No-Arbitrage Bounds on Contingent Claims Prices with Convex Constraints on the Dollar Investments of the Hedge Portfolio*

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Abstract. With constrained portfolios, contingent claims do not generally have a unique price, for which there are no arbitrage opportunities. We generalize earlier results of El Karoui and Quenez (1995) and Cvitanić and Karatzas (1993) by showing that there is an interval of no-arbitrage prices, when there are convex constraints on the dollar investments in the assets in the hedge portfolio. We also show that the bounds of the no-arbitrage interval can be found by solving two stochastic control problems, and we demonstrate how to solve these problems numerically.

Keywords. Contingent claims pricing, constrained dollar investments, no-arbitrage bounds, numerical solution.

JEL classification. C61; G13.

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1 Introduction

The pricing and hedging of contingent claims is a matter of immense interest and importance to the financial industry. The literature contains a plethora of results on various contingent claims in various models. The general approach taken is to replicate the contingent claim perfectly by a dynamic, self-financing trading strategy in the primary assets, i.e. the underlying asset, riskless borrowing/lending, etc. The price of the contingent claim must then, to avoid an obvious arbitrage, equal the cost of initiating the replicating strategy. This is the idea behind the famous European stock option pricing formula of Black and Scholes (1973) and Merton (1973) as well as behind the general results on contingent claims pricing given by Harrison and Kreps (1979) and Harrison and Pliska (1981).

In a complete market, all contingent claims are perfectly replicable and can, hence, be priced as indicated. It is broadly recognized that financial markets are not complete. Asset prices are affected by non-tradable factors, and the investors are often restricted in the portfolios they are allowed to hold. Both facts imply that, in general, it is not possible to replicate a contingent claim perfectly.

El Karoui and Quenez (1995) found a lower and an upper bound on the price of a contingent claim in a market with non-tradable assets. The basic idea is to find a *super-replicating* (or *dominating*) portfolio, that is, a portfolio generating at least the payoff of the contingent claim and, maybe, allowing the investor to withdraw funds from the portfolio. The upper price bound for the contingent claim is then defined as the lowest price of a super-replicating portfolio. It can be computed by solving an optimal control problem. The lower bound is defined and can be computed similarly. Cvitanić and Karatzas (1993) extend the analysis of El Karoui and Quenez (1995) to the case of convex constraints on the proportions of wealth invested in the risky assets.

We extend the result further to the case, where there are convex constraints on the dollar amounts invested in the assets. This way of modeling portfolio constraints, which was introduced by Cuoco (1997), contains the models of El Karoui and Quenez and Cvitanić and Karatzas as special cases. We show that the bounds can be computed by solving two stochastic control problems. Although the bounds can be computed explicitly in some cases, generally, one must resort to numerical methods. We demonstrate how the stochastic control problems, which define the bounds, can be solved numerically and provide examples.

The outline for the rest of the paper is as follows. In Section 2, we describe the model, which we will work within. In Section 3, we derive our main analytical results. The numerical method is discussed in Section 4 followed by some examples in Section 5. Finally, Section 6 concludes.

2 The Financial Market Model

Let w be a d-dimensional Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t | t \in [0,T]\}$ is a right-continuous filtration with $\mathcal{F}_t$ being the augmentation of the σ-algebra generated by $\{w(u) | 0 \le u \le t\}$. It is assumed that $\mathcal{F}_0$ is the σ-algebra generated by the zero sets of P and that $\mathcal{F} = \mathcal{F}_T$. All statements below involving stochastic variables are assumed to hold *almost surely* with respect to $\mathbb{P}$, and all stochastic processes are assumed to be adapted to $\mathbb{F}$.

The agents in the economy have access to continuous trading in an instantaneously riskless asset called *the savings account* with price $P_0(t)$ satisfying

$$P_0(t) = \exp\left\{ \int_0^t r(u)\, du \right\},$$

where r is the continuously compounded short-term spot interest rate process, and d risky assets with prices given by the vector $P(t) = (P_1(t), \ldots, P_d(t))^\top$ satisfying

$$P(t) = P(0) + \int_0^t \operatorname{diag}(P(u)) b(u)\, du + \int_0^t \operatorname{diag}(P(u)) \sigma(u)\, dw(u). \tag{2.1}$$

Here, b is an $\mathbb{R}^d$-valued stochastic process of expected rates of return, and σ is an $\mathbb{R}^{d\times d}$-valued stochastic process of volatilities. The processes r, b, and σ are assumed to be progressively measurable with respect to $\mathbb{F}$ and uniformly bounded in $[0,T] \times \Omega$.

The volatility process $\sigma(t)$ is assumed to satisfy the non-degeneracy assumption

$$\exists \varepsilon > 0\ \forall (x,t) \in \mathbb{R}^d \times [0,T] : x^\top \sigma(t)\sigma(t)^\top x \ge \varepsilon \|x\|^2. \tag{2.2}$$

As a consequence of condition (2.2), σ has full rank d implying the dynamic completeness of the market, at least potentially. As discussed below, the market can be incomplete due to restrictions on the set of admissible portfolio processes. Define the process λ_0 by

$$\lambda_0(t) = \sigma(t)^{-1}(b(t) - r(t)\mathbf{1}).$$

λ_0 is called the *relative risk process*, since it measures expected excess rate of return relative to volatility. Note that λ_0 is bounded.[1]

A *trading strategy* is a pair (α, θ), where α is an adapted one-dimensional and $\theta = (\theta_1, \ldots, \theta_d)^\top$ is a progressively measurable d-dimensional stochastic process. $\alpha(t)$ denotes the dollar amount invested in the savings account at time t. $\theta_i(t)$ is the dollar amount invested at time t in the i'th risky asset.

Let $\mathcal{K}$ be a non-empty, closed, convex subset of $\mathbb{R}^{d+1}$. A trading strategy (α, θ) is called

[1] See Remark 2.1 of Cvitanić and Karatzas (1993).

$\mathcal{K}$-admissible, if the following conditions are satisfied[2]

(i) the processes $\{\alpha(t)P_0(t)r(t)\}_{t\in[0,T]}$ and $\{\theta(t)^\top b(t)\}_{t\in[0,T]}$ belong to $\mathcal{L}^1[0,T]$ and the process $\{\theta(t)^\top \sigma(t)\}_{t\in[0,T]}$ to $\mathcal{L}^2[0,T]$,

(ii) $(\alpha(t),\theta(t)) \in \mathcal{K}$, $\ell \times \mathbb{P}$-almost everywhere.

$\mathcal{K}$ is called the *portfolio constraint set*. The set of $\mathcal{K}$-admissible trading strategies is denoted by $\mathcal{P}(\mathcal{K})$.

By restricting the individuals to $\mathcal{K}$-admissible trading strategies, a number of interesting situations can be examined. The interested reader is referred to Cuoco (1997) for examples. It turns out that the so-called support function of $-\mathcal{K}$ plays an important role. Let $\nu = (\nu_0, \underline{\nu}) \in \mathbb{R} \times \mathbb{R}^d$. Then the support function $\delta : \mathbb{R}^{d+1} \to \mathbb{R} \cup \{-\infty, +\infty\}$ of $-\mathcal{K}$ is defined by

$$\delta(\nu) = \sup_{(\alpha,\theta)\in\mathcal{K}} \left(-\alpha\nu_0 - \theta^\top \underline{\nu}\right).$$

The effective domain of δ, i.e. the set of $\nu \in \mathbb{R}^{d+1}$ for which $\delta(\nu) < \infty$, is denoted by $\tilde{\mathcal{K}}$. Next, we list a few interesting properties of δ and $\tilde{\mathcal{K}}$. See, e.g., Rockafellar (1970, Sect. 13) for more on support functions.

(i) $\tilde{\mathcal{K}}$ is a closed convex cone, called the barrier cone of $-\mathcal{K}$.

(ii) If $\mathcal{K}$ is a cone, then $\delta \equiv 0$ on $\tilde{\mathcal{K}}$.

(iii) δ is subadditive, that is

$$\delta(\nu_1) + \delta(\nu_2) \geq \delta(\nu_1 + \nu_2),$$

which follows from the corresponding property of the supremum operator.

(iv) If $(\alpha, \theta) \in \mathcal{K}$ and $\nu \in \tilde{\mathcal{K}}$, then

$$\alpha\nu_0 + \theta^\top \underline{\nu} + \delta(\nu) \geq 0. \tag{2.3}$$

Of course, this follows trivially from the definition of δ.

We need to impose the following assumption on $\mathcal{K}$.

Assumption 2.1 *$\mathcal{K}$ is such that*

[2]Here ℓ denoted Lebesgue measure, and

$$\mathcal{L}^1[0,T] = \left\{ z \text{ adapted} \;\middle|\; \int_0^T \|z(u,\omega)\|\, du < \infty \right\},$$
$$\mathcal{L}^2[0,T] = \left\{ z \text{ adapted} \;\middle|\; \int_0^T \|z(u,\omega)\|^2\, du < \infty \right\}.$$

(i) δ is bounded from above on $\tilde{\mathcal{K}}$, or, equivalently, δ is non-positive on $\tilde{\mathcal{K}}$,

(ii) $\nu_0 \geq 0$ for all $\nu \in \tilde{\mathcal{K}}$.

Let $\mathcal{H}$ denote the set of RCLL progressively measurable processes $H : [0,T] \times \Omega \to \mathbb{R}$ with $H(0) = 0$ and $|H(T)| < \infty$. Suppose an investor trades according to a portfolio process $(\alpha, \theta) \in \mathcal{P}(\mathcal{K})$ and, furthermore, adds/withdraws funds according to a process $H \in \mathcal{H}$, where $H(t)$ denotes the cumulative sum of additional amounts (positive or negative) invested by the agent in the $d+1$ assets between time 0 and time t. The value $X_x^{\alpha,\theta,-H}(t)$ of the agent's total investments, given an initial investment $x > 0$, follows the process

$$X_x^{\alpha,\theta,-H}(t) = x + \int_0^t \left(\alpha(u)r(u) + \theta(u)^\top b(u)\right) du + \int_0^t \theta(u)^\top \sigma(u)\, dw(u) + H(t).$$

If $H \equiv 0$, the trading strategy is *self-financing.*

Let $\mathcal{H}'$ denote the set of non-decreasing (and hence non-negative) processes in $\mathcal{H}$. If $H = -C$ for some process $C \in \mathcal{H}'$, the strategy is called a *trading strategy with consumption.* $C(t)$ is the cumulative amount withdrawn by the agent up to time t. A triple $(\alpha, \theta, C) \in \mathcal{P}(\mathcal{K}) \times \mathcal{H}'$, is called a $\mathcal{K}$-admissible trading strategy with consumption and an initial investment $x > 0$, if $X_x^{\alpha,\theta,C}(t) \geq 0$ for all $t \in [0,T]$. The set of such triples is denoted by $\mathcal{A}_c(x;\mathcal{K})$. If $H = D$ for some process $D \in \mathcal{H}'$, the strategy is called a *trading strategy with savings.* $D(t)$ is the cumulative amount injected by the agent up to time t. A triple $(\alpha, \theta, D) \in \mathcal{P}(\mathcal{K}) \times \mathcal{H}'$ is called a $\mathcal{K}$-admissible trading strategy with savings and an initial investment $x > 0$, if $X_x^{\alpha,\theta,-D}(t) \geq 0,\ \forall t \in [0,T]$. The set of such triples is denoted by $\mathcal{A}_s(x;\mathcal{K})$.

