

PHENOMENOLOGY OF THE INTEREST RATE CURVE.

Jean-Philippe Bouchaud^{1,2,†}, Rama Cont^{2,1}, Nicole El-Karoui⁴
Marc Potters² and Nicolas Sagna^{3,*}

¹ *Service de Physique de l'état Condensé
Centre d'Etudes de Saclay
91191 Gif-sur-Yvette Cedex, France*

² *Science & Finance, 109-111 rue Victor-Hugo, 92532 Levallois,
France*

³ *Laboratoire de Probabilités, URA CNRS 224
Université de Paris VI, Tour 56, 4 place Jussieu
75252 Paris 05, France*

⁴ *Centre de Mathématiques Appliquées
École Polytechnique, 91 128 Palaiseau CEDEX.*

[†] *Corresponding author.*

^{*} *Present address: Credit Suisse First Boston
Fixed Income Research Group
One Cabot Square London E14 4QJ, U.K.*

Abstract

This paper contains a statistical description of the whole U.S. forward rate curve (FRC), based on data from the period 1990-1996. We find that the *average* deviation of the FRC from the spot rate grows as the square-root of the maturity, with a prefactor which is comparable to the spot rate volatility. This suggests that forward rate market prices include a risk premium, comparable to the probable changes of the spot rate between now and maturity, which can be understood as a 'Value-at-Risk' type of pricing. The *instantaneous* FRC however departs from a simple square-root law. The distortion is maximum around one year, and reflects the market anticipation of a local trend on the spot rate. This anticipated trend is shown to be calibrated on the past behaviour of the spot itself. We show that this is consistent with the volatility 'hump' around one year found by several authors (and which we confirm). Finally, the number of independent components needed to interpret most of the FRC fluctuations is found to be small. We rationalize this by showing that the dynamical evolution of the FRC contains a stabilizing second derivative (line tension) term, which tends to suppress short scale distortions of the FRC, suggesting an analogy with the motion of a vibrating string. This shape dependent term could lead, in principle, to arbitrage. However, this arbitrage cannot be implemented in practice because of transaction costs. We suggest that the presence of transaction costs (or other market 'imperfections') is crucial for model building, for a much wider class of models becomes eligible to represent reality.

Electronic addresses :

bouchaud@amoco.saclay.cea.fr

cont@ens.fr

elkaroui@cmapx.polytechnique.fr

nsagna@csfp.co.uk

potters@cfm.fr

1 Introduction

Finding an adequate statistical description of the dynamics of financial assets has a long history which is probably reaching a climax, now that enormous sets of time series are readily available for analysis. The case of the interest rate curve is particularly complex and interesting, since it is not the random motion of a point, but rather the history of a whole curve (corresponding to different maturities) which is at stake. The need of a consistent description of the whole interest rate curve is furthermore enhanced by the rapid development of interest rate derivatives (options, swaps, options on swaps, etc) [13]. Present models of the interest rate curve fall into two categories [12]: the first one is the Vasicek model and its variants, which focuses on the dynamics of the short term interest rate, from which the whole curve is reconstructed (see [21] for a recent review). The second one, initiated by Heath, Jarrow and Morton [10] (see also [11]), takes the full forward rate curve as dynamical variables, driven by (one or several) continuous-time Brownian motion, multiplied by a maturity dependent scale factor. Most models are primarily motivated by their mathematical tractability rather than by their ability to describe the data. For example, the fluctuations are often assumed to be gaussian, thereby neglecting ‘fat tail’ effects. More importantly, as we shall discuss below, the empirical *shape* of the interest rate curve can only be captured by these models if one includes an additional parameter, the so called ‘market price of risk’, which takes a rather large value.

The aim of this paper is threefold. We first present an empirical study of the forward rate curve (FRC), where we isolate several important features which a good model should be asked to reproduce. Rather than aiming at a precise statistical determination of parameters (which is known to be particularly difficult for all financial data), we wish to offer a qualitative description of the dynamics of the FRC and try to decipher the information contained in its shape, in terms of an ‘anticipated risk’ and an ‘anticipated bias’. Based on empirical evidence, we argue that the market fixes the interest rate curve through a Value-at-Risk type of condition, rather than through an averaging procedure, which is the starting point of the classical models alluded to above. Furthermore the ‘anticipated bias’ is found to be strongly correlated with the past trend on the spot rate itself. We then present a general class of string models, inspired from statistical physics, which describe the motion of an elastic *curve* driven by noise, and discuss how these models offer a natural framework to account for some of the empirical results, in particular the small number of independent factors needed to describe the evolution of the FRC. Finally, we discuss the general concept of arbitrage and its relation with model

building. We suggest that the argument of absence of arbitrage opportunities, usually used to restrict possible models of reality, is much too strong: in the presence of transaction costs, qualitatively new classes of models can be considered.

As an important preliminary remark, we want to stress that the adequacy of a model can be assessed from two rather different standpoints. The first one, which we shall adopt here, is to see how well a given model describes the dynamics of the primary object, namely the forward rate curve. The second one, more concerned with derivative markets, asks how consistent are the determination of the model's parameters across several different derivative products, irrespective of the ability of these parameters to reproduce the statistics of the underlying asset. A well known example is the case of the Black-Scholes theory of options: while there should be no volatility smile in a Black-Scholes world, it is still an interesting question from an applied point of view to know whether the implied volatility on plain vanilla options can be used to price exotic options (within a Black-Scholes scheme). We will not consider however in the present paper the consequences of our model to the problem of derivative pricing.

2 Statistical analysis of the forward rate curve

2.1 Presentation of the data and notations

The forward interest rate curve (FRC) at time t is fully specified by the collection of all forward rates $f(t, \theta)$, for different maturities θ . It allows for example to calculate the price $B(t, \theta)$ at time t of a (zero-coupon) bond, which pays 1 at time $t + \theta$. It is given by:

$$B(t, t + \theta) = \exp - \int_0^\theta du f(t, u) \quad (1)$$

$r(t) = f(t, \theta = 0)$ is called the 'spot rate'. Note that in the following θ is always a time difference; the maturity *date* T is $t + \theta$.

2.2 The data set

Our study is based on a data set of daily prices of Eurodollar futures contracts. The Eurodollar futures contract is a futures contract on an interest rate, as opposed to the Treasury bill futures contract which is a futures contract on the price of a

Treasury bill. The interest rate underlying the Eurodollar futures contract is a 90-day rate, earned on dollars deposited in a bank outside the U.S. by another bank. When interest rates are fixed, a well known arbitrage argument [5] implies that forward and futures contracts must have the same value. But when interest rates are stochastic in principle forward contracts and future contracts are no longer identical – they have different margin requirements – and one may expect slight differences between futures and forward contracts [20], which we shall neglect in the following. We have furthermore checked that the effects discussed below are qualitatively the same as those observed when FRC is reconstructed from swap rates. The interest of studying forward rates rather than yield curves is that one has a direct access to a ‘derivative’ (in the mathematical sense – see Eq.(1)), which obviously contains more precise informations.

In practice, the futures markets price three months forward rates for *fixed* expiration dates, separated by three month intervals. Identifying three months futures rates to instantaneous forward rates, we have available time series on forward rates $f(t, T_i - t)$, where T_i are fixed dates (March, June, September and December of each year), which we have converted into fixed maturity (multiple of three months) forward rates by a simple linear interpolation between the two nearest points such that $T_i - t \leq \theta \leq T_{i+1} - t$. Days where one contract disappears are not included, to remove possible artefacts. Between 1990 and 1996, we have at least 15 different Eurodollar maturities for each market date; longer maturities are in general less liquid than short maturities; we however believe that such a difference in liquidity does not affect the qualitative conclusions drawn in this paper (this is partly confirmed by our analysis of the swap rates). Between 1994 and 1996, the number of available maturities rises to 30 (as time grows, longer and longer maturity forward rates are being traded on future markets); we shall thus often use this restricted data set. Since we only have daily data, our reference time scale will be $\tau = 1$ day. The variation of $f(t, \theta)$ between t and $t + \tau$ will be denoted as $\delta f(t, \theta)$:

$$\delta f(t, \theta) = f(t + \tau, \theta) - f(t, \theta) \quad (2)$$

It would obviously be very interesting to extend the present analysis to intra-day fluctuations.

2.3 Quantities of interest and data analysis

The description of the FRC has two, possibly interrelated, aspects :

(i) what is, at a given instant of time, the *shape* of the FRC as a function of the maturity θ ?

(ii) what are the statistical properties of the increments $\delta f(t, \theta)$ between time t and time $t + \tau$, and how are they correlated with the shape of the FRC at time t ?

The two basic quantities describing the FRC at time t is the value of the short term interest rate $f(t, \theta_{\min})$ (where θ_{\min} is the shortest available maturity), and that of the short term/long term *spread* $s(t) = f(t, \theta_{\max}) - f(t, \theta_{\min})$, where θ_{\max} is the longest available maturity. The two quantities ¹ $r(t) \simeq f(t, \theta_{\min})$, $s(t)$ are plotted versus time in Fig. 1; note that

– The volatility σ of the spot rate $r(t)$ is equal to ² $0.8\%/\sqrt{\text{year}}$. This is obtained by averaging over the whole period. However, as previously noticed [4, 1], there seems to be a systematic correlation between σ and r , which can be parametrized as $\sigma \propto r^\beta$. For the period considered (1990-96), we find a much smaller value $\beta \simeq 0.4$ than the one reported in [4] $\beta = 1.5$, based one monthly data from 1964 to 1989, or $\beta = 1.1$ [1], based on weekly data from 1984 to 1993.

