On the Dynamic Foundation of Evolutionary Stability in Continuous Models[*]

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Abstract

We show in this paper that none of the existing static evolutionary stability concepts (ESS, CSS, uninvadability, NIS) is sufficient to guarantee dynamic stability in the weak topology with respect to standard evolutionary dynamics if the strategy space is continuous. We propose a new concept, evolutionary robustness, which is stronger than the previous concepts. Evolutionary robustness ensures dynamic stability for replicator dynamics in doubly symmetric games.

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Key words: replicator dynamics, evolutionary stability, ESS, CSS.

Very preliminary, comments welcome!

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1 Introduction

Continuous models play an important role in all formal sciences although the observed reality is measured in discrete quantities. In economics, prices and quantities are usually modeled as continuous variables although both lie on a finite grid in reality. In game theory, choices of timing, of quality, of effort etc. are frequently assumed to belong to a real interval. Modeling parameters as continuous variables is justified in most cases because first, the continuous model provides a good approximation to reality and, second, it opens the door for the powerful tools of analysis.

In evolutionary game theory, however, models in which strategies are continuous variables are still the exception. Yet, if evolutionary game theory is to be extended beyond the well understood field of games with a small number of strategies, one must be able to deal with continuous models. Important applications will include, for example, bargaining games, games of timing, oligopoly games, public good games, the evolution of preferences and the evolution of markets. The set of possible actions is usually quite a large set and is, therefore, best modelled through a continuous approach.[1]

The first problem encountered in this area is the lack of a convincing static stability concept. In the finite case, one uses the notion of Evolutionarily Stable Strategy (ESS) as a stability concept. In games with a *finite* number of pure strategies the ESS definition has gained strong support through the fact that ESS is a sufficient condition for dynamic stability with respect to the replicator dynamics. Unfortunately, in games with a *continuum* of pure strategies this fact ceases to be true. One of the contributions of the present paper is to propose a static stability concept for continuous evolutionary models, called Evolutionary Robustness ($\mathcal{ER}$), and to provide a dynamic foundation for this concept.

The basic idea of dynamic stability is that a population which is slightly perturbed from the equilibrium should not be driven further away through the evolutionary dynamics. The critical question is which perturbations are

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[1] For some examples of applications with continuous strategy evolution see Bester and Güth [2], Heifetz and Spiegel [9], Huck, Kirchsteiger, and Oechssler [11], Ok and Vega–Redondo [14] or Oechssler and Riedel [13] and the references cited therein.

to be considered "close" to the original equilibrium. Thus, as pointed out by Eshel [7] the whole issue crucially depends on the choice of topology for the space of populations.[2]

In the finite case, the choice of topology is inconsequential since all sensible measures of distance yield the same result: If ε is chosen small enough, then a population P is close to population Q if and only if Q can be written as $Q = (1-\varepsilon)P + \varepsilon R$ for small $\varepsilon > 0$ and some population R. In the case of a continuum of strategies this notion of distance seems unduly restrictive (see also Eshel et al., [8] for this argument). Why should a monomorphic population playing a strategy arbitrarily close to the equilibrium strategy not be considered close to the equilibrium population? To give an example, if the strategy of some animal is the length of its tail, are then two populations, one featuring a tail of length x and the other a length of $x+\varepsilon$, far apart even if ε is arbitrarily small? In economic applications this may be even less convincing.

A further argument in favor of this view could be made as follows. Suppose a population is optimal against a given environment but a small change in the environment occurs such that the entire payoff function – and with it, the optimal action – shifts somewhat. The new optimal strategy may now be ε away from the current population. For stability it seems reasonable to require that the population does not drift away from the new optimum any further.

Based upon these arguments, we choose the weak topology as the way to measure how different two populations are. The weak topology captures two aspects: it allows both for a *large* change of strategic play by a *small* fraction of players as well as a *small* change of strategic play by a *large* fraction of the population.

We show that all previously suggested static definitions – whether ESS, Vickers and Cannings' [20] uninvadability, Bomze's [3] strong uninvadability, Eshel and Motro's [5] continuously stable strategy (CSS), or Apaloo's [1] neighborhood invader strategy (NIS) – fail to guarantee dynamic stability

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[2] We always identify a population with the aggregate play of its members, i.e. with the probability measure over the set of (pure) strategies.

(in the above sense) in the continuum case. If one accepts our notion of distance, then one also has to accept the fact that all known static evolutionary concepts are insufficient to guarantee even Lyapunov stability with respect to the replicator dynamics.

The concept we introduce, Evolutionary Robustness or $\mathcal{ER}$, is a natural generalization of ESS for the continuous model with the weak topology – and in fact, in the finite case, the two notions coincide. A population is called evolutionarily robust if it obtains a higher than average payoff against all possible populations which are close in the sense of the weak topology. Hence, the set of states to check is larger than required, e.g., by uninvadability. To give an example. Suppose that the strategy set is the unit interval and the homogeneous population which places all mass on zero is the equilibrium candidate. For uninvadability, one must only check the populations in which a large part of the population still plays zero whereas a small fraction does something else. For $\mathcal{ER}$, one must additionally check populations in which every player uses a different strategy which is close to zero but not equal to zero. This latter requirement is more in the spirit of the concepts CSS and NIS proposed in the biology literature. In fact, $\mathcal{ER}$ implies uninvadability as well as CSS and NIS and unifies, in this sense, the previously suggested concepts.

Bomze [3] has established a weaker kind of dynamic stability for (strongly) uninvadable populations. He shows that replicator dynamics converge weakly to the uninvadable population if the set of admissible starting points for the replicator dynamics is restricted to populations having small cross entropy relative to the equilibrium population. For homogeneous equilibrium populations, this means that admissible initial populations must be close to the equilibrium population in the variational topology, whereas dynamics converge in the weak topology. The same approach is used in our own previous work [13]. Although this is a very weak kind of stability, it has the advantage of allowing to work with discontinuous payoff functions. In contrast, if one adopts the weak topology, then payoff functions must be continuous.

Seymour [15] shows that replicator dynamics converge in the variational topology to the equilibrium population if the payoff function has a discontinuous upward jump at the equilibrium strategy. There must be a positive gap between the equilibrium payoff and all other competing strategies in order to ensure this strong type of stability.

