A NOTE ON FUNAKI AND YAMATO'S TRAGEDY OF THE COMMONS

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Abstract. In the model of Funaki and Yamato (1999) the tragedy of the commons can be avoided with pessimistic players, while this does not hold for optimistic players. We propose a new core concept to overcome this puzzle and provide numerical simulations of simple games where the conclusions coincide or are less sensitive to behavioural assumptions.

1. Introduction

Funaki and Yamato (1999) define a game with a common pool resource and tackle the question whether or not the tragedy of the commons can be avoided. Their answer strongly depends on the expectations regarding the coalition formation: if players have pessimistic expectations the answer is positive. On the other hand with optimistic expectations the tragedy of the commons cannot be avoided in games with at least 4 players.

In partition function form games a coalition's payoff depends on the entire coalition structure and so a deviation by a coalition invokes a response from the rest of the players. On the other hand the payoff the deviating coalition obtains after the deviation depends on this reaction, so the deviation happens with certain *beliefs* or expectations about the reaction. While it is possible that the deviating coalition is indifferent about this reaction – as is the case in the less general *characteristic function games* – such beliefs cannot be removed completely. The optimistic and pessimistic assumptions of Funaki and Yamato (1999) are extreme in the sense that players can be misled by an unlikely, though very favourable or infavourable outcome. In a footnote they acknowledge this fact and suggest to use intermediate cases by allowing some coalitions to be optimistic and others to be pessimistic. In this note we take an even more general approach. We impose less restrictive assumptions on the expectations, thereby allowing not only optimism and pessimism to be present in a game, but also that these be present in a gradual way. The corresponding core concepts refine their analysis.

First we give a brief overview of the model, then propose our new solution and close our paper with some numerical simulations explaining the difference between our results and those of Funaki and Yamato.

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2. The Model

Funaki and Yamato (1999) consider a game with a set $N = \{1, \ldots, n\}$ of identical fishermen. Fisherman j exerts an effort l_j to catch fish[1]. The total amount of labour is given by $l_N = \sum_{j \in N} l_j$. The production function f specifies the number of fish caught for each value of the total amount of labour l_N. We assume that $f(0) = 0$, $f'(l_N) > 0$, $f''(l_N) < 0$, and $\lim_{l_N \to \infty} f'(l_N) = 0$, that is, there are decreasing returns to labour. The distribution of fish is proportional to the effort of the individual fishermen. The price of fish is normalised to 1, and the personal cost of labour is q. We assume that $0 < q < f'(0)$. The income of fisherman j is given by:

$$m_j(l_1, l_2, \ldots, l_n) = \frac{l_j}{l_N} f(l_N) - qx_j,$$

where $m_j(0, 0, \ldots, 0) = 0$. We further assume that fishermen are allowed to form coalitions and then for coalition S the coalitional income is simply $m_S = \sum_{j \in S} m_j$. For any given coalition structure $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$ the vector of efforts $(l_1^*, l_2^*, \ldots, l_k^*)$ aggregated per coalition is an *equilibrium under the strategy vector* $\mathcal{P}$ if it is a Nash-equilibrium with the coalitions as composite players.

Given this model Funaki and Yamato (1999) prove the existence of a unique equilibrium for any given coalition structure.

Lemma 1 (Funaki and Yamato, (1999), Theorem 1)**.** *For any coalition structure* $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$*, there exists a unique equilibrium vector* $\left(l_{S_1}^*, l_{S_2}^*, \ldots, l_{S_k}^*\right)$ *given by*

$$f'(l_N^*) + (k-1) f(l_N^*)/l_N^* = kq, \tag{1}$$

$$l_{S_i}^* = l_N^*/k \quad \text{for all } i = 1, \ldots, k, \text{ and} \tag{2}$$

$$l_{S_i}^* > 0 \quad \text{for all } i = 1, \ldots, k. \tag{3}$$

3. Our proposal

The existence and uniqueness of this equilibrium enables us to define a partition function form game. In this game a group of fishermen *deviates* if, unhappy with the given payoff configuration, they form a new fishing firm. Since the deviation changes the payoffs of the remaining coalitions it induces a reaction from the rest of the players. These players, called *residuals* face a problem that can be modelled by a partition function form game. Since the deviating players are considered "put" they only affect the game by their externalities, and so the *residual game* has less players than the original one. Generally the deviation creates a totally new situation and so the residual solution does not have to depend on the pre-deviation payoffs of the residuals. In order to be consistent we assume –and this is our only assumption– that the residual solution is in the residual core, provided it is not empty. Since the core concept used here is a generalisation of the coalition structure core, it may implement more than one partition. Thus uncertainty and expectations cannot entirely be removed, but optimism and pessimism leads here to less extreme results.

[1]Instead of x we use l for the amount of labour.

In the following we give a formal definition of the core concept. This definition is inductive and is done if four steps. For a single-player game the core is straightforward. Given the definition for all at most $k-1$ player games we define domination for k player games. First we give the definition for the pessimistic case, and then a slightly modified version gives the optimistic core. As usual, the definition is based on the domination relation.

