Abstract

In this paper we present a solution to the problem of pricing guarantees of minimum returns on pension fund contributions. These guarantees exist by law in Colombia and cover all pension fund contributions made to the country's private pension fund administration companies (AFPs). As of September 1997, the funds were collecting contributions of 2.3 million affiliates with an accumulated capital of 1.5 billion dollars starting from zero in 1994. Two types of guarantees exist: on obligatory contributions and on voluntary contributions. The solutions are based on a discrete martingale approach . We show that both guarantees are equivalent to an "option to exchange." However, in the case of voluntary contributions a *ceiling* on the payoff s must be added. Using a discrete martingale framework and a binomial solution we develop all aspects of the model that are necessary for its practical application in the context of the pension fund guarantees. Binomial formulas are obtained for both forms of guarantees. Besides solving the problem of pricing the guarantees offered by FOGAFIN, the contributions in terms of options theory of this paper are: i) we adapt the binomila model of Rubinstein [10] to relate the relevant parameters of the same to a continuous-time lognormal process; ii) we provide a binomial solution to the problem of an option to exchange *with a ceiling* . We then investigate the incentives that the current fixed-price system introduces and propose possible systems of incentives that can be used to encourage higher-risk investment by the AFP's and a shift of the fund's portfolio to risky equity and debt. Given the country's effort to encourage capital markets development and the financing of the real sector via private financial markets, this strategy appears to be desirable from the social and economic point of view.

A Discrete Martingale Model of Pension Fund Guarantees in Colombia: Pricing and market effects

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1 Introduction

The purpose of this paper is to present a discrete-time solution to the problem of pricing a financial guarantee on pension funds in Colombia and to evaluate the potential effects of these guarantees on market developments. In Colombia contributors to privately managed pension funds, the *Administradoras de Fondos de Pensiones (AFP,* administrators of pension funds), are the beneficiaries of a guarantee. This guarantee assures contributors that they will earn on their contributions a return at least as large as that of a "synthetic portfolio" constructed by the country's Superintendent of Banks. The guarantee is provided by the Fondo de Garantías de Entidades Financieras (FOGAFIN, henceforth simply "the fund"), a fund that also provides insurance to depositors of the banking system (and other financial institutions) under a system of independently managed reserve funds.

This somewhat unusual arrangement is the result of the process of structural reforms that were introduced in Colombia at the beginning of the 1990s, including the pension system reform. The pension system reform switched the country from a system of "pay-as-you-go" to one based on accumulated savings and transferred most of the responsibilities of managing the accumulating funds to private sector business. However, the same legislation made special provision making the state responsible for guaranteeing a minimum performance and the security of the system. This minimum performance is guaranteed (a put option issued by the AFPs) with the capital and a special reserve held by the AFPs. When the capital and reserves of the AFPs are insufficient to guarantee the minimum return, the state steps in with a reinsurance (a second put option issued by the state). The main purpose of this is to make the system "safe" for pension fund contributors. The government, in turn, passed the responsibility of providing the reinsurance to the fund. For this reinsurance, the AFPs have to pay a premium. This rather complicated situation will be described in detail below. The purpose of this paper is to present a pricing mechanism of these guarantees based on options theory. As we will see, since there are two forms of contributions and the guarantee coverage are different and there is also the option issued by the AFPs, I will present three pricing mechanisms. All are based on the notion of an "exchange option." Throughout I use a discrete martingale framework with numerical solutions based on the binomial distribution.

Any pricing mechanism introduces incentives to exploit the contract in the agent advantage. Thus, two very relevant aspects to consider are: the incentives to which are subject to the AFPs under the present pricing system, and the incentives that the guarantor would like to build into an eventual alternative pricing mechanism. The AFP's have already built a considerable portfolio of assets, and are expected to grow rapidly in the future making them the largest single source of accumulated savings in the economy ready to be invested in risky financial assets. If a change is made from the current fixed-rate pricing to a variable "fair pricing" mechanism should take into consideration the incentives that will encourage the AFP's to shift their portfolios assets into more or less risky types of assets. A change in the pricing mechanism can thus be viewed as an opportunity to introduce incentives that are consistent with some socially desirable objectives.

The organization of the paper is as follows: in section 2 I present a few institutional aspects of the guarantees analyzed in this paper and some technicalities about the synthetic portfolio; in sections 3 and 4 I develop the actual pricing exercise of two different types of pension fund contributions; in section 5 I price the option issued by the AFPs to their members; in section 6 I analyze the sensitivity of this latter option to the main state variables, investigate the incentives the current pricing structure introduce in the operation of the AFPs and propose schemes that can be used to introduce " virtuous incentives" to the guarantee; in section 7 I present conclusions and policy recommendations to the study.

1.1 The contributions and the guarantees issued by FOGAFIN

There are three types of contributions made to the AFPs. Each of these types of contributions are covered by a different form of guarantee:

- *Obligatory contributions to obligatory funds (OCOF)*: These are contributions that are part of the pension fund arrangements that every employer must offer to its employees. The employee must pay 25% of the contribution out of its salary, and the remaining 75 must be provided by the employer. These contributions are guaranteed in 100% of their value including a returns that must be equal or exceed those of a synthetic portfolio defined by the superintendent of financial institutions that supervise the AFP's.

- *Voluntary contributions to obligatory funds (VCOF)*: These contributions are those that the employee makes voluntarily, over and above the obligatory component without participation of the employer. These contributions are guaranteed up to 150 times the country's minimum wage, not including returns on contributions. That is, there is an upper limit on the guarantee.

- *Voluntary contributions to voluntary funds (VCVF):* These are contributions to the pension fund that are not covered by any guarantee. The difference between these contributions and the previous ones is that they can be withdrawn without penalty within 6 months and are managed in a separate fund.

As of December 1997, the AFP covered about 2.5 million members making OC with an accumulated capital of US$2.0 billion, starting from a base of zero in April 1994. In the last year alone the number of affiliates has increased by 300,000 and the capital accumulated more than doubled by 0.8 billion dollars. Of the 2.5 million members, approximately 1.5% of affiliates make VCOF and have also been increasing rapidly (320% in the last year, reaching US$ 340 million). The accumulated capital of VCVF is negligible up to now. In any event, they are not covered by any guarantee, thus they can be considered ordinary investments in mutual funds. Currently, the fund charges a fixed rate (i.e. not risk adjusted) of .17% per year for the guarantee. The AFP's are supervised by the Superintendent of Banks to whom they must present monthly reports based on a Plan Unico de Cuentas (PUC, uniform accounting plan). Depending on the results of the quarter, the AFP's must make a number of transactions between their capital, reserves and the accumulated assets that will be described below.

1.2 The pricing of the guarantees

It has become increasingly popular to use contingent claims approach to price different forms of guarantees. In the context of banking the most classical example is the applications of options theory to pricing deposit insurance by Merton [9]. More recently the approach has been making incursions into more traditional fields of insurance such as property-liability-insurance (Doherty and Garven, [5]). This is the approach used in this paper.

The guarantee on OCOF consists of a promise to the AFP contributors that they will have earned at the end of each quarter of the year, an *accumulated annual return* (realized yield) over the last 36 months (aar_p, p=3 years) at least as large as that of a risky "synthetic portfolio" defined by the Superintendent of Banks (SPSB). This guarantee is to some extent similar to some of the so-called Maturity Guarantees, whose solution was pioneered by Brennan and Schwartz (Br&Sw, [3]) and generalized by Banicello and Ortu [1]. This type of guarantee ensure that the payoff to the investor , at maturity, will be of at least a certain stated amount. They are particularly valuable in case of downward markets. In the Br&Sw model , on a single amount invested at time zero, the investor is guaranteed to receive at expiration at least the guaranteed amount. Hence the value of the investment at expiration is equal to the actual value of the funds based on market returns plus the value of the guarantee. However, there are some important differences in the way these two guarantees work. In the case of the guarantee on the AFPs, the guarantee is not a fixed amount but a minimum return on the "synthetic portfolio." This particular feature makes the guarantee on the AFPs similar to an exchange option. Exchange options have been studied by Margrabe [8] (the continuous time version) and by Rubinstein [10]. Besides solving the problem of pricing the AFP guarantee offered by the fund, the contribution in terms of options theory of this paper is the following: i) I adapt the binomial model of Rubinstein [10] to relate the relevant parameters of the same to a continuous-time lognormal process, something omitted in the presentation of Rubinstein [10]; ii) I provide a binomial solution to the problem of an option to exchange *with a ceiling* (the guarantee on the VCOF) that may or may not accept a continuous-time solution. But first let us present the SPSB.