A continuous model should only be used if it closely approximates the discrete model as the number of strategies grows large. We show that this is satisfied in our case. When the parameter space is discrete and strategies lie on a finite grid, the corresponding discrete replicator dynamics converge to the continuous replicator dynamics in the weak topology as grid size vanishes. In particular, this result shows that one can use numerical procedures in order to study the behavior of the continuous model.

The paper is organized as follows. In the next section we introduce the evolutionary dynamics. Section 3 describes the static evolutionary concepts which have so far been proposed for the continuous model. In Section 4 we then show by examples that the proposed concepts do not ensure dynamic stability with respect to the weak topology. In Section 5 we suggest our new concept of evolutionary robustness, show that this implies all previous concepts, and establish dynamic stability for doubly symmetric games. Finally, in Section 6 we show that the continuous model is indeed the limit of the discrete model if the grid size of the strategy set vanishes.

2 Evolutionary dynamics

We consider symmetric two–player games with (Borel) strategy set $S \subset \mathbb{R}$ and continuous payoff function $f(x, y)$. A *population* is identified with the aggregate play of its members and is described by a probability measure P on the measure space $(S, \mathcal{B})$, where $\mathcal{B}$ denotes the Borel σ–algebra on S. We denote by $\Delta(S)$ the set of all populations (probability measures or mixed strategies) on S, and by $\rho(P, Q)$ the distance of two populations measured with respect to the weak topology,

$$\rho(P,Q) := \inf \left\{ \begin{array}{c} \varepsilon > 0 : \forall s \in S,\ Q\left((-\infty, s-\varepsilon]\right) - \varepsilon \leq \\ P\left((-\infty, s]\right) \leq Q\left((-\infty, s+\varepsilon]\right) + \varepsilon \end{array} \right\}.$$

Thus, population Q is considered ε–close to a Dirac measure with mass on u, δ_u, if the mass on both, $(-\infty, u-\varepsilon)$ and $(u+\varepsilon, \infty)$, is at most ε.

Throughout the paper we assume that the average payoff of population P against population Q is bilinear in P and Q, i.e.

$$E(P,Q) = \int_S \int_S f(x,y)Q(dy)P(dx)\,. \tag{1}$$

Since f is continuous, the bilinear functional of expected payoffs, $E(P,Q)$, is continuous with respect to the weak topology.

We assume that evolutionary dynamics[3] are regular ($\dot{P}(S) = 0$) and payoff monotonic, i.e. that (set of) types with higher average fitness have higher growth rates ($\hat{P}(A) := \dot{P}(A)/P(A)$ denotes the growth rate of $P(A)$), or formally

Definition 1 *A regular dynamic is called* payoff monotonic *if for all A and $A' \in \mathcal{B}$*

$$\hat{P}(A) > \hat{P}(A')$$

if and only if

$$\frac{1}{P(A)} \int_A E(\delta_x, P)P(dx) > \frac{1}{P(A')} \int_{A'} E(\delta_x, P)P(dx).$$

Let P^* be an equilibrium point of the dynamics, that is $\dot{P}^*(A) = 0$ for all Borel subsets A of S. P^* is called *Lyapunov stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all initial populations $Q(0)$ with $\rho(Q(0), P^*) < \delta$ we have $\rho(Q(t), P^*) < \varepsilon$ for all $t > 0$. P^* is called *asymptotically stable* if it is Lyapunov stable and there exists $\varepsilon > 0$ such that for all initial populations $Q(0)$ with $\rho(Q(0), P^*) < \varepsilon$ we have $\rho(Q(t), P^*) \to 0$.

A particular type of dynamics which we shall consider below as a special case are the well known replicator dynamics (Taylor and Jonkers [17]). Let

$$\sigma(x,P) := E(\delta_x, P) - E(P,P)$$

denote the differential fitness of pure strategy x when matched against population P, i.e. the difference between the payoff of strategy x and the average population payoff. The idea of the replicator dynamics is that the relative

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[3] That evolutionary dynamics can be meaningfully defined in the continuous context is shown in Bomze [3] and Oechssler and Riedel [13].

increment of the frequency of a set of strategies is given exactly by the average differential fitness of strategies in that set. Formally, the replicator dynamics for the continuous case are defined as

$$\dot{P}(t)(A) = \int_A \sigma\left(x, P\right) \; P(t)(dx). \tag{2}$$

Note that by taking $A = \{x\}$ we get the usual formulation of the replicator dynamics for the finite strategy case.

3 Static evolutionary concepts

The classical definition of an ESS (Maynard Smith [12]) requires that for all mutant populations R there exists an ε such that the original population P does better against the mixed population $(1-\varepsilon)P + \varepsilon R$ than R does. In this definition some invasion barrier exists for each R.

Vickers and Cannings [20] strengthened the definition by requiring that there be a uniform invasion barrier for all mutant populations R. This requirement, which was later termed *uninvadability*, is stronger than ESS if we leave the realm of finite matrix games (see Vickers and Cannings [20]).

Definition 2 (uninvadability) *A population P is called* uninvadable *if there is a uniform invasion barrier, that is, an $\varepsilon > 0$ such that*

$$E(P, (1-\eta)P + \eta Q) > E(Q, (1-\eta)P + \eta Q) \tag{3}$$

holds for all $Q \neq P$ and all $0 < \eta \leq \varepsilon$.

Yet a stronger definition was introduced by Bomze [3]. *Strong uninvadability* requires a uniform invasion barrier for all populations Q close to P in the variational norm. The latter definition coincides with uninvadability if the equilibrium population state is homogeneous (see Lemma 1 of Oechssler and Riedel, [13]).

In the above definitions a population is considered close to a homogeneous population $P = \delta_u$ if the mass on u remains at least $1-\varepsilon$. Elsewhere, we have shown [13] (see also Bomze [3]) that if this definition of "closeness" is employed, uninvadability is a sufficient condition for dynamic stability. However, as we have argued above the variational norm is unduly restrictive and the weak topology should be used instead. This view was first advocated by Eshel and Motro [5] who introduced the following definition.

