

SIMULATION OF BEHAVIOR AND INTELLIGENCE

Stable Coalitions in Monotonic Games

(Reconsidered version, January 2000, Comment reg. Rawls principle of justice, March 2005)

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Abstract

A formal scheme is described for coalition formation in a game of interconnected participants with monotonic utility functions. Special coalitions are studied which have an advantage over the rest in the sense of higher utility for each of the participants taken separately.

Comment regarding Rawls second principle of justice.

It seems that coalition formation principle in current analysis is very much like as follows:

“Rawls second principle of justice: The welfare of the worst-off individual is to be maximized before all others, and the only way inequalities can be justified is if they improve the welfare of this worst-off individual or group. By simple extension, given that the worst-off is in his best position, the welfare of the second worst-off will be maximized, and so on. The difference principle produces a lexicographical ordering of the welfare levels of individuals from the lowest to highest.” Cit. Public Choice III, Dennis C. Mueller (2003: 600)

Keywords: coalition, game, utility, monotonic

АКАДЕМИЯ НАУК СССР

АВТОМАТИКА
И
ТЕЛЕМЕХАНИКА

(ОТДЕЛЬНЫЙ ОТТИСК)

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УСТОЙЧИВЫЕ КОАЛИЦИИ В МОНОТОННЫХ ИГРАХ

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(Таллин)

Описывается формальная схема образования коалиций в игре взаимосвязанных участников с монотонными функциями полезности. Изучаются особые коалиции, имеющие преимущество перед всеми остальными в смысле большей полезности для каждого из ее участников в отдельности.

1. Введение

В играх многих лиц [1] под коалицией понимается подмножество участников. Среди всех коалиций обычно выделяются рациональные коалиции: участник такой коалиции извлекает из взаимодействия в коалиции удовлетворяющую его пользу. Кроме того, иногда дополнительно требуется, чтобы извлечение этой пользы было обеспечено независимо от действий игроков, не вошедших в коалицию. В работе строятся различные разновидности «замечательных» в смысле рациональности коалиций игроков и указываются соотношения между ними. Описываются также конструктивные процедуры их отыскания.

Предложенный в работе класс игр подчинен дополнительному условию монотонности, изученному ранее в [2] (хотя для чтения настоящей статьи знания [2] не требуется). Отличия формальной схемы настоящей работы от [2] по существу нет, а отличие в интерпретации состоит в том, что абстрактные показатели взаимосвязи элементов системы понимаются здесь как показатели полезности. Развитый подход в одном частном случае позволил установить возможность для рациональных коалиций находиться в состоянии индивидуального равновесия по Нэшу. Пример такого типа приведен в конце статьи.

2. Формальные определения и понятия

Рассматривается множество n игроков, обозначаемое через I . Каждому игроку $j \in I$ ($j=1, 2, \dots, n$) сопоставлено множество R_j , из которого игрок j может выбирать элементы. Предполагается, что множества R_j конечны и не пересекаются. Их объединение образует множество $W=R_1 \cup R_2 \cup \dots \cup R_n$. Отобранные игроком j элементы из R_j составляют множество $A^j \subseteq R_j$. Множество A^j называется выбором игрока j , а набор $\langle A^1, A^2, \dots, A^n \rangle$ — совместным выбором. Случай $A^k = \emptyset$ не исключается и называется отказом k -го игрока от выбора.

Введем функции полезности элементов $w \in A^j$. Предположим, что осуществился некоторый совместный выбор $\langle A^1, A^2, \dots, A^n \rangle$. Пусть по результату выбора однозначно определяется совокупность чисел $\pi_w \geq 0$, приписываемых элементам $w \in A^j$, $j=1, 2, \dots, n$; на остальных элементах W числа не определяются. Числа π_w называются показателями полезности, или просто полезностями, и, по определению, являются в общем слу-

1. Introduction

In many-persons games [1], by a coalition we shall understand a subset of participants. Among all coalitions we usually single out rational coalitions – a participant in such coalition extracts from the interaction in the coalition a benefit, which satisfies him. In addition, sometimes it is further stipulated that extraction of this benefit is ensured independently of the actions of the players not entering into the coalition. In this paper we construct different varieties of coalitions of players that are “outstanding” in the sense of rationality, and indicate relations between such coalitions. Also, constructive processes for discovering them are described.

The class of games proposed in this paper is subjected to an additional monotonic condition, which has been studied earlier in [2] (although the knowledge of [2] here is not presupposed). There is no difference between the formal scheme of the present paper and that of [2] in essence; the difference involved in interpretation is in abstract indices of interconnection of elements of the system, which are understood as utility indices. The approach developed enables us to establish, in one particular case, the possibility of finding rational coalitions in the state of individual equilibrium according to Nash. An example of such a type is presented at the end of the paper.

2. Formal Definitions and Concepts

We consider a set of n players denoted by I . Each player $j \in I$ ($j = 1, 2, \dots, n$) is matched by a set R_j from which the player j can select elements. It is assumed that the sets R_j are finite and do not intersect. Their union forms a set $W = R_1 \cup R_2 \cup \dots \cup R_n$. The elements selected by the player j from R_j compose a set $A^j \subseteq R_j$. The set A^j is called the choice of the player j , while the collection $\langle A^1, A^2, \dots, A^n \rangle$ is called the joint choice. The case $A^k = \emptyset$ is not excluded and is called the refusal of k -th player from the choice.