1.3 The return on the synthetic portfolio of the Superintendent of Banks (SPSB).

The SPSB is composed of a basket of actually traded securities and several indices that are representative of the country's financial system. The return on this SPSB is calculated using the following formula:

$$\widetilde{R}_{SP} = \frac{1}{2}\left[0.9\widetilde{R}_{AFP} + \Big(w_S(0.9\widetilde{R}_S) + (1-w_S)(0.95\widetilde{R}_D)\Big)\right] \tag{1}$$

where

- $\widetilde{R}_{SP} =$ is the minimum aar_p required on contributors' funds;
- $\widetilde{R}_{AFP} =$ the weighted aar_p of the AFP operating in the country, weighted by the capitalization of the AFP;
- $\widetilde{R}_S =$ the aar_p on an index of the three stock exchanges in the country (Bogota, Cali and Medellin);
- $\widetilde{R}_D =$ the aar_p on a portfolio of traded debt securities selected by the Superintendent of Banks and rebalanced periodically, some of these securities consist of risk free assets issued by the Colombian government and thus, with known payoff, R_f, at the beginning of the reporting period.
- $w_S = [0..5\%]$, the maximum of 5% or the proportion of stocks held by the AFP in the three stock exchanges;

1.4 The guarantees on the OCOF

At the end of each reporting period, t, the return on the contributors assets, $A_{j,t}$, are computed as

$$\widetilde{R}_{j,t} = \left[\frac{\widetilde{A}_{j,t}}{A_{j,t-p}} - 1\right]^{\frac{1}{p}}$$

if $R_{j,t} < R_{SP}$, the AFP must transfer resources to members accounts up to the point of setting $R_{j,t} \geqslant R_{SP}$. These resources come first from the company's contingency reserves of 1% of total member's assets that the AFPs must hold by law. Once this reserve is exhausted AFPs must use its capital. The ratio of minimum required capital to risky members' assets is (1 : 40). In case of shortfall of resources in these two accounts (reserve and capital), the company is liquidated, the fund covers the shortfall and the contributors assets are transferred to other AFPs in the market. In all cases, the value of the contributors' assets at the end of the inspection period must be such that the following relation is satisfied:

$$\left[\frac{\widetilde{A^{SP}_{j,t}}}{A_{j,t-p}} - 1\right]^{\frac{1}{p}} = \widetilde{R}_{SP,t}$$

where $\widetilde{R}_{SP,t}$ is the minimum aar_p. Thus, when closing a contract with the AFP, according to the regime established by Law 100, affiliates buys a number of services including:

- fund management services;
- the value of the guarantee sold by the AFP;
- the price of the (reinsurance) guarantee bought by the AFP from FOGAFIN the benefice of the members;[1]

Members acquire this services and guarantees through a fixed commission paid as a percentage of the monthly contributions made to the AFPs. As we will see later, this fixed pricing scheme introduces considerable distortions in the functioning of the system. We will explore these distortions in detail.

The two consecutive guarantees provided by the AFPs and the fund can be viewed as a put option sold by the guarantors (the short positions) to the members of the AFP (the long position). There are two differences between the two options: one is the exercise price and the other is that while the option issued by the fund is a standard option to exchange, the one issued by the AFP is an option to exchange *with a ceiling payoff*. These two options to exchange assure the affiliate that, at the end of each period, the return will be at least equal to the aar_p. If $\widetilde{A}^{SP}_{j,t} < \widetilde{A}_{j,t}$ nothing happens and the option expires worthless. If $\widetilde{A}^{SP}_{j,t} > \widetilde{A}_{j,t}$ the flows to the members are $(\widetilde{A}^{SP}_{j,t} - \widetilde{A}_{j,t})$.The payoffs of the option at expiration for the long position is simply:

$$Max\left[0, \widetilde{A}^{SP}_{j,t} - \widetilde{A}_{j,t}\right] \tag{2}$$

where $\widetilde{A}^{SP}_{j,t}$, unlike in the model of Br&Sw, represents a stochastic "exercise price" of the option. This stochastic "exercise price" makes this guarantee the equivalent of the option to exchange.

The idea of an option to exchange was first proposed by Margrabe [8] in the context of designing options to reward superior performance. Solutions for these type of options were provided by Margrabe [8] and Rubinstein [10]. Stulz [12] examines similar European options on the minimum or maximum of two risky assets. Margrabe [8] examines the value of an option to exchange one non-dividend-paying risky asset for another. One problem preventing the direct application of Margrabe's solution to the AFP guarantee is the assumption that the underlying assets pay no dividends. With dividends, no analytical solution for the continuous time *American* option exists. Rubinstein [10], provides a binomial methodology to the exchange option problems with dividends. It is on this model that I build the solution to the AFP guarantee. Given that the guarantee on OCOF and VCOF are both essentially European options, the solution of Margrabe could be applied to the OCOF. However, the ceiling feature of the option issued by the APF and on the VCOF prevents its application to these.

[1]The AFPs are also required to buy to the benefice of the members a disability and survival insurance.

On the other hand, the solution by Rubinstein does not provide the link between a continuous time process and the binomial model nor does it solve the problems of pricing an exchange option with ceiling. For these reasons I start from scratch and find a binomial solution to both the guarantees with and without ceiling and establish its relation with a continuous time lognormal process.

Exchange options are options that confer the long position the right to choose between the delivery of either of two assets. One of the classical examples are dual currency bonds. The owner of such a bond possesses de right (long a call), embedded in the dual-bond, to choose accepting payment in either currency Y or currency X depending upon the relative values of the two currencies. Other areas of application are mortgage standby commitments, stock tender offers, investment performance incentive fees (the application originally proposed by Margrabe, [8]), bond future contracts, etc.

Since we are interested in pricing the guarantee issued by the fund, I will focus our attention on this. The problem of pricing the option issued by the AFPs will be touched upon tangentially in the section about guarantees on VCOF. The reason is that both are options to exchange with a *ceiling payoff* where the only difference resides in the exercise price and the level of the ceiling..

How does the option to exchange fit our problem? The contributor of an AFP, at expiration of the guarantee has the right to accept delivery of either, $\widetilde{A}_{j,T}$ if $\widetilde{A}_{j,T}^{SP} < \widetilde{A}_{j,T}$ or $\widetilde{A}_{j,T}^{SP}$ in the contrary case. In other words, the AFP contributor has the right to obtain asset $\widetilde{A}_{j,T}^{SP}$ in "exchange" for $\widetilde{A}_{j,T}$ or viceversa. Consider the option of obtaining $\widetilde{A}_{j,T}^{SP}$ in exchange of asset $\widetilde{A}_{j,t}$. Starting with our equation 2 this can be represented as

$$P(T) = Max\left[0, \widetilde{A}_{j,T}^{SP} - \widetilde{A}_{j,T}\right] \tag{3}$$

Assume the option is of European type. From the payoff expression it is evident that the exchange option can be viewed as either a call on $\widetilde{A}_{j,T}^{SP}$ with a strike price equal to the price of $\widetilde{A}_{j,t}$ (i.e. the right to acquire an asset with market value $\widetilde{A}_{j,T}^{SP}$ at a price of $\widetilde{A}_{j,t}$), or a put on $\widetilde{A}_{j,t}$ with a strike price equal to the price of $\widetilde{A}_{j,T}^{SP}$. Although there exist a perfect equivalent of both formulations, for our context it sounds more "natural" to think in therms of the second formulation since most guarantees (and deposit insurance in particular) are usually presented as puts. At maturity, the option is worth at least zero and no more than $\widetilde{A}_{j,T}^{SP}$. If assets $\widetilde{A}_{j,T}^{SP}$ and $\widetilde{A}_{j,t}$ are at least zero, then

$$0 \leq P(T) \leq \widetilde{A}_{j,T}^{SP}$$

By thinking of $\widetilde{A}_{j,T}^{SP}$ *as a nummeraire*, the solution of the problem can be simplified considerably. This idea was suggested to Margrabe [8] by Stephen Ross.[2] With $\widetilde{A}_{j,t}$

[2] The advantages of converting equation 2 into equation 4 are more than trivial. To start with, if both assets are close in price the difference is near zero, a difficult number to handle, specially in the binomial context. For example, the binomial procedure of building a tree with small up and down steps would collapse. For a starting value of zero, no tree can be built. For either a small positive or negative difference, the tree would not be able to cross the zero barrier. Thus a pricing

expressed in units of $\widetilde{A}_{j,t}^{SP}$ the value of the contributors assets become a relative $\widetilde{A}_{j,t}/\widetilde{A}_{j,t}^{SP}$ with a variance $s^2 = \sigma_j^2 - 2\rho\sigma_j\sigma_{SP} + \sigma_{SP}^2$ (where ρ represents the correlation between series $\widetilde{A}_{j,t}$ and $\widetilde{A}_{j,t}^{SP}$) and exercise price of $1 = \widetilde{A}_{j,T}^{SP}/\ \widetilde{A}_{j,T}^{SP}$. More specifically, dividing equation 3 by $\widetilde{A}_{j,T}^{SP}$ yields

$$\widetilde{A}_{j,T}^{SP} \times Max\left[0, 1 - \frac{\widetilde{A}_{j,T}}{\widetilde{A}_{j,T}^{SP}}\right] \tag{4}$$

a put option with a "subjacent" $\widetilde{V}_{j,t} = \widetilde{A}_{j,t}/\widetilde{A}_{j,t}^{SP}$ (which I will call "the surplus") and "exercise price" $X = 1$. Due to the regulatory restriction that the AFPs make up any shortfall in the return of the contributors' funds at the en of the inspection period, $\widetilde{V}_{j,0} \geqslant 0$, i.e. the option is, at the beginning of the following inspection period always "out-of-the-money".I explain the situation in detail with the help of Figure 1.

