The Potential Approach to Bond and Currency Pricing

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Abstract

In this paper, we begin the modeling of bond and currency prices from the modeling of the state-price density satisfying basic properties of a potential. We provide extensive examples and show their implications on bond and currency pricing. Most classic short rate models are special cases of this general approach. We also investigate the connection to the Heath, Jarrow, and Morton model. One advantage of the potential approach resides in its ease in simultaneously modeling the yield curves of many countries and their exchange rates. The properties of exchange rates under each example are derived and we illustrate their possibility in explaining the forward premium puzzle.

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1 Introduction

The pricing kernel, or the state-price density, which relates future cash flows to today's price, is the fundamental building block of modern asset pricing theory. In abstract, the state-price density process can be regarded as a positive supermartingale, or, under some regularity conditions, a *potential*. The theory of Markov processes provides a rich framework for the generation of examples of potentials. In this paper, we begin the modeling of bond and currency prices from the modeling of the state-price density satisfying basic properties of a potential. We provide extensive examples on the potential modeling of the state-price densities and their implications on bond and currency pricing. We show that most classic interest rate models are special cases of this general approach. We also investigate the connection between the potential approach and the Heath et al. (1992) approach (henceforth: HJM) widely used in the finance area. One advantage of the potential approach resides in its great ease in modeling the yield curves of many countries at the same time, together with the exchange rates between them. We derive the properties of exchange rates under each example and illustrate their possibility in explaining the forward premium puzzle.

The bulk of literature on bond pricing focuses on two approaches. One stream of literature, the *short rate models*, specifies the instantaneous interest rate process directly and then comes up with expressions for the prices of zero-coupon bond and other interest rate derivatives. Cox et al. (1985), Beaglehole and Tenney (1991), Black et al. (1990), Brennan and Schwartz (1979), Duffie and Kan (1994), Fong and Vasicek (1991), Hull and White (1990), Longstaff and Schwartz (1991), Richard (1978), Schaefer and Schwartz (1987), Vasicek (1977), are just some of the many papers which study different models for the spot rate process and explore the consequences of this model choice. All these models are based either on a general (e.g. Cox et al. (1985)) or partial equilibrium (e.g. Hull and White (1990)) framework.

Another approach to interest rate modeling, the *forward rate models*, began with the paper of Ho and Lee (1986) and was thoroughly analyzed in the continuous time setting by Babbs (1990) and Heath et al. (1992) (see Jamshidian (1988) and Sommer (1996) for the continuum limit of the discrete-time Ho and Lee model and Amin and Jarrow (1991) for an extension of the Heath et al. model to an international economy). The idea of this approach is to model the forward rate process (or, equivalently, to model the movement of the yield curve) directly.

Only recently, a less developed approach has emerged in the bond-pricing literature. These models are based on the direct specification of the state-price process ξ_t. Refer to the bond price equation (1.1), the positivity of bond prices implies that the state-price density process ξ_t is a *positive supermartingale*. If additionally we assume the economically reasonable condition $P(0,t) \to 0$ as $t \to \infty$, then the state-price density ξ_t is what is known as a *potential*.[1] Therefore, this approach has also been referred to as the *potential approach*.

[1] A positive supermartingale tending to 0 in expectation is called a potential because of the very close links with the Markov process concept of a potential. See for example, Bhattacharya and Waymire (1990).

To our knowledge, the earliest published reference to the state-price density approach appears to be Constantinides (1992), where it is used to generate a fairly general squared-Gaussian model. Backus et al. (1998a,b,c) illustrate the role of pricing kernel in bond and currency pricing in a discrete-time setup. Rogers (1997) formalizes the potential approach and illustrates the application of a *resolvent* representation in modeling the potential (a special case of Rogers' approach is presented in Flesaker and Hughston (1996)). In this paper, we follow the procedure of Rogers (1997) and apply the resolvent representation to bond and currency pricing.

Given a state price density ξ_t, the price of a zero-coupon bond is given by

$$P(t,T) = E_t\left[\xi_T\right]/\xi_t, \tag{1.1}$$

where the expectation is taken under the objective probability measure. Using the above equation as the starting point has several advantages compared to the traditional approaches of term structure modeling. First, it offers great flexibility for the construction of interest rate models with strictly positive interest rates. Requiring nonnegative interest rate at the cost of analytical complexity may not be appropriate for some classes of instruments; however, for the calculation of long term contracts and most structured products there is a substantial increase in accuracy if a positive interest rate model is used (see e.g. Rogers (1995,1996)).

Secondly, state-price models based on (1.1) offer great simplifications when modeling international term structures. The exchange rate between two countries equals the ratio of their state-price densities. This observation was made by Saá-Requejo (1993) and Backus et al. (1998b) in a discrete-time version and by Ahn (1997) in the continuous-time framework. Starting from the state price density, we can therefore simultaneously model the term structures of interest rates in any two countries and the exchange rate between them. Due to this very ease of the potential approach in modeling the international term structures, we are enabled to explore implications on the *forward premium puzzle* in great detail and clarity under different specifications. We are able to point directions in model specification to account for the forward premium puzzle.

Lastly, since the state-price density is more closely related to the equilibrium of an economy, the correct specification of the state-price density also serves as a benchmark for future equilibrium modeling of the economy under the sense of reverse engineering.

The paper is structured as follows. In the next section, we describe the resolvent procedure for the specification of the potentials and show how interest rates, bond prices, and exchange rates can be derived from the state-price density. Section 3 analyzes several examples and explores their correspondence to the classic interest rate models. In section 4, we analyze the connection between the state-price density approach and the HJM approach. Since the key to the construction of interest rate models within the HJM framework is the specification of the forward rate volatility, we will elaborate on the connection of the forward rate volatility and the volatility of the pricing kernel process. Section 5 extends the pricing kernel model from a one-country economy to a multi-country economy. The flexibility of the state-price density approach allows us to construct international term structure models in very convenient ways and provides us with insights in model specifications to account for the forward premium anomaly of currency prices. Section 6 concludes.

2 The Resolvent Representation

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space equipped with a filtration $F = (\mathcal{F}_t)_{t \leq 0}$ satisfying the usual conditions of right-continuity and completeness. The fundamental building block of modern asset pricing theory is the *state-price density*, ξ_t, or the *pricing kernel*, $\xi_{t,T} = \xi_T/\xi_t$, which relates future cash flows, denoted as K_T, to today's price, denoted as p_t:

$$p_t = E\left[\frac{\xi_T}{\xi_t} K_T | \mathcal{F}_t\right] = E_t\left[\frac{\xi_T}{\xi_t} K_T\right], \tag{2.1}$$

where $E[\cdot|\mathcal{F}_t] = E_t[\cdot]$ is the conditional expectation operator defined on $(\Omega, \mathcal{F}, \mathbf{P})$. The existence and uniqueness of a positive state-price density is guaranteed by the assumption of an arbitrage-free and complete market. We refer to Duffie (1996) for details.

For any Markov process $\{X\}_{t \geq 0}$ with *resolvent* $(R_\lambda)_{\lambda > 0}$, we can take $\alpha > 0$ and a positive function g on the state space $x \in \mathcal{X}$ and define the state-price density ξ_t as

$$\xi_t = e^{-\alpha t} R_\alpha g(X_t). \tag{2.2}$$

The resulting state price density ξ_t is a potential.[2] Different choices of g and α give a wide range of possible potentials, even within the context of a fixed Markov process. Using this

[2] See Appendix (A.1) for details and proof.

framework, the spot rate can be given as

$$r(t) = \left. \frac{\partial E\left[\xi_T | F_t\right]}{\partial T} \right|_{T=t} = \frac{g(X_t)}{R_\alpha g(X_t)} \,. \tag{2.3}$$

The bond price corresponds to

$$P(t,T) = \frac{E\left[e^{-\alpha\tau} R_\alpha g(X_T) | F_t\right]}{R_\alpha g(X_t)}, \tag{2.4}$$

where $\tau = T - t$ is the maturity of the bond.

Since the resolvent of a Markov process is hard to specify in a usable closed form, we can use the *resolvent operator*,

$$R_\lambda = (\lambda - L)^{-1}$$

where L is the infinitesimal generator of the Markov process. If we pick some positive function $f : X \to (0,\infty)$ and then define g via

$$g = (\lambda - L) f,$$

we have $R_\lambda g = f$, and, provided g is everywhere non-negative, we have the state price density, as in (2.2), now given by

$$\xi_t = e^{-\alpha t} R_\alpha g(X_t) = e^{-\alpha t} f(X_t).$$

From (2.3), we derive the instantaneous interest rate as

$$r(t) = \frac{g(X_t)}{R_\alpha g(X_t)} = \frac{(\alpha - L) f(X_t)}{f(X_t)} \,,$$

and from (2.4) we obtain the bond price as

$$P(t,T) = \frac{E_t\left[e^{-\alpha\tau} f(X_T)\right]}{f(X_t)}.$$

2.1 The Markov process

Before choosing any functional forms for the f function, we need to specify the Markov process. Of interest are the following class of d-dimensional stationary Markov processes:

$$dX = \kappa(\theta - X)\,dt + \sqrt{V(X^{\eta})}dW \tag{2.5}$$

where κ is a $d \times d$ matrix capturing the speed of mean-reversion. The $d \times 1$ vector θ captures the long-run mean of X. The instantaneous variance matrix $V(X^{\eta})$ is assumed to be diagonal with $V_{ii} = a_i + b_i X_i^{\eta}$. The infinitesimal generator of X is given by

$$(Lf)(X) = (\theta - X)'\,\kappa'\,(\nabla f) + \frac{1}{2}\mathrm{tr}\left[V(X^{\eta}) \cdot (Hf)\right], \tag{2.6}$$

where ∇f denotes the vector gradient, Hf the Hessian matrix of f, and "$\cdot$" the dot-product. To simplify, we can re-scale the Markov process, with no loss of generality, such that $a_i = 0$ and $b_i = 1$, and therefore $V_{ii} = X_i^{\eta}$.

