The Journal of Derivatives

Vacation Options under Stochastic Volatility

Marcos Alexandre Bezerra de Melo
Instituto de Matemática Pura e Aplicada - IMPA
Estrada Dona Castorina, 110 Jardim Botânico 22460-320 Rio de Janeiro, Rio de Janeiro-Brasil.
e-mail: mmelo@impa.br, *Tel*: (55) 21 25547415
Marcos Alexandre Bezerra de Melo is currently enrolled in the Master of Mathematical Finance at IMPA.

ABSTRACT

This paper analyzes the vacation option introduced by Shreve and Večeř [2000] when the volatility follows the generalized markovian process proposed by Hoffman et al [1992] and as an illustration of this model, the classical model of Hull and White [1987] is derived. Finally I also simulate an example of the Hoffman's model where the volatility is past dependent and stochastic and compare with the results obtained by Henderson and Hobson [2000a] for the passport option case. An interesting result is that the common practice of raise prices due to stochastic volatility is not so generally valid for the vacation option.

1. Introduction

The first attempt to model the stock price evolution in the financial market was made by the french mathematician Louis Bachelier. In his thesis, Bachelier [1900], he stated that the stock price evolves according to the following SDE

$$dS_t = \mu dt + \sigma dW_t. \tag{1.1}$$

Although the original character of this model, it has a drastic economic drawback. If W is assumed, by hypothesis, to be a brownian motion and μ and σ are positive constants; it is easy to show that there is a positive probability of the price S been negative. In 1965 Paul Samuelson, see Samuelson [1965], proposed the geometric brownian motion as an alternative to the Bachelier's model. The Samuelson model imposes the stock price to satisfy

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{1.2}$$

The works of Samuelson and Merton, see Merton [1990] and the references therein, on continuous-time finance served as one of the most important basis for the development of the Balck-Scholes seminal paper on option pricing, Black and Scholes [1973].

Since the foundation of the Chicago Board Option Exchanges (CBOE) in 1973 and the publication of Black-Scholes paper, the trading of options contracts have been enormously increasing throughout the years. In accordance with this trading growth, many kinds of financial contracts were developed; such as the exotic options: Asian, Lookback, Barrier and the recent one Passport option.

The passport option was introduced by Bankers Trust in 1997 and published in a paper of Hyer et al [1997]. A passport option is a call option on the wealth of a trading account; where its owner gains the possible profits of the portfolio at the expiry and the seller bears the losses if it occurs. The portfolio is composed, in the simple case, by only one asset that can be traded during the life of the option and restrictions on the number of shares, time between transactions, etc., give raise to many types of passport options, see Anh et al [1998] and Chan [1999]. A straightforward application of this option is on active portfolio management and if we limit the holder to trade only once during the life of the option it can be used, in a suitable sense, for portfolio insurance purposes. In this case the holder's best strategy is to trade at time $t = 0$. See Anh et al [1998] for more details.

In the original paper, Hyer, Lipton-Lifschitz and Pugachevsky derive the Hamilton-Jacobi-Bellman equation for the European passport option, and then obtain a closed form solution when the investment interest rate coincides with the risk-free rate (the well known symmetric case). The subsequent papers, Andersen et al [1998], Anh et al [1998], Penaud et al [1998] and Chan [1999] extend the plain vanilla European passport option to incorporate various exotic features using PDE arguments. In 1998, Nagayama [1998] uses the skorohod theorem to derive the symmetric European passport case. More recently Henderson and Hobson [2000] uses local time to construct a clever relationship between passport and lookback option and Shreve and večeř [2000] derive a general version of options on trading account that englobes most of the cases hereby mentioned.

Notwithstanding the apparent good features of the passport option, it has some practical disadvantages: the price and the transaction costs due to holder's optimum strategy. The vacation option can be seen as a partition of the passport option in the sense that the holder can be long in the stock or zero, for the vacation call; and short or zero, for the vacation put; instead of been short or long in stock as proposed by the passport option. Thus, the lower possibilities of trading provides a lower price and allows the holder to choose the region of trading according to his expectations of the stocks's movements. One of the puzzles of this valuation option's is that the holder strategy is not known *a priori* and so the option writer must be able to hedge against all of the holder's strategy possibilities. If the holder trades optimally he must switch his position in the underlying asset every time his portfolio value crosses zero, but if the portfolio starts at zero the probabilistic model assures it will cross zero infinitely many times in an arbitrarily small interval of time. In spite of the theoretical result suggestion of been "short when ahead, long when behind" instantaneously, the holder always trades suboptimally due to transaction costs. The advantage of the vacation option in this case is to reduce the transaction costs by reducing the range of trading possibilities.

The main purpose of this paper is to analyze the behavior of the vacation option under a rather general markovian stochastic volatility model proposed by Hoffman et al [1992]. The rest of the paper is organized as follows. In section 2, I describe the main results of the Shreve-Večeř model for vacation options, section 3 deals with option pricing under stochastic volatility and the problem of market completeness. In section 4, I proceed with numerical experiments and simulation with crude Monte Carlo method.

2. Vacation Options

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis. Let W be a brownian motion defined on this basis and let $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be the $\mathbb{P}$-augmentation of the filtration generated by W. Suppose the underlying asset follows the SDE

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{2.1}$$

where σ is the volatility of the stock. Generally we can assume S to be an exchange rate, future prices, etc., but for the sake of simplicity lets assume S is a nondividend paying stock. r is the interest rate. Define a portfolio, or strategy π, which value X_t^π on time $t \in [0, T]$ evolves according to

$$dX_t^\pi = \pi_t dS_t + r(X_t^\pi - \pi_t S_t)dt; \quad (2.2)$$

$$X_0^\pi = X_0, \quad (2.3)$$

where X_0 is the initial wealth of the investor. The owner of a generalized option on trading account will receive the payoff $[X_T^\pi]^+$ at time T and the seller should be able to hedge against the holder strategy, whatever this strategy can be. This way, the rational price, V, of this contract at time $t \in [0, T]$ should be

$$V^{[\alpha,\beta]}(t, S_t, X_t) = \sup_{\pi_u \in \{\alpha,\beta\}} e^{-r(T-t)} \mathbb{E}[[X_T^\pi]^+ | \mathcal{F}_t]. \quad (2.4)$$

The varieties of this general model relies on the restrictions imposed on the values that π can assume. $\pi_t = \gamma_t$, where $\{\gamma_t\}_{t\in[0,T]}$ is $\mathcal{F}_{t-1}$-measurable and $\pi_t \in \{\alpha, \beta\}$. π_t is the number of shares held at time t and $\alpha \leq \beta$.