The figure represents the payoff of the holder of a guarantee of minimum return on the contributions made to the AFPs that consists of two consecutive options to exchange. The "underlying asset" for these two potions is the surplus $\widetilde{V}_{j,t}$. The first option has an exercise price of $X^{AFP} = \widetilde{V}_{j,T} = 1.0$. When $\widetilde{V}_{j,T} \leq 1.0$, the AFP must cover the difference first with reserves (R) and then capital (C). Once these accounts have been exhausted the fund continues to cover the losses up to the point of reestablishing the relation $\widetilde{V}_{j,T} = 1.0$.[3] I have drawn the ceiling on the AFPs' payments with the horizontal AB. From the moment that $\widetilde{V}_{j,T}$ falls below

$$X^{AFP} = \widetilde{V}_{j,T} = \left(\widetilde{A}_{j,T}^{SP} - (C+R)\right)/\widetilde{A}_{j,T}^{SP} = 1 - \frac{C+R}{\widetilde{A}_{j,T}^{SP}},$$

it is the fund that compensates the members. In figure 1 I have also represented the value of the options at the beginning of the semester. Curve $\pi\pi'$ represents the value of the option issued by the AFP and $\pi\pi''$ the one issued by the AFP plus the one issued by the fund. The area between $\pi\pi'$ and $\pi\pi''$ represents the net value of the option issued by the fund. The option issued by both the AFPs and the fund are exactly of the same type, the only difference being the exercise price and that the one issue by the AFPs' have a ceiling at $(C + R)$.[4]

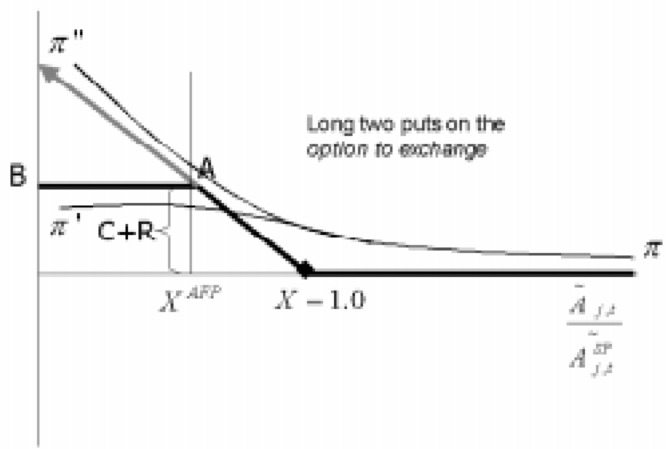
formula would yield a discontinuity around zero. For a slight positive value the option would never be exercised and thus have a value of zero, and for a slight negative starting value the option would always be exercised yielding a positive value to the option.

[3] This implies also that, in contrast to bank stocks or stocks of companies in the real sector, shares of AFPs are in effect puts instead of calls.

[4] A "solution" to the problem of pricing the option issued by the AFPs could be obtained by a round-about way. It is evident that the total value of the long position consists of the summation of the value of the option issued by the AFPs (with a ceiling) plus the value of the option issued by the fund (without ceiling). This could be represented as follows. Let the value of an option as a function of its exercise price and its ceiling be represented as $p_i(exercise\ price, ceiling)$. Then the situation just described can be written down as

$$p_T(1.0, \infty) = p_{AFP}(1.0, C+R) + p_F(1 - \frac{C+R}{\widetilde{A}_{j,T}^{SP}}, \infty)$$

Figure 1: LONG POSITION ON TWO PUTS/GUARANTEES (OPTIONS TO EXCHANGE): The figure represents the payoff of the holder of a guarantee of minimum return on the contributions made to the AFPs that consists of two consecutive options to exchange. The "underlying asset" for these two potions is the ratio, $\widetilde{V}_{j,t} = \widetilde{A}_{j,t}/\widetilde{A}^{SP}_{j,t}$, between the actual value of the contributor's assets, and that they would have if invested in the synthetic portfolio SPBSB ("the excedent").The first option has an exercise price of $X^{AFP} = \widetilde{V}_{j,T} = 1.0$. The second option, that is exercised when the AFP has exhausted its reserves and capital, has an exercise price of $X^{AFP} = \widetilde{V}_{j,T} = \left(\widetilde{A}^{SP}_{j,T} - (C+R)\right)/\widetilde{A}^{SP}_{j,T}$.



A Numerical Example

The situation can be made easier to understand with a numerical example. I will use this example to illustrate each step of the pricing process for the rest of the paper. Assume that the nominal value of the contributors' assets were 100 pesos at the beginning of last quarter and the contributions were 1 pesos per month for the quarter,that is at a rate of γ=0.01 (1%) per month. The return on the risky SPSB turned out to be 7%.[5]. I also assume that at the beginning of the inspection period $\widetilde{V}_{j,0} = 1.0$.[6] For simplicity I suppose that $C + R = 0.0$, that is, $X^{AFP} = X = \left(\widetilde{A}^{SP}_{j,T} - 0.0\right) / \widetilde{A}^{SP}_{j,T} = 1$. The value of the members' assets plus contributions for the period, if invested in a portfolio equivalent to the SPSB would have been:

$$\widetilde{A}^{SP}_{j,T} = \left(100 \times (1 + 0.07 + 0.0303)^{3/12}\right) = 102.418$$

Assume now two possible outcomes for the realized value of the contributions. A high return of 12% and a low return of 5%. This yields asset values (including contributions), $\widetilde{A}_{j,t}$, at the end of the inspection period of \$103.566 and \$101.950, respectively. In each case the value of the option per peso of $\widetilde{A}^{SP}_{j,T}$ at maturity will be

$$\begin{aligned} P_{12\%,T} &= Max(0, 102.418 - 103.566) \\ &= Max(0, \mathbf{1} - (103.566 \div 102.418)) = 0.0 \\ P_{5\%,T} &= Max(0, 102.418 - 101.950) \\ &= Max(0, \mathbf{1} - (101.950 \div 102.418)) = 0.00457 \end{aligned}$$

In the first case the option is not exercised. in the second case it is with a payoff of 0.00457pesos per peso of $\widetilde{A}^{SP}_{j,T}$ to the long position of the option.$\blacksquare$

2 Pricing the fund's guarantee on OCOF

I will now focus on the reinsurance or put option bought by the AFPs from FOGAFIN (the short position in the put) to the benefit of their members (the long position in the put). I start with this option because it is the simplest. Assume an economy in discrete time. The time space is:

The term on the l.h.s and the second term on the r.h.s. are both standard options to exchange without ceiling whose values can be computed using the formulas developed below. The first term on the r.h.s. represents the option issued by the AFPs with a ceiling equal to $C + R$. The price of the latter can be obtained solving for $p_{AFP}(1.0, C + R)$. That is, the solution of the option with ceiling is equal to the difference between two other option without ceiling:

$$p_{AFP}(1.0, C + R)p_T(1.0, \infty) = p_T(1.0, \infty) - p_{AFP}(1.0, C + R) + p_F(1 - \frac{C + R}{\widetilde{A}^{SP}_{j,T}}, \infty)$$

We will, nonetheless find an explicit binomial formulae for this pricing problem.

[5] We are assuming absence of inflation.

[6] This situation would arrive if i) the aaa_p of the AFP's portfolio was exactly equal to the one of the SPSB, or ii) the return was below and the AFP was forced to "make-up" the shortfall at the beginning of the period.

$$\{0, 1, 2, ..., T\}$$

and the probability space Ω is finite of the form

$$\Omega = \{\omega_1, \omega_2, ..., \omega_N\}$$

On this space, endowed of a σ-algebra $(\Omega, \mathfrak{F})$ I have defined a probability space **P,** such that :

$$\mathbf{P}(\{\omega_i\}) > 0 \qquad \forall \qquad i = 1, ..., N$$

The probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ in turn is endowed of a filtration, F, that is of a family of increasing σ-algebras in $\mathfrak{F}$ representing the (increasing) information available to investors at each point in time. In this environment I assume the existence of portfolios $\widetilde{A}_{j,t}$ and $\widetilde{A}_{j,t}^{SP}$, modeled by discrete stochastic process adapted to the filtration F, and a risk free asset A_t^f, with $r_t = 1/A_t^f$ representing the risk free rate of return.

