\ 0 & \text{otherwise}\end{cases}
\end{aligned}\tag{24}$$

$$\begin{aligned}
\beta(t) &= Pr(\sigma^M_{k\in n(j\in n(i))}(t)=-1\mid\forall i\in A)\\
&\simeq \begin{cases}\frac{b}{2}\{1+\text{sign}[\hat{U}_B(t)-\hat{U}_A(t)]\} & \text{if } \mathrm{p_B}(1)\neq 0\\ 0 & \text{otherwise}\end{cases}
\end{aligned}\tag{25}$$

$$\gamma(t) = Pr(\sigma^M_{k\in n(j\in n(i))}(t) = +1 \mid \forall i \in C)$$
$$\simeq \begin{cases} k(1-p_C(t)) & \text{if } \mathrm{p_A}(1) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (26)$$

$$\delta(t) = Pr(\sigma^M_{k\in n(j\in n(i))}(t) = -1 \mid \forall i \in C)$$
$$\simeq \begin{cases} k'(1-p_C(t)) & \text{if } \mathrm{p_B}(1) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (27)$$

$$\epsilon(t) = Pr(\sigma^M_{k\in n(j\in n(i))}(t) = +1 \cup -1 \mid \forall i \in C)$$
$$= \gamma(t) + \delta(t) \quad (28)$$

where a, b, k, and k' ($0 < a, b, k, k' < 1$) are parameters that controls the magnitude of transition probabilities, and sign[0] is assumed to be -1, that is, $\alpha(t)$ and $\beta(t)$ are zero when $\hat{U}_A(t) = \hat{U}_B(t)$ holds. From Eqs (9) and (10), the global average utilities for the A and B consumers, $\hat{U}_A(t)$ and $\hat{U}_B(t)$ in Eqs. (24) and (25), are approximated as

$$\hat{U}_A(t) = n_0\{r(+1) + s(+1)p(+1)p_A(t-1)\} \quad (29)$$
$$\hat{U}_B(t) = n_0\{r(-1) + s(-1)p(-1)p_B(t-1)\} \quad (30)$$

where $p(+1)$ and $p(-1)$ ($0 \leq p(\pm) \leq 1$) are positive parameters that represent the mean values of $p_i(+1,t)$ and $p_i(-1,t)$ over i, respectively. These mean values could approximately be fixed as nearly unity over time since those who have either A or B essentially form clusters in order to survive no matter what values $p_A(t)$ and $p_B(t)$ are. Only exceptions are the cases where either $p_A(t)$ or $p_B(t)$ are nearly zero or zero. However, in those cases the values of the parameters $p(+1)$ and $p(-1)$ do not matter since $p_A(t)$ and $p_B(t)$ are nearly zero or zero.

The last columns in Tables II, III, and IV show equilibria that are approximated by numerically solving Eqs. (21) to (23) with Eqs. (24) to (28). The parameters a (= b), k (= k'), and $p(+1)$ (= $p(-1)$) are chosen as 0.08, $\lambda/2$=0.16, and 1, respectively. The parameter k (= k') is set as $\lambda/2$ so that $1 - p_C(t)$ (= $p_A(t) + p_B(t)$) can be described by the logistic equation in Eq. (1) whose parameter is λ. One can see that the mean-field theory with the above parameters successfully approximates the equilibria the system most likely reaches.

The trajectory of $\{p_A(t), p_B(t)\}$ and overall mean utilities, $\hat{U}_A(t)$ and $\hat{U}_B(t)$, which are approximated by the above mean-field theory, are shown in Figs. 3(a) and (b), respectively. These figures correspond to the case in Figs. 2(a) and (b), respectively, which are obtained as a result of a simulation. It is observed that the generous features of Figs. 2(a) and (b) are successfully recovered by Figs. 3(a) and (b).

VI. DISCUSSION

In this paper the diffusion phenomena of two competitive, interchangeable, and durable goods have been studied based on the framework of the spatial 3×3 symmetric coordination like-game. The payoff matrix of the game contains the positive effects of the network externalities that affect the payoff matrix itself dynamically by providing feedbacks to the system from the system itself.

