THE CORE OF A PARTITION FUNCTION GAME

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Abstract. We consider partition function games and introduce new definitions of the core that include the effects of externalities. We assume that all players behave rationally and that all stable outcomes arising are consistent with the appropriate generalised concept of the core. The result is a recursive definition of the core where residual subgames are considered as games with fewer players and with a partition function that captures the externalities of the deviating coalition. Some properties of the new concepts are discussed.

1. Introduction

The core has become one of the most popular solutions in coalitional game theory [1, 2, 7, 12, 13]. Informally, the core is the set of undominated outcomes. An outcome is not in the core if there is a coalition that can profitably deviate from it. The core is determined by the inspection of the characteristic function.

Recently there is a strong revival of interest in games in the more general partition function form [6, 10, 16, 18]. In partition function form games (PFGs), introduced by Thrall and Lucas [15], coalitional payoffs are defined as a function of the entire coalition structure or *partition*.

In PFGs a deviation by a coalition or a set of coalitions typically affects the payoffs of the residual players thus invoking a response from them. Such a response can change the worth of the deviation dramatically, as the externalities, in general, go both ways making the definition domination a lot less obvious. Already the introductory paper [15] defined domination in this new context and their definition is what we will call later as the (classical) pessimistic approach where it is assumed that in a deviation residual players minimise the payoff of the *deviators*. This idea has been used in most of the early papers [8, 9] together with its complement, the optimistic approach where residuals maximise the deviators' payoffs [14, 17]. Funaki and Yamato [6] use both and find that the two approaches lead to contradictory conclusions.

Cornet [4] in his summary describes a further one; the status quo approach assumes that residual players do not react, which is clearly inconsistent with the partition function form often used explicitly to model externalities.

Tulkens and Chander [16] use a pre-determined residual partition assuming that the residual coalitions break up to singletons. D'Aspremont, Jacquemin, Gabszewicz and Weymark [5] assume exactly the opposite: when players leave a cartel the remaining members stay together. These being the most extreme cases, in the present paper we allow residuals to chose any of *their* most preferred partitions.

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Our model therefore generalises these approaches. In another respect it is a refinement of the optimistic/pessimistic cores and we will also incorporate the status quo approach to some extent.

We propose a new pair of optimistic and pessimistic approaches, but together, as an "interval" rather than as individual concepts[1]. These differ crucially from previous definitions in the treatment of the residual players. In our model they are equally rational players who maximise their payoff in the same fashion as the deviators. The result is a range that is much narrower than before, in particular, a previously empty optimistic core may have elements in the new model.

The refinement will be done in two steps. After the introduction of some basic concepts and the necessary notation, we give our first definition of dominance. Here we assume that as a reaction to a deviation residuals form an outcome that is a member of the core in the residual subgame. In our second definition we also use the status quo, that is, pre-deviation partition of the residual players and this extra piece of information enables us to reduce the set of possible payoffs arising as results of the deviations further.

2. Basic definitions

Let $N = \{1, \ldots, n\}$ be a set of players. Nonempty subsets of N are called *coalitions*. A *partition* $\mathcal{P}$ is a set of disjoint coalitions, $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$, so that their union is N. The *set of partitions* is Π the set of partitions of a given subset S of N is $\Pi(S)$.

The *partition function*

$$\begin{array}{rcl} V : 2^N \times \Pi & \longrightarrow & \mathbb{R} \\ (S, \mathcal{P}) & \longmapsto & V(S, \mathcal{P}) \, \text{whereS} \in \mathcal{P}. \end{array} \tag{2.1}$$

is a mapping that assigns a value to each coalition in every partition.

Given a game (N, V) an *outcome* is an ordered pair $(x, \mathcal{P})$, where $x \in \mathbb{R}^N$ denotes the *vector of payment allocations* x_i to the individual players $i \in N$ and $x_S \in \mathbb{R}^S$ its restriction to $S \subseteq N$. The vector of payment allocations must satisfy a feasibility constraint namely that given a partition $\mathcal{P}$, for all $S \in \mathcal{P}$ we have $\sum_{i \in S} x_i \leq V(S, \mathcal{P})$.