Let (N, Π, V) denote a partition function form game, where N is the set of players, Π is the set of partitions of the players and V is the partition function. Then $(x, \mathcal{P})$ shall denote an *outcome*, where x is a *feasible* payoff vector *under* partition $\mathcal{P}$ such that $\mathcal{P} \in \Pi$. Set subscripts to relation symbols between real vectors mean a restriction of the relation. Let $x, y \in \mathbb{R}^N$ and $S \subseteq N$. Then we write $x >_S y$ if $x_i \geq y_i$ for all $i \in S$ and $x_S > y_S$. Similarly, we write $x =_S y$ if $x_i = y_i$ for all $i \in S$.

First we define a residual game:

Definition 1 (Residual Game)**.** Let (N, Π, v) be a game. Let S be a coalition and R be its complement in N. Let $\mathcal{S}$ be a multi-coalition deviation[2], a partition of S. Given the deviation $\mathcal{S}$ the residual game $(R, \Pi(R), v_{\mathcal{S}})$, where $\Pi(R)$ is the set of partitions of R, is the partition function form game over the player set R and with the partition function

$$\begin{aligned} v_{\mathcal{S}} \ &: \ \Pi(R) \longrightarrow (2^R \longrightarrow \mathbb{R}) \\ &\qquad \mathcal{R} \longmapsto v_{\mathcal{S}}(\mathcal{R}) \\ v_{\mathcal{S}}(\mathcal{R}) \ &: \ C \longmapsto \begin{cases} v(C, \mathcal{R} \cup \mathcal{S}) & \text{if } C \in \mathcal{R} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Definition 2 (Core)**.** The definition consists of four steps. Let (N, Π, v) be a game.

Step 1. The core of a trivial game.
The *core* of a game with $N = \{1\}$ is the efficient outcome with the trivial partition, $\mathcal{P}_0 = \{\{1\}\}$:

$$C_{-}(\{1\}, \{\mathcal{P}_0\}, v) = \{(v(1, \mathcal{P}_0), \mathcal{P}_0)\}.$$

Step 2. Inductive assumption.
Given the definition of the core for every game with at most $k-1$ players we can define dominance for a game of k players.

Step 3a. Dominance (Pessimistic case)
For any set of deviating players S the deviating coalitions $\mathcal{S}$ represented as a partition of S induces a residual game $(R, \Pi(R), v_{\mathcal{S}})$. Then the outcome $(x, \mathcal{P})$ is dominated via a set of coalitions $\mathcal{S}$ if either

(1) the core of the residual game is *empty* and *for all* residual outcomes the deviation $\mathcal{S}$ is profitable. Formally: for all partitions $\mathcal{Q}$ containing $\mathcal{S}$ there exists an outcome $(y, \mathcal{Q})$ such that $y >_S x$. Or

[2]For games in a partition function form the Pareto-efficiency of the undominated outcomes can only be attained by allowing simultaneous deviations by more than one coalitions.

(2) the core of the residual game is *not empty* and *for all* residual *core* outcomes the deviation $\mathcal{S}$ is profitable. Formally: for all residual core outcomes $(y_R, \mathcal{R})$ there exists an outcome $(y, \mathcal{S} \cup \mathcal{R})$ with a payoff vector $y =_S y_R$ and $y >_S x$.[3]

Step 3b. Dominance (Optimistic case).
For any set of deviating players S the deviating coalitions $\mathcal{S}$ represented as a partition of S induces a residual game $(R, \Pi(R), v_{\mathcal{S}})$. Then the outcome $(x, \mathcal{P})$ is dominated via a set of coalitions $\mathcal{S}$ if either

(1) the core of the residual game is *empty* and *there exists* a residual outcome that makes the deviation $\mathcal{S}$ profitable. Formally: there exists an outcome $(y, \mathcal{Q})$, such that $\mathcal{Q}$ contains $\mathcal{S}$ and $y >_S x$.
(2) the core of the residual game is *not empty* and *there exists* a residual *core* outcome that makes the deviation $\mathcal{S}$ profitable. Formally: there exists a residual core outcome $(y_R, \mathcal{R})$, such that there exists an outcome $(y, \mathcal{S} \cup \mathcal{R})$, such that $y =_S y_R$ and $y >_S x$.

The outcome $(x, \mathcal{P})$ is dominated if it is dominated via a set of coalitions.

Step 4. The core (Pessimistic & optimistic cases).
The *core* of a game $(N, \Pi, \{v_{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ of k players is the set of undominated outcomes and we denote it by $C_-(N, \Pi, \{v_{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ and $C_+(N, \Pi, \{v_{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ respectively for the pessimistic and optimistic approaches.

The properties of the core together with its relation to other existing core and core-like concepts are discussed in Kóczy (2000) and to a greater extent in Kóczy (forthcoming). Here we recall only one result. If $C_{\min}$ and $C_{\max}$ denote the pessimistic and optimistic cores used by Funaki and Yamato (1999), then $C_{\max} \subseteq C_+ \subseteq C_- \subseteq C_{\min}$ when applied to the same game.