Both the simulations and the mean-field approximation have shown that the existence of the positive effects of the network externality makes the system inherent three stable equilibria, A*, B*, and P*, and if there is a difference in initial fraction between agents who adopt A and who does B, the difference is eventually amplified that decides which equilibrium the system reaches. On the other hand, without the effects of the network externality the slight initial difference is not enlarged and both superior and inferior products are observed to coexist by forming local clusters in the market. Additionally, from the study on the model with an innovation factor, it is shown that both the timing and size of the innovation factor matter for an inferior product in order to retake the market.

In the future, we hope to introduce a random connection between consumers into the model.

Tables I to IV

Table I

	Strategy A $(+1)$	Strategy B (-1)	Strategy C (0)
Strategy A $(+1)$	$R_i(+1) + S(+1, t)$	$R_i(+1)$	$R_i(+1)$
Strategy B (-1)	$R_i(-1)$	$R_i(-1) + S(-1, t)$	$R_i(-1)$
Strategy C (0)	0	0	0

Payoff matrix for a row player. The definitions of $R_i(\pm 1)$ and $S(\pm 1, t)$ are given in Eq. (3), and (4) and (5), respectively.

Table II

Conditions			Case	Frequencies of Equilibria			M.F.
ic	r	s	No.	A*	P*	B*	Approx.
(ic-1)	(r-1)	(s-1)	1	0	1	0	P*
		(s-2)	2	0.38	0.36	0.26	P*
		(s-3)	3	0	0	1	B*
		(s-4)	-	-	-	-	-
	(r-2)	(s-1)	4	0	0	1	B*
		(s-2)	5	0	0	1	B*
		(s-3)	6	0	0	1	B*
		(s-4)	7	0	0	1	B*
(ic-2)	(r-1)	(s-1)	8	0	1	0	P*
		(s-2)	9	0.06	0.16	0.78	B*
		(s-3)	10	0	0	1	B*
		(s-4)	11	1	0	0	A*
	(r-2)	(s-1)	12	0	0	1	B*
		(s-2)	13	0	0	1	B*
		(s-3)	14	0	0	1	B*
		(s-4)	15	0	0	1	B*
	(r-3)	(s-1)	16	1	0	0	A*
		(s-2)	17	1	0	0	A*
		(s-3)	18	1	0	0	A*
		(s-4)	19	1	0	0	A*

Frequencies of the equilibria A*, B*, and P*. The symbol A* denotes the equilibrium where the product A takes over the whole market, and B* for the product B. The symbol P* stands for a polymorphic equilibrium where the product A and B coexist. The parameter sets are constructed as the combinations of the following three conditions, (ic), (r), and (s). The condition (ic) has two categories that are (ic-1) $\{p_A(1), p_B(1), p_C(1)\} = \{10/101^2, 10/101^2, 1{-}20/101^2\}$ and (ic-2) $\{p_A(1), p_B(1), p_C(1)\} = \{9/101^2, 11/101^2, 1-20/101^2\}$. The condition on the parameters $r(\pm1)$ has three categories that are (r-1) $\{r(+1), r(-1)\} = \{1, 1\}$, (r-2) $\{r(+1), r(-1)\} = \{1, 2\}$, and (r-3) $\{r(+1), r(-1)\} = \{2, 1\}$. The condition on the parameters $s(\pm1)$ has four categories that are (s-1) $\{s(+1), s(-1)\} = \{0, 0\}$, (s-2) $\{s(+1), s(-1)\} = \{1, 1\}$, (s-3) $\{s(+1), s(-1)\} =$

$\{1, 2\}$, and (s-4) $\{s(+1), s(-1)\} = \{2, 1\}$. All the combinations of the above three conditions count 24. However, only 19 cases excluding one side of symmetric cases are examined. The last columns show equilibria that are approximated by the mean-field theory in Sec. V.