Define the following set of stochastic processes

$$\mathcal{N} = \left\{\nu \in \mathcal{L}^2[0,T] | \nu(t,\omega) \in \tilde{\mathcal{K}},\ \forall (t,\omega) \in [0,T] \times \Omega \right\}.$$

Define for $\nu \in \mathcal{N}$ the processes β_ν, λ_ν, Z_ν, and ζ_ν by

$$\begin{aligned}
\beta_\nu(t) &= \exp\left\{-\int_0^t (r(s) + \nu_0(s)))\, ds\right\}, \\
\lambda_\nu(t) &= \sigma(t)^{-1}(b(t) + \underline{\nu}(t) - (r(t) + \nu_0(t))\mathbf{1}), \qquad (2.4)\\
Z_\nu(t) &= \exp\left\{-\int_0^t \lambda_\nu(s)^\top dw(s) - \frac{1}{2}\int_0^t \lambda_\nu(s)^\top \lambda_\nu(s)\, ds\right\}, \\
\zeta_\nu(t) &= \beta_\nu(t) Z_\nu(t).
\end{aligned}$$

Note that Z_ν is a local martingale for $\nu \in \mathcal{N}$. Introduce the following subset of $\mathcal{N}$:

$$\mathcal{N}^* = \{\nu \in \mathcal{N} | Z_\nu \text{ is a martingale}\}.$$

Since λ_0 is bounded, Z_0 is a martingale, so $0 \in \mathcal{N}^*$.

We shall work under the equivalent measures $\mathbb{P}_\nu$, $\nu \in \mathcal{N}^*$, defined by $d\mathbb{P}_\nu/d\mathbb{P} = Z_\nu(T)$. For that purpose, recall that, by Girsanov's theorem, the process w_ν defined by

$$w_\nu(t) = w(t) + \int_0^t \lambda_\nu(s)\, ds \tag{2.5}$$

is a $\mathbb{P}_\nu$-Wiener process. Let $\mathrm{E}^\nu[\cdot]$ denote expectation under the measure $\mathbb{P}_\nu$, and let $\mathrm{E}_t^\nu[\cdot]$ be short for $\mathrm{E}^\nu[\cdot|\mathcal{F}_t]$. For later use, note that an application of Itô's lemma yields

$$d\left(\beta_\nu(t) X_x^{\alpha,\theta,C}(t)\right) = \beta_\nu(t)\left\{\theta(t)^\top \sigma(t)\, dw_\nu(t) - \left(\alpha(t)\nu_0(t) + \theta(t)^\top \underline{\nu}(t)\right)\, dt - dC(t)\right\}. \tag{2.6}$$

In particular,

$$\beta_0(t) X_x^{\alpha,\theta,C}(t) + \int_0^t \beta_0(s)\, dC(s) = x + \int_0^t \beta_0(s)\theta(s)^\top \sigma(s)\, dw_0(s), \tag{2.7}$$

and hence

$$\mathrm{E}^0\left[\beta_0(T) X_x^{\alpha,\theta,C}(T) + \int_0^T \beta_0(t)\, dC(t)\right] \le x. \tag{2.8}$$

3 No-Arbitrage Bounds on Contingent Claims Prices

In this section, we will define and characterize the arbitrage buying and selling prices of a contingent claim.

3.1 Basic Definitions

We will discuss the pricing and hedging of contingent claims, which are encompassed by the following definition.

Definition 3.1 *A contingent claim is a positive $\mathcal{F}_T$-measurable random variable B with*

$$0 < \mathrm{E}^0\left[\beta_0(T)B\right] < \infty.$$

Obviously, we constrain ourselves to European-type contingent claims. The by now well-known fact that the time 0 no-arbitrage price of B in a complete market is

$$U_B(0) = \mathrm{E}^0\left[\beta_0(T)B\right]$$

is demonstrated in Theorem 3.1.

Any investor will be willing to sell B at a price x, if at the same price she can buy a trading strategy, which on the expiration time of the contingent claim is at least as valuable as the contingent claim and perhaps enables the investor to withdraw funds (for consumption or alternative investment) on or before the expiration time.

Definition 3.2 *A process u is called a* price process admissible for sellers, *if u is an RCLL non-negative optional process with $u(T) \geq B$ for which a trading strategy with consumption $(\alpha, \theta, C) \in \mathcal{A}_c(x; \mathcal{K})$ exists, such that*

$$u(t) = X_x^{\alpha,\theta,C}(t), \quad \forall t \in [0, T].$$

If a lowest price process admissible for sellers exists, then it is called the arbitrage selling price *of B and denoted by $\bar{u}_B$.*

Note that, in particular,[3]

$$\bar{u}_B(0) = \inf\left\{x > 0 \,\middle|\, \exists(\alpha, \theta, C) \in \mathcal{A}_c(x; \mathcal{K}) : X_x^{\alpha,\theta,C}(T) \geq B\right\}.$$

Any investor will be willing to buy B at a price x, if at the same price she can sell a trading strategy, which on the expiration time of the contingent claim is worth no more than the contingent claim and perhaps involves a flow of funds to the investor.

Definition 3.3 *A process u is called a* price process admissible for buyers, *if u is an RCLL non-negative optional process with $u(T) \leq B$ for which a trading strategy with savings $(\alpha, \theta, D) \in \mathcal{A}_s(x; \mathcal{K})$ exists, such that*

$$u(t) = X_x^{\alpha,\theta,-D}(t), \quad \forall t \in [0, T].$$

If a highest price process admissible for buyers exists, then it is called the arbitrage buying price *and denoted by $\underline{u}_B$.*

Note that, in particular,[4]

$$\underline{u}_B(0) = \sup\left\{x > 0 \,\middle|\, \exists(\alpha, \theta, D) \in \mathcal{A}_s(x; \mathcal{K}) : X_x^{\alpha,\theta,-D}(T) \leq B\right\}.$$

In a complete market, we have the following standard result.

Theorem 3.1 *Let B be a contingent claim and suppose $\mathcal{K} = \mathbb{R}^{d+1}$. Define $U_B(0) = \mathrm{E}^0[\beta_0(T)B]$. Then*

$$U_B(0) = \bar{u}_B(0) = \underline{u}_B(0),$$

and a self-financing trading strategy (α_0, θ_0) exists, such that $X_0 \equiv X_{u_0}^{\alpha_0,\theta_0,0}$, where the process X_0 is given by

$$X_0(t) = \frac{1}{\beta_0(t)} \mathrm{E}_t^0 \left[\beta_0(T)B\right].$$

In particular, $U_B(0) = \mathrm{E}^0[\beta_0(T)B]$.

[3] With the convention $\inf \emptyset = \infty$.

[4] With the convention $sup \emptyset = 0$.

Proof: If $X_x^{\alpha,\theta,C}(T) \geq B$ for some $x \in (0,\infty)$ and some trading strategy with consumption (α,θ,C), then, from (2.8), we have $x \geq U_B(0)$, and, thus, $\bar{u}_B(0) \geq U_B(0)$. From the martingale representation theorem, a progressively measurable process $\eta_0 \in \mathcal{L}^2[0,T]$ exists, such that

$$X_0(t) = \frac{1}{\beta_0(t)}\left[u_0 + \int_0^t \eta_0(s)^\top \, dw_0(s)\right]. \tag{3.1}$$

The pair (α_0, θ_0) defined by $\theta_0(t) = \beta_0(t)^{-1}(\sigma(t)^\top)^{-1}\eta_0(t)$, $\alpha_0(t) = X_0(t) - \theta_0(t)^\top \mathbf{1}$ is an admissible trading strategy. Comparing (3.1) with (2.7), we see that the processes X_0 and $X_{u_0}^{\alpha_0,\theta_0,0}$ are identical. Therefore, $U_B(0) \geq \bar{u}_B(0)$ and, thus, $U_B(0) = \bar{u}_B(0)$. The equality $U_B(0) = \underline{u}_B(0)$ can be demonstrated in a similar manner. □

λ_ν would be the relative risk process and ζ_ν the unique state-price density in a hypothetical unconstrained market $\mathcal{M}_\nu$, where the interest rate is $r(t) + \nu_0(t)$, the drift vector of the risky return process is $b(t) + \underline{\nu}(t)$, and where all investors receive a cash flow at the rate $\delta(\nu(t))$. The unconstrained hedging price of a contingent claim B in that auxiliary market would be

$$U_B^\nu(0) = \mathrm{E}^\nu\left[\beta_\nu(T)B - \int_0^T \beta_\nu(t)\delta(\nu(t))\, dt\right]. \tag{3.2}$$

We shall show in the following that, in the original constrained market $\mathcal{M}$, the arbitrage selling price of B is $\sup_{\nu\in\mathcal{N}^*} U_B^\nu(0)$, and the arbitrage buying price of B is $\inf_{\nu\in\mathcal{N}^*} U_B^\nu(0)$.

3.2 Characterization of the Arbitrage Selling Price

Define $\mathcal{S}$ as the set of all $[0,T]$-valued $\mathbb{F}$-stopping times. For $\tau_1, \tau_2 \in \mathcal{S}$ with $\tau_1 \leq \tau_2$, define

$$\mathcal{S}_{\tau_1,\tau_2} = \{\tau \in \mathcal{S} \mid \tau_1(\omega) \leq \tau(\omega) \leq \tau_2(\omega),\ \forall \omega \in \Omega\}.$$

Define for $\tau \in \mathcal{S}$ the random variable

$$V_B(\tau) = \operatorname*{ess\,sup}_{\nu\in\mathcal{N}^*} \mathrm{E}_\tau^\nu\left[-\int_\tau^T \beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\, ds + B\beta_0(T)\gamma_{\tau,T}(\nu_0)\right], \tag{3.3}$$

where

$$\gamma_{t,s}(\nu_0) = e^{-\int_t^s \nu_0(u)\, du}.$$

Note that

$$V_B(0) = \sup_{\nu\in\mathcal{N}^*} \mathrm{E}^\nu\left[-\beta_\nu(s)\delta(\nu(s))\, ds + B\beta_\nu(T)\right],$$

which is equal to the supremum of $U_B^\nu(0)$ defined in (3.2).

In the next theorem, we make use of the following definition. For $\tau_1, \tau_2 \in \mathcal{S}$ with $\tau_1 \leq \tau_2$, define $\mathcal{N}_{\tau_1,\tau_2}^*$ to be the restriction of $\mathcal{N}^*$ to the stochastic interval $[\tau_1, \tau_2]$.