Definition 3 (CSS) *A strategy u is a* continuously stable strategy (CSS) *if (1) it is a ESS and (2) there exists an $\varepsilon > 0$ such that for all v with $|v-u| < \varepsilon$ there exists an $\eta > 0$ such that for all x with $|v-x| < \eta$*

$$f(v,x) > f(x,x) \text{ if and only if } |v-u| < |x-u|.$$

A slightly stronger version of CSS (2) was later termed m^*–stability by Taylor [18] and convergence stability by Christiansen [4]. As shown by Eshel [6] if f is twice differentiable, a necessary condition for an ESS u to be a CSS is that

$$f_{xx}(u,u) + f_{xy}(u,u) \leq 0. \tag{4}$$

Condition (4) is sufficient if the weak inequality is replaced by a strict one.

Finally, a similar concept was introduced by Apaloo [1].

Definition 4 (NIS) *A strategy u is a* neighborhood invader strategy (NIS) *if there exists an $\varepsilon > 0$ such that for all x with $|x-u| < \varepsilon$*

$$f(u,x) > f(x,x).$$

It can easily be checked that a necessary condition for u to be a NIS is that

$$f_{xx}(u,u) + 2f_{xy}(u,u) \leq 0. \tag{5}$$

Again, condition (5) is sufficient if the weak inequality is replaced by a strict one.

Remark 1 *In the following, we will sometimes also say that a homogeneous population δ_u is CSS or NIS, meaning that the corresponding strategy u is CSS (or NIS, respectively). The fact that CSS and NIS are only defined for pure strategies, and that a natural extension to mixed strategies is not immediate, is certainly a drawback of those definitions.*

In continuous models, there are two aspects of mutations. On the one hand, an equilibrium population must do well against the mutation that a small fraction of the population plays a different strategy. This is the aspect which is emphasized by uninvadability. On the other hand, an equilibrium population should also do well against the mutation by which the whole population is shifted slightly. The latter is the point stressed by CSS and NIS. The concept $\mathcal{ER}$ we propose below captures both aspects.

4 Why stronger conditions are needed

A static evolutionary stability concept makes sense only if it presents a short-cut to a truly dynamic analysis. The following two examples will demonstrate that none of the above static stability conditions are strong enough to guarantee Lyapunov stability with respect to the replicator dynamics.

Example 1 *Consider a game with strategy set* $S = \mathbb{R}$ *and payoff function* $f(x, y) = -x^2 + 4xy$. *The pair* $(0, 0)$ *is a strict Nash equilibrium, and hence the homogeneous population* δ_0 *is an ESS. Furthermore, it is uninvadable. However, it is not Lyapunov stable with respect to the replicator dynamics.*

Proof. That $(0, 0)$ is a strict Nash equilibrium is obvious. Denoting by $\mu(Q)$ the mean value of Q and by $V(Q)$ its variance, we see that δ_0 is also uninvadable since $E(\delta_0, (1-\varepsilon)\delta_0 + \varepsilon Q) = 0$ and

$$\begin{aligned} E(Q, (1-\varepsilon)\delta_0 + \varepsilon Q) &= (1-\varepsilon)E(Q, \delta_0) + \varepsilon E(Q, Q) \\ &= 4\varepsilon\mu(Q)^2 - \mu(Q)^2 - V(Q). \end{aligned}$$

Hence, (3) is always satisfied for $\varepsilon < 1/4$.

Strategy $u = 0$ is, however, not a CSS because

$$f_{xx}(0, 0) + f_{xy}(0, 0) > 0.$$

Next, we apply the replicator dynamics. Suppose we start with an initial population $Q(0)$ which is normally distributed with mean $\mu(0) > 0$ and variance $V(0) > 0$. In the Appendix we show that all $Q(t)$ will also be normally distributed with some mean $\mu(t)$ and variance $V(t)$. This makes it possible to reduce the infinite–dimensional replicator dynamics to a two–dimensional ordinary differential equation since normal distributions are characterized through their mean and variance. The ODEs for the mean value $\mu(t)$ and the variance $V(t)$ are

$$\begin{aligned} \dot{\mu}(t) &= 2\mu(t)V(t) \\ \dot{V}(t) &= -2V^2(t). \end{aligned}$$

Starting from any initial population $Q(0)$ with positive mean value, which can be arbitrarily close to δ_0 in the weak topology, it is easy to see that the mean $\mu(t)$ diverges to infinity. Hence, δ_0 is not Lyapunov stable. A formal proof of this statement is given in the Appendix. ■

Example 2 *Consider a game with strategy set* $S = \mathbb{R}$ *and payoff function* $f(x,y) = -x^2 + x^2y^2$. *Despite* δ_0 *being an ESS, a CSS, and a NIS, it is not Lyapunov stable with respect to any payoff monotone dynamic.*

Proof. Since $(0,0)$ is a strict equilibrium, δ_0 is an ESS. Furthermore, since $f_{xx}(0,0) = -2$ and $f_{xy}(0,0) = 0$, it is also a CSS and a NIS.

However, it is not uninvadable because for all $\varepsilon > 0$ there exists an x such that with $Q = (1-\varepsilon)\delta_0 + \varepsilon\delta_x$,

$$E(Q,Q) = \varepsilon(1-\varepsilon)f(x,0) + \varepsilon^2 f(x,x) = \varepsilon^2 x^4 - \varepsilon x^2 > 0 \qquad (6)$$

but $E(\delta_0, Q) = 0$.

To see that it is not Lyapunov stable with respect to any payoff monotone dynamic consider a starting population $Q = (1-\varepsilon)\delta_0 + \varepsilon\delta_x$, which is ε–close to δ_0. Since $E(\delta_x, Q) > E(\delta_0, Q)$ for all x such that (6) is satisfied, the mass on x increases to one. ■

Examples 1 and 2 show that the existing static evolutionary concepts fail to guarantee dynamic stability. Example 1 shows that uninvadability misses the point stressed by CSS and NIS, namely that the pure equilibrium strategy must do well also against pure strategies close to it. Example 2 shows that CSS or NIS alone miss the uniform invasion barrier property required by uninvadability. An appropriate static stability concept must therefore bring these two points together.

We conclude this section with an example showing that the number of uninvadable populations can be uncountable with a continuum of strategies.