Table III

Conditions				Frequencies of Equilibria			M.F.
ic	r	s	s(−1), s(+1)	A*	P*	B*	Approx.
(ic-1)	(r-2)	(s-4)	1 , 2	0	0	1	B*
			1 , 3	0	0	1	B*
			1 , 4	0.04	0	0.96	B*
			1 , 5	0.34	0.06	0.6	B*
			1 , 6	0.76	0.06	0.18	A*
			1 , 7	0.98	0	0.02	A*
			1 , 8	1	0	0	A*
			1 , 9	1	0	0	A*
			1 , 10	1	0	0	A*

The results of the intensive parameter running for the case 7 in Table II. The parameter $s(+1) = 6$ is large enough for the product A to take over the market most likely. The last columns show equilibria that are approximated by the mean-field theory in Sec. V.

Table IV

Conditions			Case	Frequencies of Equilibria			M.F.
ic	r	s	No.	A^*	P^*	B^*	Approx.
(ic-1)	(r-2)	(s-2)	5(a)	0	0	1	B^*
			5(b)	0.04	0	0.96	B^*
			5(c)	0.72	0.06	0.22	A^*
			5(d)	0	0	1	B^*

Frequencies of the three equilibria, A^*, B^*, and P^* when an innovation factor is introduced in the case 5 in Table II. The innovation factor 5(c) $s(+1, t \geq 25) = 6$ is large enough for the product A to retake the market while 5(a) $s(+1, t \geq 25) = 2$ and 5(b) 4 are too small. In the case 5(d), the product B still always takes over the market, that is, the time $t = 35$ is too late for the product A with the innovation factor given as 6 or the innovation factor $s(+1, t \geq 35) = 6$ is too small for introducing at $t = 35$ to retake the market. These results suggest that both the timing and size of the innovation factor matter for an inferior product in order to retake the market. The last columns show equilibria that are approximated by the mean-field theory in Sec. V.

Figure Captions

FIGURE 1

Numerically solved $p_X(t)$ (square dots) in Eq. (1) with the initial value $p_X(1)$=20/101^2 and the parameter λ=0.32 and the value $p_X(t)$ (black circles) that is obtained as a result of the cellular-automata based simulation with the initial value $p_X(1)$=20/101^2 and the parameters ν=1 and $\xi = 0.5$. The two curves are almost identical, and it suggests that $p_X(t)$ obtained from the simulation could be described by the logistic equation in Eq. (1). This fact will be utilized later in Sec. V where a mean-field theory is conducted.

FIGURE 2

The results from the simulations in the case 5 with the innovation factor. (a) The trajectories of $\{p_A(t), p_B(t)\}$ on the $p_A(t)$-$p_B(t)$ plane with (triangle dots) and without (black circles) the innovation factor $s(+1, t \geq 25) =$ 6, respectively. (b) The overall mean utilities corresponding to the case with the innovation factor in Fig. 2(a). The ex and square dots are for $U_A(t)$ and $U_B(t)$, respectively.

FIGURE 3

The results from the mean-field approximation that are corresponding to the case in Figs. 2. The parameters a (= b), k (= k'), and $p(+1)$ (= $p(-1)$) are chosen as 0.08, λ/2=0.16, and 1, respectively. (a) The trajectories of $\{p_A(t), p_B(t)\}$ to A* and B* on the $p_A(t)$-$p_B(t)$ plane with and without the innovation factor $s(+1, t \geq$ 25) = 6, respectively. (b) The overall mean utilities corresponding to the Fig. 3(a) where the product A successfully retakes the market.