We introduce the utility functions of elements $w \in A^j$. We assume that certain joint choice $\langle A^1, A^2, \dots, A^n \rangle$ has been carried out. Let there be uniquely determined, with the respect to the result of the choice, a collection of numbers $\pi_w \geq 0$ that are assigned to the elements $w \in A^j$, $j = 1, 2, \dots, n$; on the remaining elements of W the numbers are not determined. The numbers π_w are called utility indices, or simply utilities, and by definition, are in general case functions $\pi_w(X_1, X_2, \dots, X_n)$ of n variables. The value of the variable X_j is the choice A^j of the player j .

We shall single out utility functions possessing a special monotonic property.

Definition 1. A set of utilities π_w is called monotonic, if for any pair of joint choices $\langle L^1, L^2, \dots, L^n \rangle$ and $\langle G^1, G^2, \dots, G^n \rangle$ such that $L^j \subseteq G^j$, $j = 1, 2, \dots, n$

$$\pi_w(L^1, L^2, \dots, L^n) \leq \pi_w(G^1, G^2, \dots, G^n) \quad (1)$$

is fulfilled for any $w \in L^j$ ¹.

We now turn to the problem of coalition formation. We shall call any nonempty subset of the set of players a coalition. Let there be given a coalition V , and let its participants have made their choices. We compose from the choices A^j of the participants of the coalition V a set-theoretic union H , which is called the choice of the coalition V : $H = \bigcup_{j \in V} A^j$ ².

To determine the degree of suitability of the selection of an element $w \in R_j$ for the player j , a participant of the coalition, we introduce an index of guaranteed utility. With this aim we turn our attention to the dependence of the utility indices on the choice of the players not entering into coalition. It is not difficult to note that as a consequence of the monotonic condition of the functions π_w the worst case for the participants of the coalition will be when all players outside the coalition V reject the choice: $A^k = \emptyset$, $k \notin V$, so that all elements outside H will not be chosen by any of the players who are capable of making their choices. In other words, the guaranteed (the least value) of utility π_w of an element w chosen by a player in the case of fixed choices $H \cap R_j$ of his partners in the coalition equals $\pi_w(H \cap R_1, \dots, A^j, \dots, H \cap R_n)$.

The quantity

$$g_j(H) = \min_{w \in A^j} \pi_w(H \cap R_1, \dots, A^j, \dots, H \cap R_n)$$

is called the guarantee of the participant j in the coalition V for the choice H .

¹ We note that fulfilment of (1) is not required for the element $w \notin L^j$. Furthermore, even the numbers π_w themselves may not be defined for $w \notin L^j$.

² A choice H without indication about the coalition V , which has effected it, is not considered, and if somewhere the symbol V is omitted, then under a coalition we understand a collection of players such and only such for which $H \cap R_j \neq \emptyset$.

We assume that according to the rules of the game, for each chosen element $w \in A^j$ a player $j \in V$ must make a payment u° . It is obvious that under condition of the payment u° the selection of each element $w \in A^j$ is profitable or at least without loss to the player $j \in V$ if and only if $\pi_w \geq u^\circ$. In the calculation for the worst case this thus reduces to the criterion $g_j(H) \geq u^\circ$. In reality we shall be interested, in relation to the player $j \in V$, in all three possibilities: a) $g_j(H) > u^\circ$, b) $g_j(H) = u^\circ$ and c) $g_j(H) < u^\circ$. We shall say that a participant of the coalition V is above u° , on the level of u° , and below u° , if the conditions a), b), and c) are fulfilled respectively. The size of the payment is further considered as a parameter u of the game being described and is called the threshold. We shall say that a coalition V , having made a choice H , functions on the level $u[H] = \min_{j \in V} g_j(H)$.

Definition 2. A coalition V is called stable with the respect to a threshold $u^\circ = u[H]$ if for a certain choice H all participants of the coalition are not below u° while someone in the coalition $k \cup V$ is below u° if any participant $k \notin V$ outside the coalition V makes a nonempty choice $A^k \neq \emptyset$.

The set of numerical values being attained by the function $u[H]$ on stable coalitions will be called the spectrum. Each value of the function $u[H]$ will be called the spectral level (or simply the level). The entire construction described above will be called a monotonic parametric game on W .

Subsequently we will be interested in stable coalitions functioning on the highest possible spectral level. It is obvious that the spectrum of each monotonic game on a finite set W is bounded, and therefore there exists a maximum spectral level $u^\mu = \max_{H \subseteq W} u[H]$.

Definition 3. A stable coalition V^* such that for a certain choice H^* the level $u^\mu : u[H] = u^\mu$ is attained is called the kernel of the monotonic parametric game on W .

Theorem 1. If V_1^* and V_2^* are kernels of the monotonic game on W , then one can always find the minimum kernel (in set-theoretic sense) V_c^* such that $V_c^* \supseteq V_1^* \cup V_2^*$. The proof is presented in the appendix.

Theorem 1 asserts that the set of kernels in the sense indicated by the binary operation of coalitions is closed. The closeness of a system of kernels allows us looking at the largest (in the set-theoretic sense) kernel, i.e., a kernel K^\ominus such that all other kernels are included in it. From the Theorem 1 it follows the existence of the largest kernel in any finite monotonic parametric game.

The rest of the paper is devoted to the description of constructive methods of setting up coalitions that are stable with the respect to the threshold u° , including those stable with the respect to the threshold u^μ , i.e., the kernels coalitions. In particular, a method of constructing the largest kernel is suggested.