Example 3 *Consider a game with strategy set* $S = [-1, 1]$ *and payoff function* $f(x, y) = -x^2 + 2xy$. *The best reply function is* $r(y) = y$, *hence there are uncountably many strict Nash equilibria* (x, x), *hence uncountably many ESS. The populations* δ_x *are even uninvadable since for* $Q = (1 - \eta)\delta_x + \eta R$ *we have for* $R \neq \delta_x$

$$\begin{aligned} E(\delta_x, Q) - E(R, Q) &= (1-\eta)\, x^2 + \eta\left(-x^2 + 2x\mu(R)\right) \\ &\quad - (1-\eta)\left(-V(R) - \mu^2(R) + 2x\mu(R)\right) \\ &\quad -\eta\left(-V(R) + \mu^2(R)\right) \\ &= (1 - 2\eta)\,(x - \mu(R))^2 + \eta V(R) > 0. \end{aligned}$$

5 Evolutionary robustness

Now we introduce the concept which we propose to use with continuous strategy spaces.

Definition 5 *A population* $P^* \in \Delta(S)$ *is* evolutionarily robust *($\mathcal{ER}$) if there exists an invasion barrier* $\varepsilon > 0$ *such that for all* $Q \neq P^*$ *with* $\rho(Q, P^*) < \varepsilon$ *we have* $E(P^*, Q) > E(Q, Q)$.

Note that $\mathcal{ER}$ is based on the same condition as uninvadability. However, the set of ε–close populations Q is much larger since the weak topology is used instead of the variational topology. As we have mentioned above, $\mathcal{ER}$ captures two aspects of stability against mutations. An evolutionarily robust population is strong against *large* deviations by a *small* fraction of the population on the one hand and against *small* shifts by a *large* fraction of the population on the other hand.

We show next that $\mathcal{ER}$ is in fact stronger than the previously defined concepts. First, since a population which is close in the variational norm is also close in the weak topology, we clearly have the following

Proposition 1 *If* P^* *is* $\mathcal{ER}$*, then it is strongly uninvadable and, hence, uninvadable and an ESS.*

Since CSS and NIS are only defined for pure strategies, we consider only pure $\mathcal{ER}$ in the following propositions.

Proposition 2 *Let* δ_u *be a homogeneous* $\mathcal{ER}$*. Then* u *is a NIS.*

Proof. Follows directly from the definition of $\mathcal{ER}$ by setting $Q = \delta_x$. ■

Thus, $\mathcal{ER}$ implies ESS and NIS and, as we show next, those two combined imply CSS for most twice differentiable payoff functions.

Proposition 3 *Let* u *be a NIS and a ESS. If* f *is twice continuously differentiable with* $f_{xx}(u,u) \neq 0$ *or* $f_{xy}(u,u) \neq 0$*, then* u *is also CSS.*

Proof. NIS states that the function $h(y) = E(\delta_u, \delta_y) - E(\delta_y, \delta_y) = f(u,y) - f(y,y)$ has a strict local minimum in u. Thus, $h^{'}(u) = 0$ and $h^{''}(u) \geq 0$, or equivalently

$$f_x(u,u) = 0$$

and

$$f_{xx}(u,u) + 2f_{xy}(u,u) \leq 0.$$

Since (u,u) is an ESS, we have $f_{xx}(u,u) \leq 0$. If $f_{xx}(u,u) < 0$, then

$$2f_{xx}(u,u) + 2f_{xy}(u,u) < 0,$$

which is a sufficient condition for CSS (see (4)).

If $f_{xy}(u,u) > 0$ then, we have also $f_{xx}(u,u) < 0$, thus as before

$$2f_{xx}(u,u) + 2f_{xy}(u,u) < 0.$$

If $f_{xy}(u,u) < 0$, then because of $f_{xx}(u,u) \leq 0$, $f_{xx}(u,u) + f_{xy}(u,u) < 0$. ■

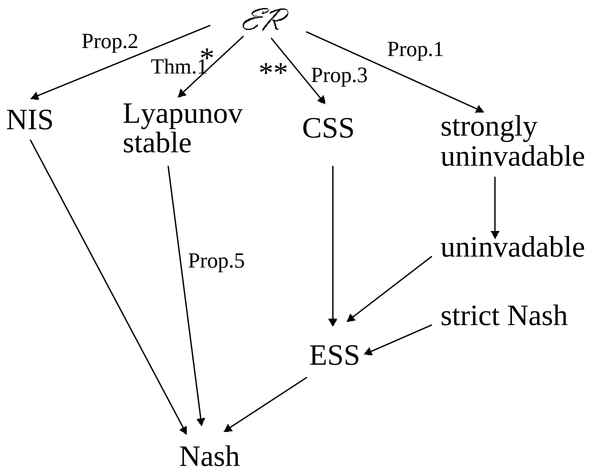

Figure 1: A familiy tree of stability concepts
An arrow means implication, * for doubly symmetric games, ** for payoff functions with $f_{xx} \neq 0$ or $f_{xy} \neq 0$.

Example 4 *There are populations that satisfy uninvadability and CSS but not* $\mathcal{ER}$. *For* $f(x, y) = -x^2 + xy$, δ_0 *is an uninvadable CSS which is not* $\mathcal{ER}$.

Proof. It is straightforward to check that 0 is an uninvadable CSS. But 0 is not NIS, hence it cannot be $\mathcal{ER}$. ■

Figure 1 summarizes the relations between the various stability concepts in a "familiy tree".

It is well known from the finite case that ESS need not exist. The same holds obviously true for $\mathcal{ER}$. On the other hand, ESS has the advantage to be a very strong refinement of the Nash equilibrium concept. In particular, there are at most finitely many ESS is finite models. This appealing property holds no longer true for ESS and uninvadability in the continuous model, see Example 3, but it does hold true for the proposed concept $\mathcal{ER}$.[4]

Proposition 4 *If S is compact, then the number of $\mathcal{ER}$ is finite.*

Proof. Suppose not. Then there is, due to compactness of Δ in the weak topology, a sequence of $\mathcal{ER}$ (P^n), which converges weakly to a $\mathcal{ER}$ P. For large n, P^n is arbitrarily close to P, and since P is $\mathcal{ER}$, we must therefore have $E(P, P^n) > E(P^n, P^n)$. On the other hand, since every P^n is a Nash equilibrium, $E(P^n, P^n) \geq E(P, P^n)$, a contradiction. ■

5.1 Stability of $\mathcal{ER}$ in symmetric games

The main problem in providing a general proof for dynamic stability is that the standard Lyapunov function for the finite case, namely cross entropy, is not well defined in the continuous case unless densities exist. Moreover, cross entropy is not continuous with respect to the weak topology. Nevertheless, we state the following as a conjecture which is based, in particular, on our inability to find a counter example.

