Quantum Market Games

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Abstract

We propose a quantum-like description of markets and economics. The approach has roots in the recently developed quantum game theory.

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Introduction One of the biggest scientific revolutions was caused by the emergence of quantum physics and the consequent falsification of the idea of the possibility of distinction between observer and observed phenomenon. Quantum Zeno effect is the most famous case in point [1]. This fact is often used (without any reference to quantum theory) by social scientists to argue that their theories are beyond mathematical description. Here we would like to propose a quantum-like approach to economics whose usefulness, we think, can be verified only by further investigation. The approach has roots in the recently developed quantum game theory [2]. For simplicity and clearness of the exposition, we shall suppose that there are only two assets on the market. The first one, denoted by \$, fulfills the role of money and its unit is 1\$. The second one, denoted by $\mathfrak{G}$, is distributed in units $1\mathfrak{G}$ of price c in \$. We shall also suppose that both assets can be exchanged in arbitrary ratio. The traders will be numbered by $k \in \mathbb{N}$, $\mathbb{N}$ being the natural numbers. The k-th trader implements a strategy (called pure) which is denoted by $|\psi\rangle_k$ and declares that his whole capital consisting of s_k units of $\mathfrak{G}$ and

d_k monetary units will be engaged in market transactions. A player can be present on the market in the shape of several traders what allows him to engage various parts his capital in different ways. The opposite situation is also possible: several traders can form a coalition. Such a clique can be described as correlated strategies of different traders. The actual participation of the k-th trader in the market turnover would be decided by an arbiter $\mathcal{A}$ who considers the data $\{|\psi\rangle_k, s_k, d_k\}$ coming from all traders. The role of the arbiter $\mathcal{A}$ is performed by an appropriate clearinghouse who acts in a fully deterministic way according to rules that are established by the law or/and tradition. The arbiter $\mathcal{A}$ can be identified with the algorithm used for clearing. Market transactions, that is capital flows, are settled according to the strategies put forward by the traders $|\psi\rangle_k$ and the algorithm $\mathcal{A}$. In this way a common price c is set for all transaction in a given turn. If one considers several markets connected by traders strategies then simultaneous transaction with different prices are possible (e.g. arbitrage is possible). Note that the trader is not obliged to know the market rules but such knowledge is necessary if he or she wants to act in a rational way.

The quantum model of market Let the real variable q

$$q := \ln c - E(\ln c) \tag{1}$$

denotes the logarithm of the price at which the k-th player can buy $\mathfrak{G}$ shifted so that its expectation value in the state $\mid \psi >_k$ vanishes. The expectation value of x is denoted by $E(x)$. The variable p

$$p := E(\ln c) - \ln c \tag{2}$$

describes the situation of a player who is supplying the asset $\mathfrak{G}$ according to his strategy $|\psi\rangle_k$. Supplying $\mathfrak{G}$ can be regarded as demanding \$ at the price c^{-1} in the $1\mathfrak{G}$ units and both definitions are equivalent. Note that we have defined q and p so that they do not depend on possible choices of the units for $\mathfrak{G}$ and \$. For simplicity we will use such units so $E(\ln c) = 0$. The strategies $|\psi\rangle_k$ belong to Hilbert spaces H_k. The state of the game $|\Psi\rangle_{in} := \sum_k |\psi\rangle_k$ is a vector in the direct sum of Hilbert spaces of all players, $\oplus_k H_k$. We will define canonically conjugate hermitian operators of demand $\mathcal{Q}_k$ and supply $\mathcal{P}_k$ for each Hilbert space H_k analogously to their physical counterparts position and momentum. This can be justified in the following way. Let $\exp(-p)$ be a definite price, where p is a proper value of the operator $\mathcal{P}_k$. Therefore, if one have already declared the will of selling exactly at the price $\exp(-p)$ (the strategy given by proper the state $|p\rangle_k$) then it is pointless to put forward any opposite offer for the same transaction. The capital flows resulting from an ensemble of simultaneous transactions correspond to the physical process of measurement. A transaction consists in a transition from the state of traders strategies $|\Psi\rangle_{in}$ to the describing the capital flow state $|\Psi\rangle_{out} := \mathcal{T}_\sigma|\Psi\rangle_{in}$, where $\mathcal{T}_\sigma := \sum_{k_d} |q\rangle_{k_d\,k_d}\langle q| + \sum_{k_s} |p\rangle_{k_s\,k_s}\langle p|$ is the projective operator given by the division σ of the set of traders $\{k\}$ into two separate subsets $\{k\} = \{k_d\} \cup \{k_s\}$, that is buying at the price $e^{q_{k_d}}$ and selling at the price $e^{-p_{k_s}}$ in the round of transaction in question. The role of the algorithm $\mathcal{A}$ is to determine the division of the market σ, the set of price parameters $\{q_{k_d}, p_{k_s}\}$ and the values of capital flows. The later are settled by the distribution