– $s(t)$ has varied between 0.53% and 4.34%. Contrarily to some european interest rates on the same period, $s(t)$ has always remained positive, a situation we shall refer to as ‘normal’. (This however does not mean that the FRC is increasing monotonously, see below).

In order to rationalize the empirical data, we shall postulate that the whole FRC oscillates around an average shape, and *parametrize* $f(t, \theta)$ as ³:

$$f(t, \theta) = r(t) + s(t) \frac{Y[\theta]}{Y[\theta_{\max}]} + \xi(t, \theta) \quad (3)$$

where Y is a certain (time independent) function and $\xi(t, \theta)$ represents the deviation from the (empirical) average curve, in the sense that

$$\langle \xi(t, \theta) \rangle = 0 \quad (4)$$

where $\langle \dots \rangle$ represents a time average. By definition, $\xi(t, \theta_{\min}) = \xi(t, \theta_{\max}) \equiv 0$.

Fig. 2 shows the scaling function $Y(u)$, determined by averaging the difference $f(t, \theta) - r(t)$ over time. Interestingly, the function $Y(u)$ is rather well fitted by a simple \sqrt{u} law. This means that on average, the difference between the forward rate with maturity θ and the spot rate is equal to $A\sqrt{\theta}$, with a proportionality constant $A = 0.85\%/\sqrt{\text{year}}$ which turns out to be nearly identical to the spot rate volatility. We shall propose a simple interpretation of this fact below.

¹We shall from now on take the three month rate as an approximation to the spot rate $r(t)$.

²The dimension of r should really be % per year, but we conform to the habit of quoting r simply in %. Note that this can sometimes be confusing when checking the correct dimensions of

Figure 1

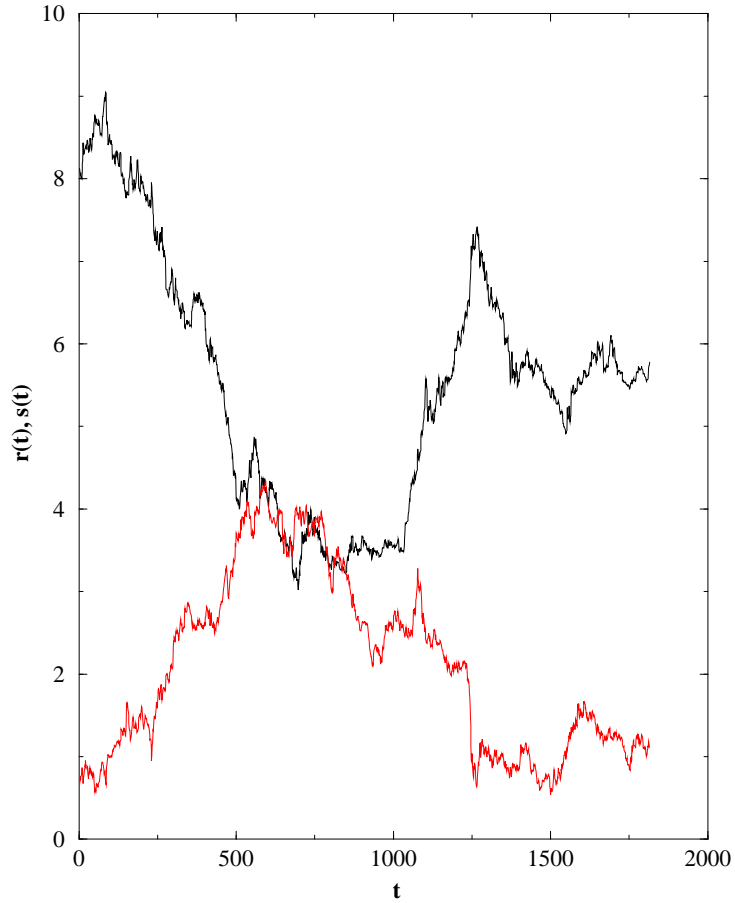


Figure 1: The historical time series of the spot rate $r(t)$ from 1990 to 1996 (top curve) – actually corresponding to a three month future rate (dark line) and of the ‘spread’ $s(t)$ (bottom curve), defined with the longest maturity available over the whole period 1990-1996 on future markets, i.e. $\theta_{\max} =$ four years. (In restricted period (94-96), this maturity grows to $\theta_{\max} = 8$ years.)

Fig 2

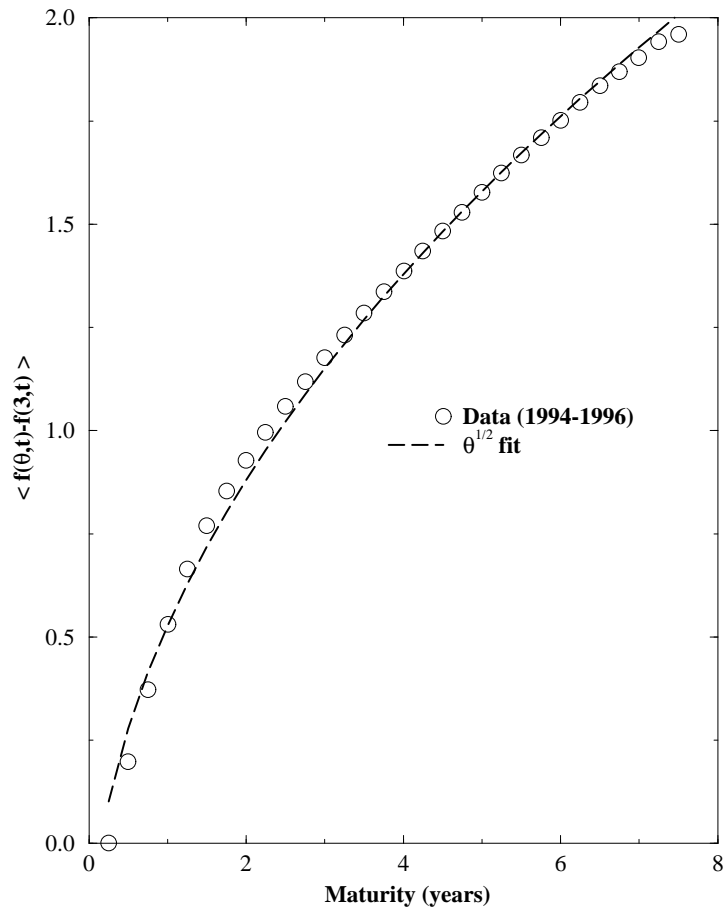


Figure 2: The average FRC in the period 94-96, as a function of the maturity θ . We have shown for comparison a one parameter fit with a square-root law, $A(\sqrt{\theta} - \sqrt{\theta_{\min}})$. The same $\sqrt{\theta}$ behaviour actually extends up to $\theta_{\max} = 10$ years, which is available in the second half of the time period.

Fig 3

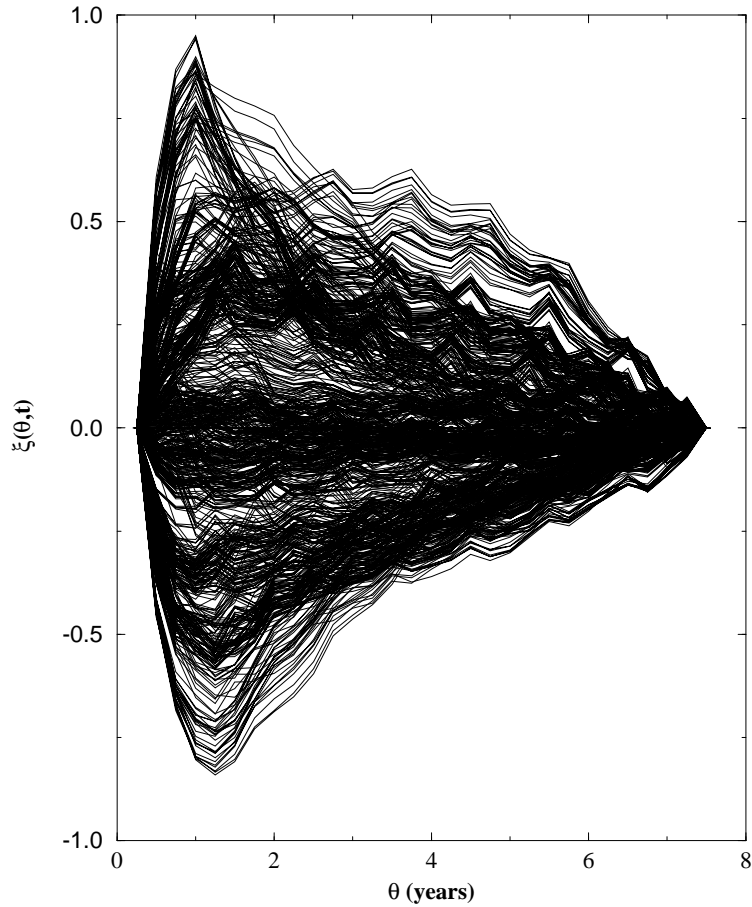


Figure 3: This shows the deviation $\xi(t, \theta)$ as a function of θ for different times t . Note that this resembles the motion of an elastic string, with a maximum around $\theta = 1$ year.