Conjecture 1 *$\mathcal{ER}$ is a sufficient condition for dynamic stability.*

In the following, we prove this conjecture for a special case. The general case seems to require different techniques. We restrict ourselves to the case of (doubly) symmetric payoff functions. Let S be compact and assume that $f : S \times S \to \mathbb{R}$ is continuous and $f(x, y) = f(y, x)$. Symmetry implies that $E(P, Q) = E(Q, P)$ for all P, Q. As in the finite case, we are able to prove that the *Fundamental Law of Natural Selection* holds in this case.[5]

Let P^* be $\mathcal{ER}$ and $\Lambda(Q) = E(P^*, P^*) - E(Q, Q)$ the difference between mean fitness in the equilibrium population and mean fitness in some arbitrary population Q. As the next theorem shows, Λ constitutes a Lyapunov function for the replicator dynamics.

Theorem 1 *Let P^* be $\mathcal{ER}$. The function $\Lambda(Q)$ is a Lyapunov function with respect to the replicator dynamics. Thus, P^* is Lyapunov stable.*

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[4] See e.g. Van Damme [19], Lemma 9.2.6, for the equivalent result in the finite case.

[5] The law states that average fitness of a population increases over time. For the finite case see e.g. Hofbauer and Sigmund [10, p.226].

Proof. see Appendix. ■

Remark 2 *Lyapunov's Theorem is usually stated for* $\mathbb{R}^n$ *only and cannot simply be extended to the space of measures on* $(S, \mathcal{B})$*. For example, Lyapunov's method does not work with the variational norm because the unit ball is not compact in this topology. With the weak topology, Lyapunov's Theorem holds true because the set of populations* $\Delta(S)$ *is compact in this topology. We prove Lyapunov's theorem for the present case in the Appendix.*

From the finite case it is well known (see e.g. Weibull, 1995) that the converse to the above Theorem need not be true. As in the finite case a necessary condition for a population P to be Lyapunov stable is that (P, P) is a Nash equilibrium.

Proposition 5 *If* P *is Lyapunov stable in some payoff monotone dynamic, then* (P, P) *is a Nash equilibrium.*

Proof. The proof is equivalent to the finite case (see e.g. Prop. 4.8 of Weibull, 1995). ■

5.2 Asymptotic stability for homogeneous $\mathcal{ER}$

For homogeneous $\mathcal{ER}$ we can strengthen the previous result to obtain asymptotic stability for the case that the initial population Q_0 places some (if small) mass on the equilibrium population.

Let δ_0 be $\mathcal{ER}$ and stable. Choose $\delta > 0$ such that when we start in $Q(0)$ with $\rho(Q(0), \delta_0) < \delta$, then $\rho(Q(t), \delta_0) < \varepsilon$ for all t where $\varepsilon > 0$ is the invasion barrier.

Theorem 2 *A homogeneous, Lyapunov stable* $\mathcal{ER}$ δ_0 *is asymptotically stable in the following sense. If* $Q(0)(\{0\}) > 0$ *and* $\rho(Q(0), \delta_0) < \delta$*, then*

$$\rho(Q(t), \delta_0) \rightarrow_{t \rightarrow \infty} 0.$$

Proof. By stability we know that we stay close to δ_0. Hence,

$$\dot{Q}(t)(\{0\}) = \sigma(0, Q(t))\, Q(t)(\{0\}) \geq 0.$$

Therefore, the weight on 0 increases, thus the equicontinuous function $Q(t)(\{0\})$ converges to some $q > 0$. This in turn implies that $\dot{Q}(\{0\})$ goes to zero. Therefore, also $\sigma(0, Q(t)) \to 0$.

Since $\Delta(S)$ is compact in the weak topology, every sequence $(Q(t_n))$ with $t_n \to \infty$ has accumulation points. Let $\overline{Q}$ be such an accumulation point. Since σ is continuous with respect to the weak topology, it follows that $\sigma\left(0, \overline{Q}\right) = 0$. By stability, $\overline{Q}$ is close to δ_0. Since δ_0 is $\mathcal{ER}$, this implies $\overline{Q} = \delta_0$. Since the accumulation point of every converging subsequence of $(Q(t_n))$ is equal to δ_0, the trajectory converges to δ_0. ■

Remark 3 *Note that double symmetry of the fitness function f is not required for Theorem 2 to hold.*

6 Finite approximation of an infinite strategy space

If the continuous model is to be used, it must be a good approximation to discrete models when the number of strategies grows large. We show that this is in fact the case in this section. We consider the following discretization of the strategy space $S = [a, b] \subset \mathbb{R}$ for arbitrary fixed grid size $1/n$, $n \in \mathbb{N}$.[6]

$$S_n := \left\{ \frac{k}{n} : \frac{k}{n} \in S, k \in \mathbb{Z} \right\}.$$

The payoff function f can be approximated, for example, by the following step function.

$$f_n(x, y) = f(\frac{k}{n}, \frac{l}{n}) \text{ if } x \in [\frac{k}{n}, \frac{k+1}{n}) \text{ and } y \in [\frac{l}{n}, \frac{l+1}{n}).$$

If the payoff function is continuous, the approximations f_n converge uniformly to f as $n \to \infty$. As the following theorem shows, the corresponding replicator dynamics converge also.

---

[6] In this section, we consider only compact strategy spaces. Under some additional boundedness assumption, the results can be extended to half-spaces and the whole real line.

