A Dynamic Model of the Incorporation of New Information into Prices

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Abstract:

A representative investor in a competitive financial market is uncertain about the true state of the economy. This uncertainty is reflected by a probability distribution of values which the investor forms subjectively based on information that is arriving randomly. The subjective distribution of values is updated by the investor's learning process, which systematically lowers the perceived probability of events economically different from the latest news and increases the believed likelihood of values consistent with this information. By following this learning process, the investor's subjective distribution becomes identical (in an expected sense) to the true distribution of values. The model shows that if there is a regime shift in the true world, the subjective distribution begins an adjustment process which ends again with the true and subjective distributions equal. In increase in real volatility causes no change in the expected subjective mean, but induces an increase in the volatility of the subjective probability function. An increase in the mean of the true world causes the subjective mean to increase, but also causes a temporary increase in the subjective volatility.

Most studies of financial markets have concentrated on the observed characteristics of the prices generated by the market. Some authors relate their results *ex ante* to a sensible description of the macro or micro forces that appear to be driving the results. This paper takes the reverse tact. We begin with a sparse model of the dynamic internalization of information into the market. The model predicts market responses to exogenous forces, which can be verified by follow-up empirical studies.

Evaluation of Current Information

Consider a representative investor who is considering purchasing a share in a portfolio that has been formed to match a particular index of the market.[1] The investor intends to make the purchase if the market price of the portfolio is less than the investor's assessment of its proper value. However, the investor is uncertain as to the exact state of the economy and, thus, value of the portfolio. Represent this uncertainty by defining the set of all possible states of the economy, $\{S_j\}$. Associated with each state, S_i, is the investor's subjective probability that this actually is the condition of the economy. Therefore, we have a large set of pairs – states coupled with probabilities. $\{S_j,\pi_j\}$.

The representative investor evaluates each state and assigns a present value to current and future prospects stemming from that particular state. Denote the valuation function and the resulting value as: $\upsilon(S_j) = v_j$. The investor includes all available information in forming the evaluation including, if appropriate, the present and expected future prices of the assets in the portfolio. The investor accepts the evaluation process as a reasonable mapping of a particular point in the state space into the value space. The uncertainty faced by the investor does not come from failure of the evaluation process, but from the lack of full knowledge as to the true condition of the economy. The evaluation process is viewed as a determinate mathematical function which is defined for each point in the state space. The state space provides the values for the parameters used in the evaluation function.

The problem can be simplified further by collapsing the set of all states into a set of pairs defined by the equivalence class of those states which have the same present value. This step reflects the notion that the economic outcome to the investor is the same whether the investor selects the correct state or selects another state which happens to have the identical economic value. A specific present value and the probability that the actual state of the world is among

[1] This discussion is cast in terms of the market as a whole represented by an index. The analysis applies equally to the evaluation of a single asset purchased in a competitive market.

those states with that value generates a probability distribution defined by the pairs of value and probability.

$$\{ (v_i, \pi_i) : \text{for all j for which } \upsilon(S_j) = v_i \, , \, \pi_i = \Sigma_j \, \pi_j \}$$

This distribution will be called the *subjective distribution of values* because the probability associated with each partition of the state space was determined subjectively. The subjective value distribution of subjective probabilities clearly depends on the information available at each time, Θ_t. Then, the complete representation of the subjective distribution function is:

$\pi_{it} = f(v_i \mid \Theta_t)$ The set over which the index **I** varies can be finite or infinite.

Incorporation of New Information

Information is arriving randomly over each time period. If the information is "good" then those states associated with higher dividends, growing market share, etc. (and consequently higher evaluations) are now taken by the investor to have a higher probability of being the true economic condition of the world; and the low value states will be assigned smaller probabilities by the investor. The investor systematically incorporates arriving information into the investment decision by reassessing probabilities subjectively assigned to states, which causes the distribution of evaluations and probabilities to change. This is a somewhat different, but ultimately equivalent, approach from the view that an equity exists and that the uncertainty stems from the assignment of the value to that equity. The approach of this model begins with an *attribute* space modulo a definitive present value function. This formulation is in contrast with a model based on a *commodity* space of portfolio shares valued indeterminately.

Formalize the mechanism by which information is incorporated into the subjective distribution of values by:

$$f(v_i \mid \Theta_{t+1}) = (1 - \delta) \; f(v_i \mid \Theta_t) + s(\dot{\upsilon}) \qquad (1)$$

Where,

$\dot{\upsilon}$ is the value of state S_i and this state is the condition of the economy which is indicated by the information which arrived at time **t + 1**.

δ is a positive constant not greater than one.

and

$$s(\dot{\upsilon}) = \delta / 2\varepsilon \text{ when } \dot{\upsilon} - \varepsilon \leq \dot{\upsilon} \leq \dot{\upsilon} + \varepsilon$$

$$\begin{cases} \\ \\ 0 \quad \text{otherwise} \end{cases}$$

and ε is an arbitrarily selected, positive constant.

The parameter δ reflects the rate at which the investor incorporates newly arriving information into the currently held subjective distribution of values. δ also is the amount of probability mass that is shifted from the entire distribution and concentrated in the neighborhood of the value consistent with the just-arrived information. The term, $(1 - \delta)\, f(v_i \mid \Theta_t)$, represents a proportional reduction in subjective probability attached to all states, and equivalently to all equivalence classes of the states of the economy. $(1 - \delta)$ can be interpreted as the degree to which the investor's confidence in the prior subjective distribution has been decreased by the arrival of new information.

The new information is consistent with the belief that state S_i represents the true state of the economy. This state has value ῡ, which is interpreted as the true value of the portfolio indicated by the marginal current information. A neighborhood in the value space about the point ῡ is defined by ε. This region is ῡ± ε. The probability mass **δ** that was taken from the entire distribution is uniformly distributed over this interval.