Let us now turn to an analysis of the *fluctuations* around the average shape. Fig. 3 shows for different instants of time the whole ‘deviation’ $\xi(t, \theta)$ as a function of θ .

These fluctuations are actually similar to that of a vibrated elastic string. The average deviation $\Delta(\theta)$ can be defined as:

$$\Delta(\theta) := \sqrt{\langle \xi(t, \theta)^2 \rangle} \quad (5)$$

a formula.

³The relation between this decomposition and the more standard Principal Component Analysis is clarified below. Note in particular that $r(t)$, $s(t)$ and $\xi(t, \theta)$ are *a priori* not independent.

Fig 4

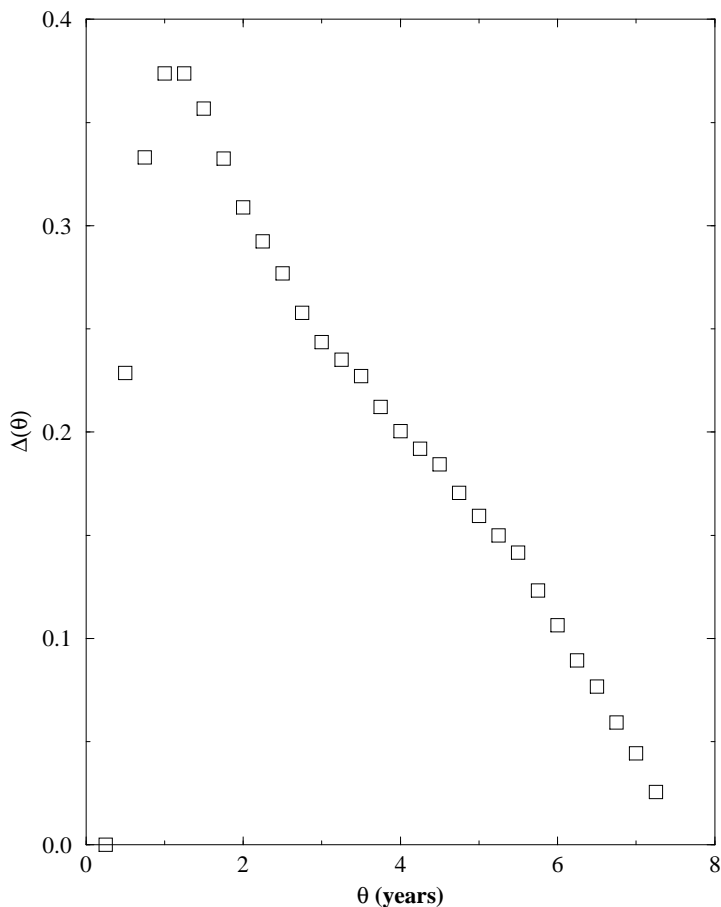


Figure 4: Root mean square deviation $\Delta(\theta)$ from the average FRC as a function of θ . Note the maximum for $\theta^* = 1$ year, for which $\Delta \simeq 0.38\%$.

which is plotted in Fig. 4, for the period 94-96. The maximum of Δ is reached for a maturity of $\theta^* = 1$ year.

A more precise study of the fluctuation modes consists in studying the equal time correlation matrix $M(\theta, \theta')$ defined as:

$$M(\theta, \theta') = \langle \xi(t, \theta) \xi(t, \theta') \rangle \quad (6)$$

Note that $M(\theta, \theta) \equiv \Delta(\theta)^2$. The eigenvalues M_q^2 of this matrix are plotted in Fig. 5, as a function of their rank $q = 1, 2, \dots$, in log-log coordinates. We have shown for comparison a q^{-2} and q^{-4} decay (see section 5), indicating that these eigenvalues are decreasing very fast with q . We have furthermore calculated the same eigenvalues in the presence of some ‘artificial’ noise (i.e. a random variable

of zero mean and width 0.04% added to each $f(t, \theta)$ independently). One sees that it only affects the high q modes, while leaving the modes with $q < 9$ relatively stable, and which are thus statistically meaningful. The deviation from the average shape can be written as:

$$\xi(t, \theta) = \sum_q M_q \xi_q(t) \Psi_q(\theta) \quad (7)$$

where $\langle \xi_q(t) \xi_{q'}(t) \rangle = \delta_{q,q'}$. Exactly as an elastic string, the first mode Ψ_1 has no nodes (and is found to be nearly proportionnal to $\Delta(\theta)$), while the second Ψ_2 has one node.

2.4 Statistics of the daily increments

We now turn to the statistics of the *daily increments* $\delta f(t, \theta)$ of the forward rates, by calculating their volatility $\sigma(\theta) = \sqrt{\langle \delta f(t, \theta)^2 \rangle}$, their excess kurtosis

$$\kappa(\theta) = \frac{\langle \delta f(t, \theta)^4 \rangle}{\sigma^4(\theta)} - 3 \quad (8)$$

and the following ‘spread’ correlation function:

$$C(\theta) = \frac{\langle \delta f(t, \theta_{\min}) (\delta f(t, \theta) - \delta f(t, \theta_{\min})) \rangle}{\sigma^2(\theta_{\min})} \quad (9)$$

which measures the influence of the short term interest fluctuations on the other modes of motion of the FRC.

Fig. 6 shows $\sigma(\theta)$ and $\kappa(\theta)$. Somewhat surprisingly, $\sigma(\theta)$, much like $\Delta(\theta)$ has a maximum around $\theta^* = 1$ year, a feature already noticed in, e.g., [15, 18]. The order of magnitude of $\sigma(\theta)$ is $0.05\%/\sqrt{\text{day}}$, or $0.8\%/\sqrt{\text{year}}$. The kurtosis $\kappa(\theta)$ is rather high (of the order of 5), and only weakly decreasing with θ .

Finally, $C(\theta)$ is shown in Fig. 7; its shape is again very similar to those of $\Delta(\theta)$ and $\sigma(\theta)$, with a pronounced maximum around $\theta = 1$ year. This means that the fluctuations of the short term rate are *amplified* for maturities around one year. We shall come back to this important point below. Note that $C(\theta)$ goes to zero for large maturities, which means that the short term rate and the short term/long term spread evolve, to a first approximation, in an uncorrelated manner. A simple fit of $C(\theta)$, shown in Fig. 7, is given by $C(\theta) = c\theta \exp(-\Gamma\theta)$, with $c = 1.5 \text{ year}^{-1}$ and $\Gamma = 1.0 \text{ year}^{-1}$. An interpretation of the values of c and Γ will be given below.

Fig 5

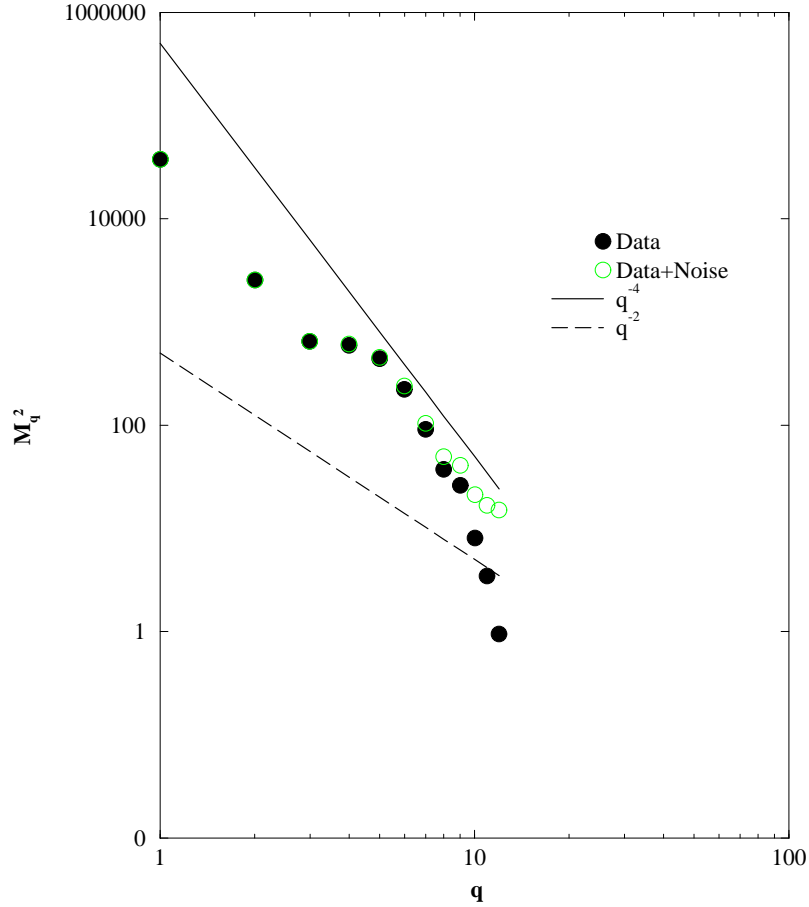


Figure 5: The eigenvalues M_q^2 of the instantaneous fluctuation matrix $M(\theta, \theta')$, as a function of their rank $q = 1, 2, \dots$, in log-log coordinates. We have shown for comparison a q^{-2} and a q^{-4} decay (see section 5), indicating that these eigenvalues are decreasing very fast with q . Also shown for comparison the eigenvalues of $M(\theta, \theta')$ when an artificial noise (uniform in $[-0.02, +0.02]$) is independently added to each point of the FRC, to estimate the relevance of the information contained in the high q eigenvalues.