The following three commonly used examples for the Markov process fit into the specification of (2.5), with η equal to 0, 1, and 2, respectively.

Example 2.1 (*An Ornstein-Uhlenbeck process with mean-reversion:*)

$$dX = \kappa(\theta - X)\,dt + dW. \tag{2.7}$$

This is one of the most simple cases for a stationary process, corresponding to (2.5) with $\eta = 0$. With this process, X is multi-variate normal with

$$\mu_t \quad = \quad E_t[X_T] = \left(I - e^{-\kappa\tau}\right)\theta + e^{-\kappa\tau}X_t;$$

$$V_t \quad = \quad E_t\left[(X_T-\mu_t)^2\right] = \int_t^T e^{-\kappa(T-s)}\left(e^{-\kappa(T-s)}\right)' ds; \tag{2.8}$$

where μ_t denotes the conditional mean and V_t the conditional variance. The characteristic function is

$$\phi_t(X_T, s) = \exp\left(is'\mu - \frac{1}{2}s'V_t s\right). \tag{2.9}$$

Refer to Stuart and Ord (1987) for the derivation. ♣

Example 2.2 (*A mean-reverting Bessel process:*)

$$dX = \kappa(\theta - X)\,dt + \sqrt{X}dW, \tag{2.10}$$

corresponding to (2.5) with $\eta = 1$. With such a Bessel process, X is a linear combination of non-central chi-square variates. For any affine transformations of X: $y = b'x$, we have the following characteristic function

$$\phi(y,\tau;s) = \left|(I - isC)^{\bullet(-A)}\right| \exp\left(isb'D(I - isC)^{-1}x\right), \tag{2.11}$$

where "•" denotes "dot power operator," and A, C, and D are diagonal matrices with $A_{ii} = 2\hat{\kappa}_i\theta_i$, $C_{ii} = (b_i/2\hat{\kappa}_i)(1-e^{-\hat{\kappa}_i\tau})$, $D_{ii} = \exp(-\hat{\kappa}_i\tau)$, and $\hat{\kappa}_i = (\kappa' b)_i/b_i$. The conditional mean $\mu(y_\tau)$ and variance $V(y_\tau)$ are

$$\begin{aligned} \mu(y_\tau) &= b'\left(I - e^{-\tau\hat{\kappa}}\right)\theta + b'e^{-\tau\hat{\kappa}}x; \\ V(y_\tau) &= \sum_{i=1}^{d}\theta_i\frac{b_i^2}{2\hat{\kappa}_i}\left(1-e^{-\hat{\kappa}_i\tau}\right)^2 + 2\sum_{i=1}^{d}x_i\frac{b_i^2}{2\hat{\kappa}_i}\left(1-e^{-\hat{\kappa}_i\tau}\right)e^{-\hat{\kappa}_i\tau}. \end{aligned} \tag{2.12}$$

The unconditional moments are

$$\mu(y) \quad = \quad b'\theta; \quad V(y) = \sum_{i=1}^{d}\theta_i\frac{b_i^2}{2\hat{\kappa}_i}.$$

Refer to Appendix A.3 for the derivation. ♣

Example 2.3 (*A geometric Brownian motion process:*)

$$dX = -\kappa X dt + X dW, \tag{2.13}$$

corresponding to (2.5) with $\eta = 2$ and $\theta = 0$. When κ is diagonal such that the elements of X are independent from each other, X_T are conditionally log normal: $\ln X_T \sim N(\mu_t, V_t)$ with

$$\begin{aligned} \mu_t &= e^{-(\frac{1}{2}+\kappa)\tau} X_t; \\ V_t &= \int_t^T e^{-(\frac{1}{2}+\kappa)s} \left(e^{-(\frac{1}{2}+\kappa)s}\right)' ds. \end{aligned} \tag{2.14}$$

♣

2.2 The f function

One requirement for f is that it has to be a positive function: $f : \mathcal{X} \to (0, \infty)$. The most common example is that of an exponential form

$$f(x) = \exp\left(b_0 + b_1' x + x' B x\right) \tag{2.15}$$

where b_0 is a scalar, b_1 is a $d \times 1$ vector, and B is a $d \times d$ matrix. We assume B is a symmetric matrix with no loss of generality. The gradient vector and the Hessian matrix of f are, respectively,

$$\begin{aligned} \nabla f &= f(x) B_1; \\ H f &= f(x) \left[2B + B_1 B_1'\right]; \end{aligned}$$

with

$$B_1 = b_1 + 2Bx,$$

being a $d \times 1$ vector. Applying the resolvent operator, we can obtain the g function:

$$\begin{aligned} g(X_t) &= (\alpha - L)\, f(X_t) \\ &= f(X_t)\left[\alpha - (\theta - X_t)'\kappa' B_1 - \frac{1}{2}\mathrm{tr}\left[(X_t^{\eta} X_t^{\eta\prime}) \cdot \left(2B + B_1 B_1'\right)\right]\right]. \end{aligned} \qquad (2.16)$$

2.3 The state price density and asset pricing

Given the Markov process in (2.5) and the f function in (2.15), we can obtain the state price density ξ_t as,

$$\xi_t = e^{-\alpha t} R_\alpha g(X_t) = e^{-\alpha t} f(X_t),$$

from which we can price contingent claims. Specifically, the instantaneous rate can be expressed as

$$\begin{aligned} r(t) &= \frac{(\alpha - L)\, f(X_t)}{f(X_t)} \\ &= \alpha - (\theta - X_t)'\kappa' B_1 - \frac{1}{2}\mathrm{tr}\left[(X_t^{\eta} X_t^{\eta\prime}) \cdot \left(2B + B_1 B_1'\right)\right]. \end{aligned} \qquad (2.17)$$

The zero-coupon bond prices,

$$P(t,T) = \frac{E_t[\xi_T]}{\xi_t} = \frac{E_t\left[e^{-\alpha\tau} f(X_T)\right]}{f(X_t)}.$$

contain expectation operation on the f function that needs to be worked out before one can obtain an analytical form. In general, any contingent claim p_t with time-T payoff

$K_T = K(X_T)$ can be priced under such a framework as

$$p_t = E_t\left[\frac{\xi_T}{\xi_t}K(X_T)\right] = \frac{E_t\left[e^{-\alpha\tau}f(X_T)K(X_T)\right]}{f(X_t)}.$$

Obviously, a combination of different specifications of the f function and the Markov process X can generate a wide range of asset pricing models. As we will show later, all diffusion-based interest rate models can de derived from this potential framework.

3 Bond Pricing: Examples

In this section we provide extensive examples where the bond prices can be worked out in reasonably simple forms. We also derive many traditional models as our special cases.

3.1 Exponential quadratic f functions with Ornstein-Uhlenbeck process for X_t

The exponential quadratic f function can be rewritten as

$$f(x) = \exp\left[\frac{1}{2}(x-c)'Q(x-c)+\gamma\right], \tag{3.1}$$

where $Q = 2B$, $c = -\frac{1}{2}B^{-1}b_1$, and $\gamma = b_0 - \frac{1}{4}b_1'B^{-1}b_1$. The pricing kernel is therefore

$$\xi_{t,T} = e^{-\alpha\tau}\frac{f(X_T)}{f(X_t)} = \exp\left[-\alpha\tau + \frac{1}{2}(X_T-c)'Q(X_T-c) - \frac{1}{2}(X_t-c)'Q(X_t-c)\right].$$

Note that γ drops out of the pricing kernel. As a result, we can assume $\gamma = 0$ by setting $b_0 = \frac{1}{4}b_1'B^{-1}b_1$ with no loss of generality. Further, we assume that Q is diagonal for simplicity.

The $g(x)$ function is of the form

$$g(X_t) = f(X_t)\left[\alpha + (X_t-\theta)'\kappa' Q\,(X_t-c) - \frac{1}{2}\mathrm{tr}\,(Q) - \frac{1}{2}\,(X_t-c)'\,Q^2\,(X_t-c)\right].$$

The instantaneous interest rate can thus be simplified to

$$r\,(t) = \frac{1}{2}\,(X_t-\widetilde{c})'\,\widetilde{Q}\,(X_t-\widetilde{c}) + \widetilde{\gamma},$$

with

$$\begin{aligned}
\widetilde{Q} &= \kappa' Q + Q'\kappa - Q^2; \\
\widetilde{c} &= \widetilde{Q}^{-1}\left(\kappa' Qc + Q\kappa\theta - Q^2 c\right); \\
\widetilde{\gamma} &= \alpha - \frac{1}{2}\mathrm{tr}\,(Q) + \theta'\kappa' Qc - \frac{1}{2}c' Q^2 c - \frac{1}{2}\widetilde{c}'\widetilde{Q}\widetilde{c}.
\end{aligned}$$

The interest rate is therefore a quadratic function of the normally distributed variates X_t. The properties of quadratic functions of normal variates are well-documented by, for example, Holmquist (1996) and Searle (1971).