The entire process captured by equation 1 asserts that the subjective likelihood is reduced for states differing in value from the state most consistent with new information. The state indicated by the most recent information is **not** taken to be the true condition; however, the probability associated with that state or states quite similar, as to economic value, are assigned higher levels of subjective probability.

It certainly is accurate to assert that this is neither the most encompassing nor specific model possible. We could attach all of the transferred probability mass to the value ῡ consistent with the new information. (This would make the most sense if value is measured by discrete numbers.) Alternatively, it might be more realistic to assert that a large amount but not all probability mass should be shifted to ῡ and that a positive but smaller mass should be shifted to values near ῡ. As the distance from ῡ in the value space increases, the amount of probability mass which is added will decrease. Also, states with values quite different from ῡ are economically quite distinct from that state which is consistent with the new information; therefore, those distant states are unlikely to be the true state and a large portion of transferred probability mass is taken from these values. In other words, the transfer of probability mass should be less discrete and should be proportional to the economic distance from the state indicated by the latest news.

Another amplification to the model might be to attach a measure of reliability to each new piece of information. That message deemed to be more likely to be correct would be assigned a higher learning rate. All of these modifications to the model produce exactly the same qualitative results but involve much more complexity. Therefore, the model employed will be

taken to be a useful representative of a large class of similar constructions all of which accomplish the same ultimate result.

Unification of subjective and actual probability distributions

The subjective distribution of values comes into consistency with the information that is arriving using the following adjustment mechanism. Assume that the true world has not changed, but that the investor has not chosen subjective probabilities which are consistent with the true word. In particular, assume that the investor is too pessimistic. This is reflected by a subjective distribution with a mean which less than the true value.[2] In the ensuing periods some news will be bad and some good. However, an abnormally high proportion of new information will be better than the perceived average and associated with valuations which are higher than the undervalued subjective distribution mean value. Bad news, relative to the mean of the subjective distribution tends to lower the subjective mean and good news raises the mean. As long as the subjective mean differs from the mean suggested by the information flow feedback, on the average, will adjust the perceived mean toward the true value.

The above discussion describes a divergence between the true environment and the perceived condition captured by the subjective distribution of values. We might ask whether or not the true environment is a single state, and of course associated with that state a unique true value. If there were a single true state, then all new information should signal a single value – the true value. Investors would immediately detect any change in the true state of the economy from the first new signal and would respond with a new subjective distribution that assigns all likelihood only to that unique value. We do not believe that such a model accurately captures the uncertainty associated with attaching a value to a financial asset. There is some uncertainty in

[2] Here and in the following the "true" value is the mean of the distribution of values which are observed. That characteristic which does **not** possess "the quality of being observable" will in this paper be assumed to not have an impact on market prices. Thus, "truth" and "observability" are inextricably joined in this model of information flow and subsequent economic actions.

the process, the issue is how to capture this in the most realistic fashion.[3]

If information were perceived with added noise or if information arrived with some degree of inaccuracy, then a distribution of information signals would be a more appropriate model. If there were a distribution of noise terms added to each new unit of information, then it can be shown that the noise must have mean zero if investors use prior information and, conditioned on past information, new noise must be generated by an unchanging random process. This means that if the information has value even when there is noise mixed in, there must be a specific stochastic structure to the random noise. However, this begs the question of the true source of this noise or error and, even more restrictively, what forces must prevail to give the noise the certain characteristics required to preserve the economic value of the information contained in the news. Finally, since noise must have this structure, if there is any averaging associated with information processing by the investor, the noise term will disappear in the limit. We believe that "noise-driven" model has too many implicit restrictions that have not been shown to mirror the normal decision making processes or an identified source of error. We have chosen not to employ a "noise" type of model.

We might assert that the economy is in a single true state at any moment, but that there are constant minor fluctuation about that level. As the economy moves to adjacent points in the state space, it is most likely that there are similar small movements in the value space. Similarly, there is a relatively small chance of a large deviation from the original state; and therefore, small probabilities of observing values much different from the value of the "base" state. However, this explanation does not give a reason for the causes of the minor deviations from the basic state condition; nor is there an obvious reason for the economy to return to the original state, although that is necessary for such a model to operate. We have chosen not to base our model on randomness in the state space either.

Rather than focus on errors in information or deviations in the state space, a better approach would be to consider the functioning of news. A specific message gives information about only a small portion of the many dimensions of the state space. For example, suppose a drug company reports higher earnings. Current earnings are only one aspect of this single firm. Even at the firm level, each message does not define the state space perfectly. This message of high earnings is consistent with numerous states, not all economically equivalent. For example, a good earnings report would be consistent with many revenue growth paths, with a cost containment or efficiency program within the company, with market penetration of a new product, etc.

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[3] If, however, the best description of the investor's reaction to new news is to accept the latest data as completely accurate, then that representative investor would be captured by setting δ equal to one and ε (in the limit) to zero.

Among the family of states consistent with the current message is a mean state or a most probable state and associated with that state is a specific value. This is $\dot{v}$ in equation 1. Other messages reaching the investor may be reiterations of the prior message, but probably relate to other aspects of the economy. The level of interest rates, the clearance of a competitor's drug, an unexpected change in the number of people that are in a population group likely to use this firm's products, etc. All of this information relates to but is distinct from the original message, yet each piece of news is likely to indicate similar but not identical true values.

To continue this line of thinking, assume that the true state of the economy were unchanged. Each piece of information received by the investor can be viewed as a relatively accurate projection of the state space onto a subspace defined by the thrust or focus of the news. Those news items which relate to a specific aspect of a particular firm can be thought of as projecting onto the real line representing the measurement in the state space of that attribute of that firm. The news will be most consistent with a particular value along that line. The news-revised value for this attribute may have an impact on other values in the state space. Together, the revised values represent the new most consistent state of the economy. From that state is computed the value $\dot{v}$. The news could be more complex or could affect many attributes in the state space; however, the state most consistent with the news, everything considered, produces the newly-computed value. If the world has not changed, then the frequency distribution of values signaled by each new piece of news produces the *true observed density function*.