Theorem 3.2 *If*

$$\sup_{\nu \in \mathcal{N}^*} \mathrm{E}^\nu \left[- \int_0^T \beta_\nu(s)\delta(\nu(s))\, ds + \beta_\nu(T) B \right] < \infty,$$

then the family of random variables $\{V_B(\tau)\}_{\tau \in \mathcal{S}}$ *has the dynamic programming property*

$$V_B(\tau) = \operatorname*{ess\,sup}_{\nu \in \mathcal{N}^*_{\tau,\vartheta}} \mathrm{E}^\nu_t \left[- \int_\tau^\vartheta \beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\, ds + V_B(\vartheta)\gamma_{\tau,\vartheta}(\nu_0) \right], \quad \forall \vartheta \in \mathcal{S}_{\tau,T}.$$

Proof: Let $\vartheta \in \mathcal{S}_{\tau,T}$. It is easy to prove one of the inequalities:

$$\begin{aligned} V_B(\tau) &= \operatorname*{ess\,sup}_{\nu \in \mathcal{N}^*_{\tau,T}} \mathrm{E}^\nu_\tau \Bigg[- \int_\tau^\vartheta \beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\, ds \\ &\qquad + \gamma_{\tau,\vartheta}(\nu_0)\, \mathrm{E}^\nu_\vartheta \Big[- \int_\vartheta^T \beta_0(s)\gamma_{\vartheta,s}(\nu_0)\delta(\nu(s))\, ds + B\beta_0(T)\gamma_{\vartheta,T}(\nu_0) \Big] \Bigg] \\ &\le \operatorname*{ess\,sup}_{\nu \in \mathcal{N}^*_{\tau,T}} \mathrm{E}^\nu_\tau \left[- \int_\tau^\vartheta \beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\, ds + \gamma_{\tau,\vartheta}(\nu_0) V_B(\vartheta) \right]. \end{aligned}$$

To prove the other inequality, we pick $\mu \in \mathcal{N}^*$ and aim to show that

$$V_B(\tau) \ge \mathrm{E}^\mu_\tau \left[- \int_\tau^\vartheta \beta_0(s)\gamma_{\tau,s}(\mu_0)\delta(\mu(s))\, ds + V_B(\vartheta)\gamma_{\tau,\vartheta}(\mu_0) \right].$$

First, we establish an auxiliary result. Define

$$J^\nu_B(\vartheta) = \mathrm{E}^\nu_\vartheta \left[- \int_\vartheta^T \beta_0(s)\gamma_{\vartheta,s}(\nu_0)\delta(\nu(s))\, ds + V_B(T)\gamma_{\vartheta,T}(\nu_0) \right].$$

Let $\nu_1, \nu_2 \in \mathcal{N}^*$. Then a $\nu \in \mathcal{N}^*$ exists with

$$J^\nu_B(\vartheta) = J^{\nu_1}_B(\vartheta) \wedge J^{\nu_2}_B(\vartheta). \tag{3.4}$$

To see this, let $A = \{(t,\omega) | J^{\nu_2}_B(t,\omega) \ge J^{\nu_1}_B(t,\omega)\}$ and define

$$\nu = \nu_1 1_A + \nu_2 1_{A^c}.$$

Then $\nu \in \mathcal{N}^*$ and (3.4) holds. Due to this property, it follows from Neveu (1975, p. 121) that a sequence $\{\nu_k\}_{k\in\mathbb{N}} \subset \mathcal{N}^*$ exists with $\{J^{\nu_k}_B\}_{k\in\mathbb{N}}$ increasing and

$$V_B(\vartheta) = \lim_{k\to\infty} J^{\nu_k}_B(\vartheta).$$

Now fix $\mu \in \mathcal{N}^*$ and let $\mathcal{N}^*_\mu[\tau,\vartheta]$ be the set of processes in $\mathcal{N}^*$, which agree with μ on the stochastic interval $[\tau,\vartheta]$. Obviously,

$$\begin{aligned} V_B(\tau) &\ge \operatorname*{ess\,sup}_{\nu \in \mathcal{N}^*_\mu[\tau,\vartheta]} \mathrm{E}^\nu_\tau \left[- \int_\tau^T \beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\, ds + B\beta_0(T)\gamma_{\tau,T}(\nu_0) \right] \\ &= \operatorname*{ess\,sup}_{\nu \in \mathcal{N}^*_\mu[\tau,\vartheta]} \mathrm{E}^\nu_\tau \left[- \int_\tau^\vartheta \beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\, ds + \gamma_{\tau,\vartheta}(\nu_0) J^\nu_B(\vartheta) \right]. \end{aligned}$$

Hence, for every $\nu \in \mathcal{N}^*_\mu[\tau,\vartheta]$,

$$\begin{aligned}
V_B(\tau) &\geq \mathrm{E}^\nu_\tau \left[-\int_\tau^\vartheta \beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\,ds + \gamma_{\tau,\vartheta}(\nu_0)J^\nu_B(\vartheta)\right] \\
&= \mathrm{E}_\tau \left[\frac{Z_\nu(\vartheta)}{Z_\nu(\tau)} \left(-\int_\tau^\vartheta \beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\,ds + \gamma_{\tau,\vartheta}(\nu_0)J^\nu_B(\vartheta)\right)\right] \\
&= \mathrm{E}_\tau \left[\frac{Z_\mu(\vartheta)}{Z_\mu(\tau)} \left(-\int_\tau^\vartheta \beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\,ds + \gamma_{\tau,\vartheta}(\mu_0)J^\nu_B(\vartheta)\right)\right] \\
&= \mathrm{E}^\mu_\tau \left[-\int_\tau^\vartheta \beta_0(s)\gamma_{\tau,s}(\mu_0)\delta(\mu(s))\,ds + \gamma_{\tau,\vartheta}(\mu_0)J^\nu_B(\vartheta)\right].
\end{aligned}$$

Note that $J^\nu_B(\vartheta)$ only depends on ν on the stochastic interval $[\vartheta, T]$, so the sequence $\{\nu_k\}$ can be taken to lie in the set $\mathcal{N}^*_\mu[\tau,\vartheta]$. Therefore,

$$\begin{aligned}
V_B(\tau) &\geq \lim_{k\to\infty} \mathrm{E}^\mu_\tau \left[-\int_\tau^\vartheta \beta_0(s)\gamma_{\tau^s}(\mu_0)\delta(\mu(s))\,ds + \gamma_{\tau,\vartheta}(\mu_0)J^{\nu_k}_B(\vartheta)\right] \\
&= \mathrm{E}^\mu_\tau \left[-\int_\tau^\vartheta \beta_0(s)\gamma_{\tau,s}(\mu_0)\delta(\mu(s))\,ds + \gamma_{\tau,\vartheta}(\mu_0)V_B(\vartheta)\right],
\end{aligned}$$

by Lebesgue's monotone convergence theorem. □

Note that it follows from Theorem 3.2 that for any $\tau \in \mathcal{S}$, $\vartheta \in \mathcal{S}_{\tau,T}$, and $\nu \in \mathcal{N}^*$, we have

$$\begin{aligned}
(3.5)\quad V_B(\tau)\gamma_{0,\tau}(\nu_0) &- \int_0^\tau \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\,ds \\
&\geq \mathrm{E}^\nu_\tau \left[V_B(\vartheta)\gamma_{0,\vartheta}(\nu_0) - \int_0^\vartheta \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\,ds\right].
\end{aligned}$$

Theorem 3.3 *The process $\{V_B(t)\}_{t\in[0,T]}$ of Theorem 3.2 can be considered in its RCLL modification. For all $\nu \in \mathcal{N}^*$, V is the smallest adapted RCLL process for which*

(i) the process

$$\left\{ V_B(t)\gamma_{0,t}(\nu_0) - \int_0^t \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\,ds \right\}_{t\in[0,T]}$$

is an RCLL $\mathbb{P}_\nu$-supermartingale,

(ii) $V_B(T) = B\beta_0(T)$.

Proof: First, we show that the process V can be considered RCLL. Let $\mathbb{T}$ denote the intersection of $[0,T]$ with the set of rational numbers, $\mathbb{Q}$. Consider the positive, adapted process $\{V_B(t)\}_{t\in\mathbb{T}}$. From (3.5), the process

$$\left\{ V_B(t)\gamma_{0,t}(\nu_0) - \int_0^t \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\,ds \right\}_{t\in\mathbb{T}}$$

is a $\mathbb{P}_\nu$-supermartingale on $\mathbb{T}$. From Karatzas and Shreve (1988, Prop. 1.3.14), this process has almost surely finite limits both from the right and from the left, and, hence, we can define

$$V_B(t+) = \begin{cases} \lim_{s\downarrow t, s\in\mathbb{Q}} V_B(s), & \text{for } 0 \le t < T, \\ V_B(T), & \text{for } t = T. \end{cases}$$

The resulting process is finite and adapted. The process

$$\left\{ V_B(t+)\gamma_{0,t}(\nu_0) - \int_0^t \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\, ds \right\}_{t\in[0,T]}$$

is an RCLL $\mathbb{P}_\nu$-supermartingale for all $\nu \in \mathcal{N}^*$. We want to show that $\{V_B(t+)\}_{t\in[0,T]}$ and $\{V_B(t)\}_{t\in[0,T]}$ are modifications of each other. First, note that by the supermartingale property

$$V_B(t+) \ge \mathrm{E}_t^\nu \left[V_B(T)\gamma_{t,T}(\nu_0) - \int_t^T \beta_0(s)\gamma_{t,s}(\nu_0)\delta(\nu(s))\, ds \right]$$

for all $\nu \in \mathcal{N}^*$, and hence, by the definition (3.3), $V_B(t+) \ge V_B(t)$ a.s. Conversely, Fatou's lemma yields

$$\begin{aligned} V_B(t+) &= \mathrm{E}_t^\nu \left[\lim_{n\to\infty} V_B(t+1/n)\gamma_{t,t+1/n}(\nu_0) - \int_t^{t+1/n} \beta_0(s)\gamma_{t,s}(\nu_0)\delta(\nu(s))\, ds \right] \\ &\le \lim_{n\to\infty} \mathrm{E}_t^\nu \left[V_B(t+1/n)\gamma_{t,t+1/n}(\nu_0) - \int_t^{t+1/n} \beta_0(s)\gamma_{t,s}(\nu_0)\delta(\nu(s))\, ds \right] \\ &\le V_B(t), \end{aligned}$$

almost surely, for any $\nu \in \mathcal{N}^*$. Thus, $V_B(t+) = V_B(t)$ a.s.

Suppose that the process $\check{V}$ is such that

$$\left\{ \check{V}(t)\gamma_{0,t}(\nu_0) - \int_0^t \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\, ds \right\}$$

is a $\mathbb{P}_\nu$-supermartingale with $\check{V}(T) = B\beta_0(T)$ for all $\nu \in \mathcal{N}^*$. Then

$$\begin{aligned} \check{V}(t) &\ge \mathrm{E}_t^\nu \left[\check{V}(T)\gamma_{t,T}(\nu_0) - \int_t^T \beta_0(s)\gamma_{t,s}(\nu_0)\delta(\nu(s))\, ds \right] \\ &= \mathrm{E}_t^\nu \left[B\beta_0(T)\gamma_{t,T}(\nu_0) - \int_t^T \beta_0(s)\gamma_{t,s}(\nu_0)\delta(\nu(s))\, ds \right] \end{aligned}$$

for all $\nu \in \mathcal{N}^*$ and hence $\check{V}(t) \ge V_B(t)$. □

Theorem 3.4 *For any contingent claim B,*

$$\bar{u}_B(0) = V_B(0).$$

If $V_B(0) < \infty$, a trading strategy with consumption $(\alpha^*, \theta^*, C^*) \in \mathcal{A}_c(V_B(0); \mathcal{K})$ exists, such that $X_{V_B(0)}^{\alpha^*,\theta^*,C^*}(T) = B$.

Proof: We will prove $\bar{u}_B(0) \ge V_B(0)$ and $\bar{u}_B(0) \le V_B(0)$ separately.