The expectation of an exponential quadratic $f(X_T)$ function, such as (3.1) (with $\gamma = 0$), of normal variates can be shown to equal

$$E_t[f(X_T)] = |I - QV_t|^{-1/2}\exp\left[\frac{1}{2}(\mu_t-c)'(I-QV_t)^{-1}Q(\mu_t-c)\right]. \tag{3.2}$$

Refer to Appendix A.2 for the derivation. The bond prices are therefore given by

$$P(t,T) = |I-QV_t|^{-1/2}e^{\left[-\alpha\tau + \frac{1}{2}(\mu_t-c)'(I-QV_t)^{-1}Q(\mu_t-c) - \frac{1}{2}(X_t-c)'Q(X_t-c)\right]}, \tag{3.3}$$

which is exponential-quadratic in X_t. Constantinides (1992) developed a model similar to this example. Ahn (1998) developed a general equilibrium that sustains such a model. As shown in the following examples, this class of models fit in the affine class of Duffie and Kan (1996) only under very specific conditions. Ahn (1998) shows how this general model can be reduced to parameterized Cox et al. (1985) model under special conditions.

Example 3.1 (*Affine cases:*) We can show that when κ and Q are both scalars and $\theta = c$ this class of models (exponential-quadratic f functions with Gaussian X_t process) can be reduced to an affine structure. The instantaneous interest rate becomes

$$r(t) = \left(\alpha - \frac{1}{2}Q\right) + \left(\kappa Q - \frac{1}{2}Q^2\right)(X_t - \theta)'(X_t - \theta),$$

which follows a square-root process

$$dr = \hat{\kappa}\left(\hat{\theta} - r\right)dt + \sqrt{\hat{\alpha} + \beta r}d\hat{W}$$

where $d\hat{W}$ is a one-dimensional Brownian motion and where

$$\begin{aligned} \hat{\kappa} &= 2\kappa; \\ \hat{\theta} &= \alpha - \frac{1}{2}Q + d(2\kappa Q - Q^2)/(2\kappa); \\ \hat{\alpha} &= -(2\kappa Q - Q^2)(2\alpha - Q); \\ \beta &= 4\kappa Q - 2Q^2, \end{aligned}$$

and d is the dimension of X_t. The pricing kernel is

$$\xi_{t,T} = \exp\left[-\alpha\tau + (2\kappa - Q)^{-1}(r_T - r_t)\right],$$

which is exponential-affine in r. As stated in Duffie and Kan (1996), an affine interest rate with an exponential-affine pricing kernel generates an affine model of bond pricing. That is, the bond prices $P(t,T)$ will be exponential affine functions of the instantaneous interest rate $r(t)$:

$$P(t,T) = \exp\left(-a_\tau - b_\tau r(t)\right), \tag{3.4}$$

with

$$\begin{aligned} a_\tau &= \frac{1}{2}\log\left(1 - QV_t\right) + \alpha\tau - \left[1 - (1 - QV_t)^{-1} e^{-2\kappa\tau}\right](2\kappa - Q)^{-1}\left(\alpha - \frac{1}{2}Q\right); \\ b_\tau &= \left[1 - (1 - QV_t)^{-1} e^{-2\kappa\tau}\right](2\kappa - Q)^{-1}. \end{aligned}$$

Note that $V_t = \int_t^T e^{-2\kappa(T-s)} ds$ is reduced to a scalar now that κ is a scalar. ♣

Example 3.2 (*Beaglehole and Tenney's (1991) univariate quadratic model*):

When $d = 1$ and $\theta = \widetilde{\gamma} = 0$, we have the interest rate

$$r(t) = \frac{1}{2}\widetilde{Q}\left(X_t - \widetilde{c}\right)^2,$$

with

$$\widetilde{Q} = (2\kappa - Q)\,Q; \qquad \text{and } \widetilde{c} = \frac{(\kappa - Q)\,c}{2\kappa - Q}.$$

Apply Ito's lemma, we have the following stochastic process

$$dr = \left(\alpha - \beta\sqrt{r} - \gamma r\right)dt + \sigma\sqrt{r}dW,$$

where $\alpha = \frac{1}{2}\widetilde{Q}$; $\beta = \kappa\widetilde{c}\sqrt{2\widetilde{Q}}$; $\gamma = 2\kappa$; and $\sigma = \sqrt{2\widetilde{Q}}$. This is exactly the model proposed by Beaglehole and Tenney (1991). ♣

Example 3.3 (*Longstaff (1989) Double Square Root Model*): When $d = 1$ and $\kappa = 0$, but $\kappa\theta = \mu \neq 0$, $Q = iq$, and $\tilde{\gamma} = 0$, we have

$$\begin{aligned} dX_t &= \mu dt + dW; \\ r(t) &= \frac{1}{2}q^2 \left(X_t - \widetilde{c}\right)^2, \end{aligned}$$

where $\widetilde{c} = c - \mu/Q$. The interest rate is then given by

$$dr = \left(\alpha + \beta\sqrt{r}\right) dt + \sigma\sqrt{r} dW$$

where $\alpha = \frac{1}{2}q^2$, $\beta = \sqrt{2}\mu q$, and $\sigma = \sqrt{2}q$. This is just the double-square-root model developed by Longstaff (1989). ♣

3.2 Exponential-linear f functions with affine X_t: Affine models

With an exponential-linear f function of the form

$$f(x) = \exp(a + b'x),$$

we have the pricing kernel of the form,

$$\xi_{t,T} = \exp\left[-\alpha\tau + b'\left(X_T - X_t\right)\right],$$

which is also exponential linear in X. The constant term a drops out of the pricing kernel and thus can be set to zero with no loss of generality. The $g(x)$ function is then

$$g(X_t) = f(X_t)\left[\alpha - (\theta - X_t)'\kappa' b - \frac{1}{2}b'V(X_t^{\eta})b\right], \tag{3.5}$$

and the instantaneous interest rate is

$$r(t) = \alpha - (\theta - X_t)'\kappa' b - \frac{1}{2} b' V(X_t^{\eta}) b,$$

which is a linear function of X_t as long as the instantaneous variance matrix $V(X_t^{\eta})$ of the Markov process is affine in X_t, that is, as long as $\eta = 0$ or 1. Recall that $V(X_t^{\eta})$ is re-scaled to a diagonal matrix with $V_{ii} = X_i^{\eta}$. This affine interest rate, together with the exponential-affine pricing kernel, will generate exponential-affine bond prices, as stated in Duffie and Kan (1996). Specifically, the bond prices are given by

$$P(t,T) = \phi_t(X_T, b)\exp(-\alpha\tau - b' X_t).$$

where $\phi_t(X_T, b) = E_t[e^{b' X_T}]$ denotes the moment generating function of X_T conditional on time t information and with moment generating parameter b. Note, however, that when $V(X^{\eta})$ is not affine in X, for example, when $\eta = 2$, the interest rate $r(t)$ may no longer be Markovian. We will confine ourselves to the affine structure.

Example 3.4 When X_t follows a Bessel process, that is, when $\eta = 1$, referring to the characteristic function of $y = b'x$ in (2.11), we have

$$\phi_t(X_T, b) = \left| (I - C)^{\bullet(-A)} \right| \exp\left(b' D (I - C)^{-1} X_t \right).$$

The bond price is therefore

$$P(t,T) = \left| (I - C)^{\bullet(-A)} \right| \exp\left[-\alpha\tau - b' \left(I - D(I - C)^{-1} \right) X_t \right], \tag{3.6}$$

which is exponential affine in X_t. The continuously compounded yields

$$y(t,T) = \frac{-\log P(t,T)}{\tau} = -\log\left| (I - C)^{\bullet(-A)} \right| + \alpha + \frac{1}{\tau} b' \left(I - D(I - C)^{-1} \right) X_t,$$

are thus affine functions of the Markov process X_t. This corresponds to a generalized multi-factor Cox et al. (1985) model as studied by, among others, Dai and Singleton (1997) and Backus et al. (1998d). ♣

Example 3.5 When $\eta = 0$ and thus X_t follows an Ornstein-Uhlenbeck process, referring to (2.9), we have the moment generating function,

$$\phi_t(X_T, b) \quad = \quad \exp\left(b'\left(I - e^{-\kappa\tau}\right)\theta + b' e^{-\kappa\tau} X_t + \frac{1}{2} b' V_t b\right).$$

with $V_t = \int_t^T e^{-\kappa(T-s)}\left(e^{-\kappa(T-s)}\right) ds$. The bond price is then

$$P(t,T) = \exp\left(-\alpha\tau + b'\left(I - e^{-\kappa\tau}\right)\theta + \frac{1}{2} b' V_t b - b'\left(I - e^{-\kappa\tau}\right) X_t\right), \tag{3.7}$$

which is also exponential-affine in X_t. ♣

The extensive examples provided in this section illustrate the flexibility of the potential approach in bond pricing. By varying the specifications of the f function and the Markov process X, we can generate a wide variety of bond pricing models which virtually incorporate all traditional interest rate models as special cases or examples. In the next section, we derive the links between this potential approach and the widely used HJM approach in term structure modeling.