2.1. **The Option Price.** According to Shreve and Večeř [2000] the price of an option on trading account can be stated in the following Theorem

Theorem 2.1. *Assume $\alpha < \beta$, let $0 \leq t \leq T$ be given and let $\overline{\sigma} = \sigma\sqrt{T-t}$. When $0 \leq \alpha < \beta$ or when $\alpha \leq 0 \leq -\alpha \leq \beta$, the value of the option is given by*

$$\begin{aligned} V^{[\alpha,\beta]}(t, S_t, X_t) &= \frac{(\alpha+\beta)^2 s}{4(\beta-\alpha)} - \frac{\alpha x}{\beta-\alpha} - \frac{1}{4}(\beta-\alpha)\overline{\sigma} s d_- + \frac{1}{4\sqrt{2\pi}}(\beta-\alpha)\overline{\sigma} s e^{(\frac{-1}{2}d_-^2)} \\ &+ \frac{\beta}{\beta-\alpha}(x-\alpha s)N(d_+) - \frac{1}{4}(\beta-\alpha)(1-\overline{\sigma} d_-)sN(d_-), \end{aligned}$$

for $x \geq \frac{1}{2}(\alpha+\beta)s$, where

$$d_\pm = \frac{1}{\overline{\sigma}} \log\left(\frac{2}{\beta-\alpha}\left(\frac{x}{s}-\alpha\right)\right) + \frac{1}{\overline{\sigma}} \log\left(\frac{2\beta}{\beta-\alpha}\right) \pm \frac{\overline{\sigma}}{2}$$

and

$$\begin{aligned} V^{[\alpha,\beta]}(t, S_t, X_t) &= x + \frac{1}{4\sqrt{2\pi}}(\beta-\alpha)\overline{\sigma} s e^{(\frac{-1}{2}e_{+-}^2)} \\ &- \frac{\beta}{\beta-\alpha}(x-\beta s)N(-e_{++}) - \frac{1}{2}(\beta-\alpha)(\overline{\sigma} e_{+-}+1)sN(-e_{+-}) \\ &- (x-\beta s)N(e_{-+}) - \beta s N(e_{--}), \end{aligned}$$

for $x \leq \frac{1}{2}(\alpha+\beta)s$, where

$$e_{\pm\pm} = \frac{1}{\overline{\sigma}} \log\left(\frac{2}{\beta-\alpha}\left(\beta - \frac{x}{s}\right)\right) \pm \frac{1}{\overline{\sigma}} \log\left(\frac{2\beta}{\beta-\alpha}\right) \pm \frac{\overline{\sigma}}{2}.$$

When $\alpha < \beta \leq 0$ or when $\alpha \leq 0 \leq \beta \leq -\alpha$, the value of the option is given by

$$\begin{aligned} V^{[\alpha,\beta]}(t,s,x) &= \frac{1}{4\sqrt{2\pi}}(\beta-\alpha)\overline{\sigma}se^{(\frac{-1}{2}d_{+-}^2)} \\ &- \frac{\alpha}{\beta-\alpha}(x-\alpha s)N(-d_{++}) - \frac{1}{4}(\beta-\alpha)(\overline{\sigma}d_{+-}+1)sN(-d_{+-}) \\ &- (x-\alpha s)N(d_{-+}) + \alpha s N(d_{--}), \end{aligned}$$

for $x \geq \frac{1}{2}(\alpha+\beta)s$, where

$$d_{\pm\pm} = \frac{1}{\overline{\sigma}} \log\left(\frac{2}{\beta-\alpha}\left(\frac{x}{s} - \alpha\right)\right) \pm \frac{1}{\overline{\sigma}} \log\left(-\frac{2\alpha}{\beta-\alpha}\right) \pm \frac{\overline{\sigma}}{2}$$

and

$$\begin{aligned} V^{[\alpha,\beta]}(t,s,x) &= \frac{(\alpha+\beta)^2 s}{4(\beta-\alpha)} - \frac{\alpha x}{\beta-\alpha} - \frac{1}{4}(\beta-\alpha)\overline{\sigma}se_{-} + \frac{1}{4\sqrt{2\pi}}(\beta-\alpha)\overline{\sigma}se^{(\frac{-1}{2}e_{-}^2)} \\ &+ \frac{\alpha}{\beta-\alpha}(x-\beta s)N(e_{+}) - \frac{1}{4}(\beta-\alpha)(1-\overline{\sigma}e_{-})sN(e_{-}), \end{aligned}$$

for $x \leq \frac{1}{2}(\alpha+\beta)s$, where

$$e_{\pm} = \frac{1}{\overline{\sigma}} \log\left(\frac{2}{\beta-\alpha}\left(\beta - \frac{x}{s}\right)\right) + \frac{1}{\overline{\sigma}} \log\left(-\frac{2\alpha}{\beta-\alpha}\right) \pm \frac{\overline{\sigma}}{2}.$$

In the above Theorem if $\alpha = \beta = 1$ the option reduces to a European call, $\alpha = \beta = -1$ is a European put and when the owner is allowed to be short or long one share of stock, $\alpha = -1, \beta = 1$ we have the standard passport option. An interesting case occurs when $\alpha = 0, \beta = 1$, the vacation call and when $\alpha = -1, \beta = 0$, the vacation put. For the case of vacation option the above Theorem simplifies to