3. Search of Stable Coalitions

We consider a monotonic parametric game with n players. Below we bring together a system of concepts, which allows us constructively to discover stable coalitions with respect to an arbitrary threshold u° if they exist. In the monotonic game only a limited portion of subsets of the set W have to be searched in order to discover the largest stable coalition. With this aim in the following we study coalitions V whose participants do not refuse from a choice: for $j \in V$ the choice $A^j \neq \emptyset$. Such a coalition, which has effected a choice H , is denoted by $V[H]$. From here on, for the motive of simplicity of notation of guaranteed utility $\pi_w(H \cap R_1, \dots, A^j, \dots, H \cap R_n)$, where H is a subset of the set W , we use $\pi(w; H)$.

Definition 4. A sequence $\bar{\alpha}$ of elements $\langle \alpha_0, \alpha_1, \dots, \alpha_{m-1} \rangle$ (m is the number of elements in W) from W is said to be in concord with respect to the threshold u° , if in a sequence of subsets of the set W

$$\langle N_0, N_1, \dots, N_{m-1}, N_m \rangle,$$

where $N_0 = W$, $N_{i+1} = N_i \setminus \alpha_i$, $N_m = \emptyset$, there exists a subset N_p such that:

- a) The utility $\pi(\alpha_i; N_i) < u^\circ$ for all $i < p$;
- b) For each $w \in N_p$ the condition $u^\circ \leq \pi(w; N_p)$ is fulfilled, or, this being equivalent, for each $j \in V(N_p)$ the condition $u^\circ \leq g_j(N_p)$ ³ is fulfilled.

³ By definition $g_j(N_p) = \min_{w \in N_p \cap R_j} \pi(w; N_p)$.

A sequence $\bar{\alpha}$, in concord with the respect to the threshold u° , uniquely defines the set N_p . This fact is written in the form $N(\bar{\alpha}) = N_p$.

Definition 5. A set $S^\circ \subseteq W$ is said to be in concord with the respect to a threshold u° , if there exists a sequence $\bar{\alpha}$ of elements of W , in concord with respect to the threshold u° and such that $S^\circ = N(\bar{\alpha})$, while the coalition $V(S^\circ)$ is said to be in concord with respect to the threshold u° .

The following two statements are derived directly from Definitions 4 and 5.

- A. In the case where the set $S^\circ = W$ is in concord with the respect to the threshold u° , all players $j \in I$ are not below u° : $g_j(W) \geq u^\circ$.
- B. If the set S° , in concord with the respect to the threshold u° , is empty, then there exists a chain of constructing sets

$$\langle N_0, N_1, \dots, N_{m-1}, N_m \rangle,$$

such that for each player $j \in I$, commencing with a certain N_t , in all those coalitions $V(N_i)$, $t \leq i$, where the player j enters, this player is below u° .

Theorem 2. Let S° be a set that is in concord with respect to the threshold u° . Then any stable coalition V functioning on the level not less than u° makes a choice H , which is a subset of the set S° : $H \subseteq S^\circ$.

The proof is given in the appendix.

Corollary 1. The set S° , in concord with respect to the threshold u° , is unique. Indeed, if we assume that there exists a set S' , in concord with the respect to the threshold u° and different from S° , then from theorem 2 $S' \subseteq S^\circ$. But analogously at the same time the inverse inclusion $S' \supseteq S^\circ$ must also be satisfied, which bring us to conclusion that $S' = S^\circ$.

Corollary 2. As the spectral levels of functioning of coalitions in the monotonic parametric game grow, one can always find a chain of stable coalitions, included in one another and being in concord with respect to each increasing spectral level, as with respect to the growing threshold.

Indeed, from the formulation of the theorem it follows that a stable coalition, in concord with the respect to a spectral level $\lambda < \mu$, satisfies the relation $V(S^\lambda) \supseteq V(S^\mu)$, since in a set-theoretic sense $S^\lambda \supset S^\mu$.

Below we arrange a certain sequence $\bar{\alpha}$, which use up all elements of W . After the construction we formulate a theorem about the sequence $\bar{\alpha}$ thus constructed being in concord with respect to the threshold u° . The arrangement proves constructively the existence of a sequence of elements of W that is necessary in the formulation of the theorem.

Construction. Initial Step.

Stage 1. We consider a set of elements W . Among this set we search out elements γ_0 such that

$$\pi(\gamma_0; W) < u^\circ, \quad (2)$$

and order them in any arbitrary manner in the form of a sequence $\bar{\gamma}_0$. If there are no such elements, then all elements of W are ordered arbitrarily in the form of a sequence $\bar{\alpha}$, and the construction is completed. In this case W is assumed to be the set $N(\bar{\alpha})$.

Stage 2. Subsequently we examine the sequence $\bar{\gamma}_0$. When considering the t -th element $\gamma_0(t)$ of this sequence $\bar{\gamma}_0$, the sequence $\bar{\alpha}$ is supplemented by the element $\gamma_0(t)$, which is denoted by the expression $\bar{\alpha} \leftarrow \langle \bar{\alpha}, \gamma_0(t) \rangle$, while the set W is replaced by $W \setminus \bar{\alpha}$. After the last element of $\bar{\gamma}_0$ is examined we go over to the recursive step of the construction.

Recursive Step k .

Stage 1. Before constructions of the k -th step there is already composed a certain sequence $\bar{\alpha}$ of elements from W . Among the set $W \setminus \bar{\alpha}$ we seek out elements γ_k such that

$$\pi(\gamma_k; W \setminus \bar{\alpha}) < u^\circ, \quad (3)$$

and order them in any arbitrary manner in the form of a sequence $\bar{\gamma}_k$. Analogously to the initial step, if there happen to be no elements γ_k , the construction is ended. In this case in the role of the set $N(\bar{\alpha})$ we choose $W \setminus \bar{\alpha}$ while $\bar{\alpha}$ is completed in an arbitrary manner with all remaining elements from W .