4 Potential versus HJM Approach

In this section, we connect the potential approach to the well-know Heath-Jarrow-Morton (HJM) framework. These two approaches are in sharp contrast in that the HJM approach makes use of the information contained in the current forward curve and intends to avoid specifying the market price of risk, which is incorporated in the forward curve, while the potential approach directly specifies the pricing kernel, and thus the market price of risk. These two approaches have pros and cons of their own. For example, the HJM framework is better suited for fixed-income derivatives pricing since the only unobservable input is the volatility structure of the forward rates, which can also be directly estimated from the forward rate data. However, the potential approach works best for modeling term structures of different countries and the exchange rates between them at the same time. It also provides more insight to the underlying economy since the pricing kernel is more closely related to the equilibrium of the economy and the underlying macroeconomic fundamentals such as preference, inflation, and monetary policy. In any case, it will be interesting to see the links between these two frameworks.

4.1 Forward rates and the pricing kernel

The state price density relates to the instantaneous interest rate by

$$\xi_t = \exp\left(-\int_0^t r(s)ds\right) \cdot Z_t.$$

Since we have

$$P(t,T) = \frac{E_t\left(\xi_T\right)}{\xi_t} = E_t^*\left(\exp\left(-\int_t^T r(s)\,ds\right)\right), \tag{4.8}$$

we can interpret the variable Z_t as the *Radon-Nikodým derivative*, which takes us from the objective measure $\mathbf{P}$ to the risk-neutral measure $\mathbf{P}^* \sim \mathbf{P}$ defined as

$$Z_t = \frac{d\mathbf{P}^*}{d\mathbf{P}} = E\left(-\int_0^t \gamma(s) \cdot dW(s)\right),$$

where E denotes the *Doléans exponential*

$$E(U_t) = \exp\left(U_t - \frac{1}{2}\langle U\rangle_t\right)$$

and $\gamma(t)$ is an F_t-adapted process. We can now rewrite the expression for the pricing kernel as

$$\xi_t = P(0,t)E\left(-\int_0^t \gamma(s) \cdot dW(s)\right),$$

or in differential notation

$$\frac{d\xi_t}{\xi_t} = -r(t)dt - \gamma(t) \cdot dW.$$

Let $B(t)$ be the money market account defined as $B(t) = \int_0^t r(s)\,ds$. Since the discounted price process of the zero bond $P(t,T)/B(t)$ is a martingale under the risk-neutral measure $\mathbf{P}^*$, by the abstract version of Bayes formula the expression $Z(t)P(t,T)/B(t)$ is a martingale under the objective measure $\mathbf{P}$. From the martingale representation theorem, there exists an adapted process $\gamma(t,T)$ such that

$$Z(t)P(t,T) = B(t)P(0,T)E\left(-\int_0^t \gamma(s,T) \cdot dW(s)\right).$$

Hence, application of Ito's Lemma yields the bond price dynamics under the objective measure $\mathbf{P}$ as

$$\frac{dP(t,T)}{P(t,T)} = (r(t) - v(t,T) \cdot \gamma(t))\,dt + v(t,T) \cdot dW(t),$$

where for the bond price volatility we have

$$\begin{aligned}
v(t,T) &= \gamma(t) - \gamma(t,T) \\
&= -\int_t^T \frac{\gamma(t,T)}{\partial T}\Big|_{T=s} ds \\
&= -\int_t^T \sigma(t,s)\, ds
\end{aligned} \tag{4.9}$$

where $\sigma(t,s)$, as will be clear right away, is the instantaneous volatility of the forward rate $f(t,s)$. Equation (4.8) allows us to express the forward rate in terms of the pricing kernel. Using the stochastic version of Fubini's Theorem,[3] the representation of the forward rate $f(t,T)$ is

$$f(t,T) = \frac{1}{E_t\left(\xi_T\right)} E_t \left(\frac{\partial \xi_T}{\partial T}\right). \tag{4.10}$$

Obviously, the forward rate and short rate dynamics obey

$$\begin{aligned}
d f(t,T) &= \left(\sigma(t,T) \cdot \left(\int_t^T \sigma(t,s) - \gamma(t)\right)\right) dt + \sigma(t,T) \cdot dW \\
dr(t) &= \left(\frac{\partial f(t,T)}{\partial T}\Big|_{T=t} - \sigma(t,t) \cdot \gamma(t)\right) dt + \sigma(t,t) \cdot dW.
\end{aligned}$$

Using the expression for the bond price volatility in (4.9), the forward rate dynamics simplifies to

$$d f(t,T) = \frac{\partial \gamma(t,T)}{\partial T} \cdot \gamma(t,T)\, dt + \frac{\partial \gamma(t,T)}{\partial T} \cdot dW. \tag{4.11}$$

It is interesting to compare the forward dynamics in (4.11) with the forward rate dynamics under the risk-neutral measure derived in the HJM framework. Under the HJM framework, the arbitrage-free drift of the forward rate under the risk-neutral measure, is *completely* determined by the forward rate volatility $\sigma(t,T)$ and its integrals. In the state-price model,

[3] See Baxter (1997) for a derivation of the stochastic version of the Fubini Theorem.

the drift is fully determined by the adapted process $\gamma(t,T)$ and its derivative with respect to time-of-maturity. The interpretation of the term $\gamma(t,T)$ becomes obvious when we compare equation (4.11) with the forward rate representation in equation (4.10). Application of Ito's Lemma gives us

$$\frac{d\left[E_t\left(\xi_T\right)\right]}{E_t\left(\xi_T\right)} = -\gamma(t,T)\cdot dW.$$

Therefore, $-\gamma(t,T)$ is equal to the diffusion term of the process of the pricing kernel's time-t expectation value for time T and $\gamma(t,t) = \gamma(t)$ is often labeled as market price of risk.

4.2 The Volatility Structure of Forward Rates

The volatility of the forward rate plays a crucial role in the HJM framework: Not only does it determine the arbitrage-free drift of the forward rate under the risk-neutral measure, but also through the choice of a specific volatility function we generate a particular interest rate model, which can be used to price interest rate contingent claims. Since starting from the HJM framework it is not obvious which forward rate volatility structure gives rise to the interest rate models presented in section 3, we do the reverse: starting from the potential approach we derive the volatility structure of the forward rate which would yield the same interest rate model within the HJM framework. In what follows we delineate the recipe on how the volatility structure can be derived.

By Ito's Lemma, the forward rate process can be expressed as a function of the state variable and the bond price, i.e.

$$df(t,T) = -\frac{\partial^2 \log(P(t,T))}{\partial T \partial X} dX_t - \frac{1}{2} \frac{\partial^3 \log(P(t,T))}{\partial T \partial X^2} \langle dX \rangle_t - \frac{\partial^2 \log(P(t,T))}{\partial T \partial t} dt.$$

If the short rate can be identified as the state variable, i.e. $r(t) = X_t$, it is obvious that the volatility structure of the forward rates is

$$\frac{\partial \gamma(t,T)}{\partial T} = -\frac{\partial^2 \log(P(t,T))}{\partial T \partial r} \cdot \frac{\partial \gamma(t,T)}{\partial T}\Big|_{T=t}. \tag{4.12}$$

Otherwise, in our setting we would have

$$\frac{\partial \gamma(t,T)}{\partial T} = -\frac{\partial^2 \log(P(t,T))}{\partial T \partial X} \cdot X_t^{\eta}.$$

Assuming an affine term structure model of the form

$$P(t,T) = A(t,T) \exp\left(-B(t,T) \cdot X_t\right),$$

for some functions $A(t,T)$ and $B(t,T)$, then equation (4.12) can be further simplified to

$$\frac{\partial \gamma(t,T)}{\partial T} = \frac{\partial B(t,T)}{\partial T} \cdot \frac{\partial \gamma(t,T)}{\partial T}\Big|_{T=t}. \tag{4.13}$$

Equipped with these results, we are now able to derive the volatility structures for the calculated examples in Section 3, which would make a HJM framework equivalent of the *potential* approach considered.

4.3 Examples

4.3.1 Exponential quadratic function f and OU-process X_t

In this case, the bond price is (see equation 3.3)

$$P(t,T) = |I - QV_t|^{-1/2} e^{\left[-\alpha\tau + \frac{1}{2}(\mu_t - c)'(I-QV_t)^{-1}Q(\mu_t - c) - \frac{1}{2}(X_t - c)'Q(X_t - c)\right]}.$$

If we make the restriction that κ and Q are scalars as well as $\theta = c$ we have shown that the bond prices given in (3.3) can be reduced to an affine structure. The bond price equation then simplifies to (see equation 3.4)

$$P(t,T) = \exp\left(-a_\tau - b_\tau r(t)\right),$$

The dynamics of the pricing kernel can be stated in differential notation as

$$\frac{d\xi_t}{\xi_t} = \left(\frac{\hat{\kappa}\left(\hat{\theta} - r(t)\right)}{\hat{\kappa} - Q} - \alpha\right) dt + \left(\frac{\sqrt{\hat{\alpha} + \beta r(t)}}{\hat{\kappa} - Q}\right) dW.$$

Thus, since $Q = \frac{1}{2}(\hat{\kappa} - \sqrt{\hat{\kappa}^2 - 2\beta})$, the risk premium is has to

$$\gamma(t) = -\frac{\sqrt{\beta\left(\hat{\kappa} - 4\alpha - \sqrt{\hat{\kappa}^2 - 2\beta} + 4r(t)\right)}}{\sqrt{\hat{\kappa}^2 - 2\beta} + \hat{\kappa}}.$$

In the Cox et al. (1985) model where $\hat{\alpha} = 0$, the risk premium simplifies to

$$\gamma(t) = -\frac{2\sqrt{\beta r(t)}}{\hat{\kappa} + \sqrt{\hat{\kappa}^2 - 2\beta}} = \lambda\sqrt{r(t)}, \tag{4.14}$$

where

$$\lambda = -\frac{2\sqrt{\beta}}{\hat{\kappa} + \sqrt{\hat{\kappa}^2 - 2\beta}}$$

In the original Cox et al. (1985) model the risk premium is given $\lambda\sqrt{r(t)}$, where λ is an exogenously given constant; however, the λ in equation (4.14) is determined by the parameters of the interest rate process under the objective measure. We have therefore derived a parameterized version of the Cox et al. (1985) model.