Corollary 2.2. *The price of a vacation call and put are respectively*

$$\begin{aligned} VC(t,s,x) &= V^{[0,1]}(t,s,x) \\ &= \frac{1}{4}(1-\overline{\sigma}d_{-})sN(-d_{-}) + \frac{1}{4\sqrt{2\pi}}\overline{\sigma}se^{-\frac{1}{2}d_{-}^2} + xN(d_{+}), \end{aligned}$$

for $x \geq \frac{1}{2}s$, where $d_{\pm} = \frac{1}{\overline{\sigma}}\log\left(\frac{4x}{s}\right) \pm \frac{\overline{\sigma}}{2}$ and

$$
\begin{aligned}
VC(t,s,x) &= V^{[0,1]}(t,s,x) \\
&= \frac{1}{4\sqrt{2\pi}}\overline{\sigma} s e^{-\frac{1}{2}e_{+-}^2} - (x-s)N(e_{++}) \\
&- \frac{1}{4}(\overline{\sigma} e_{+-} + 1)sN(-e_{+-} + (x-s)N(-e_{-+} + sN(-e_{--}),
\end{aligned}
$$

for $x \leq \frac{1}{2}s$, *where* $e_{\pm\pm} = \frac{1}{\overline{\sigma}}\log\left(2 - \frac{2x}{s}\right) \pm \frac{1}{\overline{\sigma}}\log 2 \pm \frac{\overline{\sigma}}{2}$.

$$
\begin{aligned}
VP(t,s,x) &= V^{[-1,0]}(t,s,x) \\
&= \frac{1}{4\sqrt{2\pi}}\overline{\sigma} s e^{(-\frac{1}{2}d_{+-}^2)} + (x+s)[N(-d_{++}) + N(d_{-+})] \\
&- \frac{1}{4}(\overline{\sigma} d_{+-} + 1)sN(-d_{+-}) - sN(d_{--}),
\end{aligned}
$$

for $x \geq -\frac{1}{2}s$, *where* $d_{\pm\pm} = \frac{1}{\overline{\sigma}}\log\left(\frac{2x}{s} + 2\right) \pm \frac{1}{\overline{\sigma}}\log 2 \pm \frac{\overline{\sigma}}{2}$
and

$$
\begin{aligned}
VP(t,s,x) &= V^{[-1,0]}(t,s,x) \\
&= \frac{1}{4\sqrt{2\pi}}\overline{\sigma} s e^{-\frac{1}{2}e_{-}^2} + \frac{1}{4}(1 - \overline{\sigma} e_{-})sN(-e_{-}) + xN(-e_{+}),
\end{aligned}
$$

for $x \leq \frac{1}{2}s$, *where* $e_{\pm} = \frac{1}{\overline{\sigma}}\log\left(-\frac{4x}{s}\right) \pm \frac{\overline{\sigma}}{2}$

This example was borrowed from Shreve and večeř [2000]. Suppose the holder strategy is to be long in one share of the stock, $\pi \equiv 1$, and the initial wealth is given by $X_0 = S_0 - e^{-rT}K$ for a given maturity T and strike K. Thus the solution to

$$
\begin{aligned}
dX_t &= rX_t dt + \sigma S_t dW_t, &(2.5)\\
X_0 &= S_0 - e^{-rT}K, &(2.6)
\end{aligned}
$$

is $X_t = S_t - e^{-r(T-t)}K$ and therefore $X_T = S_T - K$. This way if $\pi \equiv 1$ the option on trading account, with a suitable choice of X_0, is a European call option.

The Figure 1 shows the significant difference between the passport and vacation call for different values of the volatility using the European call as a benchmark with $T = 1$,$S_0 = 100$ and $X_0 = 0$. Figure 2 shows the price of the vacation call, European call and passport for changes in the parameters x, T, t and s. The relative lower price of the vacation option relies heavily on the partition of the strategy values. The greater the restriction the lower the price and also the transaction costs in comparison with passport option.

As an immediate consequence of Corollary 2.2, one can establish the put-call parity relation

$$V^{[0,1]}(t, S_t, X_t) - V^{[-1,0]}(t, S_t, X_t) = X_t. \tag{2.7}$$

For details concerning the holder's optimum strategy and the seller's hedge, see Shreve and Večeř [2000]. The holder's optimum strategy is showed to remain optimal even in a stochastic volatility model, see Henderson and Hobson [2000a] for the passport case. Here I will assume that this result still holds for vacation options. Throughout the rest of the paper the optimum holder's strategy is assumed to satisfy $\pi^* = \mathbb{I}_{\{X^{\pi^*} < \frac{1}{2} S_t\}}$ for the vacation call, and $\pi^* = -\mathbb{I}_{\{X^{\pi^*} \geq -\frac{1}{2} S_t\}}$ for the vacation put. Where $\mathbb{I}$ is the indicator function.

3. Stochastic Volatility

Since the publication of the pathbreaking Black-Scholes paper in 1973, certainly one of the most criticized and researched aspects of their valuation formula is the constant character of the volatility, see Geske [1979], Blattberg and Gonedes [1974] and Scott [1987].

The literature on volatility models can be classified in two broad categories. The first approach describes the stock prices as a diffusion with level dependent volatility, see Geske [1979], Cox and Ross [1976] and Bensoussan et al [1994]. The second approach assumes the volatility as an autonomous diffusion driven by a second brownian motion, the first one drives the stock price. For this kind of approach see Hull and White [1987], Stein and Stein [1991], Wiggins [1987]. In a paper of 1996, Hobson and Rogers, see Hobson and Rogers [1998], proposed a new class of nonconstant volatility model which could be extended to include the level dependent models but also shared many characteristics of the second. This is done in such a way that the price and the volatility form a multidimensional Markov process. Here we exclude the models with jumps because they are beyond the scope of this paper.

One of the main puzzles of introducing stochastic volatility in the Black-Scholes model is that the market is incomplete and the underlying asset is not sufficient to hedge a given financial contract against the two sources of risk: the underlying itself and the volatility. To cope with the market completeness problem, Hobson and Rogers [1998] appeal for the use of only one source of randomness, Hull and White [1987] assumed independence of the couple of brownian motions and no systematic risk for the volatility process and finally Hoffman et al [1992] suggested a minimal martigale measure for the purpose of option pricing. In a paper of 1997, Romano and Touzi [1997] established rather general conditions a contingent claim must satisfy in order to complete the market. They proposed a slight modification of definition of market completeness in the sense of Harrison and Pliska [1981]. Notwithstanding the quite standard assumptions of their approach, the necessary and sufficient conditions in the sense of Romano and Touzi were showed to be only necessary in the sense of Harrison and Pliska.