$$\int_{-\infty}^{\ln c} \frac{|\langle q|\psi\rangle_k|^2}{{}_k\langle\psi|\psi\rangle_k} dq \tag{3}$$

which is interpreted as the probability that the trader $|\psi\rangle_k$ is willing to buy the asset $\mathfrak{G}$ at the transaction price c or lower [3]. In an analogous way the distribution

$$\int_{-\infty}^{\ln \frac{1}{c}} \frac{|\langle p|\psi\rangle_k|^2}{{}_k\langle\psi|\psi\rangle_k} dp \tag{4}$$

gives the probability of selling $\mathfrak{G}$ by the trader $|\psi\rangle_k$ at the price c or greater. These probabilities are in fact conditional because they describe the situation after the the division σ is completed.

Maximization of the capital turnover Every game is specified by its rules so to illustrate the action of the algorithm $\mathcal{A}$ let us consider an example of deterministic clearinghouse algorithm. Suppose that the clearinghouse maximize the capital turnover at a uniform price c. The assumption of unique for all traders price

$$\forall_{k_d,k_s}\ (\mathcal{Q}_{k_d} + \mathcal{P}_{k_s})|\Psi\rangle_{out} = 0$$

corresponds to entanglement in quantum theory. On the market where transaction are made between two traders, the above condition may be fulfilled only for every pair of traders separately. Let us for simplicity limit the class of admissible strategies to those whose amplitudes $\langle q|\psi\rangle_k$ ($\langle p|\psi\rangle_k$) are functions of the variable q (p) with compact supports and square-integrable. After fixing the division σ we can rescale the amplitudes of supply and demand so integrals of the squares of modules of the resulting functions $\langle q|\phi\rangle_{k_d}$ and $\langle p|\phi\rangle_{k_s}$

$$\langle q|\phi\rangle_{k_d} := \sqrt{d_{k_d}}\frac{\langle q|\psi\rangle_{k_d}}{{}_{k_d}\langle\psi|\psi\rangle_{k_d}}\ \ ,\ \ \langle p|\phi\rangle_{k_s} := \sqrt{s_{k_s}}\frac{\langle p|\psi\rangle_{k_s}}{{}_{k_s}\langle\psi|\psi\rangle_{k_s}} \tag{5}$$

measure the appropriate capital flows. For a transaction at the price c, the maximal capital flow is the lowest number from the possible total buying and selling

$$j(\sigma,c) := \min\Big\{\sum_{k_d}\int_{-\infty}^{\ln c}|\langle q|\phi\rangle_{k_d}|^2 dq\ ,c\sum_{k_s}\int_{-\infty}^{-\ln c}|\langle p|\phi\rangle_{k_s}|^2 dp\Big\}. \qquad (6)$$

The maximal value is given by the solution of the equation

$$\sum_{k_d}\int_{-\infty}^{\ln c^*}|\langle q|\phi\rangle_{k_d}|^2 dq = c^*\sum_{k_s}\int_{-\infty}^{-\ln c^*}|\langle p|\phi\rangle_{k_s}|^2 dp. \qquad (7)$$