Theorem 3 *Let* $f : S \times S \to \mathbb{R}$ *be bounded and* $f_n : S \times S \to \mathbb{R}$ *be a sequence with* $\|f - f_n\|_\infty := \sup_{x,y \in S} |f(x,y) - f_n(x,y)| \to_{n\to\infty} 0$. *Denote by* $Q_n(t)$ *the replicator dynamics induced by* f_n, *and by* $Q(t)$ *the replicator dynamics induced by* f. *Assume furthermore that all replicator dynamics start with the same population* $P \in \Delta(S)$. *Then* Q_n *converges in variational norm to* Q *uniformly on compact sets:*

$$\sup_{0\le t\le T} \|Q(t) - Q_n(t)\| \to_{n\to\infty} 0.$$

Proof. see Appendix. ■

Theorem 3 yields a very strong type of convergence. If the discretization is made fine enough, then the two trajectories of the replicator dynamics are arbitrarily close together. Note, that convergence in variational norm, i.e. convergence on every set $A \in \mathcal{B}$, of course implies weak convergence.

We can also consider what happens when we vary the initial condition P (for example, when the support of P does not belong to the set of grid points). This is done in the next theorem.

Theorem 4 *Let* $f : S \times S \to \mathbb{R}$ *be bounded and Lipschitz continuous. Assume* $P_n \to P$ *in the weak topology. Denote by* $Q_n(t)$ *the replicator dynamics starting in* P_n, *and by* $Q(t)$ *the replicator dynamics starting in* P. *Then* Q_n *converges in the weak topology to* Q *uniformly on compact sets:*

$$\sup_{0\le s\le T} \rho\left(Q(t), Q_n(t)\right) \to_{n\to\infty} 0.$$

Proof. See Appendix. ■

Theorems 3 and 4 taken together show that the continuous case is indeed the limit of successively finer approximations. Note, that the results do not depend on the particular choice of the grid. They are true for any approximation by a partition of S as long as the mesh of the partition goes to zero.

In light of the approximation results there seems to be an apparent contradiction between the stability of strict equilibria in the finite case and our counter examples from the previous section (in which strict equilibria are unstable). To see that this is not really a contradiction let us review a very simple proof for the fact that a strict equilibrium population δ_u is asymptotically stable for finite S_n. Since $f_n(u,u) > f_n(x,u)$ for all $x \neq u$, we have that $f_n(u,u) > E_n(R,\delta_u)$ for all $R \neq \delta_u$. The trick is that for any fixed grid size $1/n$ we can choose ε small enough such that only populations $Q = (1-\eta)\delta_u + \eta R$ with $\eta \leq \varepsilon$ are ε–close to δ_u. By choosing ε sufficiently small, we can find an $\omega > 0$ such that

$$\begin{aligned} E_n(\delta_u, Q) - \omega &= (1-\eta) f_n(u,u) + \eta E_n(\delta_u, R) - \omega > \\ (1-\eta)E_n(R,\delta_u) + \eta E_n(R,R) &= E_n(Q,Q). \end{aligned}$$

Hence, with any payoff monotone dynamic $Q(\{u\})$ will grow exponentially faster than any other strategy and asymptotic stability is assured. Note the important role played by the possibility to choose ε small enough for given n such that only populations Q with at least mass $(1-\varepsilon)$ are close to δ_u. It would certainly be desirable to be able to choose a uniform $\varepsilon > 0$ for all grid sizes n. However, this can not be done in general. Another way of expressing this is by saying that the basin of attraction of a strict equilibrium may vanish as $n \to \infty$.

Definition 6 *The basin of attraction of a population δ_u is given as*

$$B(\delta_u) := \{Q(0) \in \Delta : \rho(Q(t), \delta_u) \to 0\}.$$

The size of the basin is measured as

$$\sup\{\varepsilon : \forall Q(0) \in \Delta \textit{ with } \rho(Q(0), \delta_u) \leq \varepsilon,\ Q(0) \in B(\delta_u)\}.$$

Example 5 *Consider the discretization of the payoff function in Example 1. The size of the basin of attraction of the strict equilibrium population δ_0 vanishes as $n \to \infty$.*

Suppose the basin does not vanish. Then there exists a sequence of $Q_n(0)$ which converges to the Normal distribution P with $\mu(0), V(0) > 0$ and $Q_n(0) \in B(\delta_0)$. Yet, as shown in Example 1, the replicator dynamics starting from P diverge to infinity, which yields a contradiction to Theorems 3 and 4. ■

Appendix

A Example 1

First, we give a rigorous proof of the claims in Example 1.

Lemma 1 *The class of normal distributions is invariant with respect to the replicator dynamics in games with the payoff function $f(x,y) = -x^2 + 2axy$, for some parameter a.*

Proof. Simple calculations show that the fitness differential is

$$E(\delta_x, Q) - E(Q,Q) = -x^2 + 2ax\mu - \left((2a-1)\mu^2 - V\right),$$

where μ denotes the mean of population Q, and V its variance. By Theorem 6 of Oechssler and Riedel [13], the density of $Q(t)$ with respect to the initial distribution is, under replicator dynamics,

$$\frac{dQ(t)}{dQ(0)}(x) = \exp\left(\int_0^t E(\delta_x, Q(s)) - E(Q(s), Q(s))ds\right)$$

$$= \exp\left(-x^2 t + 2ax\int_0^t \mu(Q(s))ds - \int_0^t \left((2a-1)\mu\left(Q(s)\right)^2 - V\left(Q(s)\right)\right)ds\right).$$

If $Q(0)$ is a normal distribution, then $Q(t)$ is also a normal distribution, because the density of $dQ(t)/dQ(0)$ can be written in the form $\exp(\alpha x^2 + \beta x + \gamma)$. ■

Next we calculate the ODE's of the mean and the variance for the game of Example 1. By the preceding lemma, we know that $Q(t)$ is a normal distribution if $Q(0)$ is normal, that is $Q(t) = N(\mu(t), V(t))$ for some mean value $\mu(t)$ and variance $V(t)$. If $Q = N(\mu, V)$ is a normal distribution, then the fitness differential is

$$\sigma(x,Q) = E(\delta_x, Q) - E(Q,Q) = -x^2 + 4x\mu - 3\mu^2 + V.$$

The derivative of $\mu(t)$ is therefore

$$\dot{\mu}(t) = \int x\dot{Q}(t)(dx)$$

$$
\begin{aligned}
&= \int xE(\delta_x, Q(t)) - E(Q(t), Q(t))(dx) \\
&= \int \left(-x^3 + 4x^2\mu(t) - 3x\mu^2(t) + xV(t)\right) Q(t)(dx).
\end{aligned}
$$