In fact the concept of minimal equivalent martingale measure was introduced by Föllmer and Schweizer [1990] and used by Henderson and Hobson [2000a] to price the passport option with stochastic volatility. The following definition, used by Hoffman et al [1992], is slightly different of the Föllmer and Schweizer and can be seen as a localized version of their result.

Definition 3.1. An equivalent martingale measure $\mathbb{Q}$ for S is called minimal if any local $\mathbb{P}$-martingale L in which $\left\langle L, S^i \right\rangle = 0$ for $i = 1, \dots, m$ remains a local martingale under $\mathbb{Q}$.

The economic reasoning is that using $\mathbb{Q}$ for option pricing is equivalent to assuming that all nontraded risks can be diversified away and thus are unpriced. This was the approach used by Hull and White [1987] and Henderson and Hobson [2000a]. Henceforth all the expectations will be evaluated with respect to this minimal martingale measure $\mathbb{Q}$, unless the contrary is specified.

3.1. **General Markovian Model.** Lets assume the following n-dimensional diffusion process

$$dY_t^i = a^i(t, Y_t)dt + \sum_{j=1}^{n} b^{ij}(t, Y_t)dW_j^i, \tag{3.1}$$

with $i = 0, \dots, m$, $j = 1, \dots, n$ and a^i and b^{ij} measurable functions from $[0, T] \times \mathbb{R}^{m+1} \longrightarrow \mathbb{R}$. $\{W_t^i\}_{t \in [0,T]} = [W_t^1, \dots, W_t^n]^\intercal$ is an n-dimensional brownian motion defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. a^i and b^{ij} satisfy appropriate growth and Lipschitz conditions so that the solution of the Equation (3.1) is a Markov process. Indeed, this model is more general than the proposed by Henderson and Hobson [2000a] since it allows for dependence on the asset price in the coefficients a^i and b^{ij}.

3.1.1. *The Hull-White Model.* Let $\{Y_t^i\}_{t \in [0,T]} = [Y_t^0, Y_t^1, Y_t^2]^\intercal = [B_t, S_t, v_t]^\intercal$, for $i \in \{0, 1, 2\}$. So the Hull-White model can be generalized as

$$dB_t = r(t, Y_t)B_t dt, \tag{3.2}$$

$$dS_t = r(t, Y_t)S_t dt + \sqrt{v_t} S_t dW_t^1, \tag{3.3}$$

$$dv_t = \xi(t, Y_t)v_t dt + \delta(t, Y_t)v_t[\rho(t, Y_t)dW_t^1 + \sqrt{1 - (\rho(t, Y_t))^2} dW_t^2], \tag{3.4}$$

where W^1 and W^2 are independent, ρ is the correlation between the stock price S and the instantaneous variance v. For the case when $\rho = 0$ one has the classical Hull-White model and analytical approach is attainable. For the passport option case Henderson and Hobson [2000a] expose these analytical arguments in a very straightforward manner. Since the vacation option has an explicit price one can derive the new option price as follows

$$\int_{\mathbb{R}} V^{[0,1]}(t,s,x)h(\overline{v}|v_t)d\overline{v}, \tag{3.5}$$

for the case of vacation call and

$$\int_{\mathbb{R}} V^{[-1,0]}(t,s,x)h(\overline{v}|v_t)d\overline{v}, \tag{3.6}$$

for the vacation put. In the early equations $\overline{v}$ is defines as the quadratic average of the future variance

$$\overline{v} = \frac{1}{T-t}\int_t^T v_s ds, \tag{3.7}$$

where $v = \sigma^2$ is the square of volatility, h is the density of $\overline{v}$ conditioned to v, where v is a lognormal variable at the terminal time T and ξ and δ are supposed to depend on v and t but not on S. The analytical formula is derived proceeding the Equation (3.5) in a third order Taylor expansion. Changing the measure of the classical Hull-White model, we obtain

$$dS_t = \sqrt{v_t}S_t dW_t^{1,\mathbb{Q}}, \tag{3.8}$$

$$dv_t = \xi v_t dt + \delta v_t dW_t^{2,\mathbb{Q}}, \tag{3.9}$$

where $W^{1,\mathbb{Q}}$ and $W^{2,\mathbb{Q}}$ are independent brownian motions under $\mathbb{Q}$. For the case $\xi \equiv 0$ the expectation is showed to be

$$\mathrm{E}(\overline{v}) = v_0; \tag{3.10}$$

$$\mathrm{E}(\overline{v}^2) = \frac{2(e^{\delta^2 T} - \delta^2 T - 1)}{\delta^4 T^2} v_0^2; \tag{3.11}$$

$$\mathrm{E}(\overline{v}^3) = \frac{e^{3\delta^2 T} - 9e^{\delta^2 T} + 6\delta^2 T + 8}{3\delta^6 T^3} v_0^3, \tag{3.12}$$

and the expansion of the new price about the expected value of $\overline{v}$, $\overline{\overline{v}}$, is

$$V^{[0,1]}(\overline{\overline{v}}) + \frac{1}{2}\frac{\partial^2 V}{\partial \overline{v}^2}\bigg|_{\overline{\overline{v}}} \int (\overline{v} - \overline{\overline{v}})^2 h(\overline{v}) d\overline{v} + \ldots = \tag{3.13}$$

$$V^{[0,1]}(\overline{\overline{v}}) + \frac{1}{2}\frac{\partial^2 V}{\partial \overline{v}^2}\bigg|_{\overline{\overline{v}}} \mathrm{E}(\overline{v}^2) + \frac{1}{6}\frac{\partial^3 V}{\partial \overline{v}^3}\bigg|_{\overline{\overline{v}}} \mathrm{E}(\overline{v}^3) + \ldots. \tag{3.14}$$

The computation of the put price is analogous. The derivatives obtained above are possible because of the differentiability of the payoff function with respect to $\overline{v}$. This result is reinforced by the results of Henderson and Hobson [2000a] notwithstanding their approach relies on the relation between passport and lookback options. While this approach is quite elegant from a theoretical point of view it has the drawback that v does not depends on S and so the interactions between S and v are compromised. Indeed, in the original paper , Hull and White [1987], the variables ξ and δ may depend on v and t, but do not depend on S. In the classical model they are constants.