It is standard to assume a complete market in which arbitrage is not possible and where conditional assets can be uniquely priced. Following Harrison and Kreps [6] in such an economy, the value of assets are martingales[7]. Further, under these conditions, it is possible to associate a conditional right with payoff $F(Z_t)$ to a price given by

$$\pi(X) = D_t E_Q[F(Z_T)] \tag{5}$$

where E_Q is the expectations operator relative to the risk-neutral or martingale pricing measure, Q, defined on the probability space $(\Omega, \mathfrak{F}, Q)$ and D_t is some discounting process. To find the *martingale probability measure* Q defined on the probability space $(\Omega, \mathfrak{F}, Q)$ I set

[7] Rather, iff the space of martingale measures is non-empty, the market model is said to be feasible and no opportunities of arbitrage exists in the market. One could instead use a different set of assumptions that would allow us to proceed with the pricing exercise. Specifically, Brennan [2] shows (and before, Rubinstein [11] for the Black-Scholes model) that a necessary and sufficient condition for risk neutral valuation to hold for arbitrary bivariate lognormal distributions is that the representative investor exhibits constant proportional risk aversion. Alternatively, a necessary and sufficient condition for risk neutral valuation to hold for arbitrary bivariate normal distributions is that the representative investor exhibits constant absolute risk aversion. These are quite standard assumptions in the context of financial asset pricing. Finally, Margrabe [8] and Rubinstein [10] show that a hedge portfolio is possible. In contrast to vanilla options arbitrage portfolio, this one will be composed of the portfolios $\widetilde{A_{j,t}^{SP}}$ and $\widetilde{A}_{j,t}$ rather than one risky asset and one riskless asset. To replicate, the portfolio'sss value at time T must match the call's values at that time. Using n_{SP} and n_j to denote the shares of $\widetilde{A_{j,t}^{SP}}$ and $\widetilde{A}_{j,t}$ in the portfolio.. Margrabe [8] in a continuous valuation framework and Rubinstein [10] in the discrete framework, show that a short position (that of the guarantor) on an exchange option on $\widetilde{A_{j,T}^{SP}}$can be hedged by a portfolio strategy consisting of selling short $\widetilde{A}_{j,t}$ in proportion n_j and buying $\widetilde{A_{j,t}^{SP}}$ in proportion n_{SP}. The exact values of n_{SP} and n_j can be found in Rubinstein [10].[8]

$$E_Q\left[e^{-r\delta t}\widetilde{Z}_T\right] = Z_0. \tag{6}$$

Returning to the specifics of our problem, in a one period context. Assume for simplicity that $C+R=0$ and thus

$$X^{AFP} = 1 - \frac{C+R}{\widetilde{A}^{SP}_{j,T}} = 1.0.$$

The value of the put at time $t=1$ will then be

$$P_u = \widetilde{A}^{SP}_{j,u} \times Max\left[0, 1-\widetilde{V}_{j,u}\right]$$
$$P_d = \widetilde{A}^{SP}_{j,d} \times Max\left[0, 1-\widetilde{V}_{j,d}\right]$$

where u (>1) and d (<1) represent up and down steps multiplying $\widetilde{A}_{j,0}/\widetilde{A^{SP}_{j,0}}$. That is,

$$\widetilde{V_{j,u}} = \left(\widetilde{A}_{j,u}/\widetilde{A}^{SP}_{j,u}\right) = u\left(A_{j,0}/A^{SP}_{j,0}\right) \text{ and}$$
$$\widetilde{V_{j,d}} = \left(\widetilde{A}_{j,d}/\widetilde{A}^{SP}_{j,d}\right) = d\left(A_{j,0}/A^{SP}_{j,0}\right)$$

2.1 The martingale probability measure and the size of the steps

The value of $\widetilde{V}_{j,t}$ changes only at discrete times $\delta t, 2\delta t, ...,$ (non-infinitesimal time steps) up to $M\delta t$, the expiration date of the option. The size of the time step is $\delta t = (T-t)/M$. In our case $(T-t)$ will be one quarter of a year. To implement the pricing model we need three measures, the martingale probability measure, α, and the size of the up an down steps, u and d. These steps must be such that, in the limit, they approach the lognormal (or other) continuous-time distribution representative of the process that is being modeled. To do this we need three equations. The first is equation 6 that yields the martingale probability measure. For our case, equation 6 in a one period framework and including contributions (negative dividends) becomes

$$E_Q\left[e^{-r\delta t}\frac{\widetilde{A}_{j,1}\gamma_j}{\widetilde{A}^{SP}_{j,1}\gamma_{SP}}\right] = E_Q\left[e^{-r\delta t}\widetilde{V}_{j,T}\right] = V_{j,0}$$

where γ_j and γ_{SP} represent the contributions (expressed as 1 plus a contribution rate) that the affiliates have made during the quarter. γ_j represents the actual contributions made to the affiliate's funds and γ_{SP} the same amount that accumulates if these contributions would have been made to the SPSB. In a binomial framework this equation becomes

$$\left(\alpha \times u \times V_{j,0}\frac{\gamma_j}{\gamma_{SP}} + (1-\alpha) \times d \times V_{j,0}\frac{\gamma_j}{\gamma_{SP}}\right) = e^{r\delta t}V_{j,0} \tag{7}$$

Note that this equation is also the equation that sets the mean of a binomial random variable equal to the mean to a lognormal random variable in a risk neutral framework ([7], pp. 336-337). Equation 7 calls for a few observations. First, the ratio of contributions $\gamma_j/\gamma_{SP} = 1$ by definition. This is so because while the numerator refers to the actual dividends contributed by the affiliate to the AFP fund, the denominator represents the same contribution that would have been made hypothetically to the SPSB. Second, as Margrabe [8] and Rubinstein [10] have shown, the discounting rate $r = 0$.[9] In essence the reasoning goes as follows. In the setup of an arbitrage portfolio for the option to exchange, the purchases of one asset are financed with the sales of the other asset, not with sales (purchases) of the riskless asset. Thus the lender of one of the assets demands one unit of the other as repayment of principal. In consequence, he/she charges only the return on the asset as compensation and no interest on the loan. By taking these two observations into consideration and by dividing equation 7 through by $V_{j,0}$ yields:

$$(\alpha \times u + (1-\alpha) \times d) = 1 \tag{8}$$

which solving for α, the martingale probability, yields:

$$\alpha = \frac{1-d}{u-d}$$

Note that this is the same solution obtained by Rubinstein [10] using the pricing method based on the hedge portfolio, under the assumption that the ratio of the two dividend yields equals one.

To obtain the size of the up and down steps I employ the two equations that are commonly used in the development of the binomial pricing model. The first is the assumption that the variance of the binomial process equals that of a lognormal process. In our case, remembering that $\gamma_j/\gamma_{SP} = 1$,and $r = 0$, this equation will be:

$$\begin{aligned} (V_{j,0})^2 s^2 \delta t &= \alpha \times u^2 (V_{j,0})^2 + (1-\alpha) \times d^2 (V_{j,0})^2 - (V_{j,0})^2 [\alpha \times u + (1-\alpha) \times d]^2 \\ s^2 \delta t &= e^{s^2 \delta t} - 1 = \alpha \times u^2 + (1-\alpha) \times d^2 - 1^2 \end{aligned} \tag{9}$$

where $e^{s^2 \delta t} - 1$ is the variance of the lognormal distribution in a risk neutral world ([7], pp. 336-337), but with $r = 0$. The second equation is:

$$u = \frac{1}{d} \tag{10}$$

The solution of the three equations 8, 9 and 10 for u and d yield the following:[10]

$$d = 1 + \frac{1}{2}\left(b^2 + \sqrt{b^4 + 4b^2}\right) \qquad \text{and} \qquad u = \frac{1}{d}$$

[9] See Margrabe ([8], pp. 134) and Rubinstein ([10], pp. 194). See also Trigeorgis ([13], pp. 210-211).

[10] It also yields an alternative expression for α in terms of $b = e^{s^2 \delta t} - 1$:

$$\alpha = \frac{\frac{1}{2}b^2 + 2 - \frac{1}{2}\sqrt{b^4 + 4b^2}}{4 + b^2}$$

where $b = e^{s^2 \delta t} - 1$. Note that $u < 1.0 < d$.

The value of the option in a one period framework is:

$$\begin{aligned} P(\widetilde{V}_{j,t}) &= [\alpha \times P_u + (1-\alpha) \times P_d] = \\ &= \widetilde{A}_{j,T}^{SP} \left[\alpha \times Max \left[0, 1.0 - \widetilde{V}_{j,u} \right] + (1-\alpha) \times Max \left[0, 1.0 - \widetilde{V}_{j,d} \right] \right] \end{aligned} \tag{11}$$

The extension of equation 11 to the multi-period case yields the following formulae:

$$P\left(\widetilde{V}_{j,t}\right) = \widetilde{A}_{j,T}^{SP} \left[\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \alpha^j (1-\alpha)^{n-j} Max \left(0, 1.0 - u^j d^{n-j} V_{j,0}\right) \right] \tag{12}$$

where

$$\widetilde{A}_{j,T}^{SP} = \frac{\widetilde{A}_{j,T}}{u^j d^{n-j} V_{j,0}}$$

which is the well-known solution of the binomial option pricing formula in a multi-period context. Or, if expressed in units of $\widetilde{A}_{j,T}^{SP}$, 12 becomes simply

$$p\left(\widetilde{V}_{j,t}\right) = \left[\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \alpha^j (1.0-\alpha)^{n-j} Max \left(0, 1.0 - u^j d^{n-j} V_{j,0}\right) \right] \tag{13}$$

In the more general case when $C + R \geq 0$, we substitute 1.0 for $X = 1 - (C + R)/\widetilde{A}_{j,T}^{SP}$.