4. Simulations

We investigate three simple types of production functions. Depending on the choice for the cost of labour q and a parameter γ (where applicable) the tragedy of the commons can be avoided for some games. The three functions considered are the following:

$$f_1(l_N) = l_N^\gamma \tag{4}$$

$$f_2(l_N) = 1 - e^{-l_N} \tag{5}$$

$$f_3(l_N) = \frac{1}{\gamma}(l_N + 1)^\gamma - \frac{1}{\gamma} \tag{6}$$

4.1. **The function $f(l_N) = l_N^\gamma$.** This is a function that also Funaki and Yamato use to illustrate that different γ values can lead to different results for 3-player games. We show the calculation of the core for a specific example and give a summary of the results.

[3] Note the difference between the two cases: if the residual core is not empty we consider only core outcomes (first case), while if it is empty we use all outcomes (second case).

Consider a 4-player game with $\gamma = 0.2$ and $q = 0.5$. As l_N tends to 0, $f'(l_N)$ tends to infinity so the conditions on q are satisfied. First we solve Equation (2) of Funaki and Yamato (1999) for $k = 1, 2, 3$ and 4. The respective l_N^* values are 0.318, 1.256, 1.614, and 1.799. The following table summarises the payoffs.

Partition	coalitional payoff	coalition size	per-member payoff
(4)	0.636	4	0.159
(3,1)	0.209	3	0.070
(3,1)	0.209	1	0.209
(2,2)	0.209	2	0.105
(2,1,1)	0.098	2	0.049
(2,1,1)	0.098	1	0.098
(1,1,1,1)	0.056	1	0.056

We test the profitability of deviations from the grand coalition. In the grand coalition each player gets 0.159. The cooperation is threatened only by deviations that give more to the deviating players. In our example there is only one such possibility, the singleton in the partition $(3, 1)$ gets 0.209. Under the overly optimistic max-assumption he expects this amount, but is this belief realistic? We look at payoffs in the residual game consisting of the remaining 3 players

Coalition size	partition by size	value	per-member value
3	(3)	0.209	0.070
2	(2,1)	0.098	0.049
1	(2,1)	0.098	0.098
1	(1,1,1)	0.056	0.056

Should the (residual) grand coalition form, each player would get 0.070. This can be blocked by a singleton, provided the remaining two players stay together, and thus get 0.049 each. Since breaking up gives them more, 0.056, the deviation by the singleton is not profitable. Hence, the grand coalition is in the core of the 3-player residual game. Moreover, this implies that a deviation by a singleton in the original, 4-player game is also profitable. Hence the grand coalition is not in the core, and the tragedy of the commons cannot be avoided[4].

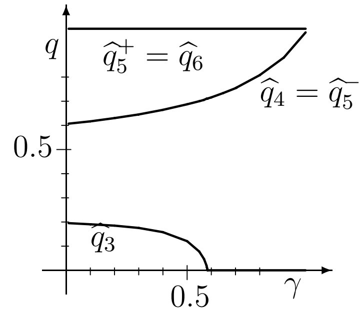
A similar argument gives the same conclusion for other values of γ and q for all four or five-player games.

4.2. **The function $f(l_N) = 1 - e^{-l_N}$.** Looking at 3, 4 and 5-player games we find that for 5-player games the grand coalition does not belong to the core; for 3 and 4 player games there exists cutoff values so that if $q < \widehat{q}$ then the grand coalition belongs to the core, while if $q > \widehat{q}$ it does not. We found $\widehat{q}_3 = 0.213$ and $\widehat{q}_4 = 0.475$.

[4]We must also note, however, that after the separation of a single player the rest of the players stay together, and hence the overfishing is only to the extent of about 30%. This structure does of course exhibit the usual instability of a cartel system.

4.3. **The function** $f(l_N) = \frac{1}{\gamma}(l_N + 1)^\gamma - \frac{1}{\gamma}$**.** In this case we have an extra parameter, γ and we find that depending on γ a similar cutoff value may exist, although here it works in the other direction: for a given γ if $q > \widehat{q}$ then the grand coalition belongs to the core, while if $q < \widehat{q}$ it does not. In particular, where the cutoff-value is 0 the grand coalition will be stable for all values of q, while if it is 1 the tragedy of the commons cannot be avoided. The following table and figure summarise our findings:

γ	$\widehat{q_3}$	$\widehat{q_4} = \widehat{q_5^-}$	$\widehat{q_5^+} = \widehat{q_6}$
0.1	0.191	0.617	1
0.3	0.175	0.645	1
0.5	0.121	0.687	1
0.584	0	0.711	1
0.7	0	0.754	1
0.704	0	0.756	1
0.9	0	0.881	1



5. Conclusions

Here, unlike with the min and the max approaches of Funaki and Yamato, the stability of the grand coalition depends highly on the function considered and the parameters used, but to a much lesser extent on the approach (optimistic or pessimistic) taken. The generality of the results of Funaki and Yamato (1999) does not hold here nor could we establish other regularity; numeric calculations for larger games become computationally too difficult.

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