If $x, y \in \mathbb{R}^N$ and $S \subseteq N$ we write $x >_S y$ to say that for all $i \in S$, $x_i \geq y_i$ and there exists $i \in S$ such that $x_i > y_i$. For better transparency we will drop some brackets when writing out partitions and thus write the partition $\{\{1, 2, 5\}, \{3, 4\}\}$ as $(125, 34)$. This should lead to no confusion. If a is a partition of players in set N_1 and B is a set of partitions of the player set N_2 such that $N_1 \cap N_2 = \emptyset$ then let $a \cup B = \{a \cup b \,|b \in B\}$.

Cornet [4, p32,37] defines the *pessimistic and the optimistic core.* He gives several definitions of domination under the two approaches and we of those we use A3 and B3, respectively:

An outcome $(x, \mathcal{P})$ is dominated via the coalition $S \subseteq N$ if

- **A3:** (pessimistic approach) there exists a partition $\mathcal{P}_S$ of S, such that for all partitions $\mathcal{P}' \supseteq \mathcal{P}_S$ there exists a feasible outcome $(x', \mathcal{P}')$ such that $x' >_S x$.
- **B3:** (optimistic approach) there exists a feasible outcome $(x', \mathcal{P}')$, such that $\mathcal{P}' \supseteq \mathcal{P}_S$ for some partition $\mathcal{P}_S$ of S and $x' >_S x$.

The pessimistic approach follows the spirit of the α-core of Aumann and Peleg [2] defined in the context of NTU CFGs, if we regard the partition formed by a set of

[1]The use of the word "interval" will be justified later when we show that the sets in question satisfy certain inclusion relations, and so among them we can define a lower and an upper bound.

players as its strategy. In this context the β-core is less appealing, but is certainly far less optimistic than the above optimistic approach.

Definition 2.1 (Residual Game)**.** Let (N, V) be a game. Let S be a coalition and R be its complement in N. Let $\mathcal{S}$ be a partition of S. The residual game $(R, V_{\mathcal{S}})$ is the game over the player set R and with the partition function

$$V_{\mathcal{S}} : 2^R \times \Pi \longrightarrow \mathbb{R} \qquad (2.2)$$
$$(Q, \mathcal{R}) \longmapsto V(Q, \mathcal{R} \cup \mathcal{S}),$$

where $\mathcal{R}$ is a partition of R and $Q \in \mathcal{R}$.

The residual game resembles to but is different from the concept of *reduced game* especially in the form given by Moulin [11]. Reduced games are influenced by the the payoff-structure of the larger game, but on the other hand do not account for the significance of the partition of the "deleted" set S that influences the residual payoffs in PFGs.

3. A recursive definition of the core

We define the *core* by induction[2] on the number of players. It is indeed a pair of definitions consisting of two core concepts based on optimistic and pessimistic assumptions regarding the residual behaviour. We denote these by C_- and C_+ respectively.

Definition 3.1. Let (N, V) be a game. The *core* of a game with $N = \{1\}$ is $C_+(\{1\}, V) = C_-(\{1\}, V) = \{(V(1,(1)),(1))\}$.

Assuming that the core is defined for every game consisting of at most $k-1$ players we define the core of a game of k players in two steps:

(1) *Dominance* is defined in a game of k players under pessimistic and optimistic assumptions giving rise to a pair of concepts of the inductive core.
   Worst case scenario: If there exists a partition $\mathcal{S}$ of a subset S of N, such that either
   (a) $C_-(R, V_{\mathcal{S}}) = \emptyset$ and *for all* partitions $\mathcal{Q} \supseteq \mathcal{S}$ there exists a payoff vector $y >_S x$, or
   (b) $C_-(R, V_{\mathcal{S}}) \neq \emptyset$ and *for all* partitions $\mathcal{Q} = \mathcal{S} \cup \mathcal{R}$ that satisfy $(y_R, \mathcal{R}) \in C_-(R, V_{\mathcal{S}})$, moreover there exists a payoff vector y such that $y >_S x$
   then the outcome $(y, \mathcal{Q})$ dominates $(x, \mathcal{P})$.
   Best case scenario: If there exists a partition $\mathcal{S}$ of a subset S of N, such that either
   (a) $C_+(R, V_{\mathcal{S}}) = \emptyset$ and *there exists* a partition $\mathcal{Q} \supseteq \mathcal{S}$ and a payoff vector $y >_S x$, or
   (b) $C_+(R, V_{\mathcal{S}}) \neq \emptyset$ and *there exists* a partition $\mathcal{Q} = \mathcal{S} \cup \mathcal{R}$ such that $(y_R, \mathcal{R}) \in C_+(R, V_{\mathcal{S}})$, and a payoff vector y such that $y >_S x$
   then the outcome $(y, \mathcal{Q})$ dominates $(x, \mathcal{P})$.