There is at least one solution of the Eq. (7) because the functions are continuous. If there are more solutions we may choose any of them. The division $\sigma^* = (k_s^*, k_d^*)$ corresponding to the maximal capital flow

$$j(\sigma^*, c^*) = \max_{\sigma} j(\sigma, c^*)$$

gives the changes in the amounts of the asset $\mathfrak{G}$

$$\frac{1}{c^*}\int_{-\infty}^{\ln c^*}|\langle q|\phi\rangle_{k_d^*}|^2 dq \quad , \quad -\int_{-\infty}^{-\ln c^*}|\langle p|\phi\rangle_{k_s^*}|^2 dp \qquad (8)$$

and the money \$

$$-\int_{-\infty}^{\ln c^*}|\langle q|\phi\rangle_{k_d^*}|^2 dq \quad , \quad c^*\int_{-\infty}^{-\ln c^*}|\langle p|\phi\rangle_{k_s^*}|^2 dp \qquad (9)$$

possessed by the traders after clearing the round of transactions. The Eq. (7) guarantees that the capital changes will be balanced, so the described game belongs to specially interesting zero sum class of game that do not generate capital surplus nor demand capital inflow (we neglect such cost as brokerage etc). Maximization of the $\mathfrak{G}$ turnover results in different price c^* and capital flows. This asymmetry can be removed by separate normalization of the strategies in Eq. (5) that is by replacing the square roots by $\sqrt{\frac{d_{k_d}}{\sum_{l_d} d_{l_d}}}$ and $\sqrt{\frac{s_{k_s}}{\sum_{l_s} s_{l_s}}}$, [4] and performing calculations for probabilities instead of capital flows. In this case the assumption of compactness of the supports is superfluous. Deutsch arguments [4] can be repeated here to show that the stochastic interpretation of the presented model is not necessary. Note that the asymmetry justifies presentation of the supply and demand curves in terms of probabilities and not capital flows [3]. Another, more quantum, algorithm can be defined if the transactions are made with weights analogous to Eq. (6) for all possible divisions σ and prices c. Such transactions should be balanced for not realized bids. This algorithm may exclude some trades from round of transactions. The transaction operator would resemble the commonly use in quantum theory scattering matrix:

$$\mathcal{T}_{\sigma\alpha} := I + \alpha_d \sum_{k_d} |q\rangle_{k_d\,k_d}\langle q| + \alpha_s \sum_{k_s} |p\rangle_{k_s\,k_s}\langle p|,$$

where $\alpha_d, \alpha_s \in \mathbb{R}_+$, and I is the identity operator in $\sum_k \oplus H_k$.

Market as a measuring apparatus When a game allows a great number of players in it is useful to consider it as a two-players game: the trader $|\psi\rangle_k$ against the Rest of the World (RW). (The player RW has a lot in common with a macroscopic measuring apparatus.) The concrete algorithm $\mathcal{A}$ may allow for an effective strategy of RW (for a sufficiently large number of players the single player strategy should not influence on the form of the RW strategy). If one considers the RW strategy it make sense to declare its simultaneous demand and supply states because for one player RW is a buyer and for another it is a seller. To describe such situation it is convenient to use the Wigner formalism [5]. The pseudo-probability $W(p,q)dpdq$ on the phase space $\{(p,q)\}$ known as the Wigner function is given by

$$\begin{aligned} W(p,q) &:= h_E^{-1} \int_{-\infty}^{\infty} e^{i\hbar_E^{-1}px}\, \frac{\langle q+\frac{x}{2}|\psi\rangle\langle\psi|q-\frac{x}{2}\rangle}{\langle\psi|\psi\rangle}\, dx \\ &= h_E^{-2} \int_{-\infty}^{\infty} e^{i\hbar_E^{-1}px}\, \frac{\langle p+\frac{x}{2}|\psi\rangle\langle\psi|p-\frac{x}{2}\rangle}{\langle\psi|\psi\rangle}\, dx, \end{aligned}$$