Since the third moment of the normal distribution $Q(t)$ is

$$m_3(t) = \int x^3 Q(t)(dx) = \mu(t)^3 + 3\mu(t)V(t),$$

it follows that

$$\dot{\mu}(t) = 2\mu(t)V(t). \tag{7}$$

By a similar calculation, using the fact that the fourth moment of the normal distribution is $m_4(t) = 3V^2(t)+6\mu^2(t)V(t)+\mu^4(t)$, one obtains for the second moment $m_2(t) = \int x^2 Q(t)(dx)$ the ODE

$$
\begin{aligned}
\dot{m}_2(t) &= \int x^2 \sigma\left(x, Q(t)\right) dx \\
&= \int \left(-x^4 + 4x^3\mu(t) - 3x^2\mu^2(t) + x^2V(t)\right) Q(t)(dx) \\
&= -m_4(t) + 4m_3(t)\mu(t) - 3m_2(t)\mu^2(t) + m_2(t)V(t) \\
&= -3V^2(t) - 6\mu^2(t)V(t) - \mu^4(t) + 4\left(\mu^3(t) + 3\mu(t)V(t)\right)\mu(t) \\
&\quad -3\left(V(t) + \mu^2(t)\right)\mu^2(t) + \left(V(t) + \mu^2(t)\right)V(t) \\
&= -2V^2(t) + 4\mu^2(t)V(t).
\end{aligned}
$$

Since the variance is given by $V(t) = m_2(t) - \mu^2(t)$, we get for the variance the dynamics

$$\dot{V}(t) = \dot{m}_2(t) - 2\dot{\mu}(t)\mu(t) = -2V^2(t). \tag{8}$$

The solution of (8) and (7) is

$$
\begin{aligned}
V(t) &= \frac{V(0)}{1 + 2V(0)t}, \\
\mu(t) &= \mu(0)\left(1 + 2V(0)t\right).
\end{aligned}
$$

This shows that for every initial normally distributed population with positive mean and positive variance, the mean diverges to infinity. Since every neighborhood of the uninvadable state δ_0 contains such normal distributions, the uninvadable state δ_0 is not Lyapunov stable.

B Lyapunov's Theorem

In this part of the appendix, we prove Lyapunov's theorem for the present case. Denote by $\mathcal{M}$ the vector space of all finite measures on the measurable space $(S, \mathcal{B}(S))$. Let P^* be a $\mathcal{ER}$ with invasion barrier $\varepsilon > 0$. Let

$$\begin{aligned} R &: \mathcal{M} \longrightarrow \mathcal{M} \\ &: Q \longmapsto \left(A \longmapsto \int_A \sigma(x, Q) Q(dx) \right) \end{aligned}$$

the function which defines replicator dynamics. Let $\Lambda : \mathcal{M} \longrightarrow \mathbb{R}$ be continuously differentiable with respect to the weak topology. We denote by $\nabla\Lambda$ the gradient of Λ. $\nabla\Lambda(Q)$ is a linear functional on $\mathcal{M}$ which we denote by $\langle \nabla\Lambda(Q), \cdot \rangle$.

Λ is called a Lyapunov function if it satisfies in an ε-neighborhood of P^*

1) $\Lambda(Q) \geq 0$ and $\Lambda(Q) = 0 \Rightarrow Q = P^*$,

2) $\langle \nabla\Lambda(Q), R(Q) \rangle \leq 0$.

Proposition 6 *If a Lyapunov function Λ exists, then P^* is a stable state for the replicator dynamics.*

Proof. The proof is exactly analogous to the proof in the finite dimensional case. We just highlight the point where our choice of the weak topology plays an important role.

Let $\eta > 0$ be given, and without loss of generality, $\eta \leq \varepsilon$. The ball $K_\eta = \{Q \in \mathcal{M} : \rho(Q, P^*) \leq \eta\}$ is compact in the weak topology. Therefore, $m = \min_{Q \in \partial K_\eta} \Lambda(Q)$ exists and is strictly positive (this is the step which does not work with the variational norm). Now choose $\delta > 0$ such that $\rho(Q, P^*) < \delta$ implies $\Lambda(Q) < m$. If we start closer than δ to P^*, we can never leave K_η. This establishes stability. ■

Note that $\Lambda(Q) = E(P^*, P^*) - E(Q, Q)$ is a continuous bilinear functional, hence continuously differentiable. The above theorem provides therefore a rigorous foundation for the stability theorem of Section 4.

C Proof of Theorem 1

In order to show that Λ is a Lyapunov function, we have to establish that for Q with $\rho(Q, \delta_u) \leq \varepsilon$

- $\Lambda(Q) \geq 0$, and $\Lambda(Q) = 0$ if and only if $Q = P^*$,
- and that $\Lambda(Q(t))$ decreases along trajectories induced by replicator dynamics as long as we stay close to P^*, or equivalently, $\langle \nabla\Lambda(Q), R(Q)\rangle \leq 0$ in the ε-neighborhood of P^*.

For the first claim, note that for Q close to P^*,

$$\begin{aligned} \Lambda(Q) &= E(P^*, P^*) - E(Q, Q) \\ &= E(P^*, P^*) - E(P^*, Q) + E(P^*, Q) - E(Q, Q) \\ &\geq E(P^*, P^*) - E(P^*, Q) \\ &= E(P^*, P^*) - E(Q, P^*) \\ &\geq 0. \end{aligned}$$

The second and the third equalities follow from symmetry of f. Note that the first inequality becomes strict unless $Q = P^*$. Finally, the second inequality is implied by the fact that every $\mathcal{ER}$ is a symmetric Nash equilibrium.

For the second assertion, note that due to symmetry

$$\langle \nabla\Lambda(Q), R(Q)\rangle = -E\left(R(Q), Q\right) - E\left(Q, R(Q)\right) = -2E\left(R(Q), Q\right).$$

But

$$\begin{aligned} E\left(R(Q), Q\right) &= \int\int f(x, y)\sigma\left(x, Q\right) Q(dx)Q(dy) \\ &= \int\int f(x, y)\sigma\left(x, Q\right) Q(dy)Q(dx) \\ &= \int E\left(\delta_x, Q\right)\sigma\left(x, Q\right) Q(dx) \\ &= \int \left(E\left(\delta_x, Q\right) - E(Q, Q)\right)\sigma\left(x, Q\right) Q(dx) \\ &= \int \sigma\left(x, Q\right)^2 Q(dx) \geq 0, \end{aligned}$$

where the fourth equality is due to the fact that $\int \sigma(x,Q)\,Q(dx) = 0$, hence one can introduce the constant $E(Q,Q)$ without changing the value of the expression. ■

D Proof of Theorem 3.