Stage 2. Here we carry out constructions, which are analogous to stage 2 of the initial step. The entire sequence of elements $\bar{\gamma}_k$ is examined element by element. While examining the t -th element $\gamma_k(t)$ the sequence $\bar{\alpha}$ is complemented in accordance with the expression $\bar{\alpha} \leftarrow \langle \bar{\alpha}, \gamma_k(t) \rangle$. After examining the last element $\gamma_k(t)$ of the sequences $\bar{\gamma}_k$ we return to stage 1 of the recursive step.

On a certain step p , either initial or recursive, at stage 1 there are no elements γ , which are required by the inequalities (2) or (3), and the construction could not continue any more.

Theorem 3. *A sequence $\bar{\alpha}$ constructed according to the rules of the procedure is in concord with the respect to the threshold u° .*

The proof is presented in the appendix.

In the current section, in view of the use, as an example, of the concepts just introduced, we consider a particular case of a monotonic parametric game in which the difference in the individual and cooperative behavior of the participants of the coalition is easily revealed. We assume that the utilities

$$\pi_w(A^1, \dots, A^{j-1}, X_j, A^{j+1}, \dots, A^n)$$

do not depend on X_j in the case that choices specified by the remaining players are fixed. In this case the j -s participant of the coalition V , under the condition that the remaining participants of it keep their choices, can limit his choice X_j to a single element $w' \in R_j$ on which the maximum guarantee $\max_{w' \in R_j} g_j(H)$ is attained. However, such a selection narrowing his choice down to a single-element, generally speaking, reduces the choice (in view of monotonicity of utility indices π_w) to the guarantee of the remaining participants of the coalition. Consequently, individual behavior of the participants of a coalition contradicts their cooperative behavior. In spite of this contradiction, in the general case, in the given case, using the concept of a stable coalition $V(S^\circ)$ in concord with respect to the threshold u° , and having slightly modified the criteria of “individual interests” of the players, we can convince someone that there always exists a situation in which the individual interests do not contradict the coalition interests.

We define the winnings of the j -th participant of the coalition in the form of the sum of utilities after subtraction of all payments u° , i.e., as the number

$$f_j(H) = \sum_{w \in A_j} [\pi(w; H) - u^\circ]$$

(the winnings f_k for $k \notin V$ are not defined). Having represented H as a joint choice $\langle A^1, A^2, \dots, A^{|V|} \rangle$, we can consider the behavior of each j -th participant as player in a certain non-cooperative game selecting a strategy A^j .

The situation of individual equilibrium in the sense of Nash [1] of the participants of the coalition V in the game with winnings f_j is defined as their joint choice $\bigcup_{j \in V} A_*^j = H^*$ such that for each $j \in V$

$$f_j(A_*^1, \dots, A_*^{j-1}, A^j, A_*^{j+1}, \dots, A_*^{|V|}) \leq f_j(H^*)$$

for any $A^j \subseteq R_j$. In other word, the situation of equilibrium exists if none of the participants of the coalition has any sensible cause for altering his choice A_*^j under the condition that the rests keep to their choices.

Not every choice H of participants of the coalition V is an equilibrium situation. To see this it is sufficient to consider a choice H such that in the coalition V there are players having chosen elements $w \in A^j$ with utilities $\pi(w; H) < u^\circ$; for the selection of such an element the player pays more than this element brings in winnings $f_j(H)$ and, therefore, for the player, proceeding merely on the basis of individual interests, it would be advantageous to refrain from selection of such elements. Refraining from the selection of such elements of the set H is equivalent to non-equilibrium of H in the sense of Nash.

Lemma. *Let the utilities $\pi(w; H)$ be independent of A^j . Then a joint choice S° of the participants of the stable coalition $V(S^\circ)$, in concord with the respect to the threshold u° , is a situation of individual equilibrium.*

Indeed, according to Theorem 2, S° is the largest choice in the set-theoretic sense among all choices H of the stable coalition $V(S^\circ)$, where for any $w \in H$ the relation $\pi(w; H) \geq u^\circ$ is fulfilled. Let the choice of the participants of the coalition, with an exception of that of the j -th participant, be fixed. Since the utilities $\pi(w; S^\circ)$ do not depend on A^j , the j -th participant of $V(S^\circ)$ cannot secure an increase in the winnings $f_j(S^\circ)$ either by broadening or by narrowing his choice in comparison with $R_j \cap S^\circ$.

4. Coalitions functioning on the highest spectral level

We consider the problem of search of the largest kernel. First of all we present some facts, which are required for the solution of this problem.

From the definition of the guarantee $g_j(H)$ of the participant j effecting the choice H we see that the equality

$$g_j(H) = \min_{w \in A^j} \pi(w; H) \quad (4)$$

is fulfilled. Hence, according to the definition of the level $u[H]$ of functioning of the coalition $V(H)$ it follows that

$$u[H] = \min_{w \in H} \pi(w; H)$$

If we carry out a search of the subset H^* of the set W on which the value of the maximum of the function $u[H]$ is achieved, then thereby the search of a coalition functioning on the highest level $u^\mu = u[H^*]$ of the spectrum of a monotonic parametric game is effected. Without describing the search procedure, we give the definition of a sequence of elements W allowing us to discover the largest (in the set-theoretic sense) choice H^\ominus of the largest coalition – a kernel K^\ominus .