(2) The *core* of a game of k players is the set of undominated outcomes.

3.1. **Notes and properties.** There are two new ideas in this definition. Firstly, we consider residual games, that is, subgames restricted to some subset of the players, but still experiencing the externalities of the deviating partition. Secondly, we assume that the residual players play a game similar to the "big" game and hence end up in a set of outcomes that is similar to the core of the entire game. Since this core is contained in the outcome set of game $(R, V_{\mathcal{S}})$ the externalities exerted by $\mathcal{R}$ may cause less extreme payoffs for the coalitions in $\mathcal{S}$ and hence for

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[2]Our concept has no connection with the recursive core of Becker and Chakrabakti [3] where the recursion is in time: the core is updated as the game is played.

the individual players in S (We prove this later). Consequently our pessimistic approach is less pessimistic than A3 of Cornet, and our optimistic approach is less optimistic than B3 of Cornet. Before going any further we show that our definitions indeed generalise the core of a characteristic function game.

Lemma 3.2. *The core of a PFG is a generalisation of the core of a characteristic function form game (CFG).*

Proof. We show that if the partition function is of the form of a characteristic function, that is, there are no externalities, then the pessimistic and optimistic cores are identical to the core of a CFG with the given characteristic function. This proves the lemma.

The proof is by induction.

For a game containing a single player our lemma is true.

Assuming that the result has been shown for all games with at most $k-1$ players, we consider a game of k players. We consider an outcome $(x, \mathcal{P})$ that is dominated via $\mathcal{S}$ by $(y, \mathcal{Q})$ via partition $\mathcal{S} \subseteq \mathcal{Q}$. Since the game contains no externalities the deviation does not affect the payoff of the residual players. The worth of the deviation is independent of the behaviour of the residual players and hence the optimistic and pessimistic approaches give identical sets.

Further we note that the payoffs of the deviating coalitions in S do not depend on each other, and so if there exist rational deviations, then there exists a coalition $S \subseteq N$ so that a deviation by S is profitable. This is identical to the classical definition of the core. □

Cornet has already shown that $C_{B3} \subseteq C_{A3}$. While this relation is easy to see, the recursive definition makes the relation of the optimistic and the pessimistic core less plausible.

Theorem 3.3. *The pessimistic contains the optimistic core, that is,*

$$C_+ \subseteq C_-. \tag{3.1}$$

Proof. The proof is by induction in the number of players in a game.

For a game of a single player we have $C_+ = C_-$ and so 3.1 is satisfied.

Assuming that $C_+ \subseteq C_-$ is satisfied for all games with at most $k-1$ players, we consider a deviation $\mathcal{S}$ from an outcome $(x, \mathcal{P})$ in a game of k players. As the deviation includes at least one player, the residual game consists of at most $k-1$ players. The residual players, as a reaction, form a core-outcome in the residual game whenever this is possible and if not then form any other outcome. In either case by our assumption the best/worst case residual outcome sets $\Pi_+(R)$ and $\Pi_-(R)$ satisfy

$$\Pi_+(R) \subseteq \Pi_-(R). \tag{3.2}$$

We define a function that orders deviations by their profitability as a function of the deviating partition as well as the residual partition. A deviation may include several coalitions and so such an ordering is not obvious. However, as we only want to know whether a given deviation is profitable or not under a certain residual behaviour, the function

$$\begin{aligned} W : \Pi(S) \times \Pi(R) &\longrightarrow \mathbb{R} \\ (\mathcal{S}, \mathcal{R}) &\longmapsto W(\mathcal{S}, \mathcal{R}) = \min_{P \in \mathcal{S}} \left\{ V(P, \mathcal{S} \cup \mathcal{R}) - \sum_{i \in P} x_i \right\} \end{aligned} \tag{3.3}$$

is suitable. Then for the given deviation $\mathcal{S}$

$$\min_{\mathcal{R} \in \Pi_-(R)} W(\mathcal{S}, \mathcal{R}) \leq \min_{\mathcal{R} \in \Pi_+(R)} W(\mathcal{S}, \mathcal{R}) \leq \max_{\mathcal{R} \in \Pi_+(R)} W(\mathcal{S}, \mathcal{R}) \tag{3.4}$$