Example (Cont.)

Using the numerical example started above, the parameters to be entered into the option pricing formula would be the following:

$\widetilde{A}_{j,0}^{SP} = 100.00$

$\widetilde{A}_{j,0} = 100.00$

$r = 0.00$

$T = 0.25$

$\sigma = 0.15$

$M = 100$

Using equation (12) the value of the option (price of the guarantee), $p\left(\widetilde{V}_{j,t}\right)$, obtained at the beginning of the inspection period is 0.029 pesos per pesos of contributor funds per quarter.[11] The reader should remember that we are still holding to the assumption of zero capital and reserves and that the AFPss portfolio has not accumulated any excess returns over that of the SPSB. A nonzero positive capital and reserve (required by law) and any accumulated excess return would reduce the price of the guarantee.■

[11] Compared to today's premium of 0.00043 pesos per peso of contributor´s funds, this is rather high. A more realistic picture can be obtained if we assume that capital and reserves amount to exactly the minimum. In this case the premium falls to 0.019 pesos per peso of contributor´s funds. It is not surprising that these premiums are so high since the option, in this situation is "at the market". For an option "out of the market", for example if accumulated realized value of contributor's funds is, say 113.00 (an accumulated return of 13% over the SPSB) against a required minimum of 100.00 based on the SPSB, the premium would fall again to 0.0005, or about the value of today's premium.

3 Pricing the guarantee on VCOF

It might be recalled that the VCOF are those that the employee makes voluntarily, over and above the obligatory component without participation of the employer. These contributions are guaranteed up to 150 times the country's minimum wage, including returns at least equal to the SPSB on contributions. As before, the AFPs must keep reserves and put up capital at the ratio of 1 : 40 on assets. Equally to the OCOF, the guarantee offered by the fund enters into play once reserves and capital have been exhausted. So, now we have two consecutive options with ceilings. As before we will focus on the option issued by the fund. I will first present the payoffs that result from the existence of both options. Then, to simplify the presentation, I will focus on the pricing of a "generic" option to exchange with a ceiling. Only at the end will be return to the specific problem at hand to which I will apply the generic solution.

The payoffs at expiration resulting from the existence of guarantees on VCOF can be represented in Figure 2

The points of interest X, A and C occur at the values of the horizontal axe 1.0, $(1-(C+R)/\widetilde{A}^{SP}_{j,T})$ and $(1-(C+R+150S)/\widetilde{A}^{SP}_{j,T})$, respectively. All payoffs resulting from the guarantee to the long position can be represented in the following schema:

$$P_t = \begin{cases} 150S & \text{paid by the fund} \quad \text{if } \widetilde{A}_{j,T} \leq \widetilde{A}^{SP}_{j,T} - (C+R+150S) \\ \widetilde{A}^{SP}_{j,T} - (C+R) - \widetilde{A}_{j,T} & \text{paid by the fund} \quad \text{if } \widetilde{A}^{SP}_{j,T} - (C+R+150S) \leq \widetilde{A}_{j,T} \leq \widetilde{A}^{SP}_{j,T} - (C+R) \\ \widetilde{A}^{SP}_{j,T} - \widetilde{A}_{j,T} & \text{paid by the AFP} \quad \text{if } \widetilde{A}^{SP}_{j,T} - (C+R) \leq \widetilde{A}_{j,T} \leq \widetilde{A}^{SP}_{j,T} \\ 0 & \qquad\qquad\qquad\quad \text{if } \widetilde{A}_{j,T} \geq \widetilde{A}^{SP}_{j,T} \end{cases} \tag{14}$$

The payoffs for the long position of the option *issued by the fund* is:

$$Max\left[0, Min\left(150S, \widetilde{A}^{SP}_{j,T} - (C+R) - \widetilde{A}_{j,T}\right)\right].$$

Dividing and multiplying through by $\widetilde{A}^{SP}_{j,T}$,

$$\begin{aligned} & \widetilde{A}^{SP}_{j,T} \times Max\left[0, Min\left(\frac{150S}{\widetilde{A}^{SP}_{j,T}}, 1 - \frac{(C+R)}{\widetilde{A}^{SP}_{j,T}} - \frac{\widetilde{A}_{j,T}}{\widetilde{A}^{SP}_{j,T}}\right)\right] \\ = \; & \widetilde{A}^{SP}_{j,T} \times Max\left[0, Min\left(\frac{150S}{\widetilde{A}^{SP}_{j,T}}, X_{AFP} - V_{j,T}\right)\right]. \end{aligned} \tag{15}$$

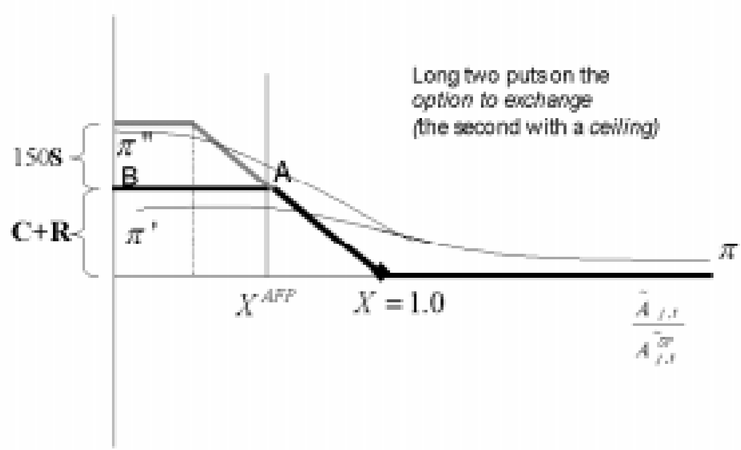
We now develop the binomial tree. For a one period case we obtain

$$\widetilde{A}_{j,1} = \begin{cases} u\widetilde{A}_{j,0} & (\omega = \omega_1) \text{ with probability } p \\ d\widetilde{A}_{j,0} & (\omega = \omega_2) \text{ with probability (1-}p\text{)} \end{cases} \tag{16}$$

At each additional step the tree unfolds and the value of the contribution is:

$$\widetilde{A}_{j,0} u^m d^{n-m} \qquad m = 0, 1, 2, ..., n.$$

Figure 2: LONG POSITION ON TWO PUTS/GUARANTEES (OPTIONS TO EXCHANGE WITH A CEILING): The figure represents the payoff of the holder of a guarantee of minimum return on the contributions made to the AFPs that consists of two consecutive options to exchange. The "underlying asset" for these two potions is the ratio, $\widetilde{V}_{j,t} = \widetilde{A}_{j,t}/\widetilde{A}^{SP}_{j,t}$, between the actual value of the contributor's assets, and that they would have if invested in the synthetic portfolio SPBSB ("the excedent").The first option has an exercise price of $X^{AFP} = \widetilde{V}_{j,T} = 1.0$ and the payoffs have a ceiling of $C+R$ (the AFPs' capital and reserves). The second option, that is exercised when the AFP has exhausted its reserves and capital, has an exercise price of $X^{AFP} = \widetilde{V}_{j,T} = \left(\widetilde{A}^{SP}_{j,T} - (C+R)\right)/\widetilde{A}^{SP}_{j,T}$ and the payoffs have a ceiling of $150S$ (minimum salaries).



Then in a one period context, the payoff of the put at time $t = 1$ will be

$$P'_u = \widetilde{A}^{SP}_{j,T} \times Max\left[0, Min\left(\frac{150S}{\widetilde{A}^{SP}_{j,T}}, X_{AFP} - V_{j,u}\right)\right]$$
$$P'_d = \widetilde{A}^{SP}_{j,T} \times Max\left[0, Min\left(\frac{150S}{\widetilde{A}^{SP}_{j,T}}, X_{AFP} - V_{j,d}\right)\right]$$

As Margrabe [8] and Rubinstein [10] have shown, the discounting rate $r = 0$. The value of the option in a one period framework is thus:

$$\begin{aligned} P'(\widetilde{V}_{j,t}) &= \left[\alpha \times P'_u + (1-\alpha) \times P'_d\right] = \\ &= \widetilde{A}^{SP}_{j,T}\left\{\alpha \times Max\left[0, Min\left(\frac{150S}{\widetilde{A}^{SP}_{j,T}}, X_{AFP} - V_{j,u}\right)\right] + \right. \\ &\left. +(1-\alpha) \times Max\left[0, Min\left(\frac{150S}{\widetilde{A}^{SP}_{j,T}}, X_{AFP} - V_{j,u}\right)\right]\right\} \end{aligned} \tag{17}$$

I relate the probability, α, as well as the size of the up and down jumps (u and d) to observable parameters. The appropriate values for α, d and u are the solutions of the three equations 8, 9 and 10. The multi-period extension of equation 17 is

$$P'\left(\widetilde{A}_{j,T}\right) = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!}\alpha^j(1-\alpha)^{n-j} Max\left[0, Min\left(\frac{150S}{\widetilde{A}^{SP}_{j,T}}, X_{AFP} - u^j d^{n-j}\widetilde{A}_{j,0}\right)\right] \tag{18}$$

which is similar to equation (12) but for the ceiling.