In the world just described, the investor receives and evaluates relatively accurate information about an aspect of the firms or customers in the market. The true state is unknown and, in this complex state space, unknowable. However, the investor is aware that there is uncertainty about the true state and (unlike uncertainty in the sense of Frank Knight) attaches tentative probabilities to each state and freely adjusts those probabilities in the light of additional information. The observed density function is based on the frequency of values produced by news which reveals accurate but only partial information about the true state. The subjective distribution of values is based on the representative investor's assignment of probabilities based on the learning mechanism defined in equation 1.

Let us denote the probability distribution of valuations that arrive, given that the true state does not change, as $\varphi(v_i)$. Note that this distribution is "true" not because the world actually follows this distribution, but because this is the distribution of values produced by various news messages all associated with the true state. No time subscript is required because the true state is not changing. Particular arriving messages produce different values, and the fact that some values are more likely to be generated than others is captured by the density function $\boldsymbol{\varphi}$.

We already have seen that if the mean of the subjective distribution μ_f is less than the mean of the true distribution μ_φ, the information incorporation process defined by equation 1 will cause the subjective mean to increase toward the true mean. In fact, the subjective distribution will approach the true distribution in the unique sense that in an arbitrarily small neighborhood of

every value $\mathbf{v_i}$, the expected value of the subjective distribution is equal to the value of the true distribution.[4] The perceived world eventually will statistically approach the true world. If the investor is "too pessimistic," subjective valuations will rise on the average. If the investor perceives the degree of uncertainty inherent in the world is greater than it actually is, that also will correct itself (in the sense of expectation).

The linkage between the subjective distribution of values and the market price

Given the set of perceived values and probabilities, the representative investor is willing to pay no more than a certain price for a share of the market portfolio. If the market price is less than this demand price, the investor will continue to purchase shares in the portfolio until the market price is equal to the demand price.

It is appropriate to assume that the representative investor is risk averse; therefore, the market price will be somewhat less than the mean of the subjective distribution of values. Without significant loss of generality, we will assume that the difference between the market price and the mean of the subjective distribution is a function of the volatility of values which will be represented by the variance of the subjective distribution.

[4] In subsequent discussion any statement about the equality of distributions or parameters of functions will be in the sense of expectation as opposed to functional equivalence or identity.

When the latest news is "good" – that is, reflects a value greater than the present subjective mean – the mean of the subjective distribution after the arrival of the news (and the partial incorporation of that information into the revised subjective distribution) must increase also. This implies that the market price increases, assuming for the moment that the variance of the subjective distribution does not change.[5]

There is connection between the subjective distribution of values and the distribution of market prices. To see this, assume that at time **t + 1** information arrives which is consistent with the market value $\dot{\upsilon}$. The expected mean of the subjective distribution is:

$$\mu_{f(t+1)} = \int_{-\infty}^{\infty} v\, f(v \mid \Theta_{t+1})\, dv$$

$$\mu_{f(t+1)} = \int_{-\infty}^{\infty} v\, f(v \mid \Theta_{t})\, dv + \int_{-\infty}^{\infty} v\, (\delta/2\varepsilon)\, dv$$

$$\mu_{f(t+1)} = (1 - \delta)\, \mu_{f(t)} + (\delta/4\varepsilon)\, v^2 \Big|_{\dot{\upsilon} - \varepsilon}^{\dot{\upsilon} + \varepsilon}$$

$$\mu_{f(t+1)} = (1 - \delta)\, \mu_{f(t)} + \delta\, \dot{\upsilon} \qquad (2)$$

The expected value of the information arriving in **t + 1** is $\mu_{f(t)}$. Therefore, the mean value of the true distribution of values is the same as the expected mean value of the subjective distribution and, if the representative consumer is risk-neutral, also equal to the expected mean value of the distribution of market prices. Since the market price, just as the subjective distribution which actually determines this price, is buffeted by arriving information, it is possible only to make assertions in terms of expected values. Finally, if the representative investor were risk averse, the market price would be less than the expected mean of subjective values by the necessary risk premium.

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[5] The relation between the mean and the variance of the subjective distribution is complex and will be considered in detail below.

Return to equation 2. If, as expected, the new information has a value equal to the existing expected mean value, there is no change in the mean price (under risk-neutrality). When information with value above the mean arrives, the new price is a linear combination of the previous subjective mean and the new value. The weight to be attached to each (the mean and the new value) depends on the rate of learning **δ** or the rate of maintaining previously-held beliefs in light of new information **(1 - δ)**.[6]

This section has provided the linkage among three distributions. The true distribution of values and the subjective distribution are equal in an expected sense. The distribution of prices has the same expected mean but deviations from that mean which are smaller by a constant factor of proportionality (up to additive shifts due to the risk premium). All of the discussion has assumed that the underlying state of the economy has not changed. The following section will describe the effects of a systematic change in the true distribution. Since the distribution of observed prices is tightly linked to the subjective distribution of value, the intermediate and long-

[6] If, instead, the investor were risk averse, there would be no change in the true or subjective distributions. The risk premium would be positive for this risk-averse investor, but the risk premium would have a constant expected value. The dispersion of prices would increase because of variations in the risk premium. The effect of increased risk aversion is a systematic decrease in the market price and an increase in variance of prices.