A profitable deviation in the pessimistic case it is profitable in the optimistic case, too, hence for the game of k players we can conclude that the set of pessimistic undominated outcomes contains that of the pessimistic ones. $\square$

Since $\Pi_+(R) \subseteq \Pi_-(R) \subseteq \Pi(R)$, using the same function W, we can show that given an outcome $(x, \mathcal{P})$ and a deviation $\mathcal{S}$:

$$\min_{\mathcal{R} \in \Pi(R)} W(\mathcal{S}, \mathcal{R}) \leq \min_{\mathcal{R} \in \Pi_-(R)} W(\mathcal{S}, \mathcal{R}) \tag{3.5}$$

and similarly

$$\max_{\mathcal{R} \in \Pi_+(R)} W(\mathcal{S}, \mathcal{R}) \leq \max_{\mathcal{R} \in \Pi(R)} W(\mathcal{S}, \mathcal{R}). \tag{3.6}$$

Therefore we have the following result:

Lemma 3.4. *The inductive core is a refinement of the classical optimistic - pessimistic core-pair:*

$$C_{B3} \subseteq C_+ \subseteq C_- \subseteq C_{A3}. \tag{3.7}$$

In fact we can provide examples where the inclusions $C_{B3} \subseteq C_+$ and $C_- \subseteq C_{A3}$ are strict.

3.2. **Example.** We define a game of 8 players. We summarise the coalitional payoffs in the following table (the rest of the payoffs are zero) :

<table>
<tr><td colspan="2">1, 2, 3</td><td colspan="2">4, 5, 6</td><td colspan="2">7, 8</td></tr>
<tr><td>partition</td><td>payoff</td><td>partition</td><td>payoff</td><td>partition</td><td>payoff</td></tr>
<tr><td>1, 2, 3</td><td>0,0,0</td><td>any</td><td>100</td><td>any</td><td>100</td></tr>
<tr><td rowspan="3">123</td><td rowspan="3">1</td><td>4, 5, 6</td><td>1,1,1</td><td>7, 8
78</td><td>2,2
0</td></tr>
<tr><td>456</td><td>3</td><td>7, 8
78</td><td>0,0
6</td></tr>
<tr><td>45, 6
56, 4
64, 5</td><td>2,1</td><td>7, 8
78</td><td>1,1
2</td></tr>
<tr><td>other</td><td>0</td><td>any</td><td>0</td><td>any</td><td>0</td></tr>
</table>

Before starting to solve the game, notice that the payoff of 1, 2 and 3 is independent of others' actions. Similarly the payoffs of 4, 5 and 6 do not depend on the partition of 7 and 8. These properties strongly simplify the calculations.

First look for the classical pessimistic core, C_{A3}. Since at all partitions any derivation carries the risk of becoming 0 for some partition of the residuals.

$$C_{A3} = \left\{ (x, \mathcal{P}) \middle| \forall P \in \mathcal{P} \in \Pi^f, \sum_{i \in P} x_i = V(P, \mathcal{P}) \right\}. \tag{3.8}$$

In the classical optimistic case, if $(1, 2, 3) \subset \mathcal{P}$ they will benefit from merging, and in every other case all of deviations by $\{4, 5, 6\}$, or $\{7, 8\}$ carry the possibility of achieving 100 and so no outcomes are undominated.

$$C_{B3} = \emptyset. \tag{3.9}$$

For inductive cores we observe that the following residual cores are identical in the optimistic and the pessimistic case:

$$\begin{aligned} C(\{1,2,3\},V_{\mathcal{Q}}) &= \left\{ (x,\mathcal{P}) \;\middle|\; \begin{array}{c} (123) \\ x_1+x_2+x_3=1 \\ x_1,x_2,x_3\geq 0 \end{array} \right\} \\ C(\{1,2,3,4,5,6\},V_{\mathcal{S}}) &= \left\{ (x,\mathcal{P}) \;\middle|\; \begin{array}{c} (123)\cup\Pi(\{4,5,6\}) \\ x_1+x_2+x_3=1 \\ x_1,x_2,x_3\geq 0 \\ x_4=x_5=x_6=1 \end{array} \right\}, \end{aligned}$$

where $\mathcal{Q}$ and $\mathcal{S}$ are partitions of $\{4,5,6,7,8\}$ and $\{7,8\}$ respectively.