Example (Cont.)

Assume, in our numerical example with VCOF rate $\gamma = 0.05$ (5%). The value of the accumulated contributions with returns amounts to 100.0. The ceiling on the guarantee is of 150.0. Assume as before two possible outcomes for the realized value of the contributions. Those obtained from a high return of 12% and a low return of 5%. This yields asset values, $\widetilde{A}_{j,t}$, at the end of the inspection period (including contributions) of 104.003 and 102.411, respectively. Assume $S = 1.0$. The value of the members' assets plus contributions for the period, if invested in a portfolio equivalent to the SPSB would have been:

$$\widetilde{A}^{SP}_{j,T} = \left(100 \times (1 + 0.07 + 0.0303)^{3/12}\right) = 102.418;$$

and $X_{AFP} = 1.0 - \frac{(C+R)}{\widetilde{A}^{SP}_{j,T}} = 1.0 - \frac{0}{102.418} = 1.0$. In each case the value of the option at maturity will be

$$\begin{aligned} P_{12\%,t} &= 102.418 \times Max\left[0, Min\left(\frac{150}{102.418}, 1.0 - \frac{104.003}{102.418}\right)\right] = 0.0 \\ P_{5\%,t} &= 102.418 \times Max\left[0, Min\left(\frac{150}{102.418}, 1.0 - \frac{102.411}{102.418}\right)\right] = 0.0007 \end{aligned}$$

The parameters to be entered into the option pricing formula would now be the following:

$\widetilde{A}_{j,T}^{SP} = 102.418$

$\widetilde{A}_{j,0} = 100.0$

$r = 0.04$

$T = 0.25$

$\sigma = 0.15$

$\Gamma_j = 80.0$

$M = 100$

$\gamma = 0.05$

Using equation (18) the value of the option (price of the guarantee), $P'\left(\widetilde{A}_{j,T}\right)$, obtained at the beginning of the inspection period is 0.0011 per hundred pesos of contributor funds, or 0.00001% per quarter, considerably lower than the premium on OCOF.■

4 Pricing the option issued by the AFPs to their members

Equation (18), but with a ceiling equal to $C+R$,would apply to the options issued by the AFPs to their members. To see this I start with the payoffs for the long position of the option *issued by the AFP* which is:

$$Max\left[0, Min\left(C+R, \widetilde{A}_{j,T}^{SP} - \widetilde{A}_{j,T}\right)\right].$$

Dividing and multiplying through by $\widetilde{A}_{j,T}^{SP}$,

$$\widetilde{A}_{j,T}^{SP} \times Max\left[0, Min\left(\frac{C+R}{\widetilde{A}_{j,T}^{SP}}, 1 - \frac{\widetilde{A}_{j,T}}{\widetilde{A}_{j,T}^{SP}}\right)\right] = \widetilde{A}_{j,T}^{SP} \times Max\left[0, Min\left(\frac{C+R}{\widetilde{A}_{j,T}^{SP}}, 1 - V_{j,T}\right)\right]. \tag{19}$$

We now develop the binomial tree. For a one period case we obtain

$$\widetilde{A}_{j,1} = \begin{cases} u\widetilde{A}_{j,0} & (\omega = \omega_1) \text{ with probability } p \\ d\widetilde{A}_{j,0} & (\omega = \omega_2) \text{ with probability (1-}p\text{)} \end{cases} \tag{20}$$

At each additional step the tree unfolds and the value of the contribution is:

$$\widetilde{A}_{j,0} u^m d^{n-m} \qquad m = 0, 1, 2, ..., n.$$

Then in a one period context, the payoff of the put at time $t = 1$ will be

$$P'_u = \widetilde{A}_{j,T}^{SP} \times Max\left[0, Min\left(\frac{C+R}{\widetilde{A}_{j,T}^{SP}}, 1 - V_{j,u}\right)\right]$$
$$P'_d = \widetilde{A}_{j,T}^{SP} \times Max\left[0, Min\left(\frac{C+R}{\widetilde{A}_{j,T}^{SP}}, 1 - V_{j,d}\right)\right]$$

The value of the option in a one period framework is thus:

$$\begin{aligned} P'(\widetilde{V}_{j,t}) &= \left[\alpha \times P'_u + (1-\alpha) \times P'_d\right] = \\ &= \widetilde{A}^{SP}_{j,T} \left\{ \alpha \times Max \left[0, Min\left(\frac{C+R}{\widetilde{A}^{SP}_{j,T}}, 1 - V_{j,u}\right)\right] + \right. \\ &\quad \left. +(1-\alpha) \times Max \left[0, Min\left(\frac{C+R}{\widetilde{A}^{SP}_{j,T}}, 1 - V_{j,u}\right)\right]\right\} \end{aligned} \tag{21}$$

I relate the probability, α, as well as the size of the up and down jumps (u and d) to observable parameters. The appropriate values for α, d and u are the solutions of the three equations 8, 9 and 10. The multi-period extension is

$$P'\left(\widetilde{A}_{j,T}\right) = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \alpha^j (1-\alpha)^{n-j} Max \left[0, Min\left(\frac{C+R}{\widetilde{A}^{SP}_{j,T}}, 1 - u^j d^{n-j} \widetilde{A}_{j,0}\right)\right] \tag{22}$$

which is similar to equation (18).

5 Comparative dynamics and analysis of incentives

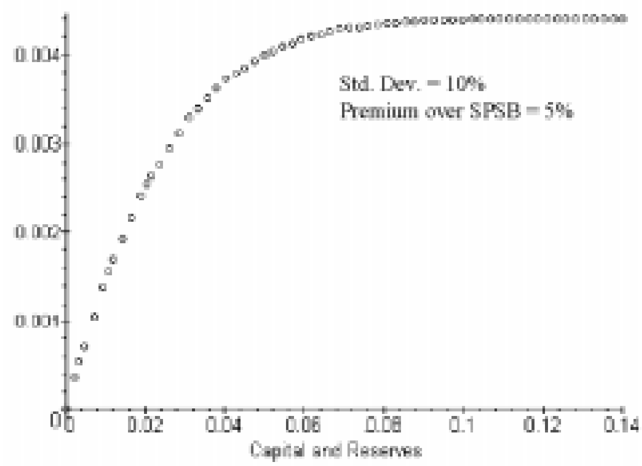
The purpose of this section is to analyze the sensitivity of the option issued by the AFP to members. One of the objectives is simply to develop a notion of how the value of the option changes with the state variables. Another use, from the economic point of view often more interesting, is to assess the incentives to which are subject the agents that engage in these contracts. It should be remembered that, with the present fixed-rate pricing structure, the AFP have an incentive to minimize the value of the option they issue to members.

5.1 Analysis of the guarantee on OCOF

The relatively young nature of the industry should be taken into consideration in this analysis. So, it is possible that the AFPs have not, as yet, developed a strategy that maximizes the value of the fixed-price option. For example, the state variable Capital and Reserves may not reflect a strategy of maximizing the value of the option simply because i) the AFPs have been set-up with a capital that reflects the medium to long term growth expectations, giving an excess capitalization for the short run; ii) Several of the AFPs were forced to recapitalize by the Superintendent of Banks due to failure to meet minimum capital (or primary capital) standards. In the latter case the Superintendent tends to demand a capitalization that largely exceeds the standard given current levels of assets.

For the case of this guarantee, the three state variables easiest to manipulate by the AFPs and that have an effect on the value of the option they issue to members are:

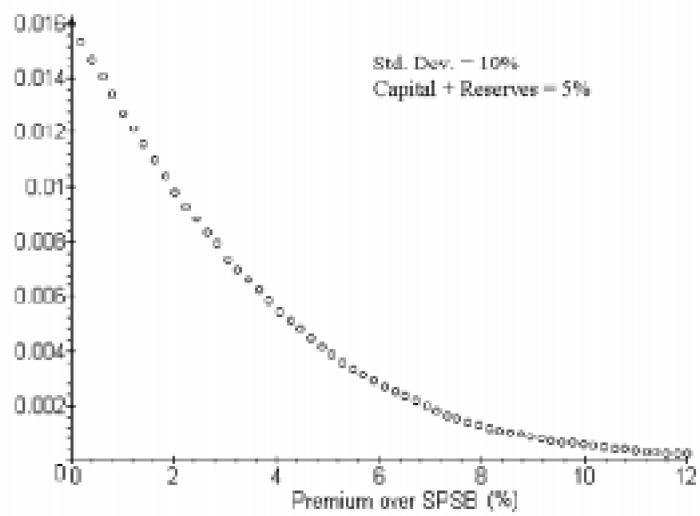
Figure 3: VALUE OF THE OPTION ISSUED BY THE AFP AS A FUNCTION OF CAPITAL AND RESERVES. This figure shows the result of plotting the value of the guarantee issued by the AFP to its members on OCOF against Capital and Reserves. Capital and Reserves held by the AFP represents the defacto ceiling on this guarantee. To construct the plot the other two relevant state variables were kept constant. The standard deviation at 10% and the premium of the affiliates contributions over the SPSB at 5%.