Proof of $\bar{u}_B(0) \geq V_B(0)$. Of course, we may assume that $\bar{u}_B(0) < \infty$. Hence, an $x \in (0, \infty)$ and a strategy $(\alpha, \theta, C) \in \mathcal{A}_c(x; \mathcal{K})$ exist with $X_x^{\alpha,\theta,C}(T) \geq B$. From (2.6), we see that

$$\beta_\nu(t) X_x^{\alpha,\theta,C}(t) + \int_0^t \beta_\nu(s)\, dC(s) + \int_0^t \beta_\nu(s) \left(\alpha(s)\nu_0(s) + \theta(s)^\top \underline{\nu}(s)\right) ds$$
$$= x + \int_0^t \beta_\nu(s)\theta(s)^\top \sigma(s)\, dw_\nu(s),$$

which obviously is a non-negative $\mathbb{P}_\nu$-local martingale, hence a supermartingale. In particular,

$$\begin{aligned} x &\geq \mathrm{E}^\nu \left[\beta_\nu(T) X_x^{\alpha,\theta,C}(T) + \int_0^T \beta_\nu(s)\, dC(s) + \int_0^T \beta_\nu(s) \left(\alpha(s)\nu_0(s) + \theta(s)^\top \underline{\nu}(s)\right) ds \right] \\ &\geq \mathrm{E}^\nu \left[\beta_\nu(T) B - \int_0^T \beta_\nu(s) \delta(\nu(s))\, ds \right] \\ &= V_B(0), \end{aligned}$$

where the last inequality follows from $X_x^{\alpha,\theta,C}(T) \geq B$, C increasing, and inequality (2.3). From the inequality $x \geq V_B(0)$, we conclude that $\bar{u}_B(0) \geq V_B(0)$.

Proof of $\bar{u}_B(0) \leq V_B(0)$. We can assume $V_B(0) < \infty$. Define the process

$$Q_B^\nu(t) = V_B(t)\gamma_{0,t}(\nu_0) - \int_0^t \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\, ds. \tag{3.6}$$

From Theorem 3.3, the Doob-Meyer decomposition, and the martingale representation theorem, we have that for all $\nu \in \mathcal{N}^*$, an $\mathbb{R}^d$-valued, progressively measurable, square-integrable process η_ν and an adapted, increasing, RCLL process A_ν with $A_\nu(0) = 0$ and $A_\nu(T) < \infty$ exist, such that

$$Q_B^\nu(t) = V_B(0) + \int_0^t \eta_\nu(s)^\top dw_\nu(s) - A_\nu(t), \quad 0 \leq t \leq T.$$

Define

$$X^*(t) = \frac{V_B(t)}{\beta_0(t)} = \frac{Q_B^\nu(t) + \int_0^t \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\, ds}{\beta_\nu(t)}.$$

Note that $X^*(0) = V_B(0)$ and $X^*(T) = V_B(T)/\beta_0(T) = B$. If we can find a strategy $(\alpha^*, \theta^*, C^*) \in \mathcal{A}_c(V_B(0); \mathcal{K})$ such that the processes X^* and $X_{V_B(0)}^{\alpha^*,\theta^*,C^*}$ are identical, we are done. We shall use the following lemma, the validity of which will be demonstrated at the end of the proof.

Lemma 3.1 *The expressions $\gamma_{0,t}(\nu_0)^{-1}\eta_\nu(t)$ and*

$$\gamma_{0,t}(\nu_0)^{-1}\big[-dA_\nu(t) + \beta_\nu(t)\left(X^*(t)\nu_0(t) + \delta(\nu(t))\right) dt$$
$$+ \eta_\nu(t)^\top \sigma(t)^{-1}\left(\underline{\nu}(t) - \nu_0(t)\mathbf{1}\right) dt\big]$$

are independent of ν.

By this lemma, we may define a trading strategy (α^*, θ^*) by

$$\theta^*(t)^\top = \frac{\eta_\nu(t)^\top \sigma(t)^{-1}}{\beta_\nu(t)},$$

$$\alpha^*(t) = X^*(t) - \theta^*(t)^\top \mathbf{1},$$

and a cumulative consumption process C^* by

$$\begin{aligned} C^*(t) &= \int_0^t \beta_\nu(s)^{-1}\, dA_\nu(s) - \int_0^t \left(X^*(s)\nu_0(s) + \delta(\nu(s))\right) ds \\ &\quad - \int_0^t \frac{\eta_\nu(s)^\top \sigma(s)^{-1}}{\beta_\nu(s)} \left(\underline{\nu}(s) - \nu_0(s)\mathbf{1}\right) ds \\ &= \int_0^t \beta_\nu(s)^{-1}\, dA_\nu(s) - \int_0^t \left(\alpha^*(s)\nu_0(s) + \theta^*(s)^\top \underline{\nu}(s) + \delta(\nu(s))\right) ds. \end{aligned} \tag{3.7}$$

Taking $\nu \equiv 0$, we have

$$C^*(t) = \int_0^t \beta_0(s)^{-1}\, dA_0(s).$$

Note that the process C^* is increasing, adapted, RCLL with $C^*(0) = 0$ and $C^*(T) < \infty$, i.e. $C^* \in \mathcal{H}'$. To prove admissibility of the defined strategy, we must show that

$$(\alpha^*(t), \theta^*(t)) \in \mathcal{K}, \quad \ell \times \mathbb{P} - \text{a.e.}$$

According to Rockafellar (1970, Thm. 13.1), it suffices to show that

$$\delta(\nu(t)) + \alpha^*(t)\nu_0(t) + \theta^*(t)^\top \underline{\nu}(t) \geq 0, \quad \ell \times \mathbb{P} - \text{a.e.}, \tag{3.8}$$

for all $\nu \in \mathcal{N}^*$. From (3.7), we have

$$\begin{aligned} A_\nu(t) &= \int_0^t \beta_\nu(s)\, dC^*(s) + \int_0^t \beta_\nu(s) \left(\alpha^*(s)\nu_0(s) + \theta^*(s)^\top \underline{\nu}(s) + \delta(\nu(s))\right) ds \\ &= \int_0^t \beta_\nu(s) \left[dC^*(s) + \left(\alpha^*(s)\nu_0(s) + \theta^*(s)^\top \underline{\nu}(s) + \delta(\nu(s))\right) ds\right] \\ &\leq k \left[C^*(t) + \int_0^t \left(\alpha^*(s)\nu_0(s) + \theta^*(s)^\top \underline{\nu}(s) + \delta(\nu(s))\right) ds\right] \end{aligned}$$

for some constant k. For given ν, define for all $t \in [0, T]$ the set

$$F_t = \{\omega \in \Omega | \alpha^*(t)\nu_0(t) + \theta^*(t)^\top \underline{\nu}(t) + \delta(\nu(t)) < 0\},$$

and let

$$\mu_n(t) = \frac{\nu(t) 1_{F_t^c} + n\nu(t) 1_{F_t}}{1 + \|\nu(t)\|}, \quad n \in \mathbb{N},$$

where F_t^c is the complement of the set F_t. Then $\mu_n \in \mathcal{N}^*$. Suppose (3.8) does not hold. Then, for n large enough,

$$\begin{aligned} \mathrm{E}\left[A_{\mu_n}(T)\right] \leq \mathrm{E}\Big[kC^*(T) &+ k\int_0^T \frac{1}{1 + \|\nu(t)\|} \left(\alpha^*(t)\nu_0(t) + \theta^*(t)^\top \underline{\nu}(t) + \delta(\nu(t))\right) 1_{F_t^c}\, dt \\ &+ nk \int_0^T \frac{1}{1 + \|\nu(t)\|} \left(\alpha^*(t)\nu_0(t) + \theta^*(t)^\top \underline{\nu}(t) + \delta(\nu(t))\right) 1_{F_t}\, dt\Big] \\ < 0, \end{aligned}$$

a contradiction. Hence (3.8) must hold.

We now have that

$$
\begin{aligned}
d\left(\beta_\nu(t)X^*(t)\right) &= dQ_B^\nu(t) + \beta_0(t)\gamma_{0,t}(\nu_0)\delta(\nu(t))\,dt \\
&= \eta_\nu(t)^\top\, dw_\nu(t) - dA_\nu(t) + \beta_0(t)\gamma_{0,t}(\nu_0)\delta(\nu(t))\,dt \\
&= \beta_\nu(t)\theta^*(t)^\top\sigma(t)\,dw_\nu(t) - \beta_\nu(t)\,dC^*(t) \\
&\quad - \beta_\nu(t)\left(\alpha^*(t)\nu_0(t) + \theta^*(t)^\top\underline{\nu}(t)\right)\,dt.
\end{aligned} \tag{3.9}
$$

Comparing this expression with (2.6), we conclude that X^* and $X_{V_B(0)}^{\alpha^*,\theta^*,C^*}$ are identical.

Proof of Lemma. For any $\nu,\mu\in\mathcal{N}^*$,

$$
\begin{aligned}
Q_B^\mu(t) = \gamma_{0,t}(\mu_0-\nu)&\left[Q_B^\nu(t) + \int_0^t \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\,ds\right] \\
&- \int_0^t \beta_0(s)\gamma_{0,t}(\mu_0)\delta(\mu(s))\,ds,
\end{aligned}
$$

and hence

$$
\begin{aligned}
dQ_B^\mu(t) &= \gamma_{0,t}(\mu_0-\nu_0)\Big[Q_B^\nu(t)\left(\nu_0(t)-\mu_0(t)\right)\,dt + dQ_B^\nu(t) + \beta_0(t)\gamma_{0,t}(\nu_0)\delta(\nu(t))\,dt \\
&\qquad + \int_0^t \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\,ds\ \left(\nu_0(t)-\mu_0(t)\right)dt\Big] \\
&\quad - \beta_0(t)\gamma_{0,t}(\mu_0)\delta(\mu(t))\,dt \\
&= \gamma_{0,t}(\mu_0-\nu_0)\Big[\eta_\nu(t)^\top\,dw_\nu(t) - dA_\nu(t) \\
&\qquad + \beta_\nu(t)\left\{X^*(t)\left(\nu_0(t)-\mu_0(t)\right) + \delta(\nu(t)) - \delta(\mu(t))\right\}\,dt\Big] \\
&= \gamma_{0,t}(\mu_0-\nu_0)\Big[\eta_\nu(t)^\top\,dw_\mu(t) - dA_\nu(t) \\
&\qquad + \eta_\nu(t)^\top\sigma(t)^{-1}\left(\underline{\nu}(t)-\underline{\mu}(t)-\left(\nu_0(t)-\mu_0(t)\right)\mathbf{1}\right)\,dt \\
&\qquad + \beta_\nu(t)\left\{X^*(t)\left(\nu_0(t)-\mu_0(t)\right) + \delta(\nu(t)) - \delta(\mu(t))\right\}\,dt\Big],
\end{aligned} \tag{3.10}
$$

where we have used the relation

$$
w_\nu(t) = w_\mu(t) + \int_0^t \sigma(s)^{-1}\left(\underline{\nu}(s)-\underline{\mu}(s)-\left(\nu_0(s)-\mu_0(s)\right)\mathbf{1}\right)\,ds,
$$

which follows from the definition of the processes w_ν and λ_ν. On the other hand,

$$
dQ_B^\mu(t) = \eta_\mu(t)^\top\,dw_\mu(t) - dA_\mu(t). \tag{3.11}
$$

Comparing the Wiener terms in (3.10) and (3.11), we have that

$$
\gamma_{0,t}(\nu_0)^{-1}\eta_\nu(t) = \gamma_{0,t}(\mu_0)^{-1}\eta_\mu(t), \quad \forall\nu,\mu\in\mathcal{N}^*,
$$

which proves the first assertion. To prove the second, we conclude from a comparison of the other terms in (3.10) and (3.11) that