Fig 6

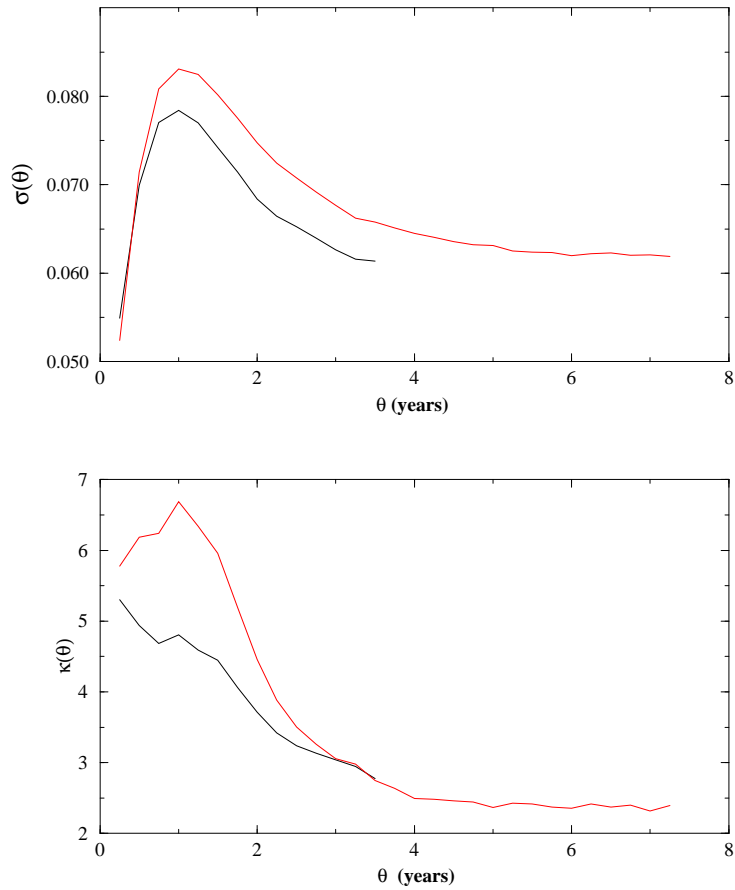


Figure 6: The daily volatility and kurtosis as a function of maturity. Note the maximum of the volatility for $\theta = \theta^*$, while the kurtosis is rather high, and only very slowly decreasing with θ . The two curves correspond to the periods 90-96 and 94-96, the latter period extending to longer maturities.

Fig 7

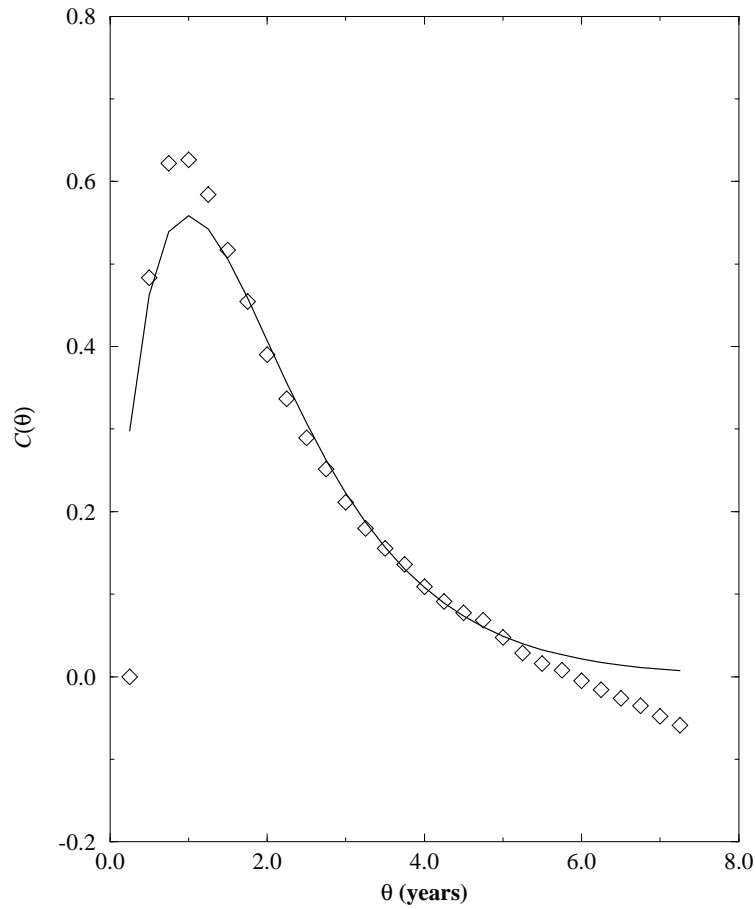


Figure 7: The correlation function $C(\theta)$ (defined by Eq. 9) between the daily variation of the spot rate and that of the forward rate at maturity θ , in the period 94-96. Again, $C(\theta)$ is maximum for $\theta = \theta^*$, and decays rapidly beyond. A simple fit using the function $c\theta\exp(-\Gamma\theta)$ is shown for comparison, which leads to $c = 1.5 \text{ year}^{-1}$ and $\Gamma = 1.0 \text{ year}^{-1}$. Note also that $C(\theta_{\max})$ is close to zero, indicating that, in a first approximation, the evolution of the spot rate and of the spread are independent.

Much as above, one can define a correlation matrix for the variations of $\xi(t, \theta)$ as:

$$N(\theta, \theta') = \langle \delta\xi(t, \theta) \delta\xi(t, \theta') \rangle \quad (10)$$

Its diagonalisation gives results very similar to the diagonalisation of M , in particular concerning the fall-off of the eigenvalues as a function of q and the shape of the low q modes.

Owing to the rapid decrease of the eigenvectors of M and N , a good approximation thus consists in retaining only the so called ‘butterfly’ mode of deformation Ψ_1 , and to write ⁴:

$$f(t, \theta) = r(t) + s(t) Y \left[\frac{\theta}{\theta_{\max}} \right] + \xi_1(t) \Psi_1(\theta) \quad (11)$$

where $Y(u) := \sqrt{u}$. That only a rather small degrees of freedom are needed to describe most of the FRC’s fluctuations was already discussed on several occasions [16, 15, 22, 7], although not exactly in the present terms.

3 Classical models

3.1 Vasicek

The simplest FRC model is a one factor model due to Vasicek [23], where the whole term structure can be ascribed to the short term interest rate, which is assumed to follow a stochastic evolution described as:

$$dr(t) = \Omega(r_0 - r(t)) + \sigma dW(t) \quad (12)$$

where r_0 is an ‘equilibrium’ reference rate, Ω describes the strength of the reversion towards r_0 (and is the inverse of the mean reversion time), and $dW(t)$ is a Brownian noise, of volatility 1. In its simplest version, the Vasicek model prices a bond maturing at T as the following average:

$$B(t, T) = \langle \exp - \int_t^T du r(u) \rangle \quad (13)$$

where the averaging is over the possible histories of the spot rate between now and the maturity, where the uncertainty is modelled by the noise W . The computation

⁴We have included the factor M_1 in the definition of Ψ_1

of the above average is straightforward and leads to (using Eq. (1):

$$f(t, \theta) = r(t) + (r_0 - r(t))(1 - e^{-\Omega\theta}) - \frac{\sigma^2}{2\Omega^2}(1 - e^{-\Omega\theta})^2 \quad (14)$$

The basic results of this model are as follows:

- Since $\langle r_0 - r(t) \rangle = 0$, the average of $f(t, \theta) - r(t)$ is given by

$$\langle f(t, \theta) - r(t) \rangle = -\sigma^2 / 2\Omega^2 (1 - e^{-\Omega\theta})^2, \quad (15)$$

and should thus be *negative*, at variance with empirical data. Note that in the limit $\Omega\theta \ll 1$, the order of magnitude of this (negative) term is very small: taking $\sigma = 1\%/\sqrt{\text{year}}$ and $\theta = 1$ year, it is found to be equal to 0.005% !

- The volatility $\sigma(\theta)$ is monotonously decreasing as $\exp -\Omega\theta$, while the kurtosis $\kappa(\theta)$ is identically zero (because W is gaussian)

- The correlation function $\mathcal{C}(\theta)$ is proportionnal to the volatility $\sigma(\theta)$, and thus does not exhibit the marked maximum shown in Fig. 7.

- The variation of the spread $s(t)$ and of the spot rate should be perfectly correlated, which is not the case (see Fig. 7): more than one factor is in any case needed to account for the deformation of the FRC.