where the positive constant $h_E = 2\pi\hbar_E$ is the dimensionless economical counterpart of the Planck constant. Recall that this measure is not positive definite except for the stated bellow cases. In the general case the pseudo-probability density of RW is a countable linear combination of Wigner functions, $\rho(p,q) = \sum_n w_n W_n(p,q)$, $w_n \geq 0$, $\sum_n w_n = 1$. According to Eq. (3) and (4) (see also Ref. [3]) the diagrams of the integrals of the RW pseudo-probabilities

$$F_d(\ln c) := \int_{-\infty}^{\ln c} \rho(p = const., q)dq \qquad (10)$$

(RW bids selling at $\exp(-p)$)
and

$$F_s(\ln c) := \int_{-\infty}^{\ln \frac{1}{c}} \rho(p, q = const.)dp \qquad (11)$$

(RW bids buying at $\exp q$) against the argument $\ln c$ may be interpreted as the dominant supply and demand curves in Cournot convention, respectively [3]. Note, that due to the lack of positive definiteness of ρ, F_d and F_s may not be monotonic functions. Textbooks on economics give examples of such departures from the low of supply (work supply) and low of demand (Giffen assets) [6]. We will call an arbitrage algorithm resulting in non positive definite probability densities *a giffen*.

Quantum Zeno effect If the market continuously measures the same strategy of the player, say the demand $\langle q|\psi\rangle$, and the process is repeated sufficiently often for the whole market, then the prices given by the algorithm $\mathcal{A}$ do not result from the supplying strategy $\langle p|\psi\rangle$ of the player. The necessary condition for determining the profit of the game is the transition of the player to the state $\langle p|\psi\rangle$ [3]. If, simultaneously, many of the players changes their strategies then the quotation process may collapse due to the lack of opposite moves. In this way the quantum Zeno effects explains stock exchange crashes. Effects of this crashes should be predictable because the amplitudes of the strategies $< p|\psi\rangle$ are Fourier transforms of $< q|\psi\rangle$ Another example of the quantum market Zeno effect is the stabilization of prices of an asset provided by a monopolist.

Eigenstates of $\mathcal{Q}$ and $\mathcal{P}$ Let us suppose that the amplitudes for the strategies $\langle q|\psi\rangle_k$ or $\langle p|\psi\rangle_k$ that have infinite integrals of squares of their modules, ($\langle q|\psi\rangle_k \notin L^2$) have the natural interpretation as the will of the k-th player of buying (selling) of the amount d_k (s_k) of the asset $\mathfrak{G}$. So the strategy $\langle q|\psi\rangle_k = \langle q|a\rangle = \delta(q,a)$ means that in the case of classifying the player to the set $\{k_d\}$, refusal of buying cheaper than at $c = e^a$ and the will of buying at any price equal or above e^a. In the case of "measurement" in the set $\{k_d\}$ the player declares the will of selling at any price. The above interpretation is consistent with the Heisenberg uncertainty relation. The strategies $\langle q|\psi\rangle_2 = \langle q|a\rangle$ (or $\langle p|\psi\rangle_2 = \langle p|a\rangle$) do not correspond to the RW behaviour because the conditions $d_2, s_2 > 0$, if always fulfilled, allow for unlimited profits (the readiness to buy or sell $\mathfrak{G}$ at any price). The demand Eq. (10) and supply Eq. (11) functions give probabilities of coming off transactions in the game when the player use the strategy $\langle p|const\rangle$ or $\langle q|const\rangle$ and RW, proposing the price, use the strategy ρ. The authors have analyzed the efficiency of the strategy $\langle q|\psi\rangle_1 = \langle q|-a\rangle$ in a two-player game when RW use the strategy with squared module of the amplitude equal to normal distribution [3]. The maximal intensity of the profit [3] is equal to 0.27603 times the variance of the RW distribution function. Of course, the strategy