$$\begin{aligned}
-dA_\mu(t) &= \gamma_{0,t}(\mu_0-\nu_0)\Big[\beta_\nu(t)\left(X^*(t)(\nu_0(t)-\mu_0(t))+\delta(\nu(t))-\delta(\mu(t))\right)\,dt-dA_\nu(t)\\
&\qquad\qquad +\eta_\nu(t)^\top\sigma(t)^{-1}\left(\underline{\nu}(t)-\underline{\mu}(t)-(\nu_0(t)-\mu_0(t))\mathbf{1}\right)\,dt\Big]\\
&= \gamma_{0,t}(\mu_0-\nu_0)\Big[\beta_\nu(t)\left(X^*(t)\nu_0(t)+\delta(\nu(t))\right)\,dt-dA_\nu(t)\\
&\qquad\qquad +\eta_\nu(t)^\top\sigma(t)^{-1}(\underline{\nu}(t)-\nu_0(t)\mathbf{1})\,dt\Big]\\
&\quad -\gamma_{0,t}(\mu_0-\nu_0)\Big[\beta_\nu(t)\left(X^*(t)\mu_0(t)+\delta(\mu(t))\right)\,dt\\
&\qquad\qquad +\eta_\nu(t)^\top\sigma(t)^{-1}(\underline{\mu}(t)-\mu_0(t)\mathbf{1})\,dt\Big]\\
&= \gamma_{0,t}(\mu_0-\nu_0)\Big[\beta_\nu(t)\left(X^*(t)\nu_0(t)+\delta(\nu(t))\right)\,dt-dA_\nu(t)\\
&\qquad\qquad +\eta_\nu(t)^\top\sigma(t)^{-1}(\underline{\nu}(t)-\nu_0(t)\mathbf{1})\,dt\Big]\\
&\quad -\left[\beta_\mu(t)\left(X^*(t)\mu_0(t)+\delta(\mu(t))\right)\,dt+\eta_\mu(t)^\top\sigma(t)^{-1}(\underline{\mu}(t)-\mu_0(t)\mathbf{1})\,dt\right],
\end{aligned}$$

from which the claim easily follows. □

3.3 Characterization of the Arbitrage Buying Price

In this section, we state results for the arbitrage buying price similar to those obtained for the arbitrage selling price in the previous section. We omit proofs, since, except for obvious changes, they are identical to those for the corresponding results for arbitrage selling prices. Basically, "supremum" is replaced by "infimum" and "supermartingale" is replaced by "submartingale". Define for $\tau\in\mathcal{S}$ the random variable

$$v_B(\tau)=\operatorname*{ess\,inf}_{\nu\in\mathcal{N}^*}\mathrm{E}^\nu_\tau\left[-\int_\tau^T\beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\,ds+B\beta_0(T)\gamma_{\tau,T}(\nu_0)\right].$$

Theorem 3.5 *The family of random variables $\{v_B(\tau)\}_{\tau\in\mathcal{S}}$ has the dynamic programming property*

$$v_B(\tau)=\operatorname*{ess\,inf}_{\nu\in\mathcal{N}^*_{\tau,\vartheta}}\mathrm{E}^\nu_t\left[-\int_\tau^\vartheta\beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\,ds+v_B(\vartheta)\gamma_{\tau,\vartheta}(\nu_0)\right],\quad\forall\vartheta\in\mathcal{S}_{\tau,T}.$$

It follows that for any $\tau\in\mathcal{S}$, $\vartheta\in\mathcal{S}_{\tau,T}$, and $\nu\in\mathcal{N}^*$, we have

$$\begin{aligned}
&v_B(\tau)\gamma_{0,\tau}(\nu_0)-\int_0^\tau\beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\,ds\\
&\qquad\qquad\le\mathrm{E}^\nu_\tau\left[v_B(\vartheta)\gamma_{0,\vartheta}(\nu_0)-\int_0^\vartheta\beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\,ds\right].
\end{aligned}$$

Theorem 3.6 *The process $\{v_B(t)\}_{t\in[0,T]}$ can be considered in its RCLL modification. For all $\nu\in\mathcal{N}^*$, v is the largest adapted RCLL process for which* *(i) the process*

$$\left\{v_B(t)\gamma_{0,t}(\nu_0) - \int_0^t \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\,ds\right\}_{t\in[0,T]}$$

is an RCLL $\mathbb{P}_\nu$-submartingale,

(ii) $v_B(T) = B\beta_0(T)$.

Theorem 3.7 *For any contingent claim B,*

$$\underline{u}_B(0) = v_B(0).$$

If $v_B(0) < \infty$, a trading strategy with savings $(\alpha^*, \theta^*, D^*) \in \mathcal{A}_s(v_B(0); \mathcal{K})$ exists with the property $X_{v_B(0)}^{\alpha^*,\theta^*,-D^*}(T) = B$.

The simple proof of the fact that there are arbitrage opportunities if the price of the contingent claim is not in the *arbitrage-free interval* $[\underline{u}_B(0), \bar{u}_B(0)]$ is given in Karatzas and Kou (1996, Sec. 5). They also show that if the contingent claim price is equal to either $\underline{u}_B(0)$ or $\bar{u}_B(0)$, there may or may not be an arbitrage opportunity.

3.4 Attainable Contingent Claims

In this subsection, we find conditions under which a contingent claim is attainable according to the following definition.

Definition 3.4 *A contingent claim B with finite arbitrage selling price, i.e. $\bar{u}_B(0) < \infty$, is called* $\mathcal{K}$-attainable, *if a self-financing $\mathcal{K}$-admissible portfolio process (α, θ) exists, such that $X_{\bar{u}_B(0)}^{\alpha,\theta,0}(T) = B$.*

A K-attainable contingent claim can be perfectly replicated with a self-financing trading strategy.

Theorem 3.8 *Given a contingent claim B with $V_B(0) < \infty$ and a $\nu^* \in \mathcal{N}^*$. The following conditions are equivalent*

(i) $\left\{V_B(t)\gamma_{0,t}(\nu_0^*) - \int_0^t \beta_{\nu^*}(s)\delta(\nu^*(s))\,ds\right\}_{t\in[0,T]}$ *is a $\mathbb{P}_{\nu^*}$-martingale,*

(ii) ν^* attains the supremum in

$$V_B(0) = \sup_{\nu\in\mathcal{N}^*} \mathrm{E}^\nu\left[\beta_\nu(T)B - \int_0^T \beta_\nu(s)\delta(\nu(s))\,ds\right],$$

(iii) B is $\mathcal{K}$-attainable by a trading strategy (α^*, θ^*) and the process

$$\left\{ \beta_{\nu^*} X^{\alpha^*,\theta^*,0}_{V_B(0)}(t) - \int_0^t \beta_{\nu^*}(s)\delta(\nu^*(s))\, ds \right\}_{0\le t\le T}$$

is a $\mathbb{P}_{\nu^*}$-martingale,

and each implies

(iv) $C^* \equiv 0, \quad \delta(\nu^*) + \alpha^*\nu_0^* + (\theta^*)^\top \underline{\nu}^* \equiv 0$ for the triple $(\alpha^*, \theta^*, C^*)$ of Theorem 3.4.

Proof: (i) $\Leftrightarrow$ (ii): The process given in (i) is a $\mathbb{P}_{\nu^*}$-martingale, if and only if

$$\begin{aligned} V_B(0) &= \mathrm{E}^{\nu^*}\left[V_B(T)\gamma_{0,T}(\nu_0^*) - \int_0^T \beta_{\nu^*}(s)\delta(\nu^*(s))\, ds \right] \\ &= \mathrm{E}^{\nu^*}\left[B\beta_{\nu^*}(T) - \int_0^T \beta_{\nu^*}(s)\delta(\nu^*(s))\, ds \right], \end{aligned}$$

which is equivalent to (ii).

(i) $\Rightarrow$ (iv): Note that the process defined in (i) is identical to the process Q_{ν^*} defined in the proof of Theorem 3.4. By the martingale property, assumed in (i), the process A_{ν^*} must be identically equal to zero. By (3.7), we have

$$C^*(t) = -\int_0^t \left(\alpha^*(s)\nu_0^*(s) + \theta^*(s)^\top \underline{\nu}^*(s) + \delta(\nu^*(s)) \right) ds,$$

which is non-positive by (2.3). On the other hand, we know that $C^*(t) \ge 0$, thus $C^*(t) = 0$, and $\alpha^*(t)\nu_0^*(t) + \theta^*(t)^\top \underline{\nu}^*(t) + \delta(\nu^*(t)) = 0$.

(i) $\Rightarrow$ (iii): Again, by the assumed martingale property, $A_\nu \equiv 0$ and $C^* \equiv 0$, so, from (3.9),

$$d(\beta_{\nu^*}(t)X^*(t)) = \beta_{\nu^*}(t)\theta^*(t)^\top\sigma(t)\, dw_{\nu^*}(t) - \beta_{\nu^*}(t)\left(\alpha^*(t)\nu_0^*(t) + \theta^*(t)^\top \underline{\nu}^*(t) \right) dt.$$

A comparison with (2.6) yields that the processes X^* and $X^{\alpha^*,\theta^*,0}_{V_B(0)}$ are identical and that $X^*(T) = B$. Also from (3.9), we see that

$$d(\beta_{\nu^*}(t)X^*(t)) - \beta_{\nu^*}(t)\delta(\nu^*(t))\, dt = \eta_{\nu^*}(t)^\top\, dw_{\nu^*}(t),$$

so $\left\{ \beta_{\nu^*}(t)X^{\alpha^*,\theta^*,0}_{V_B(0)}(t) - \int_0^t \beta_{\nu^*}(s)\delta(\nu^*(s))\, ds \right\}$ is a $\mathbb{P}_{\nu^*}$-martingale.

(iii) $\Rightarrow$ (ii): By the martingale property assumed in (iii), we have

$$\begin{aligned} V_B(0) &= X^{\alpha^*,\theta^*,0}_{V_B(0)}(0) \\ &= \mathrm{E}^{\nu^*}\left[\beta_{\nu^*}(T)X^{\alpha^*,\theta^*,0}_{V_B(0)}(T) - \int_0^T \beta_{\nu^*}(s)\delta(\nu^*(s))\, ds \right] \\ &= \mathrm{E}^{\nu^*}\left[B\beta_{\nu^*}(T) - \int_0^T \beta_{\nu^*}(s)\delta(\nu^*(s))\, ds \right]. \end{aligned}$$

$\square$

Theorem 3.9 *Let B be a contingent claim with $V_B(0) < \infty$. Suppose that for $\nu \in \mathcal{N}^*$ with*

$$\delta(\nu) + \alpha^* \nu_0 + (\theta^*)^\top \underline{\nu} \equiv 0$$

we have that

$$\left\{ V_B(t)\gamma_{0,t}(\nu_0) - \int_0^t \beta_{\nu^*}(s)\delta(\nu^*(s))\, ds \right\}_{t\in[0,T]} \text{ is of class } D[0,T] \text{ under } \mathbb{P}_\nu. \tag{3.12}$$

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v), where

(v) B is K-attainable by a portfolio (α, θ) and $\left\{\beta_0(t) X^{\alpha,\theta,0}_{V_B(0)}(t)\right\}_{0\le t\le T}$ is a $\mathbb{P}_0$-martingale.