While it is easy to make κ non zero by taking a discrete time process where dW is non gaussian, it is awkward to account for a maximum in the volatility $\sigma(\theta)$ within such an approach (see the discussion below). However, the most obvious inconsistency of this model is the fact that the average spread should be negative. A way out is to introduce the ‘market price of risk’: as shown by Vasicek [23], the probability measure over which the average (13) is performed is not necessarily the historical average. Arbitrage arguments allow a ‘change of measure’, which in the present case simply amounts to correcting the ‘true’ (i.e historical) value of r_0 to an effective value $r_0 + \lambda\sigma$, where λ is the market price of risk. With this correction, one finds that:

$$\langle f(t, \theta) - r(t) \rangle = \lambda\sigma(1 - e^{-\Omega\theta}) \quad (16)$$

A fit of the empirical data with such a formula leads to $\Omega = 6.17 \cdot 10^{-2}/\text{year}$ and $\lambda\sigma = 2.26\%$. The value of Ω corresponds to a mean reverting time of around 16 years, which is worrying since the data set is only 7 years long – it is thus not really consistent to set $\langle r_0 - r(t) \rangle$ to zero. The market price of risk q corresponds to demanding for a one year maturity bond an extra return (over the spot rate) of around 0.15%.

Fig 8

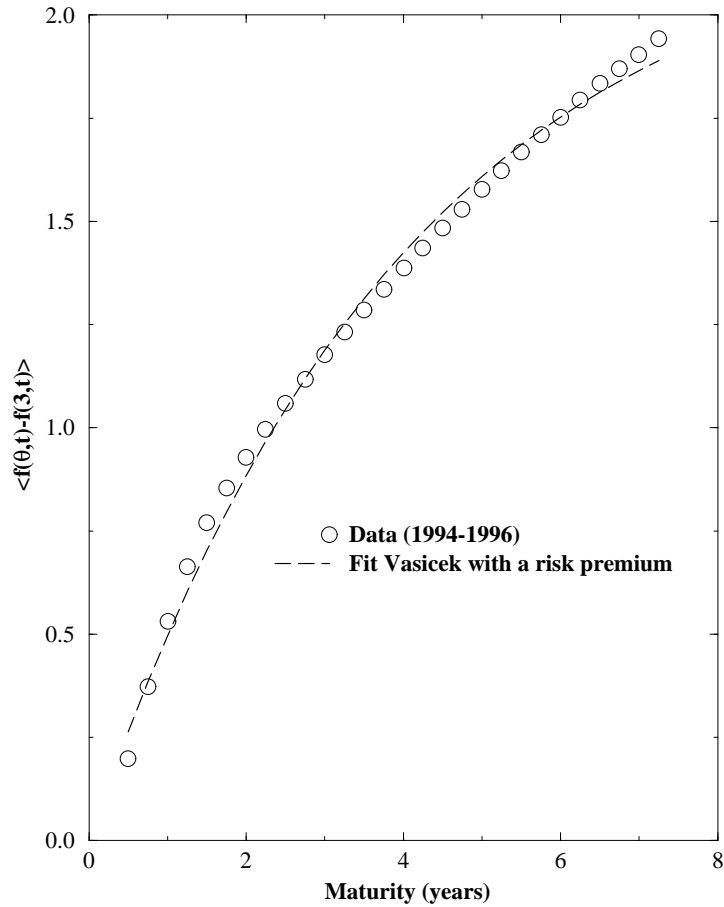


Figure 8: Fit of the FRC with a Vasicek term structure, Eq. (16) with a non zero market price of risk. This is a two parameter fit – compare to the one parameter fit in Fig. 2 – which is however found to be acceptable. One finds $\Omega = 6.17 \cdot 10^{-2}/\text{year}$ and $\lambda\sigma = 2.26\%$. Note that we neglect the difference between $r(t)$ and $f(t, \theta_{\min})$.

3.2 Hull and White

An interesting extension of Vasicek's model designed to fit exactly the 'initial' FRC $f(t = 0, \theta)$ was proposed by Hull and White [14]. It amounts to replacing the above constants Ω and r_0 by time dependent functions. For example, $r_0(t)$ represents the anticipated evolution of the 'reference' short term rate itself with time. These functions can be adjusted to fit $f(t = 0, \theta)$ exactly. Interestingly, one can then derive the following relation (for a zero market price of risk λ):

$$\left\langle \frac{\partial r(t)}{\partial t} \right\rangle = \left\langle \frac{\partial f}{\partial \theta}(t, 0) \right\rangle \quad (17)$$

up to a term of order σ^2 which turns out to be negligible, exactly for the same reason as explained above. On average, the second term (estimated by taking a finite difference estimate of the partial derivative using the first two points of the FRC) is definitely found to be *positive*, and equal to 0.8 %/year. On the same period (90-96), however, the spot rate has *decreased* from 8.1 % to 5.9 %, instead of growing by 5.6%.

In simple terms, both the Vasicek and the Hull-White model mean the following: the FRC should basically reflect the market's expectation of the average evolution of the spot rate (up to a correction on the order of σ^2 , but which turns out to be very small – see above). However, since the FRC is on average increasing with the maturity (situations when the FRC is 'inverted' are comparatively much rarer), this would mean that the market systematically expects the spot rate to rise, which it does not. It is hard to believe that the market persists in error for such a long time. Hence, the upward slope of the FRC is not only related to what the market expects on average, but that a systematic risk premium is needed to account for this increase. Within a classical framework, this is attributed to the 'market price of risk', which is however not a directly measurable quantity. In section (4), we will propose a more direct interpretation of this risk premium.

Let us however mention that some empirical results can be accommodated by the Hull-White model and its extensions [15, 21, 17]. For example, a 'humped volatility' can be obtained by choosing Ω to be time dependent in such a way that it starts being negative at time $t = 0$ (meaning that somehow the spot rate should *escape* from its reference value r_0) until a certain time T_0 beyond which it remains positive. It is easy to show in that case that the volatility is maximum for a maturity $\theta = T_0$. However, apart from the fact that the interpretation of this curious shape for $\Omega(t)$ is not clear, one can also show that in that case the average FRC should display an inflexion point at $\theta = T_0$, which is not seen empirically.

In order to fit independently the shape of the FRC and the shape of the volatility, one should turn to two-factor Hull-White models [15], in which each one of the two mean reversal process can be more easily interpreted (see our own discussion below).

3.3 Heath-Jarrow-Morton

A second, more recent, line of thought, consists in writing stochastic differential equations for each of the forward rates. In this case, today's FRC is by construction described exactly, and provides the initial condition for the stochastic evolution. The simplest, time translation invariant, one factor model reads [10, 12]:

$$df(t, \theta) = \left[\frac{\partial f(t, \theta)}{\partial \theta} + \mu(\theta) \right] dt + \sigma(\theta) dW(t) \quad (18)$$

with a certain maturity dependent volatility $\sigma(\theta)$. If the expiration date $T = t + \theta$ rather than the maturity is kept fixed, the corresponding rate $\tilde{f}(t, T) \equiv f(t, \theta = T - t)$ is such that:

$$d\tilde{f}(t, T) = df(t, \theta) - \frac{\partial f(t, \theta)}{\partial \theta} dt = \mu(\theta) dt + \sigma(\theta) dW(t) \quad (19)$$

Absence of arbitrage opportunities then impose, within this continuous time framework, a relation between μ and σ which reads:

$$\mu(\theta) = \sigma(\theta) \int_0^\theta \sigma(\theta') d\theta' + \lambda \sigma(\theta), \quad (20)$$

where λ is again the market price of risk. This model has primarily been devised to price consistently (within a no-arbitrage framework) interest rates derivatives rather than to represent faithfully the historical evolution of the FRC itself. It is however interesting to remark that the above criticism still holds if one wants to interpret data using Eq. (18): since the average value of the *slope* of the FRC can only be generated by the maturity dependent drift $\mu(\theta)$, a non zero value of λ is needed in order to reproduce its correct order of magnitude – the order σ^2 term is again much too small, and can for all practical purposes be set to zero (this was first noticed in [3]). Furthermore, Eq. (18) does not ensure that the shape of the FRC remains ‘realistic’ with time: one the contrary, the FRC tends to distort more and more as time passes when the volatility has a non trivial maturity dependence.

We shall discuss in section 5 below how the introduction of a ‘line tension’ may heal this disease.

In conclusion, the classical models are only consistent with empirical data provided a rather large market price of risk is included. In technical terms, this means that, in the context of arbitrage theories, the ‘risk-neutral’ probability measure to be used for – say – derivative pricing cannot be identified with the empirical probability. Other theoretical difficulties inherent to arbitrage theories of the interest rate curve have been discussed in [9].