Since news flow and observed market prices are random, it might be claimed that this is an additional aspect of risk to be considered. Two points are appropriate in response. (1) The investor in this model observes information and determines value. If the distribution of information and prices were important, then these are a part of the state space and would be included in the evaluations. However, these distributions have not changed; so any risk premium they must earn does not change. (2) The variations of observed prices about an unchanging expected mean is similar to the "noise" contained in market prices in models which begin with observed prices. In those models, statistical noise generally does not enter into risk measures.

term effects on either distribution will be comparable.

Effects of a Positive Regime Shift in Values

A positive regime shift will be modeled by assuming that all values of the true distribution are increased by a positive, additive factor **η**. Suppose that the true distribution in time **t = 0** is given by:

$\varphi_t(v_i) = \varphi_0(v_i)$

A regime change occurs at time **t + 1**, so that:

$\varphi_{t+1}(v_i) = \varphi_0(v_i + \eta)$

Then at time **t + 2**, the mean of the subjective distribution is:

$$\mu_{f(t+2)} = \int_{-\infty}^{\infty} v\, f(v \mid \Theta_{t+2})\, dv$$

$$\mu_{f(t+2)} = \int_{-\infty}^{\infty} v\,(1-\delta)\, f(v \mid \Theta_{t+1})\, dv + \int_{-\infty}^{\infty} v\, s(v_{t+2})\, dv$$

$$\mu_{f(t+2)} = \int_{-\infty}^{\infty} v\,(1-\delta)\, f(v \mid \Theta_{t+1})\, dv + \int_{-\infty}^{\infty} v\, s(v_{t+2})\, dv$$

$$\mu_{f(t+2)} = (1-\delta) \int_{-\infty}^{\infty} v(1-\delta)\, f(v \mid \Theta_t)\, dv + (1-\delta) \int_{-\infty}^{\infty} v\, s(v_{t+1})\, dv + \int_{-\infty}^{\infty} v\, s(v_{t+2})\, dv$$

$$\mu_{f(t+2)} = (1-\delta)^2\, \mu_{f(t=0)} + (1-\delta) \int_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon} v\, s(v_{t+1})\, dv + \int_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon} v\, s(v_{t+2})\, dv$$

Note: the $\dot{\upsilon}$ are produced by the true distribution of values *after* the regime shift.

$$\mu_{f(t+2)} = (1-\delta)^2\, \mu_{f(t=0)} + (1-\delta)(\delta/4\varepsilon)\, v^2 \Big|_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon} + (\delta/4\varepsilon)\, v^2 \Big|_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon}$$

$$\mu_{f(t+2)} = (1-\delta)^2 \ \mu_{f(t=0)} + (1-\delta)\,\delta \ \mu_{f(t=1)} + \delta \ \mu_{f(t=1)} \qquad (3)$$

Equation 3 shows the first two steps of the adjustment process. The new expected mean of the subjective distribution of prices (and the expected mean of observed prices) shifts upward toward the shifted true mean. The subjective distribution approaches the true distribution at a rate which is tied directly to the learning parameter **δ**. All expected probabilities of the subjective distribution approach those of the true distribution, and the mean of the subjective distribution approaches the mean of the true distribution.

Effect of a Volatility shift on the Mean

The previous section considered an additive shift in the mean which left the volatility of the true distribution unchanged. Suppose that in time period **t + 1** the mean of the true distribution did not change, but the volatility about the mean increased. The mean of the subjective distribution at time **t + 2** is:

$$\mu_{f(t+2)} = \int_{-\infty}^{\infty} v\, f(v \mid \Theta_{t+2})\, dv$$

$$\mu_{f(t+2)} = (1-\delta) \int_{-\infty}^{\infty} v(1-\delta)\, f(v \mid \Theta_t)\, dv + (1-\delta) \int_{-\infty}^{\infty} v\, s(v_{t+1})\, dv + \int_{-\infty}^{\infty} v\, s(v_{t+2})\, dv$$

$$\mu_{f(t+2)} = (1-\delta)^2 \ \mu_{f(t=0)} + (1-\delta) \int_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon} v\, s(v_{t+1})\, dv + \int_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon} v\, s(v_{t+2})\, dv$$

$$\mu_{f(t+2)} = (1-\delta)^2 \ \mu_{f(t=0)} + (1-\delta)\,\delta \ \mu_{f(t=0)} + \delta \ \mu_{f(t=0)} = \mu_{f(t=0)} \qquad (4)$$

A change in volatility has no effect on the expected value of the mean of subjective values. However, as we shall see below, the volatility of the subjective distribution will increase to match that of the true distribution. In turn, the mean of the observed market prices should decrease for risk-averse investors as the risk premium increases in response to the volatility change.

Effect of a Regime Shift in Volatility on the Volatility of the Subjective Distribution

Assume that the true, subjective, and price distributions all are in their stochastic steady state at time **t = 0**. Then, following Rothschild and Stiglitz [1], increase the volatility of true values by transferring probability mass from the center of the true density function equally toward each tail of the distribution. This transform, a "mean preserving spread," occurs at time **t = 1**.

Without loss of generality, we shall give specific characteristics to this transformation. Assume that the amount of probability mass shifted is $\mathbf{2\tau}$. Define a region ς about the mean of the true distribution such that:[7]

$$d\tau/dv = \text{a constant} \leq \varphi_{t=0}(v) \text{ for all } v \text{ in } \varsigma$$

and

$$\int_{\varsigma} [\varphi_{t=0}(v) - \varphi_{t=1}(v)]\, dv = 2\tau$$

$$\int_{\text{maximum } v \leq \mu_\varphi \text{ and } v \text{ not in } \varsigma}^{\mu_\varphi} [\varphi_{t=0}(v) - \varphi_{t=1}(v)]\, dv = \tau$$

These conditions require that the reduction in probability be equal on either side of the mean and that for no state value does the true probability ever become negative. In a similar fashion, the probability mass is moved to a region Λ.