$\langle p|\psi\rangle_1 = \langle p|0,27603\rangle$ has the same properties. In such games a=0.27603 is a global fixed point of the profit intensity function. This may explain the universality of markets on which a single client facing the bid makes up his/hers mind. Does it mean that such common phenomena have quantal nature? The Gaussian strategy of RW [7] can be parametrized by the temperature $T = \beta^{-1}$ (see below). Any decrease in profits is only possible by reducing the variance of RW (i.e. cooling). Market competition is the mechanism responsible for risk flow that allows the market to attain the thermodynamical balance. A warmer market influences destructively on the cooler traders and they diminish the uncertainty of market prices.

Adiabatic strategies Financial mathematics teach us that the first moments of the random variables p and q measure the expected profit from one transaction and the second moments measure the risk [8]. Therefore we will define the observable

$$H(\mathcal{P}_k,\mathcal{Q}_k) := \frac{(\mathcal{P}_k - p_{k0})^2}{2m} + \frac{m\omega^2(\mathcal{Q}_k - q_{k0})^2}{2}, \tag{12}$$

where $p_{k0} := \frac{{}_k\langle\psi|\mathcal{P}_k|\psi\rangle_k}{{}_k\langle\psi|\psi\rangle_k} \neq E(\mathcal{P}_k)$, $q_{k0} := \frac{{}_k\langle\psi|\mathcal{Q}_k|\psi\rangle_k}{{}_k\langle\psi|\psi\rangle_k}$, $\omega := \frac{2\pi}{\theta}$, and call it *the risk inclination operator*, cf Ref. [3]. θ denotes the characteristic time of transaction introduced in the MM model [3]. The parameter $m > 0$ measures the risk asymmetry between buying and selling positions. Analogies with quantum harmonic oscillator allow for the following characterization of quantum market games. The constant h_E describes the minimal inclination of the player to risk. It is equal to the product of the lowest eigenvalue of $H(\mathcal{P}_k,\mathcal{Q}_k)$ and 2θ. Note that 2θ is the minimal interval during which it makes sense to measure the profit [3]. Except the ground state all the adiabatic strategies $H(\mathcal{P}_k,\mathcal{Q}_k)|\psi\rangle = const|\psi\rangle$ are giffens [5]. Future investigations may reveal the situations foe which the existence of giffens lead to optimal strategies.

It should be noted here that in a general case the operators $\mathcal{Q}_k$ do not commute because traders observe moves of other players and often act accordingly. One big bid can influence the market at least in a limited time spread. Therefore it is natural to consider noncommutative quantum mechanics [9] where one considers

$$[x^i, x^k] = i\Theta^{ik} := i\Theta\epsilon^{ik}.$$

The analysis of harmonic oscillator in more then one dimensions [10] imply that the parameter Θ modifies the constant $\hbar_E \to \sqrt{\hbar_E^2 + \Theta^2}$ and, accordingly, the eigenvalues of $H(\mathcal{P}_k,\mathcal{Q}_k)$. This has the natural interpretation that moves performed by other players can diminish or increase one's inclination to taking risk.

Correlated coherent strategies We will define correlated coherent strategies as the eigenvectors of the annihilation operator $\mathcal{C}_k$ [11]

$$\mathcal{C}_k(r,\eta) := \frac{1}{2\eta}\Big(1+\frac{ir}{\sqrt{1-r^2}}\Big)\mathcal{Q}_k + i\eta\mathcal{P}_k,$$

where r is the correlation coefficient $r\in[-1,1]$, $\eta>0$. In these strategies buying and selling transactions are correlated and the product of dispersions fulfills the Heisenberg-like uncertainty relation $\Delta_p\Delta_q\sqrt{1-r^2}\geq\frac{\hbar_E}{2}$ and is minimal. The annihilation operators $\mathcal{C}_k$ and their eigenvectors may be parameterized by $\Delta_p=\frac{\hbar_E}{2\eta}$, $\Delta_q=\frac{\eta}{\sqrt{1-r^2}}$ i r This leads to following form of the correlated Wigner coherent strategy

$$W(p,q)dpdq = \frac{1}{2\pi\Delta_p\Delta_q\sqrt{1-r^2}}\, e^{-\frac{1}{2(1-r^2)}\big(\frac{(p-p_0)^2}{\Delta_p^2}+\frac{2r(p-p_0)(q-q_0)}{\Delta_p\Delta_q}+\frac{(q-q_0)^2}{\Delta_q^2}\big)}dpdq.$$