4 Risk-premium and the $\sqrt{\theta}$ law

4.1 The average FRC and Value-at-Risk pricing

The observation that on average the FRC follows a simple $\sqrt{\theta}$ law (i.e. $\langle f(t, \theta) - r(t) \rangle \propto \sqrt{\theta}$) suggests a more intuitive and direct interpretation. At any time t , the market anticipates either a future rise, or a decrease of the spot rate. However, the average anticipated trend is, on the long run, zero, since the spot rate has bounded fluctuations. Hence, the *average* market’s expectation is that the future spot rate $r(t)$ will be close to its present value $r(t = 0)$. In this sense, the average FRC should thus be flat (again, up to a small σ^2 correction). However, even in the absence of any trend in the spot rate, its *probable* change between now and $t = \theta$ is (assuming the simplest random walk behaviour) of the order of $\sigma\sqrt{\theta}$, where σ is the volatility of the spot rate. Hence, money lenders are tempted to protect themselves against this potential rise by adding to their estimate of the average future rate a *risk premium* of the order of $\sigma\sqrt{\theta}$ to set the forward rate at a satisfactory value. In other words, money lenders take a bet on the future value of the spot rate and want to be sure not to lose their bet more frequently than – say – once out of five. Thus their price for the forward rate is such that the probability that the spot rate at time $t + \theta$, $r(t + \theta)$ actually exceeds $f(t, \theta)$ is equal to a certain number p :

$$\int_{f(t, \theta)}^{\infty} dr' P(r', t + \theta | r, t) = p \quad (21)$$

where $P(r', t' | r, t)$ is the probability that the spot rate is equal to r' at time t' knowing that it is r now (at time t). Assuming that r' follows a simple random walk

centered around $r(t)$ then leads to ⁵:

$$f(t, \theta) = r(t) + A\sigma\sqrt{\theta} \quad A = \sqrt{2} \operatorname{erfc}^{-1}(2p) \quad (22)$$

which indeed matches the empirical data, with $p \sim 0.16$.

At this point, we thus depart both from the so-called ‘unbiased expectation hypothesis’ [3] or from Vasicek type of models, where it is assumed that today’s FRC tells us something about the *average* future evolution of the spot rate, possibly corrected by an (unmeasurable) market price of risk. This, for example, is used to calibrate the time scale Ω^{-1} which appears in the Vasicek model, Eq. (14). In our mind, the shape of today’s FRC must rather be thought of as an envelope for the probable future evolutions of the spot rate. The market seems to price future rates through a *Value at Risk* procedure (Eqs. 21, 22) rather than through an averaging procedure. In a strict sense, Eq. (22) is not acceptable from the point of view of arbitrage, since buying the spot rate and selling the forward rate would lead to a profit. However, one should keep in mind that this profit is not certain because Eq. (22) only describes the FRC in an average sense; one should also take into account the random deformation of the curve described by $\xi(t, \theta)$ (which we interpret below) which makes the above strategy risky. Nevertheless, the simple strategy of lending long-term money and borrowing short-term money is obviously a major source of income for most banks !

4.2 The instantaneous FRC. The anticipated bias.

Let us now discuss, along the same lines, the shape of the FRC at a *given instant of time*, which of course deviates from the average square root law. As discussed in section 2, the deviation $\xi(t, \theta)$ is actually well approximated by the first ‘butterfly’ mode $\xi_1(t)\Psi_1(\theta)$, which has the shape drawn in Fig. 4. We interpret this as follows: for a given instant of time t , the market actually expects the spot rate to perform a *biased* random walk. We shall argue that the market estimates the trend $m(t)$ by extrapolating the past behaviour of the spot rate itself. However, it appears that the market also ‘knows’ that trends on interest rates do not persist for ever in time; the ‘anticipated’ bias is thus in general maturity dependent. Hence, the probability distribution $P(r', t + \theta | r, t)$ used by the market is not centered around

⁵This assumption is certainly inadequate for small times, where large kurtosis effects are present. However, on the scale of months, these non gaussian effects can be considered as small [2].

$r(t)$ but rather around:

$$r(t) + \int_t^{t+\theta} dum(t, t+u) \quad (23)$$

where $m(t, t')$ can be called the anticipated bias at time t' , seen from time t .

As discussed above, it is reasonable to think that the market estimates m by extrapolating the recent past to the nearby future. Mathematically, this reads:

$$m(t, t+u) = m_1(t)G(u) \quad m_1(t) := \int_0^\infty dv K(v)\delta r(t-v) \quad (24)$$

where $K(v)$ is an averaging kernel of the past variations of the spot rate. We will call $G(u)$ the trend persistence function; it is normalized such that $G(u=0) = 1$, and describes how the present trend is expected to persist in the future. Eq. (21) then gives:

$$f(t, \theta) = r(t) + A\sigma\sqrt{\theta} + m_1(t) \int_0^\theta du G(u) \quad (25)$$

Identifying this expression with Eq. (11), one finds that:

- As discussed above, the spread s is associated to the volatility of the spot rate. Note however that other sources of risk, such as the exchange rate fluctuations for foreign investors, might also be included in the ‘effective’ volatility of the spot rate used in Eq. (21).

- The persistence function is simply the derivative of the first mode $\Psi_1(\theta)$, or $\Delta(\theta)$. This function is plotted in Fig. 9; the interesting point – related to the existence of a maximum in $\Delta(\theta)$ – is that it becomes negative (but small) beyond one year. This means that the market anticipates a trend reversion on the scale of one year.

This interpretation furthermore allows one to understand why the three functions introduced above, namely $\Delta(\theta)$, $\sigma(\theta)$ and the correlation function $C(\theta)$ have similar shapes. Indeed, taking for simplicity the kernel $K(v)$ to be $\gamma\exp[-\gamma v]$, one finds:

$$dm_1(t) = -\gamma m_1 dt + \gamma dr(t) \quad (26)$$

Hence, the correlation function $C(\theta)$ is nothing but:

$$C(\theta) = \gamma \int_0^\theta du G(u) \quad (27)$$

(we have assumed in the above expression that $C(\theta_{\max}) \simeq 0$, which is not a bad approximation – see Fig. 7). On the other hand, since from Eq. (26) $\langle m_1^2 \rangle =$

Fig 9

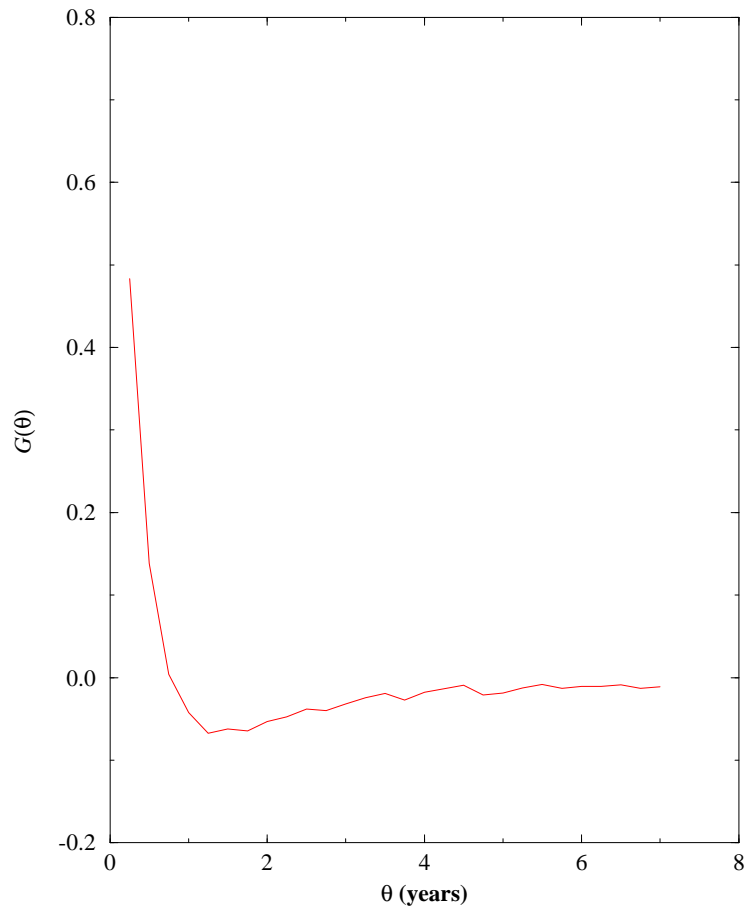


Figure 9: The ‘persistence’ function $G(\theta)$ describing how the market anticipates the persistence in the future of the observed past trend of the spot rate. Note that $G(\theta)$ becomes negative beyond one year, meaning that the market anticipates a trend reversion on the scale of one year.

$\gamma\sigma^2(0)/2$ one also finds:

$$\Delta(\theta) = \sqrt{\frac{\gamma}{2}}\sigma(0) \int_0^\theta du G(u) = \frac{\sigma(0)}{\sqrt{2\gamma}}C(\theta) \quad (28)$$

thus showing that $\Delta(\theta)$ and $C(\theta)$ are proportionnal. Actually, even the numerical prefactor predicted by this simplified description is quite good: using the value of $\gamma \equiv c$ determined by the fit shown in Fig. 7, the predicted value of $\Delta(\theta^*)$ is found to be $\simeq 0.29\%$, instead of the observed 0.38% (see Fig. 4).

Turning now to the volatility $\sigma(\theta)$, one finds that it is given by:

$$\sigma^2(\theta) = [1 + C(\theta)]^2 \sigma^2(0) + \sigma_s^2 Y(\theta)^2 \quad (29)$$

where σ_s^2 is the contribution of the spread volatility. We thus see that the maximum of $\sigma(\theta)$ is indeed related to that of $C(\theta)$. Intuitively, the reason for the volatility maximum is as follows: a variation in the spot rate changes that market anticipation for the trend $m_1(t)$. But this change of trend obviously has a larger effect when multiplied by a longer maturity. For maturities beyond one year, however, the decay of the persistence function comes into play and the volatility decreases again.