$$\varphi_{t=0}(v) - \varphi_{t=1}(v) = \text{the constant } d\tau/dv$$

$$\int_{-\infty}^{\text{maximum } v < \mu_\varphi \text{ and } v \text{ in } \Lambda} [\varphi_{t=1}(v) - \varphi_{t=0}(v)]\, dv = \tau$$

and

$$\int_{\text{minimum } v > \mu_\varphi \text{ and } v \text{ in } \Lambda}^{\infty} [\varphi_{t=1}(v) - \varphi_{t=0}(v)]\, dv = \tau$$

and

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[7] We let $\boldsymbol{\tau}$ alternatively be the cumulative density function or the value of that function. Then, $d\tau/dv$, is the density function.

$$\int_{-\infty}^{x}\int_{-\infty}^{\infty} [\,\varphi_{t=1}(v) - \varphi_{t=0}(v)\,]\,dv \geq 0 \text{ for all } x$$

The first condition insures that the probability mass removed from the region of the density function for values less than the mean value is completely shifted to another region on that side of the mean. Similarly, there is a one-for-one shift in probability mass for values above the mean. The last condition requires that the net effect of the shift is to move probability away from the mean and toward the tails of the distribution.

Now consider the expected subjective distribution in time **t = 1** after the mean preserving spread has increased volatility of the true distribution. The expected value of the subjective distribution function of values at any specific value **ω** is:

$$E(\,f(\omega|\,\Theta_{t=1}) = E\,[\,(1+\delta)\,f(\omega|\Theta_{t=0}) + s(v)\,]$$

$$E(\,f(\omega|\,\Theta_{t=1}) = (1+\delta)\,E\,[\,f(\omega|\Theta_{t=0})\,] \quad \text{if} \quad v-\varepsilon \geq \omega \text{ or } \omega \geq v+\varepsilon$$

$$E(\,f(\omega|\,\Theta_{t=1}) = (1+\delta)\,E\,[\,f(\omega|\Theta_{t=0}) + \delta/2\varepsilon\,] \quad \text{if} \quad v-\varepsilon \leq \omega \leq v+\varepsilon$$

$$E(\,f(\omega|\,\Theta_{t=1}) = (1+\delta)\,E\,[\,f(\omega|\Theta_{t=0})\,] + \delta/2\varepsilon \text{ probability}\,(\,v-\varepsilon \leq \omega \leq v+\varepsilon\,)$$

A first-order approximation of the average probability over this interval is the probability at the value ω. Since the constant ε which defines the neighborhood can be selected to be sufficiently small, this first-order approximation can be made as accurate as desired.

Then, the expected subjective function after the increase in volatility is:

$$E(\,f(\omega|\,\Theta_{t=1}) = (1+\delta)\,E\,[\,f(\omega|\Theta_{t=0})\,] + \delta/2\varepsilon\,(\,2\varepsilon\,E(\,f(\omega|\,\Theta_{t=1})\,)$$

$$E(\,f(\omega|\,\Theta_{t=1}) = (1+\delta)\,E\,[\,f(\omega|\Theta_{t=0})\,] + \delta\,(\,E(\,f(\omega|\,\Theta_{t=1})\,)$$

The probabilities at each subjective value is a linear combination of the expected probability for that value at time **t = 0** and the expected probability at time **t = 1**.

This relationship, which thus far has not been related to the mean preserving spread, reasserts our previous results. If the true density function is unchanged, the subjective density function approaches the true density in expectation. If the true density is unchanged, then the expected values of the subjective density function also are unchanged.

Now, consider the specific shape of the subjective distribution of values one time period

after the true distribution has undergone a mean preserving spread. We define four regions of the density function:

For values of ω **not** within the set ς, from which probability mass was removed by the spread transform, and not within the set Λ, to which probability mas was transferred, and not near the boundaries of these regions, then the spread neither added nor subtracted probability mass which means that $E(f(\omega| \Theta_{t=1}) = E [f(\omega|\Theta_{t=0})]$ and the linear combination of prior and subsequent expected density functions is:

$$E(f(\omega| \Theta_{t=1}) = (1 + \delta) E [f(\omega|\Theta_{t=0})] + \delta (E(f(\omega| \Theta_{t=1})))$$

$$E(f(\omega| \Theta_{t=1}) = (1 + \delta) E [f(\omega|\Theta_{t=0})] + \delta (E(f(\omega| \Theta_{t=0})) = E [f(\omega|\Theta_{t=0})]$$

This equation asserts, except for those values near the boundary, the subjective distribution is not changed by the spread for those values not affected by the spread.

Within and not near the boundaries of the region ς, that part of the true distribution from which probability mass was removed,

$$E(f(\omega| \Theta_{t=1}) = (1 + \delta) E [f(\omega|\Theta_{t=0})] + \delta [E(f(\omega| \Theta_{t=0}) - d\tau/d\omega)]$$

$$E(f(\omega| \Theta_{t=1}) = E [f(\omega|\Theta_{t=0})] + \delta (- d\tau/d\omega)$$

That part of the subjective distribution function corresponding to the region of the true density function near the mean (from which probability mass was removed) has an expected density function which is a linear combination of the prior expected density function and the amount by which the spread subtracts probability mass. Similarly, in the region (but not near the boundaries) of the subjective distribution function matching those area to which the mean preserving spread transfers probability mass, the new expected probability is a combination of the prior expected probability plus a part (depending on δ) of the mass added by the spread.

This discussion has been cluttered by phrases excluding values near the boundaries. Those values within a distance of **ε** measured along the value axis, have a fraction of a fraction of the impact of the spread added or subtracted. Suppose we consider points near the boundary of the region from which the spread subtracts mass. Within the region, the expected value of the subjective density is $E [f(\omega|\Theta_{t=0})] + \delta (- d\tau/d\omega)$, the prior expected value of the density plus a "learned" portion of the effects of the spread. Within the region not affected by the spread, the expected subjective density is $E [f(\omega|\Theta_{t=0})]$. There is a band centered on the boarder between the two regions over which the effect of the spread's actions must be further reduced to reflect the fact that some values of v within the interval from ω - ε to ω + ε are included in the band over which probability mass is transferred by the learning process but are not in the region from which

probability mass was subtracted by the mean preserving spread.