Definition 6. A sequence $\bar{\alpha}$ of elements $\langle \alpha_0, \alpha_1, \dots, \alpha_{m-1} \rangle$ (m is the number of elements in W) from W is called the defining sequence of the monotonic game, if in the sequence of sets ⁴

$$\langle N_0, N_1, \dots, N_{m-1}, N_m \rangle$$

there exists a subsequence $\langle \Gamma_0, \Gamma_1, \dots, \Gamma_p \rangle$ such that:

- a) for any element $\alpha_i \in \Gamma_k \setminus \Gamma_{k+1}$ of the sequence $\bar{\alpha}$ the utility $\pi(\alpha_i; N_i) < u[\Gamma_{k+1}]$ ($k = 0, 1, \dots, p-1$);
- b) in the stable coalition $V(\Gamma_p)$ no subcoalition exists on a level above $u[\Gamma_p]$.

From the Definition 6 one can see that the defining sequence in many ways is analogous to a sequence, which is in concord with the respect to the level u° . Since any stable coalition $V(\Gamma_k)$ functions on the level $u^k = u[\Gamma_k]$, it is not difficult to note that the defining sequence $\bar{\alpha}$ composes strictly increasing spectral levels $u[\Gamma_0] < u[\Gamma_1] < \dots < u[\Gamma_p]$ of functioning of stable coalitions $V(\Gamma_k)$ in the monotonic parametric game. As a result, we require yet another formulation.

Definition 7. A stable coalition $V \subseteq I$ is said to be determinable, if there exists a defining sequence $\bar{\alpha}$ of elements W such that among the choices of this coalition there is a choice Γ_p composed by $\bar{\alpha}$ according to Definition 6.

Theorem 4. For each monotonic parametric game a determinable coalition exists and is unique. Among the choices of the determinable coalition there is a choice on which the highest spectral level u^μ is attained.

The proof of the theorem is presented in the appendix.

Corollary to Theorem 4. The concepts of a determinable coalition and the largest kernel are equivalent.

⁴ The given sequence is constructed exactly in the same way as the one in Definition 4.

Indeed, directly from the formulation of the Theorem 4 we see that a determinable coalition always is the largest kernel. Hence, since a determinable coalition always exists, while the largest kernel is unique, it follows that the largest kernel coincides with the determinable coalition.

Thus, the problem of search of the largest kernel is solved if we construct a defining sequence $\bar{\alpha}$ of elements W . The construction of $\bar{\alpha}$ can be effected by the procedure of discovering kernels (KFP) from [2]. In conclusion we present yet another approach to the concept of “stability” of a coalition.⁵

Definition 8. A coalition \hat{V} is said to be a critical, if for a certain choice \hat{H} of it no coalition V having a nonempty intersection with the coalition \hat{V} functions on a level higher than $u[\hat{H}]$. The level $\hat{u} = u[\hat{H}]$ is called the critical level of the coalition \hat{V} , while the choice \hat{H} is called its critical choice.

From the Definition 8, in particular, it follows at once the uniqueness of the critical level of the coalition \hat{V} . Indeed, on the contrary, if were two different levels \hat{u}' and \hat{u}'' , $\hat{u}' < \hat{u}''$, then \hat{u}' could not be a critical one according to the definition: it is sufficient to consider the coalition $V = \hat{V}$ itself with the choice \hat{H}'' , which ensures $\hat{u}'' > \hat{u}'$.

It is obvious that kernels are critical coalitions. The inverse statement, generally speaking, is not true; a critical coalition is not necessarily a kernel.

We now consider the following hypothetical situation. Let \hat{V} be a critical coalition and let \hat{H} be its critical choice. We assume that this coalition is stable with respect to the threshold u° ; i.e., $u^\circ \leq u[\hat{H}]$ (see Definition 2). We assume that an increase of the threshold u° up to the level $u^\circ > u[\hat{H}]$ took place and the critical coalition \hat{V} with the critical choice \hat{H} was transformed into unstable coalition with respect to the higher threshold u° . Let the participants of the coalition \hat{V} preserving the stability of the coalition attempt to increase their guarantees. One of the possibilities for increasing the guarantee of a participant $j_0 \in \hat{V}$ is to refrain from the choice of an element $\alpha_0 \in A^{j_0}$ on which the value $g_{j_0}(H)$ - the

⁵ This approach is close to the concept of “M-stability” in cooperative n-person games [1].

minimum level of utility guaranteed for him, see (4), is attained. It is natural to assume that a participant with a level of guarantee $g_{j_0}(\hat{H}) = u[\hat{H}] < u^\circ$ will be among the participants attempting to increase their guarantees, and refrains from the selection of the element α_0 indicated above. It may happen that the refusal of α_0 gives rise, for another participant $j_1 \in V(\hat{H} \setminus \alpha_0)$, to a decrease from his guarantee $g_{j_1}(\hat{H}) > u[\hat{H}]$ to the quantity $g_{j_1}(\hat{H} \setminus \alpha_0) \leq u[\hat{H}]$. A participant $j_1 \in V(\hat{H} \setminus \alpha_0)$, acting from the same considerations as j_0 , refrains from the selection of an element α_1 on which $g_{j_1}(\hat{H} \setminus \alpha_0)$ is attained. Such a refusal of α_1 can give rise to subsequent refusals, and emerges hereby a chain of “refusing” participants $\langle j_0, j_1, \dots \rangle$ of the coalition \hat{V} .