4. Numerical Procedures

The model considered in this section was borrowed from Hoffman et al [1992] and can be viewed as a general version of the Ornstein-Uhlenbeck mean reverting Stein and Stein model [1991]. In particular it allows for volatility, at the same time, be stochastic and past dependent.

$$
\begin{aligned}
dB_t &= r(t, Y_t)B_t dt, && (4.1)\\
dS_t &= r(t, Y_t)S_t dt + \sigma_t S_t dW_t^1, && (4.2)\\
d\sigma_t &= -q(\sigma_t - \zeta_t)dt + p\sigma_t dW_t^2, && (4.3)\\
d\zeta_t &= \frac{1}{\eta}(\sigma_t - \zeta_t)dt. && (4.4)
\end{aligned}
$$

The variables σ and ζ are, respectively, the instantaneous and the weighted average volatility; q measures their speed of adjustment and p is the intensity effect of the external noise added by the second source of randomness W^2. Finally, η is the strenght of the past dependence of the average volatility. The equation for ζ can be solved explicitly as

$$\zeta_t = \zeta_0 e^{-\frac{t}{\eta}} + \frac{1}{\eta}\int_0^t e^{-\frac{(t-s)}{\eta}}\sigma_s ds; \tag{4.5}$$

explaining the exponential weighted nature of ζ and the strenght of the past dependence of the average volatility given by the parameter η. Typically, volatility is taken to be an Itô process satisfying a stochastic differential equation driven by a second brownian motion, This is the easiest way the stochastic volatility approach incorporates correlation with stock price changes. One feature that most models applied to the volatility on the original Black-Scholes model seem to have is *mean reversion*. This term refers to the typical time it takes for a process to get back to the mean level of its invariant distribution. The traditional mean reverting Ornstein-Uhlenbeck process is defined as a solution of

$$d\sigma_t = q(m - \sigma_t)dt + pdW_t, \tag{4.6}$$

which is explicitly gaussian and given by

$$\sigma_t = m + (\sigma_0 - m)e^{-qt} + p\int_0^t e^{-q(t-s)}dW_s, \tag{4.7}$$

where σ_0 is the initial volatility and m is the mean reversion parameter. This way σ is distributed as a normal $\mathcal{N}(m + (\sigma_0 - m)e^{-qt}, \frac{p^2}{2q}(1 - e^{-2qt}))$. Figures 3 and 4 show a sample path for σ and ζ and S and X. There is not evidence that the vacation and even the passport option can be well modeled with such assumptions because we do not have enough supporting empirical data for these options. Thus, the characteristics of smile, skewness, and so on, that are known as good features for a stochastic volatility model are not considered here. Notwithstanding, the use of a as general as possible model permits us to investigate the behavior of the option price in a wide class of changing parameters.

As a first attempt to calibrate the model, the parameters can be chosen in accordance with the literature. The following choice seems to be reasonable: $\sigma_0 = 0.1$,$\zeta_0 = 0.1$,$q = 1$,$p = 0.3$ and $\eta = 0.1$. Recalling the Equation (2.2) with the initial conditions: $S_0 = 100$, $T = 1$, $X_0 = 0$ and $r = 0.1$; the model can be rewritten as

$$\begin{aligned} dS_t &= 0.1S_t dt + \sigma_t S_t dW_t^1, && (4.8)\\ d\sigma_t &= -(\sigma_t - \zeta_t)dt + 0.3\sigma_t dW_t^2, && (4.9)\\ d\zeta_t &= 10(\sigma_t - \zeta_t)dt, && (4.10)\\ dX_t^{\pi^*} &= \pi_t^* dS_t + 0.1(X_t^{\pi^*} - \pi_t^* S_t)dt. && (4.11) \end{aligned}$$

The Table 1 shows the relation between the exact price of a vacation call and the simulated one. The parameters used for the model (4.8) are: $S_0 = 100$, $T = 1$, $\sigma_0 = \zeta_0 = 0.5$ and $10,000$ sample paths for crude Monte Carlo. The result seems to indicate that the overestimation of this stochastic volatility model is somehow related with the ratio of X_0 and S_0. As we would expect the option price raises with the initial wealth. For a fixed initial volatility the optimal strategy for the vacation call, $\pi^* = \mathbb{I}_{\{X^{\pi^*} < \frac{1}{2}S_t\}}$, is rewritten as $\pi^* = \mathbb{I}_{\{X^{\pi^*} < \frac{100}{2}\}}$. At the first glance, this strategy is likely to influence the option price but if we turn the initial stock price to $S_0 = 60$, see Table 2, we can not observe a strong similar behavior of underestimation of the constant volatility price and so the addition of a past dependent volatility seems not to alter significantly the results obtained by Henderson and Hobson [2000a].

The Table 3 shows that, at least for a sufficient small volatility, the lower price of the simulated model seem to be a constant but when the level of σ_0 increases this evidence is less strong, see Figure 6. By the Table 1 and Table 3, we can see that when the volatility becomes more significant the exact price responds with a higher price than the simulated one, specially for higher X_0. Whilst generally the effect of the simulated model is to decrease the option price and the results of Henderson and Hobson [2000a] reinforces this appointment, we can not infer that this is a common place.

5. Conclusion

The main result of this paper was to analyze the behavior of the vacation option under a rather general markovian model proposed by Hoffman et al [1992] and at the same time attempt to extend the model of Henderson and Hobson [2000a]. The traditional rule of raise the prices of the option when the asset is subject to a stochastic volatility seems not to be reasonable for the vacation options. Since this result appears suggestive, one have to calibrate the parameters and proceed with some reduction variance technique in order to obtain more accurate outputs.

References

Ahn, H., A. Penaud, and P. Wilmott. "Various Passport Options and their Valuation." *OCIAM, Mathematical Institute, University of Oxford Working Paper* (1998).

Andersen, L., L. Andreasen, and R. Botherton-Ratcliffe. "The Passport Option." *J. Comp. Finance*, 1 (1998), pp.15-36.