From the bond price equation, the volatility structure of the forward rate can be easily derived as

$$\sigma(t,T) = \frac{4e^{\hat{\kappa}(T-t)}\hat{\kappa}^2}{\left(\hat{\kappa}\left(1+e^{\hat{\kappa}(T-t)}\right)+\left(e^{\hat{\kappa}(T-t)}-1\right)\sqrt{\hat{\kappa}^2-2\beta}\right)}\sqrt{\hat{\alpha}+\beta r(t)}.$$

Again, if we set $\hat{\alpha}=0$, we obtain the volatility structure of the forward rates for the Cox et al. (1985) model.

Next, we want to derive the volatility structure of the Beaglehole and Tenney (1991) model. The pricing kernel given in their model is

$$\xi_t = \exp\left(-\alpha t + \frac{1}{2}Q\left(X_t-c\right)^2\right),$$

which can be written in differential form as

$$\begin{aligned} \frac{d\xi_t}{\xi_t} &= \left(-\alpha - X_t(X_t-c)Q\kappa + \frac{1}{2}\left(Q+Q^2(X_t-c)^2\right)\right)dt + (X_t-c)Q\,dW \\ &= -r(t)\,dt - \gamma(t)\,dW. \end{aligned}$$

The market price of risk becomes

$$\gamma(t) \quad = \quad \frac{cQ\kappa}{2\kappa - Q} + \sqrt{\frac{2Q}{2\kappa - Q}}\sqrt{r(t)}$$

which is proportional to the square-root of the interest rate $r(t)$.

The bond price can be derived as

$$P(t,T) = \frac{\exp\left(-(T-t)\,\alpha - \frac{Q(c-X_t)^2}{2} + \frac{Q\kappa\left(c - e^{-\kappa(T-t)}X_t\right)^2}{\left(e^{-2\kappa(T-t)} - 1\right)Q + 2\kappa}\right)}{\sqrt{\frac{\left(e^{-2\kappa(T-t)} - 1\right)Q + 2\kappa}{2\kappa}}}.$$

The volatility structure of the forward rates is then

$$\begin{aligned}\sigma(t,T) \quad &= \quad \frac{2e^{(t+T)\kappa}Q\kappa^2\left[c\left(\left(e^{2t\kappa} + e^{2T\kappa}\right)Q - 2e^{2T\kappa}\kappa\right) + 2e^{(t+T)\kappa}\left(2\kappa - Q\right)X_t\right]}{\left[\left(e^{2t\kappa} - e^{2T\kappa}\right)Q + 2e^{2T\kappa}\kappa\right]^2} \\ &= \quad A + BX_t,\end{aligned}$$

which is a linear function of the Markov process X_t and thus is proportional to the square-root of the interest rate $r(t)$. Note that the volatility of the short rate is $\sigma(t,t) = \sqrt{2\widetilde{Q}r(t)}$, where $\widetilde{Q} = (2\kappa - Q)Q$.

4.3.2 Exponential-linear f and affine X_t

In the case of an exponential-linear f function with affine X_t, the pricing kernel dynamics is given as

$$\frac{d\xi_t}{\xi_t} \quad = \quad \left(-\alpha + (\theta - X_t)'\,\kappa' b + \frac{1}{2}(b^2)' X_t^{2\eta}\right)dt + b' X_t^{\eta} dW.$$

The market price of risk is thus

$$\gamma(t) = -b \cdot X_t^{\eta}.$$

When X_t follows a Bessel process, the bond price becomes exponential affine in X_t, i.e. we have (see equation 3.6)

$$P(t,T) = \left|(I-C)^{\bullet(-A)}\right| \exp\left[-\alpha\tau - b'\left(I - D\left(I-C\right)^{-1}\right)X_t\right].$$

The market price of risk is $\gamma(t) = -b \cdot \sqrt{X_t}$. Using the bond price equation, the volatility structure of the forward rates can be derived as

$$\sigma(t,T) = \left[\frac{\partial\left(-D(I-C)^{-1}\right)}{\partial T}\right] b \cdot \sqrt{X_t}$$

When X_t follows an Ornstein-Uhlenbeck process, the bond price is, as shown in equation (3.7),

$$P(t,T) = \exp\left(-\alpha\tau + b'\left(I - e^{-\kappa\tau}\right)\theta + \frac{1}{2}b'V_t b + b'\left(I - e^{-\kappa\tau}\right)X_t\right),$$

which is also exponential-affine in X_t. The pricing kernel follows

$$\frac{d\xi_t}{\xi_t} = \left(-\alpha + (\theta - X_t)'\kappa' b + \frac{1}{2}b'b\right)dt + b'dW$$

and the market price of risk equals $\gamma(t) = -b$ and is therefore constant. Using the bond price equation, the volatility structure of the forward rates can be derived as

$$\sigma(t,T) \quad = \quad e^{-\kappa'(T-t)}\kappa' b = e^{-\kappa'(T-t)}\sigma(t),$$

where $\sigma(t) = \kappa' b$ is the volatility structure of the short interest rate. This gives rise to a Gaussian interest rate model, where the possibility of generating negative interest rates is positive.

Under the HJM framework, the volatility structure of the forward rates are exogenously specified while drift is derived from the volatility structure as a result of no-arbitrage. Market prices of risk are incorporated in the current forward curve. The potential approach, on the other hand, directly specifies the state price and thus the market price of risk. Although we can, as we just did, derive the volatility structure of the forward rate from the state price specification, except under very special cases such as the Gaussian interest rate model, the derived volatility structures are often functions of the state price density and are intertwined with the market price of risk $\gamma(t)$. That is, the volatility structures are generally not exogenous, as is the case in a HJM framework.

As illustrated in previous sections, the potential approach provides great flexibility in generating a wide variety of term structure models, yet its advantage is even more pronounced in simultaneously modeling international term structures and the exchange rate between them. In the next section, we will fully exploit this advantage and extend the analysis to exchange rate market.

5 Accounting For Forward Premium Puzzle

In this section, we will extend the analysis above to a multi-currency economy. We will focus on the most puzzling feature of currencies: high interest rate currencies tend to appreciate while one might guess, instead, that investors would demand higher interest rates on currencies expected to fall in value. This departure from uncovered interest parity, which we term the *forward premium puzzle*, has been documented in numerous studies and has spawned a second generation of papers attempting to account for it. One of the most influential studies is by Fama (1984). He attributes the behavior of forward and spot exchange rates to the time-varying risk premia which have to possess certain properties. Traditional asset pricing models have been notably unsuccessful in producing risk premiums with the desired property. The potential approach provides a consistent framework for the valuation of interest rates as well as foreign exchange rate contingent claims. We examine, from the perspective of the potential approach, model specifications that have "potentials" to explain the puzzle.

5.1 The potential approach of currency pricing

One of the biggest advantage of the potential approach lies in its great ease in simultaneously modeling international term structures and the exchange rates between them. Specifically, if we consider several countries at once and assume that they are governed by the same vector of Markov process X_t, if further we assume that at time t, one unit of country j's currency is worth S_t^{ij} units of country i's currency, then under certain technical assumptions, we have that the development of the exchange rate S_t^{ij} is governed by the ratio of the pricing kernels of the two countries:

$$\frac{S_T^{ij}}{S_t^{ij}} = \frac{\xi_{t,T}^j}{\xi_{t,T}^i}. \tag{5.15}$$

This observation was made by Saá-Requejo (1993) and Backus et al. (1998b)) in a discrete-time version and by Ahn (1997) in the continuous-time framework. Starting from the state price density, we can therefore simultaneously model the term structures of interest rates in any two countries and the exchange rate between them. Under the resolvent representation, we have

$$\frac{S_T^{ij}}{S_t^{ij}} = e^{(\alpha^i - \alpha^j)\tau} \frac{f^j(X_T)/f^j(X_t)}{f^i(X_T)/f^i(X_t)}, \tag{5.16}$$

where $\tau = T - t$. By assuming the same vector of Markov process but different f functions for different countries, we implicitly assume that the world economies share the same vector of shocks but the impacts and repercussions of these shocks are different to different countries.

The time-t forward price of the currency with maturity $\tau = T - t$ can also be written as the ratio of the conditional expectations of two state-price densities involved:

$$F^{ij}(t,T) = \frac{E_t\left[\xi_T^j\right]}{E_t\left[\xi_T^i\right]} = e^{(\alpha^i - \alpha^j)T} \frac{E_t\left[f^j(X_T)\right]}{E_t\left[f^i(X_T)\right]}. \tag{5.17}$$

In general, we can, as we do, set $\alpha^i = \alpha^j$ by further assuming that the long-run mean of exchange rate depreciation rates are zero (i.e., no currency consistently beats the other in the long run).