We write $\varepsilon_n = \|f - f_n\|_\infty$ and $\sigma_n(x,Q) = \int f_n(x,y)Q(dy) - \iint f_n(x,y)Q(dy)Q(dx)$ during the proof. Recall that the variational norm for probability measures is given by

$$\|P - Q\| := 2\sup_{A\in\mathcal{B}} |P(A) - Q(A)|\,.$$

The proof is based upon the inequality

$$|\sigma(x,P) - \sigma_n(x,Q)| \le L\left(\varepsilon_n + \|P-Q\|\right) \tag{9}$$

for a positive constant L. The inequality is proved below.

For all $A \in \mathcal{B}$ we have that

$$|Q(t)(A) - Q_n(t)(A)| = \left|\int_0^t \dot{Q}(s)(A)ds - \int_0^t \dot{Q}_n(s)(A)ds\right|$$

$$\begin{aligned}
&= \left|\int_0^t \int_A \sigma(x,Q(s))Q(s)(dx)ds - \int_0^t \int_A \sigma_n(x,Q_n(s))Q_n(s)(dx)ds\right| \\
&\le \int_0^t \int_A |\sigma(x,Q(s)) - \sigma_n(x,Q_n(s))|\,Q(s)(dx)ds \\
&\quad + \left|\int_0^t \int_A \sigma_n(x,Q_n(s))\left[Q(s)(dx) - Q_n(s)(dx)\right]ds\right|
\end{aligned}$$

Using inequality (9) and the fact that $\sigma_n(x,Q_n(s))$ is bounded in x by some constant M, we conclude

$$\begin{aligned}
|Q(t)(A) - Q_n(t)(A)| &\le \int_0^t L\left(\varepsilon_n + \|Q(s) - Q_n(s)\|\right) + M\,\|Q(s) - Q_n(s)\|\,ds \\
&\le (L+M)\int_0^t \left(\varepsilon_n + \|Q(s) - Q_n(s)\|\right)ds. \qquad (10)
\end{aligned}$$

Setting $\pi(t) = \varepsilon_n + \|Q(t) - Q_n(t)\|$, we obtain $\pi(t) \leq \varepsilon_n + (L+M)\int_0^t \pi(s)ds$. By Gronwall's Lemma, $\pi(t) \leq \varepsilon_n \exp((L+M)t) \leq \varepsilon_n \exp((L+M)T)$ for $t \leq T$. Since (10) holds for all $A \in \mathcal{B}$, we have that

$$\sup_{0 \leq t \leq T} \|Q(t) - Q_n(t)\| \leq \varepsilon_n \left(\exp((L+M)T) - 1\right) \rightarrow_{n\rightarrow\infty} 0,$$

as required. It remains to prove inequality (9).

$$\begin{aligned} |\sigma(x,P) - \sigma_n(x,Q)| \leq & \left| \int f(x,y)P(dy) - \int f_n(x,y)Q(dy) \right| \\ & + \left| \int f(x,y)P(dy)P(dx) - \int f_n(x,y)Q(dy)Q(dx) \right| \end{aligned}$$

$$\begin{aligned} \leq & \int |f(x,y) - f_n(x,y)|\, P(dy) + \int |f_n(x,y)|\, [P(dy) - Q(dy)] \\ & + \int |f(x,y) - f_n(x,y)|\, P(dy)P(dx) \\ & + \left| \int |f_n(x,y)|\, [P(dy) - Q(dy)]\, [P(dy) - Q(dy)] \right| \\ \leq & \ \varepsilon_n + \|f_n\|_\infty \|P - Q\| + \varepsilon_n + \|f_n\|_\infty \|P - Q\|. \end{aligned}$$

Since the functions f_n are uniformly bounded, inequality (9) follows. ■

E Proof of Theorem 4.

In this proof, we will use the metric $\|\cdot\|_{BL}$, induced by the space of bounded, Lipschitz continuous functions on S. This metric also metrizes the weak topology. For small $\rho(P,Q)$ the following estimate holds(cf. [16, p. 315])

$$2/3\rho^2(P,Q) \leq \|P - Q\|_{BL}. \tag{11}$$

Since f is Lipschitz continuous, we have

$$|\sigma(x,P) - \sigma(x,Q)| \leq L \|P - Q\|_{BL} \tag{12}$$

for some constant $L > 0$.

Let g be a bounded, Lipschitz continuous function on S. Then

$$\left|\int gdQ(t)-\int gdQ_n(t)\right| = \left|\int_0^t \int gd\dot{Q}(s)ds - \int gd\dot{Q}_n(s)ds\right|$$

$$\begin{aligned}
\leq & \left|\int_0^t \int g(x)\sigma(x,Q(s))Q(s)(dx)ds - \int g(x)\sigma(x,Q_n(s))Q_n(s)(dx)ds\right| \\
\leq & \int_0^t \left|\int g(x)\left[\sigma(x,Q(s))-\sigma(x,Q_n(s))\right]Q(s)(dx)\right| ds \\
& + \int_0^t \left|\int g(x)\sigma(x,Q_n(s))\left(Q(s)(dx)-Q_n(s)(dx)\right)\right| ds
\end{aligned}$$

Using the fact that g is bounded and Lipschitz continuous together with inequality (12), we conclude that the first term is of the order of $\|Q(s)-Q_n(s)\|_{BL}$. The fact that $\sigma(x,Q)$ is bounded and Lipschitz continuous, implies that the second term is of the order of $\|Q(s)-Q_n(s)\|_{BL}$. Altogether, we conclude that

$$\|Q(t)-Q_n(t)\|_{BL} \leq K\int_0^t \|Q(s)-Q_n(s)\|_{BL}\, ds$$

and the same argument as in the preceding proposition (Gronwall's lemma) yields in conjunction with (11) the claim. ■

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