Interestingly, the form (29) suggests that the volatility should increase again for very large θ , due to the contribution of the last term. This can indeed be seen on a restricted data set which allows to reach $\theta = 10$ years (Fig. 10), where one sees that $\sigma(\theta)$ reaches a *minimum* around $\theta = 8$ years. This effect can also be related to the slow decay of the kurtosis with θ shown in Fig. 6, which reflects the fact that short maturities are essentially sensitive to one factor (the spot rate fluctuations), while longer maturities are progressively affected by a second factor (the anticipated risk) – adding independent contributions indeed reduce the kurtosis.

Taking for simplicity $\int_0^\theta du G(u) = \theta \exp -\Gamma\theta$, as suggested by Fig. 7, one can now fit the instantaneous FRC by the expression (25), for example fixing Γ . One then extracts from the FRC the anticipated bias $m_1(t)$ as a function of t . We find that $m_1(t)$ is correlated with the past evolution of the spot rate, on the scale of one year (i.e. the past time scale determining m_1 , γ^{-1} , is comparable to the time scale of the persistence function, Γ^{-1}). This is perfectly compatible with the value of $\gamma = c$ extracted directly from the fit shown in Fig. 7, which gives $\gamma^{-1} = 8$ months.

Finally, we want to note that Eq. (26) is actually very similar in spirit to the second equation in Hull and White's two factor model [15], where the 'noise' term in the second equation is actually the spot rate itself. This model was constructed in an ad-hoc way to reproduce the volatility hump and its intuitive meaning was

Fig 10

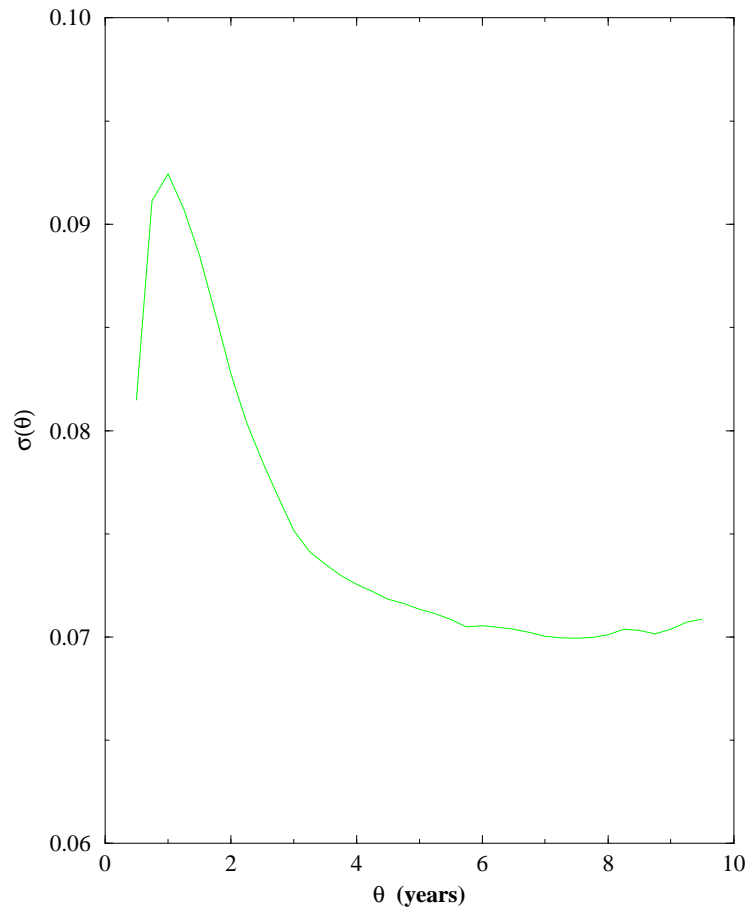


Figure 10: Volatility σ as a function of θ , as in Fig. 6, but for a restricted time period allowing to extend the range of maturities to 10 years. A volatility minimum appears around $\theta = 8$ years, beyond which the volatility of the spread becomes important.

not explicited. We believe that the above idea of an ‘autoregressive’ anticipated bias is a rather natural financial interpretation.

5 The forward rate curve as a vibrating string

In the previous section, we have thus argued that the shape of the instantaneous FRC is fixed by two distinct type of anticipations:

- An anticipated bias, whose influence is expected to decay with time, and whose amplitude is determined by the recent evolution of the spot rate.
- A anticipated risk, which gauges the potential motion of the spot rate, and which is added as a risk premium by money lenders.

Hence, as noticed by several people, the whole FRC evolves according to rather few ‘factors’ [8, 7], while in principle each maturity θ could feel the influence of some independent factor. However, one also expects that the ‘forces’ determining the evolution of each points of the FRC act as to prevent the FRC from ‘blowing apart’ with time, and to diffuse the information from one maturity to the next. A natural mechanism is the following: the dynamical evolution should be such that large deviations between nearby maturities are improbable – it is plausible that market forces tend to keep the forward rate $\tilde{f}(t, T)$ close to its local average $[\tilde{f}(t, T - \varepsilon) + \tilde{f}(t, T + \varepsilon)]/2$, where here $\varepsilon = 3$ months. The simplest choice is then to write:

$$\delta \tilde{f}(t, T) = D [\tilde{f}(t, T - \varepsilon) + \tilde{f}(t, T + \varepsilon) - 2\tilde{f}(t, T)] + \eta(t, T) \quad (30)$$

where D measures the strength of this force (‘line tension’) which could in principle be θ dependent. Eq. (30) describes a mean reversion of the forward rate towards its local average. It is the dynamical equation of a string of beads connected by springs, subject to a random force term η which can reflect, in particular, the variations of the forward rate which are specific to that given maturity (for example, the influence of a major political event which is scheduled around that particular date.).

The term proportional to D would become, in the continuum limit, a second order derivative $\partial^2 \tilde{f} / \partial T^2$, which is in principle not allowed from arbitrage arguments, at least when the ‘noise’ $\eta(t, T)$ contains a finite number of independent components. Intuitively, this can be understood in the simple case where $\eta(t, T)$ is independent of T . Then, a possible riskless winning strategy is to buy (resp. sell) one contract of maturity T if

$$\tilde{f}(t, T - \varepsilon) + \tilde{f}(t, T + \varepsilon) - 2\tilde{f}(t, T) > 0 \quad (31)$$

(resp. < 0), and then to hedge the position by buying (or selling) the appropriate number of the shortest maturity contract. Then, because of the restoring ‘string’ force, the resulting gain will be positive. The result of such a strategy (used simultaneously on all maturities) is shown in Fig. 11, in the absence of transaction costs. This curve shows indeed a clear positive slope, thereby directly showing the existence of a D term affecting the temporal evolution of the FRC. The fact that the above strategy is not riskless is due to the presence of more than one random factor η . More importantly, we have found empirically that the above growing P&L is completely destroyed by transaction costs.

Therefore, the presence or absence of shape dependent terms (such as the above second order difference term) *cannot be decided by arbitrage arguments in the presence of transaction costs*.

Still, the very existence of these terms is extremely important to understand the qualitative dynamical evolution of the FRC. In particular, the ‘line tension’ ensures that FRC becomes (and remains) smooth even if the initial condition is not, or if the term structure of the volatility is not. This is not the case within Heath-Jarrow-Morton like formulations, where $D = 0$, and where temporal evolutions tend to be ill-behaved [19].

We want to show here that the mere existence of a line tension allows one to understand qualitatively the fast decay of the eigenvalues of the correlation matrix, as observed in Fig. 5. If one introduces $\phi(t, T) = \tilde{f}(t, T) - \tilde{f}(t, t)$, and takes for notational simplicity the continuous time limit, one finds that ϕ evolves according to:

$$d\phi(t, T) = D \frac{\partial^2 \phi(t, T)}{\partial T^2} dt + d\eta(t, T) \quad \phi(t, t) \equiv 0 \quad (32)$$

Defining the eigenmodes of the operator $\frac{\partial^2}{\partial T^2}$ which vanish at $T = t$, i.e. : $\phi_q(t, T) = M_q(t) \sin(q(T - t))$, the evolution of $M_q(t)$ is then given by:

$$\frac{\partial M_q}{\partial t} = -Dq^2 M_q + \eta_q(t) \quad (33)$$

where η_q is the Fourier transform of the noise term. In this form one sees that:

- The ‘wavevector’ q is the index labelling the eigenmodes of the matrix M introduced above.
- The ‘lifetime’ of a perturbation of wavevector q is inversely proportionnal to Dq^2 . High q ’s (corresponding to short maturity differences) relax faster – this corresponds to the fact that if the FRC is distorted on very small scales, this will probably not persist in time.