If a coalition V , stable with respect to the threshold u° in the sense of Definition 2, with the choice H became unstable as the threshold u° increases, then such a coalition, generally speaking, disintegrates; i.e., some of its participants may become participants of a new coalition which already is stable with the respect to the increased threshold u° . By definition of a critical coalition, transaction of its participants into new stable coalition, when the threshold u° increases is not possible, and it disintegrates completely. The theorem presented below and proved in the appendix reflects a possible character of complete disintegration of a critical coalition in terms of the hypothetical system described above.

Theorem 5. *Let there be given a critical coalition \hat{V} having a nonempty intersection with a certain coalition V : $\hat{V} \cap V \neq \emptyset$. Let H be the choice of the coalition V and \hat{H} the critical choice of the coalition \hat{V} . Then in the coalition $\hat{V} \cap V$ there exists a sequence of its participants $\bar{j} = \langle j_0, j_1, \dots, j_{r-1} \rangle$ such that: a) in the sequence \bar{j} there are represented all participants of the coalition $\hat{V} \cap V$ (the players j_i may be repeated, r is number of elements in $\hat{H} \cup H$; b) for the sequence \bar{j} we can construct a chain of contracting coalitions*

$$\langle V(N_0), V(N_1), \dots, V(N_{r-1}) \rangle,$$

where $N_0 = \hat{H} \cup H$, $N_{i+1} \subset N_i$, so that for any $j \in V$, commencing from a certain N_t , in all those coalitions $V(N_i)$, $t \leq i$, into which the player j enters, this player is not above $u[\hat{H}]$.

5. Example of a Monotonic Game ⁶

We consider a game of n customers who at the same time are suppliers of certain goods. Let each j -th customer supply goods of j -th designation, $j = 1, 2, \dots, n$. The situation under consideration is conveniently depicted in the form of a set of arcs W of a graph G of potential deliveries of goods, and the customer–supplier, in the form of a set of its nodes. A potentially effectible delivery of goods for sum of c bank notes is depicted on the graph by a c -fold arc.

We shall assume that a “player” in the sense of the scheme of the monotonic game described above is each participant when he acts in the role of a customer and decides from whom he orders the goods required by him. We define the choice of the j -th customer in the form of a subset of arcs A^j of the set of potential arcs R_j , entering into the node j in the graph G ; $A^j \subseteq R_j$. The nodes of the graph from which $w \in A^j$ emerge are understood as the supplies of the goods, while a single arc w is interpreted as a supply, to the customer, of goods for the amount of one bank note. After all orders have been received, each j -th customer–supplier carries out the supplies.

We call any subset V of the sets of nodes I of the graph G a coalition, while the choice of a coalition is defined in the form of a set of arcs H depicting supplies of goods in bank notes $|H|$, is the money equivalent to the goods ordered by a coalition.

We assume that the participants of the coalition stimulate mutual business contacts. A supplier of goods, being a participant of a coalition, can, e.g., propose a certain rebate to his customer. Here the magnitude of rebate is appropriately set in accordance with the business activity of the supplier, having taken as a measure of its business activity the number of suppliers to himself. Taking into account what has been said, we determine the rebate in bank notes of goods supplied to the customer, in the form $\theta_w \cdot b^w$, where b^w is the number of supplies with whom the supplier concluded deals, having dispatched goods along the arc $w \in A^j$, θ_w is a coefficient of proportionality.

⁶ See also <http://www.data laundering.com/download/nonjoke.pdf>.

Let h_w be the money equivalent of the useful effect for the j -th participant of the coalition in the account in bank notes of goods being consumed, ordered along the arc $w \in A^j$ (a loss, if $h_w < 0$). With the rebate taken into account, the total useful effect amounts to

$$\pi_w = h_w + \theta_w \cdot b^w.$$

We determine utility of an order along the arc $w \in A^j$ as a quantity of money equivalent to the overall π_w per bank note of the goods ordered. The guarantee of the j -th participant of the coalition, just as in the general scheme, is quantity

$$g_j(H) = \min_{w \in A^j} \pi_w.$$

We determine the aim of the coalition as creation of a certain fund by means of deductions from utilities π_w . A rational coalition V is one which from the utility π_w per bank notes of goods ordered can deduct into the fund a certain sum of money $u^\circ > 0$, i.e., if and only if $\pi_w \geq u^\circ$ for all $w \in H$. We shall show that the concept of a rational coalition is equivalent to the concept of a stable coalition with respect to a threshold u° , if as value of the parameter of the game of customers–suppliers we take the amount deducted into the fund being created. Indeed, if $\pi_w \geq u^\circ$ for any $w \in A^j$, then $j \in V$, $g_j \geq u^\circ$, i.e., the coalition V is stable with the respect to the threshold u° (see Section 2) and visa versa.

From the results of Section 3 it follows that in the game of customers–suppliers there exists a chain of enclosed rational coalitions, which contract with the growth of the amount of deductions u° . The procedure of search of rational coalitions allows us to uncover the structure of the chain, e.g., to answer the question: is the original set of customers-suppliers a rational coalition? On the basis of Theorem 4 we can find the largest kernel – a critical coalition sustaining the maximum amount of deductions u^μ , which constitutes the main interest in this model.

Concluding, we turn our attention to the form of contradiction between the individual and co-operative behavior of the participants of a coalition in the monotone game, using the example of the game of customers-suppliers of goods. From the example it is seen that purely individual behavior with the respect to the index of guarantee would lead to situation in which each customer has a single supplier. It is obvious that a rational coalition originates in general case a more “branched” network of contacts between participants of the coalition so that the level of the index of guarantee by each of the participants will be much higher.