Finally, a check of the subjective distribution of values after the increase in volatility in the true distribution shows that the expected subjective distribution also has been transformed by a mean preserving spread. (Note that the distribution is defined by expected values and a first order approximation was employed.) If the expected probabilities were shown for the subjective distribution in time **t = 2**, further learning would be apparent. In the limit, the expected subjective distribution would approach the true distribution. For a risk-neutral representative investor, the expected distribution of market prices also would have a spread transformation that corresponds to the increased volatility in the true density function. If the representative consumer were risk averse, the risk premium would increase because the subjective distribution prior to the spread transform is superior to the transformed distribution by second degree stochastic dominance. The expected distribution of market prices would reflect the spread also, but the mean of the market price would now be even less than the mean of the subjective value distribution because volatility and risk have increased.

Effect of a Regime Shift in the Mean on Intermediate Volatility:

Volatility, inherently being a symmetric function, can change in the real world and will leave the mean of the subjective distribution and the expected (risk neutral) mean market price unchanged. Even over the period during which the subjective distribution is evolving toward the real density, the expected (risk neutral) mean price will not change although actual prices will appear randomly above and below the expected mean. In a similar fashion, increased real volatility will induce a corresponding increase in the expected volatility of market prices. There will be a transition period during which the observed volatility is less than that corresponding to the real distribution; however, as the subjective distribution continues to adapt to new information, price volatility will again tend to correspond to the volatility of real values as signaled by observed news.

A systematic increase in real value will induce a corresponding shift in market price. Once agin, during the adaption process, the expected mean price will adjust between the previous value and the increased number. It might be anticipated that an additive increase in real values would cause no change in the volatility in observed market prices for a risk neutral representative investor and only a change in market volatility associated with the effect of wealth on the risk premium for a risk-averse representative investor. However, the interaction is much more complex. Perceived risk initially will increase and then fall back to the original level.

Consider a representative investor with a subjective distribution of values which is consistent with the real distribution. That is: $E(f(v)) = \varphi(v)$ for all v. Denote an initial time period with the subscript 0. Suppose that a additive regime shift occurs in the next time period, denoted with the subscript 1. To be specific, assume that for some real number (positive, zero, or negative) Δ, $\varphi_1(v + \Delta) = \varphi_0(v)$ for every v. Then, the subjective distribution of values in time

period 1 is:

$f_1(v) = f_0(v) + s(\dot{\upsilon})$ Where $\dot{\upsilon}$ is the values observed in period 1 after the regime shift.

Continue using the subscripts 0 and 1 for the mean and variances before and immediately after the additive regime shift.

We already have seen that:

$$E(\mu_1) = E((1 - \delta)\mu_0) + \delta E(\dot{\upsilon})$$

$$E(\mu_1) = (1 - \delta) E(\mu_0) + \delta \mu_1$$

$$E(\mu_1) = (1 - \delta) E(\mu_0) + \delta (E(\mu_0) + \Delta)$$

$$E(\mu_1) = E(\mu_0) + \delta \Delta$$

The expected next-period change in the mean of the subjective distribution is the previous mean plus that amount of the shift which is "learned" in the first period.

The expression for the variance of the expected subjective distribution after the regime shift is:

$$\sigma_1^2 = \int_{-\infty}^{\infty} (v - \mu_1)^2 f_1(v)\, dv$$

$$\sigma_1^2 = \int_{-\infty}^{\infty} (v^2 - 2\mu_1 v + \mu_1^2)\, [(1 - \delta) f_0(v) + s(\dot{\upsilon})]\, dv$$

$$\sigma_1^2 = (1 - \delta) \int_{-\infty}^{\infty} (v^2 f_0(v)\, dv - 2\mu_1(1 - \delta) \int_{-\infty}^{\infty} v f_0(v)\, dv + (1 - \delta)\mu_1^2 \int_{-\infty}^{\infty} f_0(v)\, dv +$$

$$(\delta/2\varepsilon) \int_{\dot{\upsilon} - \varepsilon}^{\dot{\upsilon} + \varepsilon} v^2\, dv - 2\mu_1 (\delta/2\varepsilon) \int_{\dot{\upsilon} - \varepsilon}^{\dot{\upsilon} + \varepsilon} v\, dv + \mu_1^2 (\delta/2\varepsilon) \int_{\dot{\upsilon} - \varepsilon}^{\dot{\upsilon} + \varepsilon} dv$$

∞ ∞ ∞

$$\sigma_1^2 = (1-\delta)\int_{-\infty}^{} (v^2 f_0(v)\,dv \quad - 2\mu_1(1-\delta)\int_{-\infty}^{} v\, f_0(v)\,dv + (1-\delta)\mu_1^2 \int_{-\infty}^{} f_0(v)\,dv \quad +$$

$$(\delta/6\varepsilon)\ v^3\Big|_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon} - \mu_1(\delta/2\varepsilon)\ v^2\Big|_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon} + \mu_1^2\ (\delta/2\varepsilon)\ v\Big|_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon}$$

$$\sigma_1^2 = (1-\delta)\,\sigma_0^2 \quad - 2\mu_1(1-\delta)\ \mu_0 \ + (1-\delta)\mu_1^2\,(1) \quad +$$

$$(\delta/6\varepsilon)\ v^3\Big|_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon} - \mu_1(\delta/2\varepsilon)\ v^2\Big|_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon} + \mu_1^2\ (\delta/2\varepsilon)\ v\Big|_{\dot{\upsilon}-\varepsilon}^{\dot{\upsilon}+\varepsilon}$$