Then in the optimistic case we have

$$C_+ = \left\{ (x,\mathcal{P}) \;\middle|\; \begin{array}{c} (123,456,78) \\ x_1+x_2+x_3=1 \\ x_1,x_2,x_3\geq 0 \\ x_4=x_5=x_6=1 \\ x_7+x_8=6 \\ x_7\in[2,4] \end{array} \right\} \tag{3.10}$$

In the pessimistic case we expect the core to be at least as large as this. We have:

$$\begin{aligned} C_- = \Bigg\{ (x,\mathcal{P}) \;\Bigg|\; & \begin{array}{c} (123,4,5,6,7,8) \\ x_1+x_2+x_3=1 \\ x_1,x_2,x_3\geq 0 \\ x_4=x_5=x_6=1 \\ x_7=x_8=2 \end{array} \cup \begin{array}{c} (123,456,78) \\ x_1+x_2+x_3=1 \\ x_1,x_2,x_3,x_7,x_8\geq 0 \\ x_4=x_5=x_6=1 \\ x_7+x_8=6 \end{array} \\ & \cup \begin{array}{c} (123,ij,k,78) \\ \{i,j,k\}=\{4,5,6\} \\ x_1+x_2+x_3=1 \\ x_1,x_2,x_3,x_7,x_8\geq 0 \\ x_4=x_5=x_6=1 \\ x_7+x_8=2 \end{array} \cup \begin{array}{c} \{123,ij,k,7,8\} \\ \{i,j,k\}=\{4,5,6\} \\ x_1+x_2+x_3=1 \\ x_1,x_2,x_3\geq 0 \\ x_4=x_5=x_6=1 \\ x_7=x_8=1 \end{array} \Bigg\}. \end{aligned} \tag{3.11}$$

Indeed we find that

$$C_{B3}\subseteq C_+\subseteq C_-\subseteq C_{A3}.$$

4. Core

In the previous section we have defined the core in a way that has some very nice internal consistency: The core is an equilibrium concept and in the case of a deviation the rest of the players form a new outcome obeying the same equilibrium concept. We, however have another piece information, yet unused, namely the *status quo* residual partition: the partition that remains when the deviating players are deleted. It is clear that this partition, that we denote by $\mathcal{R}_0$ influences the set of outcomes that can be formed. For instance if the status quo is undominated, no other outcomes can arise. Of course, this definition is not an improvement if all residual cores are empty. We define the *core of a partition function game* inductively over the number of players.

Definition 4.1. Let (N,V) be a game. The *core* of a game with $N=\{1\}$ is $C_+(\{1\},V)=C_-(\{1\},V)=\{(V(1,(1)),(1))\}$. The outcome $(y,(1))$ *dominates* $(x,(1))$ if $y>x$. In this trivial case the definition of *sequential domination* is identical to that of domination.

Assuming that the core and the dominance relation have been defined for every game consisting of at most $k-1$ players, we define the core of a game of k players in three steps:

(1) *Dominance* is defined in a game of k players under pessimistic and optimistic assumptions giving rise to a pair of concepts of the core.
   Worst case scenario: If there exists a partition $\mathcal{S}$ of a subset S of N, such that either
   (a) $C_-(R, V_\mathcal{S}) = \emptyset$ and *for all* partitions $\mathcal{Q} \supseteq \mathcal{S}$ there exists a payoff vector $y >_S x$, or
   (b) $C_-(R, V_\mathcal{S}) \neq \emptyset$ and *for all* partitions $\mathcal{Q} = \mathcal{S} \cup \mathcal{R}$ that satisfy $(y_R, \mathcal{R}) \in C_-(R, V_\mathcal{S})$, moreover there exists a payoff vector y such that $(y_R, \mathcal{R})$ sequentially dominates $(x_R, \mathcal{R}_0)$ and $y >_S x$
   then the outcome $(y, \mathcal{Q})$ dominates $(x, \mathcal{P})$.
   Best case scenario: If there exists a partition $\mathcal{S}$ of a subset S of N, such that either
   (a) $C_+(R, V_\mathcal{S}) = \emptyset$ and *there exists* a partition $\mathcal{Q} \supseteq \mathcal{S}$ and a payoff vector $y >_S x$, or
   (b) $C_+(R, V_\mathcal{S}) \neq \emptyset$ and *there exists* a partition $\mathcal{Q} = \mathcal{S} \cup \mathcal{R}$ such that $(y_R, \mathcal{R}) \in C_+(R, V_\mathcal{S})$, and a payoff vector y such that $(y_R, \mathcal{R})$ sequentially dominates $(x_R, \mathcal{R}_0)$ and $y >_S x$
   then the outcome $(y, \mathcal{Q})$ dominates $(x, \mathcal{P})$.