5.2 The forward premium puzzle

Let $s_t = \ln S_t^{ij}$ and $f_t^\tau = \ln F^{ij}(t, t+\tau)$. Then $s_{t+1} - s_t$ captures the continuously compounded depreciation rate over time interval $[t, t+\tau]$, and $f_t^\tau - s_t$ captures the forward premium. By covered interest rate parity,

$$f_t^\tau - s_t = r_{t,\tau}^i - r_{t,\tau}^j$$

the forward premium equals the difference between the yields on the two countries' zero-coupon bonds with maturity τ. Consider the following regression,

$$s_{t+\tau} - s_t = \alpha + \beta(f_t^\tau - s_t) + \varepsilon_{t+\tau}. \tag{5.18}$$

The expectation hypothesis implies $\alpha = 0$ and a regression slope $\beta = 1$, yet most studies estimate β to be negative. See for example Bilson(1981), Cumby and Obstfeld (1984), Fama (1984), Hansen and Hodrick (1983), Hodrick and Srivastava (1986) and Hsieh (1984).[4] They find not only that the expectations hypothesis provides a poor approximation to the data, but that its predictions of future currency movements are in the wrong direction. In principal, equation (5.18) can be used to construct profitable investment strategies. Bekaert and Hodrick (1992) show that while such strategies are not riskless, they have positive and statistically significant average excess returns.

Following Fama (1984), we decompose the forward premium into two parts: the expected exchange rate depreciation $E_t[\delta_{t+\tau}]$ and the expected forward risk premium $E_t[p_{t+\tau}]$:

$$\begin{aligned} f_t^\tau - s_t &= E_t[s_T - s_t] + E_t[f_t^\tau - s_T] \\ &\equiv E_t[\delta_{t,T}] + E_t[p_{t,T}], \end{aligned}$$

where $T = t + \tau$. From the linear projection theorem, the slope coefficient β of the regression can be written as

$$\beta = \frac{Cov(\delta, \delta + p)}{Var(\delta + p)} = \frac{Cov(\delta, p) + Var(\delta)}{Var(\delta + p)}$$

[4]Recently, there are some preliminary evidence that the estimate of β is closer to unity when the regression is on US dollar prices of currencies of emerging markets or countries with strong capital controls.

Note that the expectation hypothesis $\beta = 1$ if and only if the forward risk premium p_t is constant over time: $Var(p) = 0$. To account for the actual negative estimate for β, Fama (1984) notes that we need (1) *time-varying* forward risk premium, (2) *negative correlation* between the risk premium (p) and the depreciation rate (δ), and (3) greater variance of the risk premium (p) than the depreciation (δ). We label this negative estimate as a puzzle because most asset pricing models so far have been notoriously unsuccessful in producing a risk premium that satisfies these properties, particularly the negative correlation between the risk premium and the depreciation rate.

Representing currency spot price S^{ij} and forward price F^{ij} in terms of the state-price density as in (5.15) and (5.17), we can rewrite the slope coefficient as

$$\beta = \frac{Cov\left[\ln \frac{\xi^j_{t,T}}{\xi^i_{t,T}}, \ln \frac{E_t\left[\xi^j_{t,T}\right]}{E_t\left[\xi^i_{t,T}\right]}\right]}{Var\left[\ln \frac{E_t\left[\xi^j_{t,T}\right]}{E_t\left[\xi^i_{t,T}\right]}\right]}. \tag{5.19}$$

Clearly, the slope coefficient β is *completely* determined once we have chosen a specific form for the pricing kernels of the two countries. To account for the forward premium puzzle, we need to specify pricing kernels that satisfy the required properties. In particular, the pricing kernels need to generate negative covariance between the depreciation rate and the forward premium.

Under the resolvent representation of currency pricing in (5.16) and (5.17), assuming zero long-run mean for exchange rate depreciation rate ($\alpha^i = \alpha^j$), we can write the depreciation rate $\delta_{t,T}$, the risk premium $p_{t,T}$ and the forward premium $f_t^\tau - s_t$ as

$$
\begin{aligned}
\delta_{t,T} &= \ln \frac{S_T^{ij}}{S_t^{ij}} = \ln f^j(X_T)/f^j(X_t) - \ln f^i(X_T)/f^i(X_t); \\
p_{t,T} &= \ln \frac{F^{ij}(t,T)}{S_T^{ij}} = \ln E_t[f^j(X_T)]/f^j(X_T) - \ln E_t[f^i(X_T)]/f^i(X_T); \\
f_t^\tau - s_t &= \ln \frac{F^{ij}(t,T)}{S_t^{ij}} = \ln E_t[f^j(X_T)]/f^j(X_t) - \ln E_t[f^i(X_t)]/f^i(X_t).
\end{aligned}
$$

As illustrated in Section 3, different combinations of the X Markov process and the f function can generate a wide range of term structure models of interest rates. Similarly, we can also obtain a wide variety of specifications for the exchange rates and forward premia, some of which hold great potentials in explaining the forward premium puzzle.

5.3 Potential models of currency pricing: examples

With the properties of the foreign exchange rate data in mind, we will examine, in the following, whether (and which category of) potential models can explain the forward premium anomaly. Specifically, we investigate what combinations of the Markov process for X and the f function can generate exchange rates and forward premia that satisfy the Fama condition.

5.4 Exponential quadratic models

When the f function is exponential quadratic of the form,

$$f^i(x) = \frac{1}{2}(x - c_i)'Q_i(x - c_i),$$

the state-price density for country i can be given as

$$\xi_t^i = \exp\left[-\alpha_i t + \frac{1}{2}(X_t - c_i)' Q_i (X_t - c_i)\right].$$

We also assume that the state variable vector X follows an Ornstein-Uhlenbeck process as specified in *Example 2.1*:

$$dX_t = \kappa(\theta - X_t)\, dt + dW(t).$$

Even within this same category, different choices of Q_i, Q_j and c_i, c_j yield a wide variety of currency pricing models. We consider two specific examples.

Example 5.6 The first example is based on the assumption that the diagonal matrix Q is equal for both countries, i.e. $Q_i = Q_j = Q$ and $c_i \neq c_j$, the foreign exchange rate becomes

$$S_t^{ij} = \exp\left[\frac{1}{2}(c_j - c_i)' Q (c_j + c_i) + (c_i - c_j)' Q X_t\right], \tag{5.20}$$

which implies that the exchange rate depreciation rate δ follows an Ornstein-Uhlenbeck process with mean reversion, and that all the interest rates are squared-Gaussian. These features should give a tractable class of international term structure models.

From (5.20), we have the depreciation rate

$$\delta_{t,T} = (c_i - c_j)' Q (X_T - X_t) = D'(X_T - X_t),$$

where $D' = (c_i - c_j)'Q$. The risk premium $p_{t,T}$ can be obtained by taking the expectation,

$$\begin{aligned} p_{t,T} &= \log \frac{E_t\left[\xi_T^j\right]\xi_T^i}{E_t\left[\xi_T^i\right]\xi_T^j} \\ &= \frac{1}{2}(c_i - c_j)' Q \left[(I - QV_t)^{-1}(2\mu_t - c_i - c_j) - (2X_T - c_i - c_j)\right] \\ &= D'\bar{Q}e^{-\kappa\tau}X_t - D'X_T + constant, \end{aligned}$$

with $\bar{Q} = (I - QV_t)^{-1}$. Refer to *Example 2.1* for μ_t and V_t.

Let V denote the unconditional variance of X, we have

$$\begin{aligned} Var[\delta_{t,T}] &= 2D'V\left(I - e^{-\kappa\tau}\right)D; \\ Var[p_{t,T}] &= D'\bar{Q}Ve^{-2\kappa\tau}\bar{Q}D + D'VD - 2D'\bar{Q}Ve^{-2\kappa\tau}D; \\ Cov[\delta_{t,T}, p_{t,T}] &= -D'V\left(I - e^{-\kappa\tau}\right)D - DVe^{-\kappa\tau}\left(I - e^{-\kappa\tau}\right)\bar{Q}D. \end{aligned}$$

We can see that the covariance of the depreciation rate and the forward risk premium is negative. However, the relative magnitude of the variances of the depreciation rate and the forward risk premium depends on the parameters values. Presumably, the exponential quadratic model can, from a theoretical viewpoint, account for the forward premium anomaly found in empirical studies. Again, from a practical viewpoint, the exponential quadratic model presented in this section is particularly attractive because it entails some attractive features with respect to the distribution of the foreign exchange rates and the interest rates. ♣