Bachelier, L. "Theorie de la Spéculation." *Ann. Sci. Ecole Norm. Sup.*, 17 (1900), pp. 21-86.

Bensoussan, A., M. Crouhy, and D. Galai. "Stochastic Equity Volatility and the Capital Structure of the Firm." *Philosophical Transactions of the Royal Society of London*, Series A, 387 (1994), pp.449-598.

Black, F., and M. Scholes. "The Pricing of Options and Corporate Liabilities." *J. Political Economy*, 81 (1973), pp. 637-659.

Blattberg, R., and N. Gonedes. "A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices." *J. Business*, 47 (1974), pp. 244-280.

Chan, S. "The Valuation of American Passport Options." *School of Business, Grainger Hall, University of Wisconsin-Madison Working Paper* (1999).

Cox, J., and S. Ross. "The Valuation of Options for Alternative Stochastic Process." *J. Financial Economics*, 3 (1976), pp.145-166.

Föllmer, H., and M. Schweizer. "Hedging of Contingent Claims under Incomplete Information." in *Applied Stochastic Analysis*, (1990), eds. M. Davis and R. Elliot. London: Gordon and Breach.

Geske, R. "The Valuation of Compund Options." *J. Financial Economics*, 7 (1979), pp.63-81.

Harrison, J., and S. Pliska. "Martingales and Stochastic Integrals in the Theory of Continuous Trading." *Stochastic Processes and Their Applications*, 11 (1981), pp. 215-260.

Henderson, V., and D. Hobson. "Local Time, Coupling and the Passport Option." *Finance and Stochastics*, 4 (2000), pp. 69-80.

Henderson, V., and D. Hobson. "Passport Options with Stochastic Volatility." *University of Bath Working Paper* (2000a).

Hobson, D., and L. Rogers. "Complete Models with Stochastic Volatility." *Finance and Stochastics*, 8 (1998), pp. 27-48.

Hoffman, N., E. Platen, and M. Scweizer. "Option Pricing under Incompleteness and Stochastic Volatility." *Mathematical Finance*, 7 (1992), pp. 399-410.

Hull, J., and A. White. "The Pricing of Options on Assets with Stochastic Volatilities." *J. Finance*, 42 (1987), pp. 281-300.

Hyer, T., A. Lipton-Lifschitz, D. Pugachevsky. "Passport to Success." *Risk*, 10 (1997), pp.127131.

Merton, R. *Continuous-time Finance* (1990), Basil Blackwell, Cambridge, MA.

Nagayama, I. "Pricing of Passport Option." *J. Math. Sci. Univ. Tokyo*, 5 (1998), pp.747-785.

Penaud, A., P. Wilmott, and H. Anh. "Exotic Passport Options." *OCIAM, Mathematical Institute, University of Oxford Working Paper* (1998).

Romano, M., and N. Touzi. "Contigent Claims and Market Completeness in a Stochastic Volatility Model." *Mathematical Finance*, 7 (1997), pp. 399-410.

Samuelson, P. "Rational Theory of Warrant Pricing." *Indus. Manag. Rev.*, 6 (1965), pp.13-31.

Scott, L. "Option Pricing when the Variance Changes Randomly: Theory, Estimation and an Application." *J. Financial Quant. Anal.*, 22 (1987), pp. 419-438.

Shreve, S., and J. Večeř. "Options on a Traded Account: Vacation Calls, Vacations Puts and Passport Options." *Finance and Stochastics*, 4 (2000), pp. 255-274.

Stein, E., and J. Stein. "Stock Price Distributions with Stochastic Volatility: An Analytic Approach." *Rev. Financial Stud.*, 4 (1991), pp. 727-752.

Wiggins, J. "Options Values under Stochastic Volatility. Theory and Empirical Estimates." *J. Financial Economics*, 19 (1987), pp. 351-372.