APPENDIX

Proof of Theorem 1. Let the level u^μ be attained for the coalitions V_1^* and V_2^* , which effect the choices H_1^* and H_2^* respectively; i.e., $u^\mu = u[H_1^*]$ and $u^\mu = u[H_2^*]$. For player $j \in I$ we consider two choices: $H_1^j = H_1^* \cap R_j$ and $H_2^j = H_2^* \cap R_j$ ⁷. By the definition of guarantee $g_j(H_1^*)$ for the participant $j \in V_1^*$ of the coalition we have

$$\min_{w \in H_1^j} \pi_w(H_1^1, H_1^2, \dots, H_1^n) = g_j(H_1^*) \geq u^\mu; \quad (\text{A.1})$$

for the participant $j \in V_2^*$ we respectively have

$$\min_{w \in H_2^j} \pi_w(H_2^1, H_2^2, \dots, H_2^n) = g_j(H_2^*) \geq u^\mu. \quad (\text{A.2})$$

We determine the choice of a participant $j \in V_1^* \cup V_2^*$ as $\Phi^j = H_1^j \cup H_2^j$. The monotonic property (1) allows us to conclude that the following inequalities are valid:

$$\min_{w \in H_1^j} \pi_w(\Phi^1, \Phi^2, \dots, \Phi^n) \geq \min_{w \in H_1^j} \pi_w(H_1^1, H_1^2, \dots, H_1^n); \quad (\text{A.3})$$

$$\min_{w \in H_2^j} \pi_w(\Phi^1, \Phi^2, \dots, \Phi^n) \geq \min_{w \in H_2^j} \pi_w(H_2^1, H_2^2, \dots, H_2^n). \quad (\text{A.4})$$

Combining (A.1) – (A.4), we obtain

$$\min_{w \in \Phi^j} \pi_w(\Phi^1, \Phi^2, \dots, \Phi^n) \geq u^\mu \quad (\text{A.5})$$

for any $j \in V_1^* \cup V_2^*$. If by Φ^* we denote the set $H_1^* \cup H_2^*$, then for the coalition $V_1^* \cup V_2^*$ effecting the choice Φ^* the inequality (A.5) is rewritten in the form

$$g_j(\Phi^*) \geq u^\mu, \quad j \in V_1^* \cup V_2^*. \quad (\text{A.6})$$

Due to the monotonic property (1) some elements $w \notin \Phi^*$ (if one can find such) may be added to Φ^* while the inequality (A.6) is still true⁸. We will denote the enlarged set by Φ^c : $\Phi^c \supseteq \Phi^*$ and obviously for $V^c = V(\Phi^c)$ we have $V(\Phi^c) \supseteq V_1^* \cup V_2^*$. By the definition of a spectral level u^μ , for the participant $j' \in V^c$, on which $u[\Phi^c]$ is attained, we have

$$g_{j'}(\Phi^c) = u[\Phi^c] \leq u^\mu, \quad (\text{A.7})$$

since u^μ is the maximum spectral level of functioning of coalitions in the monotonic game. Applying (A.7) and (A.6) to the choice Φ^c for the participant $j = j'$, we see that $g_{j'}(\Phi^c) = u^\mu$, and the coalition $V^c \supseteq V_1^* \cup V_2^*$ functions on the spectral level u^μ . The theorem is proved. ■

⁷ We note that, in the worst case, for player $k \notin V_1^*$ ($k \notin V_2^*$), $H_1^k = \emptyset$ ($H_2^k = \emptyset$).

⁸ We suppose that such elements cannot be added to Φ^c .

Proof of Theorem 2. Let S° is a subset of the set W in concord with the respect to the threshold u° ; i.e., there exists a sequence $\bar{\alpha}$, in concord with the respect to the threshold u° , such that $S^\circ = N(\bar{\alpha})$. We assume that there exists a coalition V effecting a choice $H \subset S^\circ$ and functioning on the level $u[H] \geq u^\circ$; $H \setminus S^\circ \neq \emptyset$. Let $\alpha_t \in H \setminus S^\circ$ and let α_t be an element, which is leftmost in the sequence $\bar{\alpha}$. Let p be the index of the set N_p in the sequence $\langle N_0, N_1, \dots, N_{m-1}, N_m \rangle$. It is obvious that $t < p$ and, consequently,

$$\pi(\alpha_t; N_t) < u^\circ \quad (\text{A.8})$$

in accordance with a) of the Definition 4. Since the game being considered is monotonic, $\alpha_t \in H$ and $H \subseteq N_t$ there must hold

$$\pi(\alpha_t; H) \leq \pi(\alpha_t; N_t). \quad (\text{A.9})$$

From inequalities (A.8) and (A.9) it follows

$$\pi(\alpha_t; N_t) < u^\circ \leq u[H] \quad (\text{A.10})$$

(the latter \leq by assumption). According to the inequality (A.10) and by the definition of $u[H]$ we have

$$\pi(\alpha_t; H) < \min_{j \in V} g_j(H). \quad (\text{A.11})$$

Let the element α_t be chosen by a certain q -th player; i.e., $\alpha_t \in A^q$, $q \in V$. On the basis of (A.11) we assume that

$$\pi(\alpha_t; H) < g_q(H) \quad (\text{A.12})$$

is valid. By definition $g_q(H) = \min_{w \in A^q} \pi(w; H)$. Following (A.12), we note that $\pi(\alpha_t; H) < \min_{w \in A^q} \pi(w; H)$. The last inequality is contradictory, what proves the theorem. ■