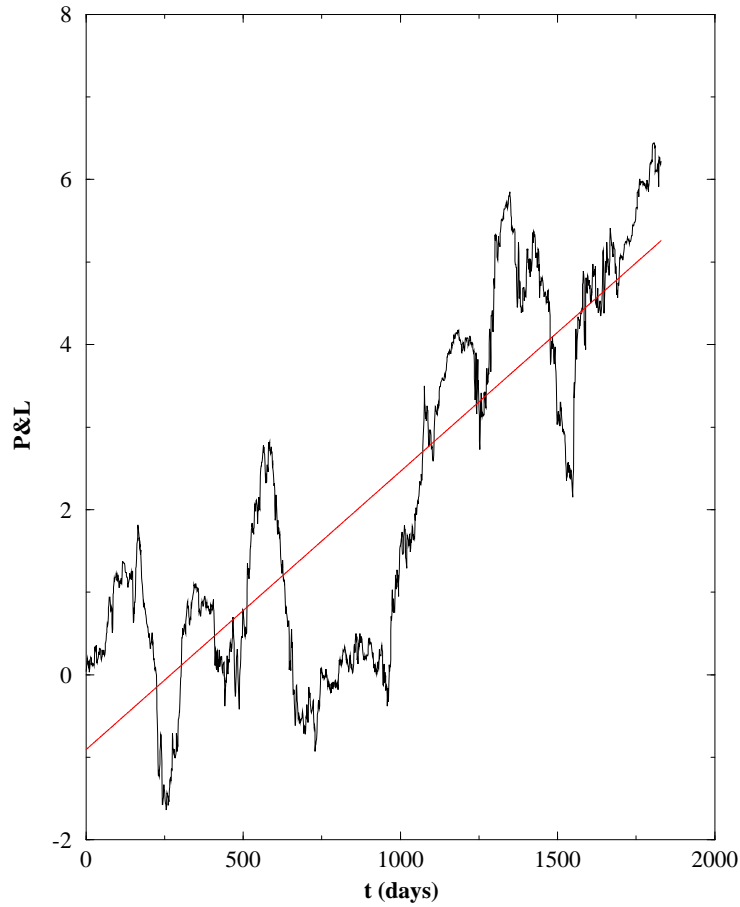


Figure 11: Empirical Profit and Loss curve corresponding to the simple ‘arbitrage’ strategy proposed in the text, which exploits the presence of a ‘string’ force proportional to the local second (discrete) derivative of the FRC. On average, this strategy clearly leads to a positive profit, although reasonable transaction costs ruins it completely. However, the presence of a non zero D is very important since it allows one to understand why the effective number of factors influencing the FRC is small (see Eq. (34) below).

- The average (over time) amplitude in mode q , proportional to the q^{th} eigenvalue of M is given by:

$$\langle |M_q|^2 \rangle = \frac{\langle |\eta_q|^2 \rangle}{Dq^2} \quad (34)$$

If $\eta(t, T)$ was independent for each maturity, $\langle |\eta_q|^2 \rangle$ would be independent of q , already leading to a q^{-2} decay of $\langle |M_q|^2 \rangle$, due to the restoring force D . The presence of correlations in the noise along the maturity axis acts to make this decay even faster. For example, if the correlations are exponentially decaying for different T 's (i.e. as $\exp -K|T - T'|$), one would find:

$$\langle |\eta_q|^2 \rangle \propto \frac{1}{K^2 + q^2} \quad (35)$$

hence leading to a much faster still decay of $\langle |M_q|^2 \rangle$. As shown in Fig. 5, the observed decay is indeed intermediate between q^{-2} and q^{-4} .

The main conclusion of this section is that a ‘line tension’ term indeed exists in the dynamical evolution of the FRC, acting as to reduce the distortions along the line. This line tension is responsible for reducing the effective number of factors needed to interpret the shape of the whole FRC, even when independent shocks affect each different maturities⁶. Arbitrage arguments in the absence of transaction costs impose that $D \equiv 0$, a conclusion that transaction costs allows one to by-pass.

6 Summary-Conclusion

The main results contained in this paper, based on the study of the whole U.S. forward rate curve since 1990 (and which we confirmed on the corresponding swap rates), are the following:

- The average departure of the forward rate curve from the spot rate grows as the square-root of the maturity, with a proportionality constant which is comparable to the spot rate volatility. This strongly suggests that forward rate market prices include a risk premium, comparable to the probable changes of the spot rate between now and maturity. This interpretation seems to us more natural (although in the same spirit) than the one based on an unobservable ‘market price of risk’.

⁶The idea of projecting the FRC on a few smooth functions of the maturity can also be found in [7, 6], although the basic mechanism underlying such a reduction, in terms of a line tension, was not discussed.

- The instantaneous FRC departs from a simple square-root law. The distortion is maximum around one year, and reflects the market anticipation of the trend on the spot rate (as in more traditional models of interest rates [3]). Somewhat surprisingly, however, this trend appears to be calibrated on the past behaviour of the spot itself: if the spot rate has decayed over the past months, the one year forward rate is below the average square root extrapolation, and vice-versa. This is consistent with the fact that the volatility is maximum for one year maturities.

- The number of independent components needed to interpret most of the FRC fluctuations is rather small, although in principle a random contribution should be assigned to each maturity. We rationalize this finding by showing that the dynamical evolution of the FRC contains a stabilizing ‘line tension’ term, which tends to suppress short scale distortions of the FRC. This shape dependent term in the dynamical evolution is not usually considered in interest rate models, because it leads, in principle, to arbitrage. However, this arbitrage cannot be implemented in practice because of residual risks and transaction costs. This is important because the statistical properties of a line subject to random forcing in the long time limit are very different when the line tension term is present (even very small) or strictly zero: the ‘line tension’ term (as all second derivative terms) lead to a smoothing of the singularities, which otherwise survive or even develop in the course of the evolution. We thus conclude that the presence of transaction costs (or other market ‘imperfections’) is crucial for model building, for a much wider class of models then becomes eligible to represent reality.

The present work should be extended in several directions. First, other interest rates should be studied, to know whether the broad features found on the U.S. market also holds in other cases. Second, the consequences of our modelling for interest rate derivatives should be worked out. Finally, it would be interesting to interpret the time dependent smile in option markets in term of a ‘forward volatility curve’, and see how the ideas expressed here transpose to this case. We hope to deal with these issues in the near future.

Acknowledgments: We want to thank J.P. Aguilar and L. Laloux for many useful discussions.

References

- [1] M. Apabhaj, K. Choe, F. Khennach, P. Willmott ‘Spot-on Modelling’, *Risk*

(November 1995).

- [2] J.P. Bouchaud, M. Potters, *Thorie des Risques Financiers*, Ala-Saclay (1997).
- [3] J.Y. Campbell (1986), “A defense of traditional hypotheses about the term structure of interest rates”, *Journal of Finance*, **XLL**, March. See also J.Y. Campbell, A.W. Lo, C. McKinlay (1997), *Econometrics of Financial Markets* Princeton University Press.
- [4] Chan K.C., Karolyi G.A., Longstaff F.A. & Sanders A.B. (1992) “An empirical comparison of alternative models of the short term interest rates”, *Journal of Finance*, **XLVII**, 3, 1209-1227.
- [5] Cox J.C., Ingersoll J.E. & Ross S.A. (1981) “The relationship between forward prices and futures prices”, *Journal of Financial Economics*, **9**, 321-46.
- [6] Chu, M. (1996) “The Random Yield Curve and Interest Rate Options”, Working Paper.
- [7] R. Douady, “Yield curve smoothing and residual variance of fixed income positions”, Working paper, June 1997.
- [8] N. El Karoui, H. Geman, V. Lacoste, “On the role of State Variables in Interest Rate Models”, Working Paper (Universit Paris VI), January 1996.
- [9] N. El Karoui, A. Frachot, H. Geman (1994) “A note on the behaviour of long zero coupon rates in no arbitrage framework”, Working Paper, May 1994.
- [10] Heath D., Jarrow R. & Morton A. (1992), “Bond pricing and the term structure of interest rates: a new methodology for contingent claims evaluation”, *Econometrica*, **60**, 77-105.
- [11] Ho T.S.Y. and Lee S.B. (1986), “Term structure movements and pricing Interest Rate contingent claims”, *Journal of Finance* **41**, 1011
- [12] see the reprints in: Lane Hughston (ed.) (1997) *Vasicek and beyond*, Risk Publications.
- [13] Hull, J. (1997) *Options, futures and other derivative securities*, Prentice-Hall.

- [14] Hull J. & White A. (1993), "One factor interest rate models and the valuation of interest rate derivative securities", *Journal of Financial and Quantitative Analysis*, **28**, 235-254.
- [15] Hull J. & White A. (1994) "Numerical procedures for implementing term structure models II: two-factor models", *Journal of Derivatives*, **Winter**, 37-48.
- [16] F. Longstaff, E. Schwartz (1992), "Interest rate volatility and the term structure: a two factor equilibrium model", *Journal of Finance*, **47**, 1259.
- [17] Moraleda J.M. & Pelsser A. (1997) "Forward vs Spot Interest-rate models of the term-structure: an empirical comparison", Working Paper, Erasmus University, Rotterdam.
- [18] Moraleda J.M. (1997) "On the pricing of interest rate options", PhD Thesis, Erasmus University, Rotterdam.
- [19] Musiela, M. "On the structure of PDEs arising in Interest Rate Models", talk given in Paris, November 1997.
- [20] Rendleman R. & Carabini C. (1979) "The efficiency of Treasury-bills futures markets", *Journal of Finance*, **34**, 895-914.
- [21] L.C.G. Rodgers (1996) "Which model for the term structure of interest rates should one use ?" in *Mathematical Finance*, IMA Volume 65, 93 (Springer-Verlag).
- [22] Schlgl E. & Sommer D. (1997) "Factor models and the shape of the term structure" AFFI Conference, Grenoble.
- [23] Vasicek, O.A. (1977) "An equilibrium characterization of the term structure", *Journal of Financial Economics*, **5**, 177-188.