Example 5.7 The second example is based on the assumption $Q_i \neq Q_j$, but $c_i = c_j = c$. Then the exchange rate becomes

$$S_t^{ij} = \exp\left[\frac{1}{2}(X_t - c)' Q_{ji} (X_t - c)\right], \tag{5.21}$$

with $Q_{ji} = Q_j - Q_i$. The depreciation rate $\delta_{t,T}$ is given as

$$\delta_{t,T} = \frac{1}{2}(X_T - X_t) Q_{ij} (X_T + X_t - 2c),$$

which is a quadratic function of normal variates. The forward risk premium $p_{t,T}$ can be derived as

$$
\begin{aligned}
p_{t,T} &= \log\left(\frac{\sqrt{|I-Q_iV_t|}}{\sqrt{|I-Q_jV_t|}}\right)+\frac{1}{2}\left(\mu_t-c\right)'\hat{Q}_{ji}\left(\mu_t-c\right) \\
&\quad -\frac{1}{2}\left(X_T-c\right)'Q_{ji}\left(X_T-c\right),
\end{aligned}
$$

with $\hat{Q}_{ji}=\left(I-Q_jV_{t,T}\right)^{-1}Q_j-\left(I-Q_iV_{t,T}\right)^{-1}Q_i$. Through messy but straightforward manipulations, we have the variances and covariance of the forward risk premium and the depreciation rate as

$$
\begin{aligned}
Var(p_{t,T}) &= \frac{1}{2}tr(\hat{Q}_{ji}Ve^{-2\kappa\tau}(\hat{Q}_{ji}-2Q_{ji})Ve^{-2\kappa\tau}+(Q_{ji}V)^2) \\
&\quad +(\theta-c)'\left[(\hat{Q}_{ji}-2Q_{ji})Ve^{-2\kappa\tau}\hat{Q}_{ji}+Q_{ji}VQ_{ji}\right](\theta-c); \\
Var(\delta_{t,T}) &= tr\left[(Q_{ji}V)^2(I-e^{-2\kappa\tau})\right]+2(\theta-c)'Q_{ji}V(I-e^{-\kappa\tau})Q_{ji}(\theta-c); \\
Cov[\delta_{t,T},p_{t,T}] &= -\frac{1}{2}tr\left[Q_{ji}^2V^2\left(I-e^{-2\kappa\tau}\right)\right]-(\theta-c)'Q_{ji}V\left(I-e^{-\kappa\tau}\right)Q_{ji}(\theta-c) \\
&\quad -\frac{1}{2}tr\left[\hat{Q}_{ji}Ve^{-2\kappa\tau}\left(I-e^{-2\kappa\tau}\right)Q_{ji}V\right] \\
&\quad -(\theta-c)'\hat{Q}_{ji}Ve^{-\kappa\tau}\left(I-e^{-\kappa\tau}\right)Q_{ji}(\theta-c).
\end{aligned}
$$

Again, the requirement for negative covariance between $\delta_{t,T}$ and $p_{t,T}$ is easily fulfilled, even for one-factor models; however, the relative magnitude of variances depends on the parameter values. ♣

5.5 Exponential Linear Models

When the f function is exponential linear, the state-price density for country i is

$$\xi_t^i = \exp\left(\alpha_i t + b_i' X_t\right).$$

The exchange rate then follows

$$S_t^{ij} = \exp\left((\alpha_j - \alpha_i)\, t + (b_j - b_i)'\, X_t\right).$$

Again, we will discuss two cases.

Example 5.8 We assume that X_t follows an Ornstein-Uhlenbeck process as specified in *Example 2.1*. The depreciation rate is given as

$$\delta_{t,T} = (b_j - b_i)'\,(X_T - X_t)\,,$$

The variance of $\delta_{t,T}$ can easily be derived as

$$Var(\delta_{t,T}) = 2\,(b_j - b_i)'\, V\left(I - e^{-\kappa\tau}\right)(b_j - b_i)\,.$$

The forward risk premium $p_{t,T}$ can be derived as

$$p_{t,T} = \frac{1}{2}\,(b_j - b_i)'\, V\,(b_j + b_i) + (b_j - b_i)'\left(e^{-\kappa\tau} X_t - X_T\right)$$

and variance of the forward risk premium is

$$Var(p_{t,T}) = (b_j - b_i)'\, V\left(I + e^{-2\kappa\tau}\right)(b_j - b_i)\,.$$

The risk premium is time varying, as required by Fama condition. However, the covariance between the depreciation rates and the forward premium is positive in such a set up. Since

$$\delta_{t,T} + p_{t,T} = \frac{1}{2}(b_j - b_i)' V (b_j + b_i) - (b_j - b_i)' \left(I - e^{-\kappa\tau}\right) X_t,$$

the covariance term of the regression slope in equation (5.18) comes down to

$$Cov[\delta_{t,T}, \delta_{t,T} + p_{t,T}] \quad = \quad (b_j - b_i)' \left(I - e^{-\kappa\tau}\right) V \left(I - e^{-\kappa\tau}\right) (b_j - b_i),$$

which is positive. Therefore, the linear exponential model with the state variable X_t following an Ornstein-Uhlenbeck process fails to generate a negative regression slope. ♣

Example 5.9 In this example we assume that X_t follows the square-root process specified in *Example 2.2*. Then we obtain a generalized multi-factor Cox et al. (1985) model. The depreciation rate is given as

$$\delta_{t,T} = (b_j - b_i)' (X_T - X_t),$$

with variance

$$Var(\delta_{t,T}) = 2(b_j - b_i)' V \left(I - e^{-\kappa\tau}\right) (b_j - b_i),$$

where V is the unconditional variance-covariance matrix of X_t. The forward risk premium is

$$\begin{aligned} p_{t,T} &= \ln E_t\left[f^j(X_T)\right]/f^j(X_T) - \ln E_t\left[f^i(X_T)\right]/f^i(X_T) \\ &= \ln \frac{\left|\left(I - C^j\right)^{-A^j}\right|}{\left|\left(I - C^i\right)^{-A^i}\right|} + \left[b_j' D_j (I - C_j)^{-1} - b_i' D_i (I - C_i)^{-1}\right] X_t - (b_j - b_i)' X_T. \end{aligned}$$

The forward premium $f^{\tau} - s_t = \delta_{t,T} + p_{t,T}$ is

$$\begin{aligned} f_t^{\tau} - s_t &= \ln E_t \left[f^j(X_T) \right] / f^j(X_t) - \ln E_t \left[f^i(X_T) \right] / f^i(X_t) \\ &= \ln \frac{\left| \left(I - C^j\right)^{-A^j} \right|}{\left| \left(I - C^i\right)^{-A^i} \right|} + \left[b_j' D_j (I - C_j)^{-1} - b_i' D_i (I - C_i)^{-1} - (b_j - b_i)' \right] X_t . \end{aligned}$$

The covariance term in the regression is therefore

$$\begin{aligned} & Cov\left(\delta_{t,T}, \delta_{t,T} + p_{t,T}\right) \\ &= \left[b_j' D_j (I - C_j)^{-1} - b_i' D_i (I - C_i)^{-1} - (b_j - b_i)' \right] V \left(I - e^{-\kappa\tau}\right) (b_j - b_i) . \end{aligned}$$

In theory, whether we can obatin a negative covariance or not depends on the arcane factor $\left[b_j' D_j (I - C_j)^{-1} - b_i' D_i (I - C_i)^{-1} - (b_j - b_i)' \right]$, the sign of which is not clear. In practice, however, Backus et al. (1998b) find extreme difficulty in trying to account for the forward premium puzzle with this class of models. ♣

While both the exponential quadratic f function and the exponential linear f function can generate affine structures for bond prices, the exponential-linear category, which has been widely used for bond pricing, has a harder time explaining the forward premium anomaly. Backus et al. (1998b), in a discrete-time set-up, find that in affine models with exponential linear pricing kernels, the forward premium anomaly either requires that the state variables have asymmetric effects on state prices in different currencies or that we abandon the requirement that interest rates be strictly positive. The exponential quadratic class of models are better suited for currency pricing for their increased degrees of freedom in specifying country-specific parameters. Since both countries share the same underlying Markov process, the only country-specific parameters in exponential-linear models is b_i in the pricing kernel, a vector with dimension equal to the number of state variables (d). However, in the exponential quadratic models, country-specific parameters include both c_i, a d-dimension vector and Q, a $d \times d$ matrix, and thus the degree of freedom is vastly increased. The advantage becomes obvious when we model international term structures although the two categories may generate similar results when one focuses on only one country.

6 Concluding Remarks

The potential approach provides a general framework for modeling interest rates and, in particular, exchange rates. Through comprehensive examples, we illustrate their correspondence, as special cases, to the classical interest rate models. We also show the relationship between the potential formulation of interest rate models and the HJM framework. The great ease of the potential approach in modeling international term structures enables us to explore the potential specifications of the state-price density to account for the forward premium puzzle in the foreign exchange market. At least two dimensions can be worked on simultaneously in the future research: On the one hand, since the direct specification of the state-price density imposes a closer link to the equilibrium economy, potential models in this paper should provide a benchmark for building practical equilibrium models in the future. On the other hand, more comprehensive empirical work needs to be done to calibrate these models to the international term structure of interest rates and exchange rates and to test which specification is best to reconcile the anomalies observed in interest rate and exchange rate markets.