Without loss of generality, assume that the mean of the distribution before the regime shift was zero. Then,

$$\sigma_1^2 = (1-\delta)\,\sigma_0^2 \ + (1-\delta)\,\mu_1^2 \quad +$$

$$(\delta/6\varepsilon)\,[\,6\dot{\upsilon}^2\ \varepsilon\,] + (\delta/6\varepsilon)\,[2\varepsilon^2\,] \ - \mu_1\,(\delta/2\varepsilon)\,[4\dot{\upsilon}\varepsilon \quad + \quad \mu_1^2\ (\delta/2\varepsilon)\,[2\varepsilon]$$

$$\sigma_1^2 = (1-\delta)\,\sigma_0^2 \ + (1-\delta)\,\mu_1^2 \quad +$$

$$\delta\dot{\upsilon}^2 \ + \ \delta\varepsilon^2/3 \ - \ 2\mu_1\,\delta\,\dot{\upsilon} \quad + \quad \mu_1^2\,\delta \qquad (5)$$

Suppose that the shift were exactly zero. There was no shift. Then the expected value of the mean of the subjective distribution after the shift is equal to the expected mean before the shift, which is equal to zero. Also, since there has been no change in the true distribution, the expected value of the variance of the subjective distribution should be unchanged.

$$E(\sigma_1^2) = E\,[\ (1-\delta)\,\sigma_0^2 \ + (1-\delta)\,\mu_1^2 \quad + \quad \delta\dot{\upsilon}^2 \ + \ \delta\varepsilon^2/3 \ - \ 2\mu_1\,\delta\,\dot{\upsilon} \quad + \quad \mu_1^2\,\delta\,]$$

$$E(\sigma_1^2) \ = \ (1-\delta)\,E(\,\sigma_0^2\,) \ + \ (1-\delta)\,E(\,\mu_1^2\,) \ + \ \delta\,E(\dot{\upsilon}^2\,)$$

$$+ \ \delta\varepsilon^2/3 \ - \ 2\mu_1\,\delta\,E(\,\dot{\upsilon}\,) \quad + E(\mu_1^2\,)\ \delta$$

$$E(\sigma_1^2) \ = \ (1-\delta)\,E(\sigma_0^2\,) + 0 \ + \ \delta\,E(\dot{\upsilon}^2\,) + \ \delta\varepsilon^2/3 \ - 0 \ + 0$$

$$E(\sigma_1^2) = (1 - \delta) E(\sigma_0^2) + \delta E(\dot{\upsilon}^2) + \delta\varepsilon^2/3$$

The last term was induced by the learning process which shifts probability mass away from the entire distribution in a proportional fashion but concentrates mass over a rectangular interval of finite width. As the width of the region to which mass is shifted is decreased, the last term becomes smaller even more rapidly. In the limit as ε approaches zero,

$$E(\dot{\upsilon}^2) = E(\sigma_0^2)$$

Therefore, if $\Delta = 0$:

$$E(\sigma_1^2) = E(\sigma_0^2)$$

We have shown that when the regime shift is zero, the post-shift variance does not differ from pre-shift variance. We now return to the expression for the variance after the regime shift. We will show that variance increases as the absolute size of the shift increases, which means that for all real shifts:[8]

$$E(\sigma_1^2) > E(\sigma_0^2)$$

The expression for the variance after the regime shift was shown as equation 5, which is repeated here for convenience:

$$\sigma_1^2 = (1 - \delta)\sigma_0^2 + (1 - \delta)\mu_1^2 +$$

$$\delta\dot{\upsilon}^2 + \delta\varepsilon^2/3 - 2\mu_1\delta\dot{\upsilon} + \mu_1^2\delta$$

$$d(\sigma_1^2)/d\Delta = 0 + (1 - \delta) 2\mu_1 \, d\mu_1/d\Delta +$$

$$2\delta\dot{\upsilon}\, d\dot{\upsilon}/d\Delta + 0 - 2\mu_1\delta\, d\dot{\upsilon}/d\Delta - 2\dot{\upsilon}\delta\, d\mu_1/d\Delta + \delta 2\mu_1\, d\mu_1/d\Delta$$

$$d(\sigma_1^2)/d\Delta = 2\mu_1 \, d\mu_1/d\Delta - \delta 2\mu_1\, d\mu_1/d\Delta + \delta 2\mu_1\, d\mu_1/d\Delta$$

$$- 2\delta\dot{\upsilon}\, d\mu_1/d\Delta + 2\delta[\dot{\upsilon} - \mu_1]\, d\dot{\upsilon}/d\Delta$$

[8] Of course, the increase in the expected variance of the subjective distribution is a transitory effect. As the learning process continues, the expected subjective distribution will approach the true distribution.

$$d(\sigma_1^2)/d\Delta = 2[\mu_1 - \delta\dot{\upsilon}]\, d\mu_1/d\Delta + 2\delta[\dot{\upsilon} - \mu_1]\, d\dot{\upsilon}/d\Delta$$

However, already it has been shown that:

$$E(d\mu_1/d\Delta) = \delta > 0$$

and

$$E(d\dot{\upsilon}/d\Delta) = 1 > 0$$

$$E[\mu_1 - \delta\dot{\upsilon}] = [E(\delta\dot{\upsilon}) - \delta E(\dot{\upsilon})] = 0$$

$$E[d(\sigma_1^2)/d\Delta] = E\{2\delta[\dot{\upsilon} - \mu_1]\}$$

$$E[d(\sigma_1^2)/d\Delta] = 2\delta E(\dot{\upsilon}) - E(\mu_1) = 2\delta(1-\delta)E(\dot{\upsilon}) \qquad (6)$$

Since the learning parameter δ was defined as: $0 < \delta \leq 1$, the sign of the expected change in variance immediately after the additive regime shift depends on the sign of the expected value of the first news-signaled observed value.[9] If there is a positive regime shift, variance rises. Then, as the new mean is identified by the learning process, the original variance also is revealed to be the true volatility and variance of the subjective distribution falls back to its original value.