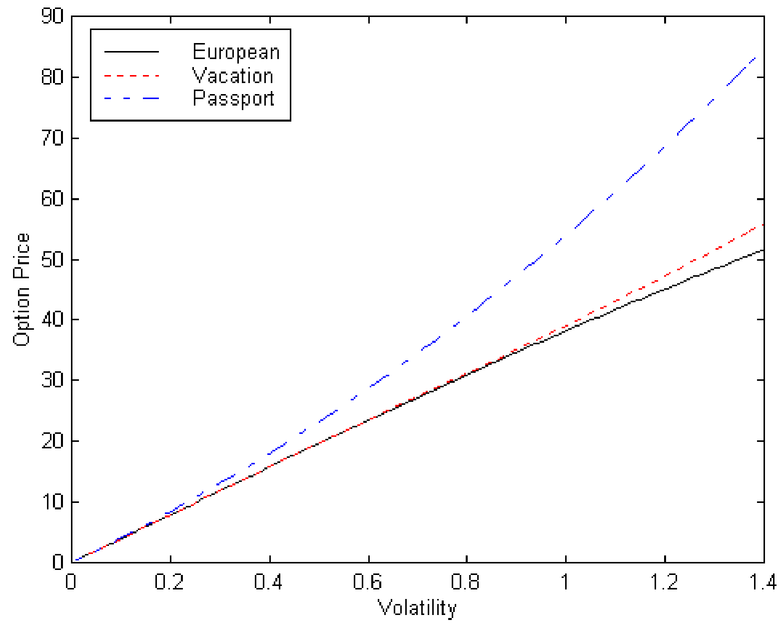

FIGURE 1. Comparison among European call, Vacation call and Passport options

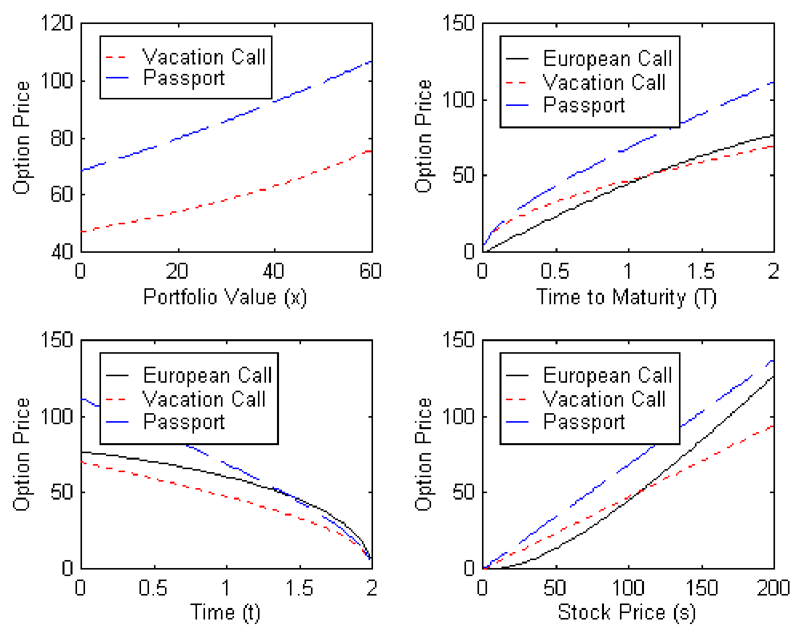

FIGURE 2. Option price vs x, T, t and s comparison among European call, Vacation call and Passport options

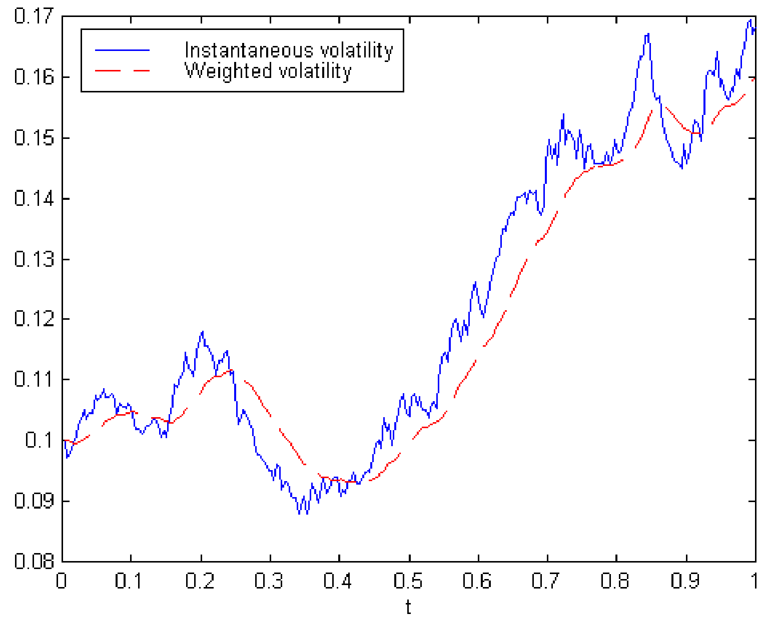

FIGURE 3. Sample path for σ_t and ζ_t

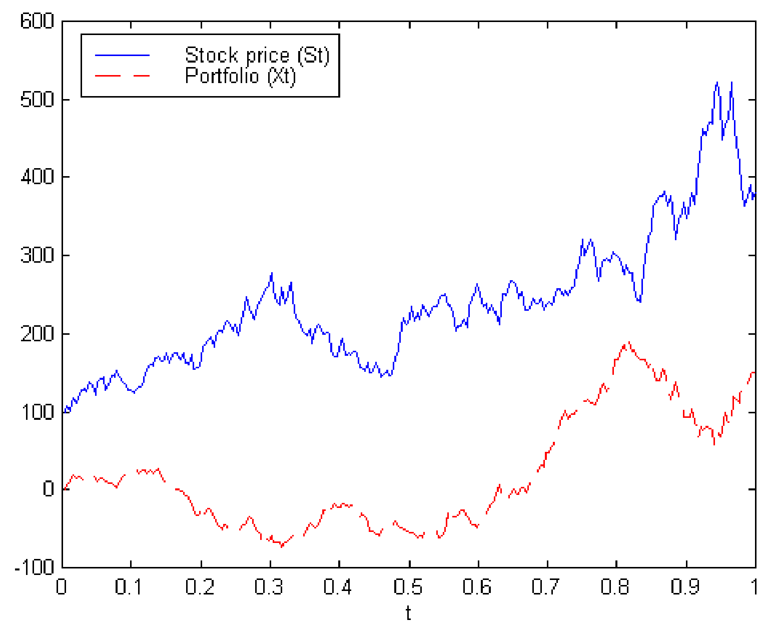

FIGURE 4. Sample path for S_t and X_t

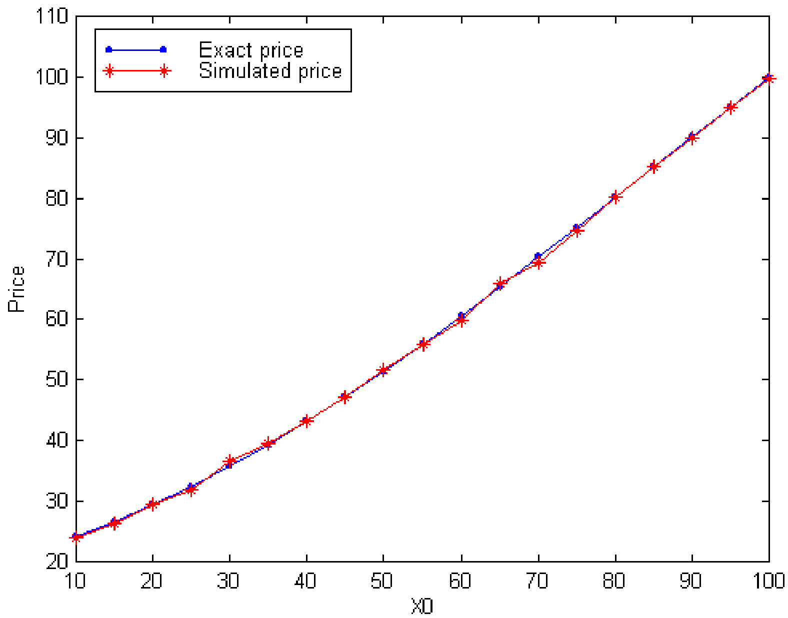

FIGURE 5. Difference between the exact price of a vacation call with constant volatility and the simulated price with stochastic and past dependent volatility