(2) We say that the outcome $(y, \mathcal{R})$ *sequentially dominates* $(x, \mathcal{P})$ if there exists a sequence of outcomes $(x, \mathcal{P}) = (x_0, \mathcal{P}_0), \ldots, (x_k, \mathcal{P}_k) = (y, \mathcal{R})$ such that for all $1 < i \leq k$ $(x_i, \mathcal{P}_i)$ dominates $(x_{i-1}, \mathcal{P}_{i-1})$.

(3) The core of a game of k players is the set of undominated outcomes of the game.

4.1. **Notes and properties.** The difference between the core and the inductive core is in the introduction of sequential dominance. Players are assumed to be myopic so that sequential domination is only possible via a sequence of (direct) dominations.

Lemma 4.2. *The above defined concept gives a generalisation of the core of a CFG.*

Proof. The proof is similar to that of Lemma 3.2 □

Lemma 4.3. *The core is refinement of the inductive core and it satisfies*

$$C_{B3} \subseteq C_+ \subseteq C_+^* \subseteq C_-^* \subseteq C_- \subseteq C_{A3}. \tag{4.1}$$

Proof. The result is established the same way as Lemma 3.4, so we only sketch the required steps. First we prove $C_+^* \subseteq C_-^*$ as in Theorem 3.3. Then prove $C_+ \subseteq C_+^*$ and $C_-^* \subseteq C$ using the same technique as in the proof of Lemma 3.4. Then, using Lemma 3.4 itself the result follows. □

4.2. **Example.** We return to our example in 3.2, and calculate the core both in the optimistic and in the pessimistic case. The calculations are similar to our previous ones except that residuals are only expected to move to a new outcome that is in the conditional core and dominates the status quo. We get:

$$C_+^* = C_-^* \quad = \quad \left\{ (x,\mathcal{P}) \;\middle|\; \begin{array}{c} \{123,4,5,6,7,8\} \\ x_1+x_2+x_3=1 \\ x_1,x_2,x_3 \geq 0 \\ x_4=x_5=x_6=1 \\ x_7=x_8=2 \end{array} \right. \cup$$

$$\left. \begin{array}{c} \{123,456,78\} \\ x_1+x_2+x_3=1 \\ x_1,x_2,x_3,x_7,x_8 \geq 0 \\ x_4=x_5=x_6=1 \\ x_7+x_8=6 \end{array} \cup \begin{array}{c} \{123,ij,k\} \cup \mathcal{P}(\{7,8\}) \\ i,j,k \in \{4,5,6\} \\ x_1+x_2+x_3=1 \\ x_1,x_2,x_3 \geq 0 \\ x_4=x_5=x_6=x_7=x_8=1 \end{array} \right\} \tag{4.2}$$

Comparing this result to the previous one we find that

$$C_{B3} \subseteq C_+ \subset C_+^* = C_-^* \subset C_- \subseteq C_{A3}.$$

5. Postscript

The two new pairs of definitions weaken our assumptions about the residual behaviour and at the same time give a refinement of earlier concepts. The internally consistency of the definition is theoretically nice and gives a lot more subtle way to understand partition function games.

Where it does not help is the case of an empty core. All four core concepts defined here are subsets of the classical pessimistic core. If we generate a characteristic function by the most commonly used method of taking a coalition's minimal obtainable worth in the partition function case to be its coalitional payoff, the pessimistic core is identical to the core of the coalitional game thus created. It is clear then that if the classical pessimistic core is empty, all our definitions produce empty cores.

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