Proof: By (iv), the process A_{ν^*} in the proof of Theorem 3.4 is identically equal to zero. From (3.6) and (3.12), we get (i). Furthermore, note that if (iv) holds for some $\nu^* \in \mathcal{N}^*$, it also holds for $\nu^* \equiv 0$. Therefore (v) follows from Theorem 3.8. □

The Theorems 3.8 and 3.9 address the arbitrage selling price, but similar results hold for the arbitrage buying price. Bensoussan and Elliott (1995) discuss the attainability of contingent claims in a Markov version of the market with non-tradable assets studied by El Karoui and Quenez (1995).

3.5 Computation of the Arbitrage Selling and Buying Prices

In this section, we will show that in a Markovian set-up, the arbitrage selling (and buying) price can be computed as the limit of a sequence of solutions to HJB equations. First, we show that in the general not necessarily Markovian set-up, the arbitrage selling (and buying) price is the limit of a sequence of processes. Define for $q \in (0, \infty)$ the set of processes

$$\mathcal{N}^*_q = \{\nu \in \mathcal{N}^* \,|\, \|\nu(t,\omega)\| \le q, \ \forall (t,\omega) \in [0,T] \times \Omega\}$$

and

$$V_{B,q}(\tau) = \operatorname*{ess\,sup}_{\nu \in \mathcal{N}^*_q} \mathrm{E}^\nu_\tau \left[-\int_\tau^T \beta_0(s)\gamma_{\tau,s}(\nu_0)\delta(\nu(s))\, ds + B\beta_0(T)\gamma_{\tau,T}(\nu_0) \right], \quad \tau \in \mathcal{S}.$$

As in Theorem 3.3, it can be shown that $V_{B,q}$ can be considered in its RCLL modification, and that $V_{B,q}$ is the smallest adapted RCLL process for which the process

$$\left\{ V_{B,q}(t)\gamma_{0,t}(\nu_0) - \int_0^t \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\, ds \right\}$$

is an RCLL $\mathbb{P}_\nu$-supermartingale, for every $\nu \in \mathcal{N}^*_q$, with $V_{B,q}(T) = B\beta_0(T)$.

Theorem 3.10 *For all $t \in [0,T]$, $V_B(t)$ is the increasing limit of $V_{B,q}(t)$ as $q \to \infty$:*

$$V_B(t) = \lim_{q\to\infty} V_{B,q}(t).$$

Proof: Define $\check{V}(t) = \lim_{q\to\infty} V_{B,q}(t)$. Since $V_B(t) \geq V_{B,q}(t)$ for all q, we have $V_B(t) \geq \check{V}(t)$. To prove the converse inequality, it suffices to show that $\check{V}$ is a price admissible for sellers, i.e., that there exists a trading strategy with consumption $(\alpha, \theta, C) \in \mathcal{A}_c(\check{V}(0); \mathcal{K})$ with $\check{V}(T) \geq B$ and $\check{V}(t) = X^{\alpha,\theta,C}_{\check{V}(0)}(t)$, $t \in [0,T]$. Since

$$\left\{ \check{V}(t)\gamma_{0,t}(\nu_0) - \int_0^t \beta_0(s)\gamma_{0,s}(\nu_0)\delta(\nu(s))\,ds \right\}_{t\in[0,T]}$$

is the increasing limit of RCLL supermartingales, it is an RCLL supermartingale itself. We can now use the same arguments as in the last part of the proof of Theorem 3.4. □

Now, let us consider the Markov model, where $r(\omega,t)$, $b(\omega,t)$, and $\sigma(\omega,t)$ can be written as $r(P(t),t)$, $b(P(t),t)$, and $\sigma(P(t),t)$. Let the contingent claim be given by $B = \varphi(P(T))$ for some non-negative function $\varphi \in C^3([0,\infty))$, such that φ and φ' satisfy polynomial growth conditions.[5] Then we have that $V_B(0) = V_B(P,0)$, where

$$V_B(P,t) = \sup_{\nu\in\mathcal{N}^*} \mathrm{E}^{\nu}_{P,t}\left[-\int_t^T \gamma_{t,s}(r+\nu_0)\delta(\nu(s))\,ds + \gamma_{t,T}(r+\nu_0)\varphi(P(T)) \right].$$

Note that the evolution of the state variables (the prices) is given by

$$dP(t) = \mathrm{diag}(P(t))\left[\left((r(P(t),t) + \nu_0(t))\,\mathbf{1} - \underline{\nu}(t)\right)dt + \sigma(P(t),t)\,dw_\nu(t)\right].$$

Since $(r+\nu_0)\mathbf{1} - \underline{\nu}$ is in general not bounded, V is not guaranteed to be a solution of the HJB equation (not even a viscosity solution), but the next result shows that it can be approximated by functions $V_{B,q}$ which are solutions of an HJB equation.

Theorem 3.11 *We have*

$$V_B(P,t) = \lim_{q\to\infty} V_{B,q}(P,t),$$

where

$$V_{B,q}(P,t) = \sup_{\nu\in\mathcal{N}^*_q} \mathrm{E}^{\nu}_{P,t}\left[-\int_t^T \gamma_{t,s}(r+\nu_0)\delta(\nu(s))\,ds + \gamma_{t,T}(r+\nu_0)\varphi(P(T)) \right], \tag{3.13}$$

and $V_{B,q}$ solves the HJB equation

$$\begin{aligned}\frac{\partial V_{B,q}}{\partial t} + \frac{1}{2}\,\mathrm{tr}\left(\mathrm{diag}(P)\sigma\sigma^\top \mathrm{diag}(P)\frac{\partial^2 V_{B,q}}{\partial P^2}\right) + r\left(P^\top \frac{\partial V_{B,q}}{\partial P} - V_{B,q}\right)&\\ + \sup_{\nu\in\tilde{\mathcal{K}},\|\nu\|\leq q}\left((\nu_0\mathbf{1} - \underline{\nu})^\top \mathrm{diag}(P)\frac{\partial V_{B,q}}{\partial P} - \nu_0 V_{B,q} - \delta(\nu)\right) &= 0, \quad t<T,\end{aligned} \tag{3.14}$$

with

$$V_{B,q}(P,T) = \varphi(P). \tag{3.15}$$

[5]The differentiability condition rules out many standard options, whose pay-off structure involves one or more bends. In that case, it is possible to approximate the pay-off function with a sufficiently smooth function φ.

Proof: The first assertion follows from the previous theorem. That $V_{B,q}$ solves the HJB equation follows from Theorem IV.4.4 in Fleming and Soner (1993). □

Likewise, we have $v_B(0) = v_B(P, 0)$, where

$$v_B(P,t) = \inf_{\nu \in \mathcal{N}^*} \mathrm{E}^{\nu}_{P,t} \left[-\int_t^T \gamma_{t,s}(r+\nu_0)\delta(\nu(s))\,ds + \gamma_{t,T}(r+\nu_0)\varphi(P(T)) \right].$$

Theorem 3.12

$$v_B(P,t) = \lim_{q \to \infty} v_{B,q}(P,t),$$

where

$$v_{B,q}(P,t) = \inf_{\nu \in \mathcal{N}^*_q} \mathrm{E}^{\nu}_{P,t} \left[-\int_t^T \gamma_{t,s}(r+\nu_0)\delta(\nu(s))\,ds + \gamma_{t,T}(r+\nu_0)\varphi(P(T)) \right], \tag{3.16}$$

and $v_{B,q}$ solves the HJB equation

$$\begin{aligned} \frac{\partial v_{B,q}}{\partial t} + \frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(P)\sigma\sigma^\top \operatorname{diag}(P)\frac{\partial^2 v_{B,q}}{\partial P^2}\right) + r\left(P^\top \frac{\partial v_{B,q}}{\partial P} - v_{B,q}\right) \\ + \inf_{\nu \in \tilde{\mathcal{K}}, \|\nu\| \le q} \left((\nu_0 \mathbf{1} - \underline{\nu})^\top \operatorname{diag}(P)\frac{\partial v_{B,q}}{\partial P} - \nu_0 v_{B,q} - \delta(\nu) \right) = 0, \quad t < T, \end{aligned} \tag{3.17}$$

with

$$v_{B,q}(P,T) = \varphi(P). \tag{3.18}$$

4 Numerical Computation of the Price Bounds

Cvitanić and Karatzas (1993) study some examples of portfolio constraints, under which they can compute the arbitrage selling price of very simple option positions. Generally, however, the no-arbitrage bounds must be found by solving the stochastic control problems (3.13) and (3.16) numerically.

4.1 General Remarks

One could consider solving the associated HJB equations (3.14)–(3.15) and (3.17)–(3.18), respectively, by first expressing the optimal control ν in terms of the value function, and then substituting this expression back into the HJB equation to get a control-independent PDE. This PDE could then be attacked with well-known finite-difference methods. The PDE will, however, be highly nonlinear, and, for such problems, the convergence properties of standard finite-difference methods are not well explored.

Instead, we shall use the so-called Markov Chain Approximation Approach. The basic idea is to approximate the original continuous-time, continuous-state diffusion control problem with a discrete-time, discrete-state Markov chain control problem. The value function for the approximating discrete problem can quite easily be computed numerically. The convergence properties of this approach are well documented. Standard references are Kushner (1990), Kushner and Dupuis (1992), and Fleming and Soner (1993, Ch. IX). For a simpler introduction to the approach the reader is referred to Munk (1997a).[6]

The state variables of the continuous-time problem are the prices of the d primary assets of the economy. From (2.1), (2.5), and (2.4), the price dynamics under the measure $\mathbb{P}^\nu$ is

$$dP(t) = \text{diag}(P(t)) \{([r(P(t),t) + \nu_0(t)]\mathbf{1} - \underline{\nu}(t)) \, dt + \sigma(P(t),t) \, dw_\nu(t)\} \tag{4.1}$$

in the Markov version of the model. To find a Markov chain approximating the process P, we use the general recipe outlined in, e.g., Kushner and Dupuis (1992, Ch. 5). To ensure non-negative transition probabilities and, hence, a well-defined Markov chain, the matrix of (instantaneous) covariances of the state variables must be diagonally dominated in all states. Of course, this can only be a problem, when $d > 1$.

In our case, the covariance matrix of P is

$$\text{diag}(P(t))\sigma(P(t),t)\sigma(P(t),t)^\top \, \text{diag}(P(t)),$$

which, in general, is not diagonally dominated for all values of $P(t)$. However, it is often possible to perform a change of variables from P to some variable y, such that the covariance matrix of y is uniformly diagonally dominated.[7] Due to space considerations, we will focus on the one-variable problem in the following.

4.2 Computation of the Price Bounds with a One-Dimensional State Variable

We approximate the one-dimensional price process $\{P(t)\}_{t\in[0,T]}$ with a Markov chain $P^{h,\tau}$ on the state space $\mathcal{R}^h = \{0, h, 2h, \ldots, Ih\}$, where we have imposed an upper bound $\bar{P} \equiv Ih$ on the state variable. The time set of the Markov chain is $\mathcal{T} = \{0, \tau, \ldots, N\tau \equiv T\}$. The parameters h and τ are called discretization parameters. We approximate the value function $V_{B,q}$ of the arbitrage

---

[6]The method has been used to solve various optimal consumption/investment problems by Fitzpatrick and Fleming (1991), Hindy, Huang, and Zhu (1997), and Munk (1997b).