- the amount of capital and reserves held by the AFP (in effect the *ceiling* on the payoffs of the option),
- the premium that the actual value of the affiliates contributions over that they would have if invested in the SPSB, at time $t = 0$,
- the variance of $\widetilde{V}_{j,t}$

In **??** I present a plot of the value of the option against Capital and Reserves of the AFPs. To construct the plot, the other two relevant state variables were kept constant. The standard deviation of $\widetilde{V}_{j,t}$ at 10% and the premium of the affiliates contributions over the SPSB at 5%. The option was priced using 200 sub periods ($M = 200$) and the number of points generated was between 60 and 100. The current average value of Capital plus Reserves for the industry is about 6%. The plot suggests that, for standard deviation and premium fixed at the indicated levels, the value of the option is highly sensitive to changes in Capital and Reserves, specially at low values. The graph suggests that the maximum gains are to be made in the region below 6%, where the slope of the curve is steepest.

Figure 4: VALUE OF THE OPTION ISSUED BY THE AFP TO MEMBERS AS A FUNCTION OF THE PREMIUM OVER THE SPSB. This figure shows the result of plotting the value of the option issued by the AFP to members with ceiling payoff equal to Capital and Reserves against the premium, at time $t = 0$, of the actual market value of the affiliates funds against the value they would have had they been invested in the SPSB. To construct the plot the other two relevant state variables were kept constant. The standard deviation at 10% and the level of Capital and Reserves at 5%.



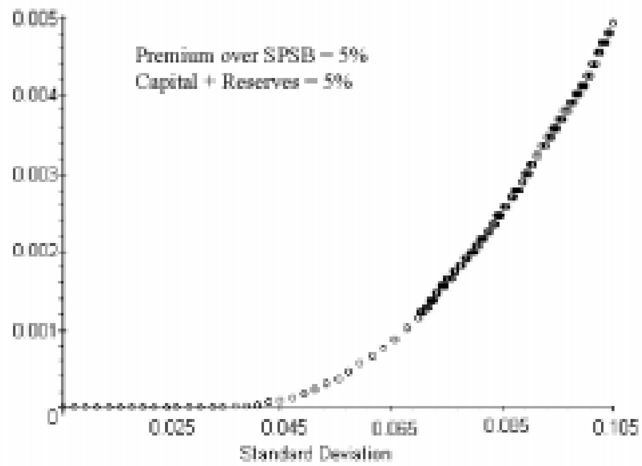
In figure 4 I plot the value of the option as a function of the premium , at time $t = 0$, of the actual market value of the affiliates funds against the value they would have had if invested in the SPSB. To construct the plot the other two relevant state variables were kept constant. The standard deviation at 10% and the level of Capital and Reserves at 5%.

The interpretation of this plot is less straight forward than the previous one. Evidently, to obtain a larger premium, AFPs must invest in more risky assets relative to the SPSB. On one side, a high premium tends to reduce the value of the option. However, a higher portfolio risk tends to increase the value of the same (see 5). Thus, AFP's will be able to play with these two state variables as to maximize the value of the option.

5.2 Incentives of the current fixed-price system

The foregoing analysis allows us to draw some interesting conclusion in terms of the incentives to which AFPs are subject under the current fixed-price system, for both the option issued by the AFPs to members and the reinsurance bought by the AFPs

Figure 5: VALUE OF THE OPTION ISSUED BY THE AFP TO MEMBERS AS A FUNCTION OF THE STANDARD DEVIATION OF $\widetilde{V}_{j,t}$. This figure shows the result of plotting the value of the option issued by the AFP to members against the standard deviation of the ratio between the market value of the affiliates funds against and value they would have had they been invested in the SPSB. To construct the plot the other two relevant state variables were kept constant. The premium over the SPSB at 5% and the level of Capital and Reserves at 5%.



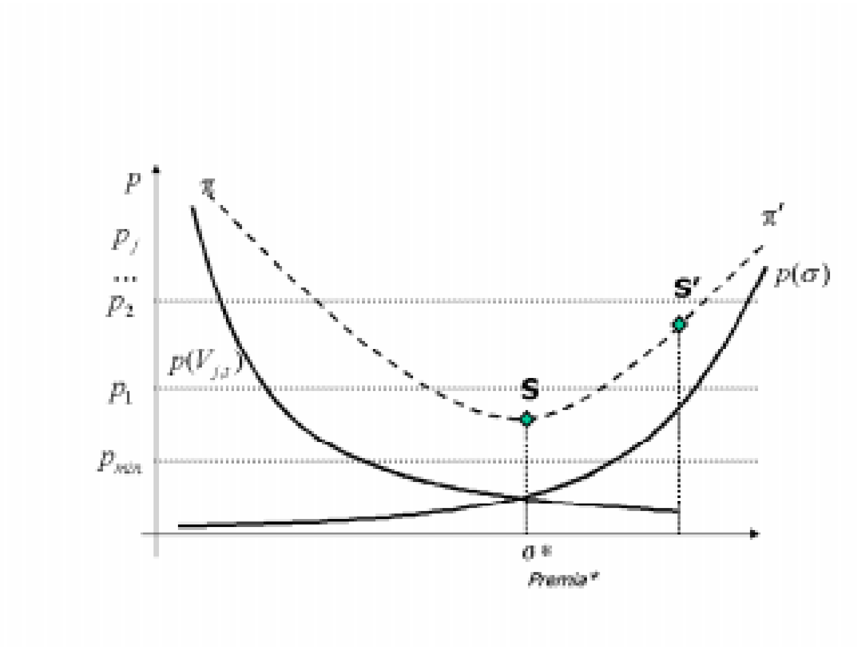
in favor of members . First, without any ambiguity, AFPs will minimize the value of the fixed-price implicit guarantee by minimizing capital and reserves. The events of the last year and half seem to confirm that AFPs indeed follow a low capitalization strategy. The Superintendent of Banks was forced to order most AFP´s to recapitalize to maintain the minimum capital required by law, in some cases repeatedly. Given the steepness of the curve over the relevant range, it is not surprising that AFPs are carrying out a hard battle to reduce the minimum capital and reserve requirements.

Second, the effect of risk on the guarantee is, unexpectedly, somewhat ambiguous in a round about way. To see this let us plot (see 6) in the same graph the value of the option as a function of the premia over SPSB and standard deviation superposed (i.e., superposing the graphs of figures **??** and **??**). In a multiperiod context, there exist a direct relationship between these two variables. In the long run, a high variance strategy will tend to increase the premium of the contributor's assets above that they would have if invested in the SPSB, $\widetilde{V}_{j,t}$. This, in turn, tends to depress the value of the option in future periods. Clearly, it is in the interest of the AFP to keep this premium as high as possible, other things constant. This two effect combined result in an incentive to the AFPs that is ambiguous. The curve for the price of the option for an AFP that takes into consideration both these variables simultaneously would most likely look something like the curve $\pi\pi'$.[12] If the AFP's investment strategy that results in a value of the guarantee somewhere to the left of the point **S**, then a higher risk strategy will first result, in future periods, in an increase in value of the option before later before it can be reduced as a result of an increase in the premia over the SPSB.

The investment strategy that have followed the AFP appears to reflect this ambiguity. As a rule, the investment strategy of the AFPs seems to follow a low-risk strategy placing most funds in government papers or papers issued by the financial sector. As of December 1996, AFPs invested only 0.3% of their funds in shares (the legal upper limit is 30%) and 11% of bonds issued by non-financial business. 47% of the assets consisted of debt (debentures, mortgage and asset-backed) issued by financial intermediaries and the remainder of government issues and papers protected by deposit insurance. Although it is true that the stock market of Colombia is small, the proportion of investment of AFPs in equities underrepresents seriously this segment of the market. What this means is that, AFPs serve predominantly to finance the government and the low-risk financial sector, while providing only marginal direct financing to the real sector. This is in a severe contrast to the funding activity of many other countries where considerably larger amounts of funds are channelled to the non-financial sector through portfolios with a strong representation of corporate debt and equity issues.

[12] This would be contour line of a surface plotting the value of the option against both state variables.

Figure 6: AN INCENTIVE SCHEME TO ENCOJURAGE RISK TAKING BY THE AFPS



5.3 A pricing system with virtuous incentives[13]

As noted before, guarantees to the AFP were included in the Law 100 that implemented the pension funds reform in Colombia. Although it is a somewhat unusual arrangement, it represents an opportunity to influence AFPs to operate in the direction of socially worthwhile objectives, i.e. "socially virtuous incentives."