The effect of a negative mean shift on short-term variance could be demonstrated in a parallel fashion. If a negative additive shift occurs, the expected variance of the subjective distribution again increases because new price signals are systematically different from the perceived mean. The expected mean of the subjective distribution drifts toward the new true mean, and expected volatility gradually returns to the original level.

Effects of Shifts in the Mean on the Volatility of Prices:

A shift in the mean of the distribution causes a temporary increase in volatility, which causes the risk premium to increase and the difference between the mean of the subjective distribution and price to increase. To be specific, suppose that there was a downward shift in the true value of the economy. The expected subjective mean would begin to fall, causing the market price to fall. In addition, the volatility of the subjective distribution will increase accelerating the rate of decrease of the market price. As the expected mean of the subjective distribution approaches the mean of the true distribution, prices will continue to fall, although more slowly, while volatility will decrease causing the risk premium to decrease from its induced higher level and placing upward pressure on the observed market price.

[9] Recall that these values are relative to the original mean which was normalized to zero.

Similarly, if the true distribution shifts rightward, the subjective distribution will begin its rightward adjustments as the average of all observed values is above the previous subjective means. Volatility rises because the new values tend to be clustered about the new, true mean, which differs from the present subjective mean. The increased volatility causes the risk premium to rise, placing downward pressure on the observed price. If the volatility shock induced by the mean shift is great enough, theory does not preclude a *decrease* in prices as the initial response to a large increase in the true state of the economy. As the perceived mean approaches the true mean of values, volatility will fall and the risk premium will return to its previous level.[10]

Initially the volatility effect of an additive regime shift reinforces the effect on prices of a negative change and counteracts a positive shift. As volatility returns to normal, the reverse condition holds. The effect of a negative mean shift, which initially increased volatility, will be associated with a decrease in volatility (relative to the just-elevated levels) which will tend to cause prices to rise (due to the volatility effect) and also tend to continue to fall (due to the shifting of the subjective distribution mean). The final effects of a negative mean shift must be that the expected subjective mean approaches the true mean; however, the path of observed prices need not be smoothly asymptotic. With a positive regime shift, the final effects of mean and volatility reinforce each other. Any conflict between the volatility and mean effects will have occurred in the initial responses to the mean shock.

We have seen that there are systematic linkages between the mean response and the induced volatility responses. The final issue is whether one effect is so large that it dominates the impact of the other on market prices.

Note that equation equation 6 is increasing in the magnitude of the regime shift. The rate of change of volatility in response to varying regime shifts is approximately linear, which implies that volatility is roughly quadratic in the size of the regime shift. This can be verified from equation 5.

Large shocks to the mean will induce very large initial increases in induced volatility. Furthermore, if the investor's utility function reflects increasing adversity to risk, as is commonly

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[10] It should be noted that no leverage or wealth effects have been involved in the linkage between the mean of true values and observed prices. The leverage effect is captured in the evaluation process. As equity prices rise, the debt-equity ratio changes. This causes the state of the firm to change. If the debt-equity ratio is important in the evaluation process, the evauation and then the market price would change. Similarly, any other financial variable describing the firm which is linked to the price of the firm's stock would be captured by the evaluation mechanism. If there is a wealth effect, one in which an increase in equity prices will cause consumers to feel more wealthy and to change their spending patterns, this also would be reflected in the state of the economy and the evaluation of that condition.

assumed, then the market effects of a mean shock-induced temporary increase in volatility may be significant.

The magnitude of the impact of a mean shift on volatility is summarized in table 1. The table indicates the proportional change in volatility of four different magnitudes of mean shock with two learning rates. There is no basis for selecting one learning rate over another, but the shock levels were selected to correspond to the monthly volatility of the value-weighted Center for Research in Security Prices (CRISP) index of New York Stock Exchange equities. The majority of monthly equity volatility levels is less than 0.02. The table shows shock values about this level.[11]

Table 1

Size of Additive Change in Mean	Learning Rate $\delta = 0.1$	Learning Rate $\delta = 0.01$
0.01	0.926	0.992
0.02	1.005	1.001
0.05	1.557	1.062
0.10	3.530	1.279

Ratio of variance in the time period after an additive regime shift to variance before the shift

[11] Volatility levels are relevant because in the steady state, even though the expected value of new information is the mean of the true (and subjective) distribution, the average difference between the mean and the value of the most recently arriving information is the order of magnitude of the standard deviation. This implies that if, on the average, the Δ shocks are about 0.02, the expected volatility of the subjective distribution will remain constant. This is the steady state. If the average shock were smaller, the expected variance would decrease in the ultimate steady state.

Changes in the true mean value of a portfolio within the ranges of five and ten percent regularly are encountered. This computation suggests that shocks of this size *can* induce complicated and conflicting forces to act on the observed market prices as they make a transition to the new equilibrium. The exact nature of the responses cannot be determined by theory because even in this very simplified model the dynamics depend on the values of several parameters which capture the way the investor revises beliefs in response to new information.

The article, "Incorporation of News in Market Prices: High Frequency Tests of a Basic Model," employs the model outlined in this paper to measure the short-run responses to regime shifts in the mean and variance. The results in that paper show that the impulse responses proscribed by this model are actually observed. The responses are fully realized by the Standard and Poor's 500 Index extremely quickly – often within seconds. One unique feature of that paper is the isolation of standardized positive and negative regime shifts. This allows the comparative testing of shocks and the measurement of asymmetries of interaction.

References

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