[7]Note that the covariances in our case are independent of the controls of the problem. If this is not so, it will, in general, not be possible to transform the state variable vector into another variable with a diagonally dominated covariance matrix. This is, e.g., the case for multi-dimensional optimal consumption/portfolio problems.

selling price problem (3.13) by

(4.2) $$V^{h,\tau}_{B,q}(P,n\tau) = \sup_{\nu\in\mathcal{N}^*_q} \mathrm{E}^{\nu}_{P,n\tau}\left[-\sum_{m=n}^{N-1} e^{-\sum_{l=n}^{m-1}\beta^{h,\tau}(P(l\tau),l\tau,\nu_0(l\tau))}\delta(\nu(m\tau))\Delta t^{h,\tau}(P(m\tau),m\tau) + e^{-\sum_{m=n}^{N-1}\beta^{h,\tau}(P(m\tau),m\tau,\nu_0(m\tau))}\varphi(P(N\tau))\right], \quad P\in\mathcal{R}^h,\ n\tau\in\mathcal{T}.$$

Here, we have introduced

$$\beta^{h,\tau}(P,t,\nu_0) = (r(P,t)+\nu_0)\Delta t^{h,\tau}(P,t),$$

and $\Delta t^{h,\tau}$ is an interpolation interval function to be specified.[8]

We will choose the approximating Markov chain and the interpolation interval function, such that $V^{h,\tau}_{B,q}$ converges to $V_{B,q}$, when $h,\tau\to 0$, and, of course, such that $V^{h,\tau}_{B,q}$ can be computed rather quickly. The basic condition for convergence is that the approximating Markov chain is *locally consistent* with the original state variable, i.e. an $\alpha>0$ exists such that

(4.3) $$\mathrm{E}^{\nu}_{P,n\tau}\left[P^{h,\tau}_{n+1} - P^{h,\tau}_{n}\right] = \Delta t^{h,\tau}(P,t)(r(P,t)+\nu_0-\nu_1) + o(h^\alpha \Delta t^{h,\tau}(P,t)),$$

and

(4.4) $$\mathrm{E}^{\nu}_{P,n\tau}\left[\left(P^{h,\tau}_{n+1} - P^{h,\tau}_{n}\right)^2\right] = \Delta t^{h,\tau}(P,t)\sigma(P,t)^2P^2 + o(h^\alpha \Delta t^{h,\tau}(P,t)),$$

for every $P\in\mathbb{R}^h$ and $\nu\in\tilde{\mathcal{K}}$ with $|\nu|\le q$. Equation (4.3) says that the expected change in the Markov chain $P^{h,\tau}$ divided by the length of the time step is approximately equal to the drift of the process $P(\cdot)$, cf. (4.1). Similarly, Equation (4.4) says that the variance of the change in $P^{h,\tau}$ divided by the length of the time step is approximately equal to the squared volatility of $P(\cdot)$.

A Markov chain locally consistent with the price process is given by the transition probabilities

(4.5a) $$\pi^{h,\tau}_{P,t}(P,t+\tau|\nu) = \frac{1/\tau}{Q^{h,\tau}(P,t)},$$

(4.5b) $$\pi^{h,\tau}_{P,t}(P+h,t|\nu) = \frac{\frac{1}{h}\left(r(P,t)+\nu_0^+ + \nu_1^-\right)P + \frac{1}{2h^2}\sigma(P,t)^2P^2}{Q^{h,\tau}(P,t)},$$

(4.5c) $$\pi^{h,\tau}_{P,t}(P-h,t|\nu) = \frac{\frac{1}{h}\left(\nu_0^- + \nu_1^+\right)P + \frac{1}{2h^2}\sigma(P,t)^2P^2}{Q^{h,\tau}(P,t)},$$

(4.5d) $$\pi^{h,\tau}_{P,t}(P,t|\nu) = 1 - \frac{\frac{1}{\tau} + \frac{1}{h}\left(r(P,t)+|\nu_0|+|\nu_1|\right)P + \frac{1}{h^2}\sigma(P,t)^2P^2}{Q^{h,\tau}(P,t)},$$

for $P\in\{h,2h,\ldots,(I-1)h\}$ and $t\in\{0,\tau,2\tau,(N-1)\tau\}$, where

$$Q^{h,\tau}(P,t) = \sup_{\nu\in\tilde{\mathcal{K}},|\nu|\le q}\left\{\frac{1}{\tau} + \frac{1}{h}\left(r(P,t)+|\nu_0|+|\nu_1|\right)P + \frac{1}{h^2}\sigma(P,t)^2P^2\right\}.$$

---

[8] Note that the discount rate $r+\nu_0$ in the problems (3.13) and (3.16) is control-dependent. Kushner (1990) and Kushner and Dupuis (1992) consider only control-independent discount rates, but the main ideas also hold with control-dependent discount rates.

$\pi_{P,t}^{h,\tau}(P',t'|\nu)$ is the probability that the state of the Markov chain is P' at time t' given that the state is P at time t and the control ν is currently applied. We assume that zero is an absorbing boundary for the price process, and let

$$\pi_{0,t}^{h,\tau}(0,t|\nu) = 1. \tag{4.5e}$$

At the artificial upper boundary $\bar{P} = Ih$, we follow Fitzpatrick and Fleming (1991) and Fleming and Soner (1993, Ch. IX) by taking

$$\pi_{\bar{P},t}^{h,\tau}(\bar{P}-h,t|\nu) = \frac{\frac{1}{h}\left(\nu_0^- + \nu_1^+\right)\bar{P} + \frac{1}{2h^2}\sigma(\bar{P},t)^2\bar{P}^2}{Q^{h,\tau}(\bar{P},t)}, \tag{4.5f}$$

$$\pi_{\bar{P},t}^{h,\tau}(\bar{P},t|\nu) = 1 - \pi_{\bar{P},t}^{h,\tau}(\bar{P}-h,t|\nu). \tag{4.5g}$$

All other transition probabilities are zero. The interpolation interval function is taken to be

$$\Delta t^{h,\tau}(P,t) = \frac{1}{Q^{h,\tau}(P,t)}.$$

The dynamic programming equation corresponding to the discrete-time, discrete-state problem (4.2) is

$$\begin{aligned} V_{B,q}^{h,\tau}(P,t) = \sup_{\nu\in\tilde{K},|\nu|\leq q} \Big\{ & e^{-\beta^{h,\tau}(P,t,\nu_0)} \sum_{P'\in\mathcal{R}^h} \pi_{P,t}^{h,\tau}(P',t|\nu) V_{B,q}^{h,\tau}(P',t) \\ & + e^{-\beta^{h,\tau}(P,t,\nu_0)} \pi_{P,t}^{h,\tau}(P,t+\tau|\nu) V_{B,q}^{h,\tau}(P,t+\tau)\Big\} \\ & - \Delta t^{h,\tau}(P,t)\delta(\nu)\Big\}, \quad P\in\mathcal{R}^h,\ t\in\mathcal{T}, \end{aligned} \tag{4.6}$$

with terminal value

$$V_{B,q}^{h,\tau}(P,T) = \varphi(P), \quad P\in\mathcal{R}^h. \tag{4.7}$$

In the following, we shall suppress the h and τ indices for ease of notation.

The DPE (4.6) is solved by backwards induction starting with the known value function, given by (4.7), at the maturity date of the contingent claim. At each point in time $t = m\tau$, we find $V_{B,q}(\cdot,t)$ by applying the policy space algorithm. The algorithm takes as input some admissible feedback control $\nu^{(0)}$, i.e. $\nu^{(0)}(P,t)$ for all $P\in\mathcal{R}^h$. A sequence of policies $\nu^{(k)}(\cdot,t)$ and functions $V_{B,q}^{(k)}(\cdot,t)$ is generated as follows. Given $\nu^{(k)}$, we get $V_{B,q}^{(k)}$ by a *policy evaluation*, i.e. $V_{B,q}^{(k)}(\cdot,t)$ is the solution to the linear system of equations[9]

$$V_{B,q}^{(k)}(P,t) = e^{-\beta(P,t,\nu_0^{(k)}(P,t))} \sum_{P' \in \mathcal{R}^h} \pi_{P,t}(P',t|\nu^{(k)}(P,t)) V_{B,q}^{(k)}(P',t)$$
$$+ e^{-\beta(P,t,\nu_0^{(k)}(P,t))} \pi_{P,t}(P,(m+1)\tau|\nu^{(k)}(P,t)) V_{B,q}^{(k)}(P,(m+1)\tau)$$
$$- \Delta t(P,t)\delta(\nu^{(k)}(P,t)), \quad P \in \mathcal{R}^h.$$

Then a new control policy $\nu^{(k+1)}$ is computed by a *policy improvement*, which amounts to letting[10]

$$\nu^{(k+1)}(P,t) = \underset{\nu \in \tilde{\mathcal{K}}, |\nu| \le q}{\arg\max} \left\{ e^{-\beta(P,t,\nu_0)} \sum_{P'} \pi_{P,t}(P',t|\nu) V_{B,q}^{(k)}(P',t) + e^{-\beta(P,t,\nu_0)} \pi_{P,t}(P,t+\tau|\nu) V_{B,q}^{(k)}(P,t+\tau) - \Delta t(P,t)\delta(\nu) \right\}.$$

The iterations are stopped the first time

$$\sup_{P \in \mathcal{R}^h} \left| V_{B,q}^{(k+1)}(P,t) - V_{B,q}^{k}(P,t) \right| < \varepsilon$$

for some tolerance level ε.[11] For more details, consult, e.g., Kushner and Dupuis (1992, Ch. 6) and Rust (1996).

5 Numerical Examples

Most of the results presented in this section could have been obtained without resorting to numerical methods, see Cvitanić and Karatzas (1993, Sect. 7). With this in mind, the following can be seen as an examination of the precision and convergence features of the proposed method on this type of problems. In these examples, we consider a single risky asset, constant market coefficients $\sigma = 0.3$ and $r = 0.1$, and options of the European type with a strike price of $\chi = 100$ and one year to maturity, i.e. $T = 1$. For numerical purposes, we impose an artificial maximum price of the underlying asset of $P_{\max} = 400$.

We consider the case of short-selling prohibition, i.e., $\mathcal{K} = \mathbb{R} \times \mathbb{R}_+$ and $\tilde{\mathcal{K}} = \{0\} \times \mathbb{R}_+$. Then $\delta \equiv 0$ on $\tilde{\mathcal{K}}$. The control problem to be solved is reduced to

$$\sup_{\nu_1 \text{ with } 0 \le \nu_1(t) \le q, \forall t \in [0,T]} \mathrm{E}^{\nu}\left[\varphi(P(T))e^{-rT}\right], \tag{5.1}$$

---

[9]With the transition probabilities (4.5), the system can be written in matrix form as

$$\mathbf{V}^{(k)} = \mathbf{M}(\nu^{(k)})\mathbf{V}^{(k)} + \Lambda(\nu^{(k)}),$$

for some tridiagonal matrix $\mathbf{M}$, where $\mathbf{V}^{(k)} = \left(V_{B,q}^{(k)}(0,t), \ldots, V_{B,q}^{(k)}(\bar{P},t)\right)^{\top}$, etc. Due to the tridiagonal structure, the system can be solved very fast.

[10]This maximization can, in general, be quite complicated, since ν_0 enters both the exponential discount factor and the probabilities. If $\nu_0 \equiv 0$, this problem disappears.