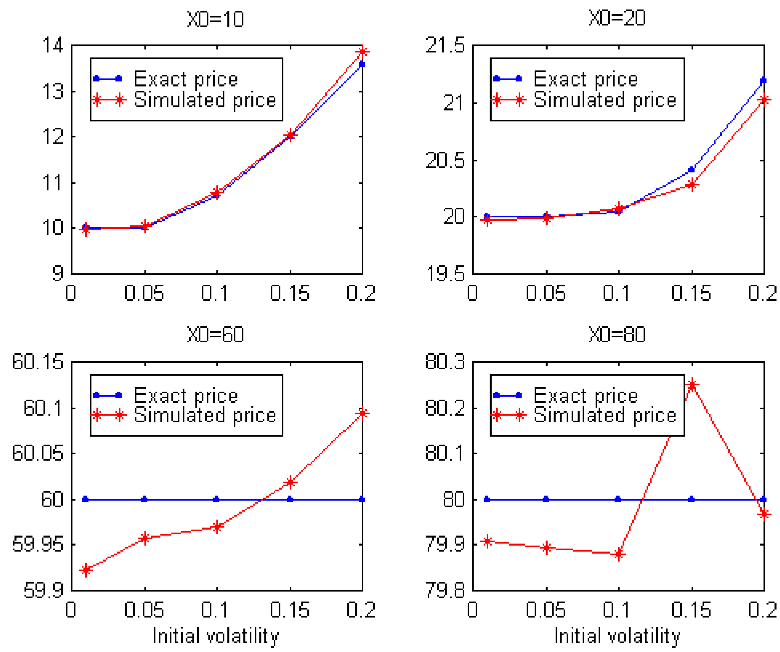

FIGURE 6. Difference between the exact price of a vacation call with constant volatility and the simulated price with stochastic and past dependent volatility for various initial volatilities

TABLE 1. Vacation call exact prices for constant volatility and simulated prices for $S_0 = 100$,$T = 1$,$r = 0.1$,$\sigma_0 = \zeta_0 = 0.5$. Upper and Lower are the bounds of a 95% confidence interval for a 10,000 samples Monte Carlo

X_0	Exact Price	Simulated	Upper	Lower	Error (%)
10	24.1660	23.9300	24.4013	23.4586	3.9395
15	26.7032	26.2303	26.7121	25.7484	3.6740
20	29.4754	29.5600	30.1113	29.0087	3.7299
25	32.4938	31.8843	32.4000	31.3685	3.2350
30	35.7671	36.6658	37.2638	36.0678	3.2620
35	39.3005	39.5031	40.0997	38.9066	3.0205
40	43.0952	43.4078	44.0545	42.7611	2.9798
45	47.1479	47.2797	47.9255	46.6338	2.7322
50	51.4514	51.6804	52.3071	51.0537	2.4252
55	55.9819	55.7887	56.4357	55.1416	2.3196
60	60.6707	59.8204	60.5018	59.1391	2.2781
65	65.4625	65.8601	66.5844	65.1357	2.1997
70	70.3219	69.4747	70.1398	68.8096	1.9146
75	75.2261	74.7019	75.4104	73.9934	1.8970
80	80.1602	80.1539	80.8522	79.4556	1.7424
85	85.1144	85.2870	85.9522	84.6217	1.5600
90	90.0824	89.9646	90.6712	89.2580	1.5708
95	95.0598	95.0800	95.7688	94.3912	1.4489
100	100.0437	99.7639	100.4403	99.0874	1.3561

TABLE 2. Vacation call exact prices for constant volatility and simulated prices for $S_0 = 60$,$T = 1$,$r = 0.1$,$\sigma_0 = \zeta_0 = 0.5$. Upper and Lower are the bounds of a 95% confidence interval for a 10,000 samples Monte Carlo

X_0	Exact Price	Simulated	Upper	Lower	Error (%)
10	16.5603	16.1854	16.3538	16.0169	2.0812
20	22.8562	22.5349	22.7459	22.3239	1.8725
30	30.8709	31.0082	31.2338	30.7826	1.4551
40	40.2457	40.1980	40.4466	39.9494	1.2369
50	50.0767	49.5460	49.7941	49.2980	1.0015
60	60.0262	59.7284	59.9879	59.4681	0.8692

Table 3. Vacation call exact prices for constant volatility and simulated prices for $S_0 = 100, T = 1, r = 0.1$. Upper and Lower are the bounds of a 95% confidence interval for a 10,000 samples Monte Carlo

X_0	σ_0	Exact Price	Simulated	Upper	Lower	Error (%)
10	0.01	10	9.9846	9.9849	9.9844	0.0047
10	0.05	10.0301	10.0429	10.0487	10.0370	0.1164
10	0.1	10.7124	10.7828	10.8022	10.7635	0.3588
10	0.15	12.0217	12.0524	12.0903	12.0145	0.6282
10	0.2	13.5891	13.8574	13.9240	13.7909	0.9606
20	0.01	20	19.9804	19.9806	19.9801	0.0025
20	0.05	20	19.9871	19.9934	19.9808	0.0629
20	0.1	20.0399	20.0760	20.0999	20.0521	0.2380
20	0.15	20.4036	20.2885	20.3390	20.2380	0.4976
20	0.2	21.1859	21.0266	21.1090	20.9442	0.7838
60	0.01	60	59.9220	59.9222	59.9217	0.0008
60	0.05	60	59.9576	59.9636	59.9515	0.0202
60	0.1	60	59.9707	59.9955	59.9460	0.0824
60	0.15	60	60.0188	60.0746	59.9629	0.1861
60	0.2	60	60.0932	60.1949	59.9629	0.3386
80	0.01	80	79.9071	79.9074	79.9069	0.0006
80	0.05	80	79.8952	79.9013	79.8891	0.0154
80	0.1	80	79.8787	79.9040	79.8534	0.0633
80	0.15	80	80.2503	80.3064	80.1942	0.1398
80	0.2	80	79.9684	80.0695	79.8672	0.2529