A very relevant question is then the incentives that the guarantor would like to build into the guarantee. Not only have the AFPs already built a considerable portfolio of assets, but the expected growth of the same threatens to make it, somewhere in the future, the largest single source of accumulated savings in the economy ready to be invested in risky financial assets. As of this date, only three years into their creation they already exceed the value of the portfolio of marketable assets held by foreign investors, which in the case of Colombia is quite considerable. Changing the current fixed-rate pricing to a variable risk adjusted pricing mechanism would be essential to introduce incentives as to encourage a shift of the AFPs' assets into more or less risky types of assets. A purely "fair price" strategy would neutralize totally incentives to moral hazard.[14] A policy of promoting a shift toward *risky assets* would

[13] A more formal and complete approach to the analysis presented below would be to evalaute the analisis of incentives in a game-theoretic framework. However such an exercise would be beyond the intended scope of this paper. For this reason we will only present the mechanisms available to the guarantor that appear most obvious.

[14] Although it would enhance incentives to increase information asymmetry between the AFP and the supervisor-guarantor. However, given that at the close of each inspection period, assets are

tend to favor investment in equities with two effects: i) increase the long term value of the affiliates assets exploiting fully the time-diversification of risk; ii) favor investment in risky equity and debt and thus Colombian business long term finance. A policy of promoting a shift toward *safe assets* would tend to favor investment in government issued assets. Such a policy would have the effect to reduce the cost of financing to the government and favor government deficit financing and the expense of the real business sector.

Two approaches could be used which I will simply call the "carrot" and the "stick" approach. Since a "carrot strategy" is generally preferable to a "stick" strategy I will develop in detail the first one, while only outlining the second. I will present strategies that will *encourage risk taking*, i.e. investments in corporate equity and bond rather than government securities. Under the current political environment of financial liberalization and promotion of financial markets in Colombia, such a strategy appears to be socially more desirable than the reverse. However, exactly the opposite effect would be achieved by reversing the strategy.

5.3.1 The "carrot" strategy

Assume that a "fair-price" risk-adjusted premium strategy is adopted, and that for management purposes the premia are structured as a scale with n steps, $p_i, i = 1, ..., n$, possibly with a minimum premia, $p_{\min}$. A carrot approach could be used in which AFP's with a higher premium over the SPSB would benefit from a discount of j steps from the rate they would have to pay according to a "fair price" strategy. Other things equal, an AFP to be able to benefit from this discount, would first be forced to implement a higher-risk strategy to be able to eventually benefit from this discount. The fact that the AFP would still have to pay a risk-adjusted (albeit discounted) premium based on the capital, reserves and standard error of $\widetilde{V}_{j,t}$, would protect the guarantor from excessive moral hazard on the side of the AFP.[15] Clearly, such a strategy would in effect be providing a *subsidy* to those AFP's that implement a high risk strategy. To the extent that the state is the ultimate guarantor (liability for which FOGAFIN provides a cushion financed with funds accumulated from premia paid by the beneficiaries of the guarantee) this subsidy could be considered a shadow price of promoting development of corporate securities markets.

This scheme can be illustrated graphically in 6. Assume that a fixed steps, $p = p_{\min}, p_1, p_2, ..., p_j$ (dotted horizontal lines in 6), rate scheme is in place. In absence of an incentive scheme, given this opportunity set the AFP would most likely choose a strategy that minimizes de insurance cost, i.e. the point **S** with premium $p(s); s = [\sigma*$,premia over SPSB*]. Assume that the pricing is done to next step below, resulting in a price of the guarantee of $p_{\min}$ and a subsidy equal to the distance from **S** to the horizontal $p_{\min}$.

valued at market prices, the possibility are minimal to engage in this form of information distortion. This situation is quite different from that a deposit institution (such as a bank) where assets are difficult to value at market prices.

[15] In a more formal framework this would mean that we would be building a multi period model of analysis.

Assume now a very simple *incentive scheme* that consists of:

- charging a price for the guarantee one step below (i.e. $j = 1$) of the fair price according to a scale, for a given increase in the premium over SPSB, and
- minimum price of $p_{\min}$.

Under these circumstances the AFP will be encouraged to increase the risk of the portfolio as to reach point $\mathbf{S}'$. In absence of an incentive scheme this would imply an increase of the price from $p_{\min}$ to p_1. However, the incentive scheme would allow the AFP to pay $p_{\min}$ and thereby increasing the subsidy to the distance from $\mathbf{S}'$ to the horizontal $p_{\min}$. Further shifts in portfolio composition can be encouraged by augmenting the subsidy at higher levels of risk taking that should manifest themselves by a higher premium over the SPSB. This mechanism, in effect, would allow AFP to *choose* the level of exposition to risk and the price they are willing to pay for the insurance. The subsidy that is offered should be sufficient to compensate for likely increase in equity that the AFP is likely to encounter as a result of the high risk strategy. On other hand, theoretically there exists the possibility for an AFP to increase continuously the risk of the portfolio, a result that might be undesirable. However. since the price of the guarantee will increase in step keeping the subsidy constant, this is unlikely to happen. Further, if an upper limit on the risk taking is desired the discount can be suspended once the premia over the SPSB reaches a certain value.[16] A system that would discourage risk taking would charge a premium on the price of the guarantee for higher premia over the SPSB. The result would be a *tax* on AFPs that engage in higher risk investment.

5.3.2 The "stick" strategy

A "stick" strategy the encourages risk taking could be based on the manipulation of capital and reserve standards. Assume, as before, that a "fair-price" risk-adjusted premium strategy is adopted, and that for management purposes the premia are structured as a scale with n steps, $p_i, i = 1, ..., n$, possibly with a minimum premia, $p_{\min}$. AFPs could be encouraged to assume more risk by increasing the minimum equity capital, or the level of reserves required to operate the AFP. To see this assume that an AFP is classified as paying a price on the guarantee of p_1. Since, as before, this price implies a subsidy, an increase in required capital will reduce the fair value of the option and thus, the subsidy obtained from the guarantee. To compensate for this loss in subsidy, the AFP can increase the risk position in its portfolio to increase again the value of the subsidy to the maximum possible before reaching the next step, p_2. This fact can be used to design a system by which the AFPs can *choose* a combination of capital and risk of their preference.

[16] It is unlikely that this will be necessary, since AFP's investment in certain high risk instruments are limited by law.

6 Conclusion and Recommendations

In Colombia affiliates to privately managed pension funds (AFP's) are the beneficiaries of a guarantee offered by the Fondo de Garantías de Institutciones Financieras (FOGAFIN). The purpose of this guarantee is to assure affiliates that they will earn on their contributions a return at least as large as that of a "synthetic portfolio" constructed by the country's Superintendent of Banks. This minimum performance is guaranteed (a put option issued by the AFPs) with the capital and a special reserve held by the AFPs. When the capital and reserves of the AFPs are insufficient to guarantee the minimum return, the state steps in with a reinsurance (a second put option issued by the state). This regime was introduced in Colombia at the beginning of the 1990s, when the system of pension funds was reformed to switch the country from a system of "pay-as-you-go" to one based on accumulated savings and transferred most of the responsibilities of managing the accumulating funds to private sector business. The main purpose of the guarantee is to make the system "safe" for pension fund contributors. Two types of contributions exist: one of obligatory contributions and one of voluntary contributions. The nature of the guarantee is different for each type of contribution. As of this moment, a fixed rate system is in place by which FOGAFIN charges the AFP's \$0.000425 per peso of affiliates funds per quarter. By the scope of funds involved, by far the most important for of guarantee is that of obligatory contributions.

This paper shows that this guarantee can be priced using a contingent claims (options) theoretic framework. More concretely, I show that the guarantee on obligatory contributions is equivalent to an "option to exchange", some aspects of which have already been studied in the literature. Using a discrete martingale framework and a binomial solution I develop all aspects of this model that are necessary to facilitate its practical application in the context of the pension fund guarantees. I also show that the guarantee on the voluntary contributions can be priced using a more conventional options framework but with an unusual way of handling the exercise price. Binomial formulas are obtained for bot forms of guarantees.

Given the importance of this form of saving for the Colombian economy and for the development of its capital markets, it is of particular importance to consider the incentives to which AFP's are subject as a result of the pricing mechanism. In a less formal analysis I show that the current pricing system encourages low levels of capital and reserves and an investment strategy that may favor government financing and financing of financial intermediaries over that of real sector business (corporate debt and equity). I also argue that the introduction of a "fair pricing" system could encourage AFPs to settle, as with the fixed-price system, on portfolios of relatively low risk (i.e. with a high proportion of government securities). This may not be a desirable result because it would undermine the development of a risky equity and debt market that would support financing of the real sector. Further, it would deprive contributors from exploiting the benefits of time diversification (the risk diversification effect of a buy-and-hold strategy) and maximizing the eventual old-age rent obtained from the system. Based on an analysis of the sensitivity of the guarantee on obligatory contributions to the main state variables, I propose two possible systems of incentives that can be used to encourage higher-risk investment by the AFP's and a shift of the fund's portfolio to risky equity and debt. Given that the country has engaged in an effort to encourage capital markets development and the financing of the real sector via private financial markets, this strategy appears to be the more desirable from the social and economic point of view. Of course, the same system, reversed, could be used to restrain AFP's from high risk investment and a shift to government securities.

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