A Derivations and Proofs

A.1 Resolvent representation of a potential

Let $f \in C(X)$ be an arbitrary continuous function on X. Consider the semigroup $\{T_t\}$ acting on $C(X)$ by $T_t f(x) = E_x f(X_t), x \in X, t \geq 0$. For $\lambda > 0$, write

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda s} T_s f(x) ds \;\; x \in X;$$

$(R_\lambda)_{\lambda>0}$ is called the *resolvent* (Laplace transform) of the semigroup $\{T_t\}$. A basic property of the resolvent is that λR_λ behaves like the identity operator on $C(X)$ for λ large in the sense that $\| f - \lambda R_\lambda f \| \to 0$ as $\lambda \to \infty$; i.e., $\lambda R_\lambda f \to f$ uniformly on X as $\lambda \to \infty$. With this property of the resolvent in mind, consider the process (Y_t) defined by

$$Y_t = e^{-\lambda t} R_\lambda f(X_t), \;\; t \geq 0,$$

where $f \in C(X)$ is fixed but arbitrary *nonnegative* function on X. Then (Y_t) is a super-martingale with respect to $F_t = \sigma\{X_s : s \leq t\}, t \geq 0$, since $E|Y_t| < \infty$, Y_t is F_t-measurable, one has $T_t f(x) \geq 0 (x \in X, t \geq 0)$, and

$$\begin{aligned} E\left[Y_{t+h}|F_t\right] &= e^{-\lambda(t+h)} E\left[R_\lambda f(X_{t+h})|F_t\right] = e^{-\lambda(t+h)} T_h R_\lambda f(X_t) \\ &= e^{-\lambda t} \int_0^\infty e^{-\lambda(s+h)} T_{s+h} f(X_t) ds = e^{-\lambda t} \int_h^\infty e^{-\lambda s} T_s f(X_t) ds \\ &\leq e^{-\lambda t} \int_0^\infty e^{-\lambda s} T_s f(X_t) ds = Y_t \;. \end{aligned}$$

A.2 Exponential-quadratic functions of normal variates

This section derives the expectation of the exponential-quadratic functions of normal variates. Specifically, when X_T is normally distributed with mean μ_t and variance- covariance matrix V_t, we want to derive the expectation of a general exponential-quadratic function of X_T,

$$f(X_T) = \exp\left[\frac{1}{2}(X_T - c)'Q(X_T - c)\right] = e^q,$$

where Q is assumed to be symmetric and positive-definite and q is defined as $\frac{1}{2}(X_T - c)'Q(X_T - c)$. First, we expand the q term as

$$q = \frac{1}{2}z'Az + D'z + C$$

where Σ is defined such that $\Sigma\Sigma' = V_t$, $z = \Sigma^{-1}(X_T - \mu_t)$ is a vector of standardized normal variates, $A = V_t Q$ is a $(d \times d)$symmetric and positive definite matrix, $D' = (\mu_t - c)'Q\Sigma$ is a $(1 \times d)$ row of coefficients, and $C = \frac{1}{2}(\mu_t - c)'Q(\mu_t - c)$ is a constant.

Then we can write the expectation of $f(X_T)$ in the integral form:

$$E[f(X_T)] = (2\pi)^{-\frac{1}{2}d}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left[\frac{1}{2}z'Az + D'z + C - \frac{1}{2}z'z\right]\prod dz_i \tag{A.1}$$

We perform an orthogonal transformation on z such that

$$z = Gx \tag{A.2}$$

where G is an orthogonal matrix: $GG' = I$. Substitute (A.2) into (A.1), we have

$$E[f(X_T)] = (2\pi)^{-\frac{1}{2}d}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left[-\frac{1}{2}x'(I - C)x + u'x + C\right]\prod dx_i, \tag{A.3}$$

where the Jacobian is unity since G is orthogonal and $u' = D'G = (\mu_t - c)' Q\Sigma G$. Since $C = G'AG$ is a diagonal matrix, the right hand side of (A.3), apart from the constant $(\exp C)$ term, factorizes into d single integrals of type

$$I_j = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}(1-c_{ii})\, x_j^2 + u_j x_j\right] dx_i,$$

which can be obtained directly from the moment generating function of the univariate normal distribution as

$$I_j = (1-c_{jj})^{-\frac{1}{2}} \exp\left(\frac{1}{2}\frac{u_j^2}{1-c_{jj}}\right). \tag{A.4}$$

Applying (A.4) to each term in (A.3) gives

$$\begin{aligned} E\left[f\left(X_T\right)\right] &= \prod_{j=1}^{d}(1-c_{jj})^{-\frac{1}{2}} \exp\left[\frac{1}{2}\sum_{j=1}^{d}\frac{u_j^2}{1-c_{jj}} + C\right] \\ &= |I - QV_t|^{-\frac{1}{2}} \exp\left[\frac{1}{2}(\mu_t - c)'\left(I - QV_t\right)^{-1} Q\left(\mu_t - c\right)\right]. \end{aligned}$$

A.3 Bessel processes

In general, a multi-dimensional (d) Bessel process can be written as

$$dX = \kappa\left(\theta - X\right) dt + \Sigma\sqrt{X} dW$$

where κ is a $(d \times d)$ matrix and Σ is a diagonal positive definite matrix (In the context, we scale $\Sigma = I$).

The properties of one-dimensional bessel process is well-known, see, for example, Cox et al. (1985). X has a conditional non-central chi-square distribution with the characteristic function given by

$$
\begin{aligned}
\phi(x,\tau;s) &= E_0[\exp(isX_\tau] \\
&= \left|1-\frac{is\sigma^2}{2\kappa}\left(1-e^{-\kappa\tau}\right)\right|^{-\frac{2\kappa\theta}{\sigma^2}} \exp\left(\frac{isxe^{-\kappa\tau}}{1-\frac{is\sigma^2}{2\kappa}\left(1-e^{-\kappa\tau}\right)}\right) \\
&= |1-2it|^{-\frac{v}{2}} \exp\left(\frac{it\delta}{1-2it}\right),
\end{aligned}
$$

with the shape parameter $v=\frac{4\kappa\theta}{\sigma^2}$, the noncentrality parameter $\delta=\frac{4\kappa x e^{-\kappa\tau}}{\sigma^2(1-e^{-\kappa\tau})}$, and $t=\frac{s\sigma^2}{4\kappa}\left(1-e^{-\kappa\tau}\right)$. With $\tau\to\infty$, we have the unconditional characteristic function

$$
\phi(x,\infty;s)=\left|1-\frac{is\sigma^2}{2\kappa}\right|^{-\frac{2\kappa\theta}{\sigma^2}}=(1-isb)^{-a},
$$

where $a=\frac{2\kappa\theta}{\sigma^2}$ and $b=\frac{\sigma^2}{2\kappa}$, which is the characteristic function of a gamma distribution with mean $ab=\theta$ and variance $ab^2=\frac{\theta\sigma^2}{2\kappa}$.

For a d-dimensional Bessel process x, the characteristic function of its linear combination can be obtained through orthogonalization:

$$
y=b'x=c'z
$$

such that the elements are z are independent from each other with

$$
dz=\hat{\kappa}\cdot(\theta-z)\,dt+\Sigma\sqrt{z}dW,
$$

where $\hat{\kappa}$ is a diagonal matrix with i-th element being $(\kappa' b)_i/b_i$. Now the problem is reduced to that of an affine function of independent Bessel processes. We have

$$
\phi(y,\tau;s) = \prod_{i=1}^{d}\phi(x_i,\tau;sb_i)
$$

$$
\begin{aligned}
&= \prod_{i=1}^{d} \left| 1 - \frac{isb_i \sigma_i^2}{2\hat{\kappa}_i} \left(1 - e^{-\hat{\kappa}_i \tau}\right) \right|^{-\frac{2\hat{\kappa}_i \theta_i}{\sigma_i^2}} \exp\left(\sum_{i=1}^{d} \frac{isb_i x_i e^{-\hat{\kappa}_i \tau}}{1 - \frac{isb_i \sigma_i^2}{2\hat{\kappa}_i} \left(1 - e^{-\hat{\kappa}_i \tau}\right)} \right) \\
&= \left| (I - isC)^{\bullet(-A)} \right| \exp\left(isb' D \left(I - isC\right)^{-1} x \right)
\end{aligned}
$$

where "$\bullet$" denotes "dot power operator," and A, C, and D are diagonal matrices with

$$
\begin{aligned}
A_{ii} &= \frac{2\hat{\kappa}_i \theta_i}{\sigma_i^2}; \\
C_{ii} &= \frac{isb_i \sigma_i^2}{2\hat{\kappa}_i} \left(1 - e^{-\hat{\kappa}_i \tau}\right); \\
D_{ii} &= \exp\left(-\hat{\kappa}_i \tau\right).
\end{aligned}
$$

The conditional mean $\mu(y_\tau)$ and variance $V(y_\tau)$ are thus

$$
\begin{aligned}
\mu(y_\tau) &= b' \left(I - e^{-\tau\hat{\kappa}}\right) \theta + b' e^{-\tau\hat{\kappa}} x; \\
V(y_\tau) &= \sum_{i=1}^{d} \theta_i \frac{b_i^2 \sigma_i^2}{2\hat{\kappa}_i} \left(1 - e^{-\hat{\kappa}_i \tau}\right)^2 + 2 \sum_{i=1}^{d} x_i \frac{b_i^2 \sigma_i^2}{2\hat{\kappa}_i} \left(1 - e^{-\hat{\kappa}_i \tau}\right) e^{-\hat{\kappa}_i \tau}.
\end{aligned}
$$

The unconditional moments are

$$
\mu(y) = b'\theta; \quad V(y) = \sum_{i=1}^{d} \theta_i \frac{b_i^2 \sigma_i^2}{2\hat{\kappa}_i}.
$$

The variance of the orthogonalized vector $z(t)$ is

$$
V(z) = \operatorname{diag}\left[\frac{b_i^3 \sigma_i^2}{2\left(\kappa' b\right)_i} \theta_i \right].
$$

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