They are not giffens. It can be shown, following Hudson [12], that they form the set of all pure strategies with positive definite Wigner functions. Therefore pure strategies that are not giffens are represented in phase space $\{(p,q)\}$ by gaussian distributions.

Mixed states and thermal strategies According to classics of game theory [13] the biggest choice of strategies is provided by the mixed states $\rho(p,q)$. Among them the most interesting are the thermal ones. They are characterized by constant inclination to risk, $E(H(\mathcal{P},\mathcal{Q}))=const$ and maximal entropy. The Wigner measure for the n-th exited state of harmonic oscillator have the form [5]

$$W_n(p,q)dpdq = \frac{(-1)^n}{\pi\hbar_E}\, e^{-\frac{2H(p,q)}{\hbar_E\omega}} L_n(\frac{4H(p,q)}{\hbar_E\omega})dpdq,$$

where L_n is the n-th Laguerre polynomial. The mixed state ρ_β determined by the Wigner measures $W_n dpdq$ weighted by the Gibbs distribution $w_n(\beta):=\frac{e^{-\beta n\hbar_E\omega}}{\sum_{k=0}^{\infty}e^{-\beta k\hbar_E\omega}}$ have the form

$$\begin{aligned}\rho_\beta(p,q)dpdq: &= \sum_{n=0}^{\infty} w_n(\beta)W_n(p,q)dpdq\\ &= \frac{\omega}{2\pi}\, x\, e^{-xH(p,q)}\Big|_{x=\frac{2}{\hbar_E\omega}\tanh(\beta\frac{\hbar_E\omega}{2})}dpdq.\end{aligned}$$

So it is a two dimensional normal distribution. It easy to notice that by recalling that $\frac{1}{1-t}\,e^{\frac{xt}{t-1}} = \sum_{n=0}^{\infty} L_n(x)t^n$ is the generating function for the Laguerre polynomials. It seems to us that the above distributions should determine the shape of the supply and demand curves for equilibrium markets. There is no giffens on such markets. It would be interesting to investigate the temperatures of equilibrium markets. In contrast to the traders temperatures [7] which are Legendre coefficients and measure "trader's qualities" market temperatures are related to risk and are positive. The Feynman path integrals may be applied to the Hamiltonian Eq. (12) to obtain equilibrium quantum Bachelier model of diffusion of the logarithm of prices of shares that can be completed by the Black-Scholes formula for pricing European options [14].

Market cleared by quantum computer When the algorithm $\mathcal{A}$ calculating in a separable Hilbert space H_k does not know the players strategies it must choose the basis in an arbitrary way. This may result in arbitrary long representations of the amplitudes of strategies. Therefore the algorithm $\mathcal{A}$ should be looked for in the NP (non-polynomial) class and quantum markets may be formed provided the quatum computation technology is possible. Is the quantum arbitrage possible only if there is a unique correspondence between $\hbar$ and h_E? Penrose ideas concerning the thinking process suggest searching for new physical phenomena (e.g. giffens) on markets, where now the clearing algorithms $\mathcal{A}$ are constructed in a non-computational way [15]. Was the hypothetical evidence given by Robert Giffen in the British Parliament [16] the first ever description of quantum phenomenon? The market exchange mechanism has inspired the so called transactional interpretation of quantum mechanics [17]. The commonly accepted universality of quantum theory should encourage physicist in looking for traces quantum world in social phenomena.

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