Proof of Theorem 3. We assume that the construction of the sequence $\bar{\alpha}$ according to the rules of the procedure ended on a certain p -th step. This means that $\bar{\alpha}$ is made up of sequences $\bar{\gamma}_k$ ($k = 0, 1, \dots, p$), and also of elements of the set N_p , found according to the rules of the procedure and being certainties for the sequences $\bar{\gamma}_k$. We consider any element α_i of the sequence thus constructed, being located on the left of the α -th element: $i < p$. The given element in the construction process falls into certain set $\bar{\gamma}_q$. By construction

$$\pi(\alpha_i; W \setminus \{\bar{\gamma}_0 \cup \bar{\gamma}_1 \cup \dots \cup \bar{\gamma}_{q-1}\}) < u^\circ. \quad (\text{A.13})$$

If to the sequence $\langle \bar{\gamma}_0, \bar{\gamma}_1, \dots, \bar{\gamma}_{q-1} \rangle$ we add the elements $\bar{\gamma}_q$, which in $\bar{\alpha}$ are on the left of the α_i -th, then this set of elements together with the added part $\bar{\gamma}_q$ composes the complement \bar{N}_i up to the set W (see Definition 4). On the basis of the monotonic property (1) we conclude that $\pi(\alpha_i; W \setminus \{\bar{\gamma}_0 \cup \bar{\gamma}_1 \cup \dots \cup \bar{\gamma}_{q-1}\}) \geq \pi(\alpha_i; W \setminus \bar{N}_i) = \pi(\alpha_i; N_i)$. The last relation in the combination with (A.13) shows that $\pi(\alpha_i; N_i) < u^\circ$. From the construction of the sequence $\bar{\alpha}$ it is also obvious that for any $j \in V(N_p)$ the guarantee $g_j(N_p) \geq u^\circ$.

The theorem is proved. ■

Proof of the Theorem 4. Theorem can be proved as follows. First, a sequence $\bar{\alpha}$, in concord with respect to the highest spectral level u^μ , in the monotonic game exists, according to Theorem 3, and is, at the same time, a defining sequence; as the subsequence $\langle \Gamma_0, \Gamma_1, \dots, \Gamma_p \rangle$ in this case we have to choose the sequence $\langle W, S^\mu \rangle$, where S^μ is a set $S^\mu \subset W$ which is in concord with respect to the highest level u^μ . The determinable coalition is $V(S^\mu)$. The uniqueness of the coalition $V(S^\mu)$ is proved in Corollary 1 to the Theorem 1. Secondly, the choice S^μ of the coalition $V(S^\mu)$, playing the part of the set Γ_p in the Definition 6, attains the maximum of the function $u[H]$, a fact which follows from Theorem 3 and b) of Definition 6; i.e., $u[S^\mu] = u^\mu$. Thirdly, the last statement of Theorem 4 is a particular case of the statement of Theorem 2, if we put $u^\circ = u^\mu$. The theorem is proved. ■

Proof of the Theorem 5. We consider a monotonic game of participants of a coalition $\hat{V} \cup V$ on the set $\hat{H} \cup H$, where \hat{H} is the critical choice of the critical coalition \hat{V} , and H is some choice of the coalition V . Below the set $\hat{H} \cup H$ is denoted by Ω , while all concepts refer to a monotonic sub-game on Ω .

Let u° be the threshold of the parameter u of the game on Ω , and let $u^\circ > u[H]$. We construct a sequence $\bar{\alpha}$ of elements Ω , which is in concord with respect to the threshold u° . Two variants could be represented: 1) the set S° , in concord with the respect to the threshold u° is empty; 2) S° is not empty. We consider them one after the other. First, in the variant 1) from a sequence of elements $\bar{\alpha}$ of elements of Ω in concord with respect to the threshold u° , we uniquely determine a sequence of participants of the coalition $\hat{V} \cup V$ choosing elements α_i from sequence $\bar{\alpha}$ and composing a certain chain $\bar{j} = \langle j_0, j_1, \dots, j_{r-1} \rangle$ (r is the number of elements Ω). Secondly, from the sequence $\bar{\alpha}$ we also uniquely determine the sequence of coalitions $\langle V(N_0), V(N_1), \dots, V(N_{r-1}) \rangle$, where $N_0 = \Omega$, $N_{i+1} = N_i \setminus \alpha_i$, with $j_i \in V(N_i)$.

In the second variant none of the participants of the coalition \bar{V} can be in a coalition, which is in concord with the respect to the threshold $u^\circ > u[H]$. This would contradict the definition of a critical coalition \bar{V} . Therefore in the chain \bar{j} thus constructed of participants of the coalition $\hat{V} \cup V$ (by the same method as in the first variant) all participants of the coalition \bar{V} are on the left of the j_p -th player; p is uniquely determined from the sequence $\bar{\alpha}$ (see Definition 4). By property a) of the Definition 4 and from the definition of the guarantee of a player $j_i \in V(N_i)$ we have

$$g_{j_i}(N_i) \leq \pi(\alpha_i; N_i) < u^\circ. \quad (\text{A.14})$$

Proceeding from the structure of the spectrum of a monotonic parametric game on Ω (see Corollary 2 to the Theorem 2) the value u° marginally close to $u[H]$ is satisfied successfully in the two variants considered. The first variant of the Theorem 5 forms the statement b) derived earlier from Definition 4 and 5 (see section 2). The second variant of the statement of the theorem is directly derived from the relation (A.14). ■

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