Figure 1

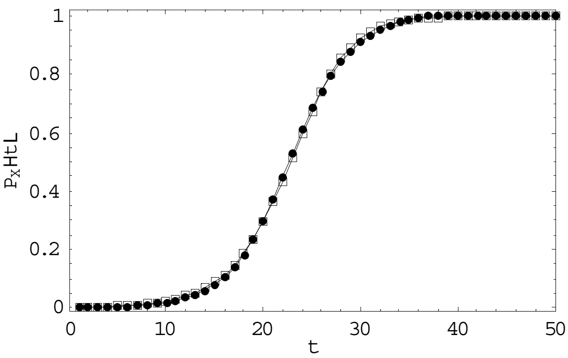
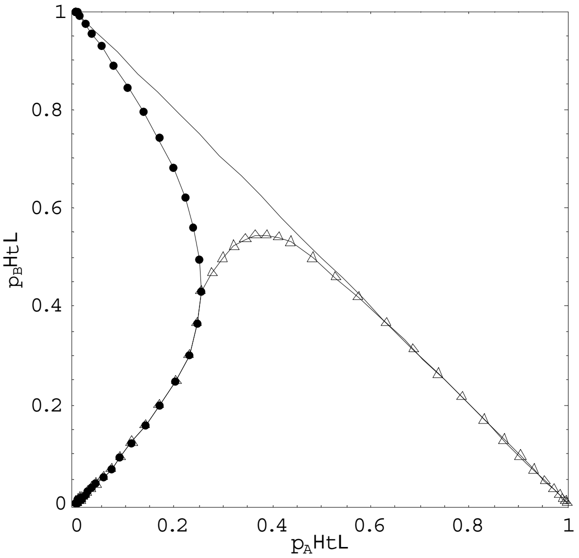


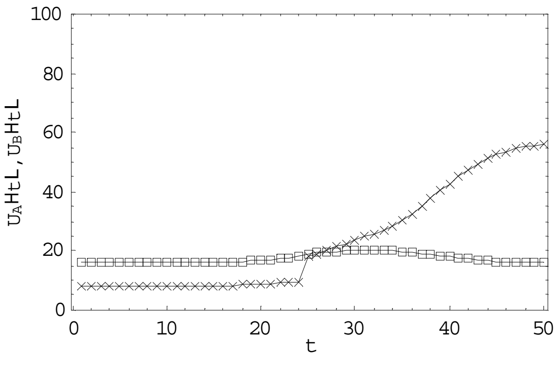
Figure 2

(a)



1
0.8
0.6
0.4
0.2
0
pBHtL
0
0.2
0.4
0.6
0.8
1
pAHtL

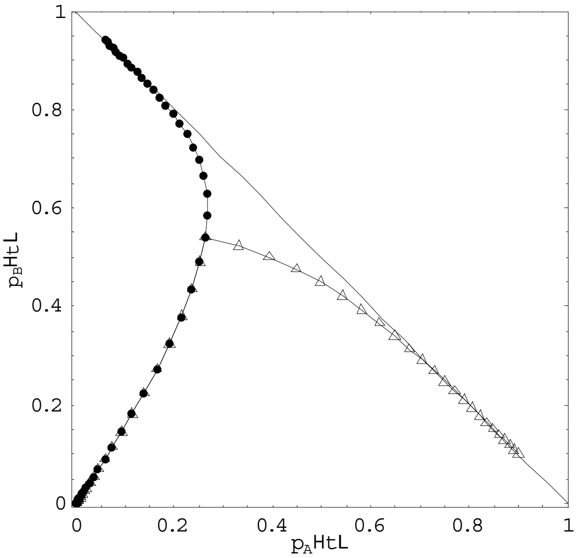
(b)



100
80
60
40
20
0
UAHtL, UBHtL
0
10
20
30
40
50
t

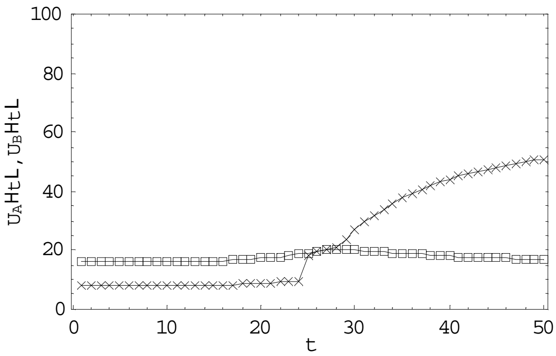
Figure 3

(a)



1
0.8
0.6
0.4
0.2
0
pBHtL
0
0.2
0.4
0.6
0.8
1
pAHtL

(b)



100
80
60
40
20
0
UAHtL, UBHtL
0
10
20
30
40
50
t

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