[11]For $m < N$, a good starting value $\nu^{(0)}(\cdot,t)$ is the control in the last iteration at time $(m+1)\tau$. Then the policy space algorithm usually converges in a few iterations.

respectively

$$\text{(5.2)} \qquad \inf_{\nu_1 \text{ with } 0 \le \nu_1(t) \le q, \, \forall t \in [0,T]} \mathrm{E}^{\nu}\left[\varphi(P(T))e^{-rT}\right],$$

where

$$dP(t) = P(t)\left[(r - \nu_1(t))\, dt + \sigma\, dw_{\nu}(t)\right], \quad P(0) = P.$$

5.1 Hedging a European Call Option under Short-Sale Prohibition

For a European call option with the terminal payoff $\varphi(P(T)) = (P(T) - \chi)^+$, it is easy to see that the supremum in (5.1) will be attained by $\nu_1 = 0$, and hence the upper bound is equal to the Black-Scholes price,

$$\mathrm{E}^0\left[(P(T) - \chi)^+ e^{-rT}\right] = P\mathfrak{N}(d_+) - \chi e^{-rT}\mathfrak{N}(d_-),$$

where $\mathfrak{N}$ is the cumulative distribution function of a standard normal random variable, and

$$d_{\pm} = \frac{\log(P/\chi) + (r \pm \sigma^2/2)T}{\sigma\sqrt{T}}.$$

Recall from Definition 3.2 that the upper bound, the arbitrage selling price, is the lowest price at which any investor will be willing to sell the contingent claim. Since the replicating portfolio of the European call option does not involve short-selling the underlying asset, any investor will be willing to sell the option at the Black-Scholes price, since she at the same price can buy the replicating portfolio. The lower bound in (5.2) is attained by $\nu_1 = q$, and by letting $q \to \infty$, the price of the underlying asset under the optimal control $\nu_1 = q$ will immediately go to zero. According to Definition 3.3, the arbitrage buying of the European call option is zero. The intuition is that an investor buying the call option would like to sell (a fraction of) the underlying asset, but since this is not allowed, the highest price admissible for buyers is zero.

Implementing the Markov chain approximation method, we get from (4.5) the transition probabilities

$$\begin{aligned}
\pi_{P,t}(P,(m+1)\tau|\nu) &= \frac{1/\tau}{Q(P)}, \\
\pi_{P,t}(P+h,t|\nu) &= \frac{\frac{1}{h}rP + \frac{1}{2h^2}P^2\sigma^2}{Q(P)}, \\
\pi_{P,t}(P-h,t|\nu) &= \frac{\frac{1}{h}\nu_1 P + \frac{1}{2h^2}P^2\sigma^2}{Q(P)}, \\
\pi_{P,t}(P,t|\nu) &= 1 - \frac{\frac{1}{\tau} + \frac{1}{h}P\,(r+\nu_1) + \frac{1}{h^2}P^2\sigma^2}{Q(P)} = \frac{\frac{1}{h}P(q-\nu_1)}{Q(P)},
\end{aligned}$$

with

$$Q(P) = \sup_{0 \le \nu_1 \le q}\left\{\frac{1}{\tau} + \frac{1}{h}P(r+\nu_1) + \frac{1}{h^2}\sigma^2 P^2\right\} = \frac{1}{\tau} + \frac{1}{h}P(r+q) + \frac{1}{h^2}\sigma^2 P^2.$$

The DPE (4.6) reduces to

$$
\begin{aligned}
V_{B,q}(P,t) = e^{-r\Delta t(P,t)} \sup_{0\le \nu_1 \le q} \Bigg\{ & \frac{\frac{1}{h}rP + \frac{1}{2h^2}P^2\sigma^2}{Q(P)} V_{B,q}(P+h,t) \\
& + \frac{\frac{1}{h}\nu_1 P + \frac{1}{2h^2}P^2\sigma^2}{Q(P)} V_{B,q}(P-h,t) \\
& + \frac{\frac{1}{h}P(q-\nu_1)}{Q(P)} V_{B,q}(P,t) + \frac{1/\tau}{Q(P)} V_{B,q}(P,t+\tau) \Bigg\}.
\end{aligned}
\tag{5.3}
$$

In the policy space solution to (5.3), the k'th improvement step in the computation of the control and the value function at time t is

$$
\nu_1^{(k+1)}(P,t) = \arg\max_{0\le \nu_1 \le q} -\frac{\nu_1 P/h}{Q(P)} D^- V_{B,q}^{(k)}(P,t) = \begin{cases} 0 & \text{if } D^- V_{B,q}^{(k)}(P,t) \ge 0, \\ q & \text{if } D^- V_{B,q}^{(k)}(P,t) < 0. \end{cases}
$$

Of course, we should have that $D^- V_{B,q}^{(k)}(P,t) \equiv V_{B,q}^{(k)}(P,t) - V_{B,q}^{(k)}(P-h,t)$ is non-negative, since both the upper and the lower arbitrage bound for a European call option are expected to be non-decreasing in the price of the underlying asset.

When computing the lower bound on the contingent claim's price, the DPE to be solved is identical to (5.3), except that 'sup' is replaced by 'inf', and the policy improvement step is

$$
\nu_1^{(k+1)}(P,t) = \arg\min_{0\le \nu_1 \le q} -\nu_1 D^- V_{B,q}^{(k)}(P,t) = \begin{cases} q & \text{if } D^- V_{B,q}^{(k)}(P,t) \ge 0, \\ 0 & \text{if } D^- V_{B,q}^{(k)}(P,t) < 0. \end{cases}
$$

For the no-arbitrage bounds on the European call option, our results show that the suggested numerical procedure converges linearly in both I, the number of intervals the price axis is divided into, and in N, the number of time steps. Furthermore, the optimal ratio I/N seems to be approximately 10. Now, Richardson extrapolation can be applied.[12] For the results presented below, we have taken the upper bound on the control to be $q = 5$.

Figure 1 shows the numerically computed bounds on the price of a call, when short sales are not allowed in the hedge portfolio. Obviously, the lower bound is zero, and the upper bound is very close to the Black-Scholes price. The numerically computed upper bound on the option price seems to be influenced by the artificial bound $\bar{P}$ on the underlying price range, but for near-the-money options the difference is small.

In Figure 2, the distance between the upper bound and the Black-Scholes price is depicted for various values of I, where N is such that $I/N = 10$. The curve tagged 'ri(100,200)' stems from an

[12] Richardson extrapolation is discussed in most introductory textbooks on numerical analysis, e.g., Buchanan and Turner (1992). Potentially, the optimal I/N ratio can be different for different values of the initial price P. We have focused on $P = \chi$. Therefore, Richardson extrapolation of the computed bounds should give the highest gain in precision for values of P near χ.

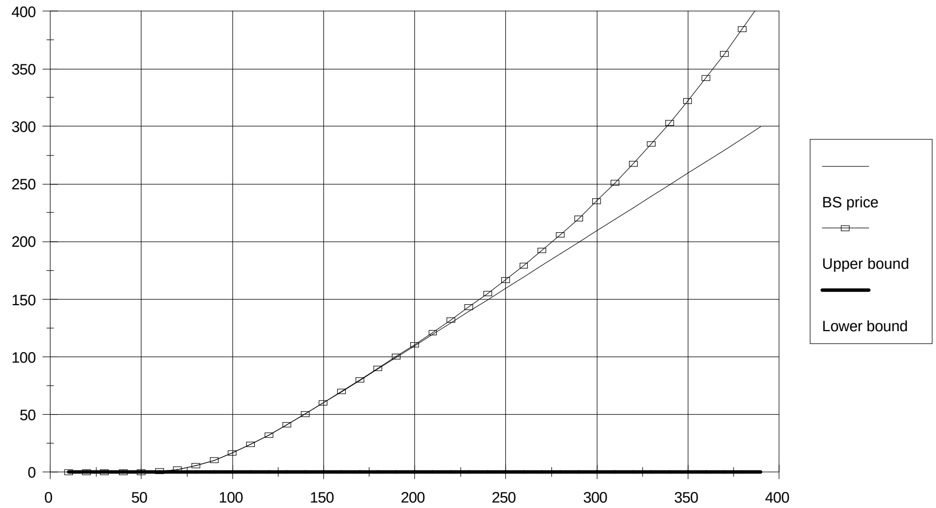

Figure 1: The arbitrage-based bounds for a European call option, when short sales are prohibited. Results from an implementation with $I = 1000$ and $N = 100$.

order one Richardson extrapolation of the results for $I = 100$ and $I = 200$. Similarly for the curve tagged 'ri(1000,2000)'. For small values of I, the numerically computed upper bound is rather imprecise for slightly out-of-the-money calls, but, again, the Richardson extrapolated results are much more precise. Even the 'ri(100,200)' results are very, very accurate for the entire range of underlying asset prices shown in the figure, and these results can be obtained very fast on even a low-memory computer.

5.2 Hedging a European Put Option under Short-Sale Prohibition

For a European put option, $\varphi(P(T)) = (\chi - P(T))^+$, it is easy to see that the supremum in (5.1) is attained by $\nu_1 = q$, and hence the arbitrage selling price is equal to χe^{-rT}, the discounted strike price. The infimum in (5.2) is attained by $\nu_1 = 0$, which implies that the arbitrage buying price of the European put is the Black-Scholes put price

$$\mathrm{E}^0\left[(\chi - P(T))^+ e^{-rT}\right] = \chi e^{-rT}\mathfrak{N}(-d_-) - P\mathfrak{N}(-d_+).$$

The intuition is that since any investor can sell the portfolio replicating the put at the Black-Scholes price, the highest price admissible for buyers will be the Black-Scholes price. On the other hand, the investors cannot *buy* the put replicating portfolio, since this would involve short-selling the underlying asset. They will accept a price of χe^{-rT} for the put, however, since investing this amount in the riskless asset to the maturity of the option can in any case cover the obligations on the written put.

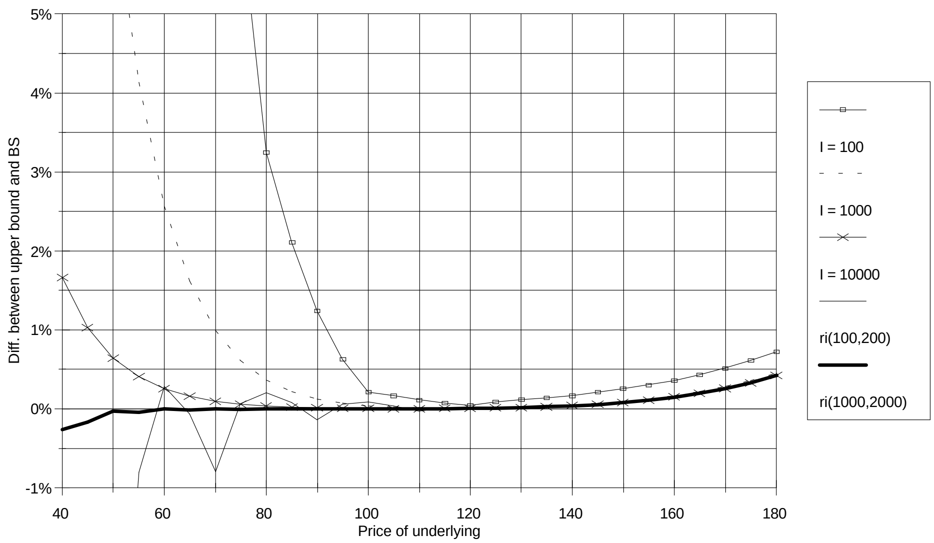

Figure 2: A comparison of the upper bound with short-sale prohibition and the Black-Scholes price for a European call option.

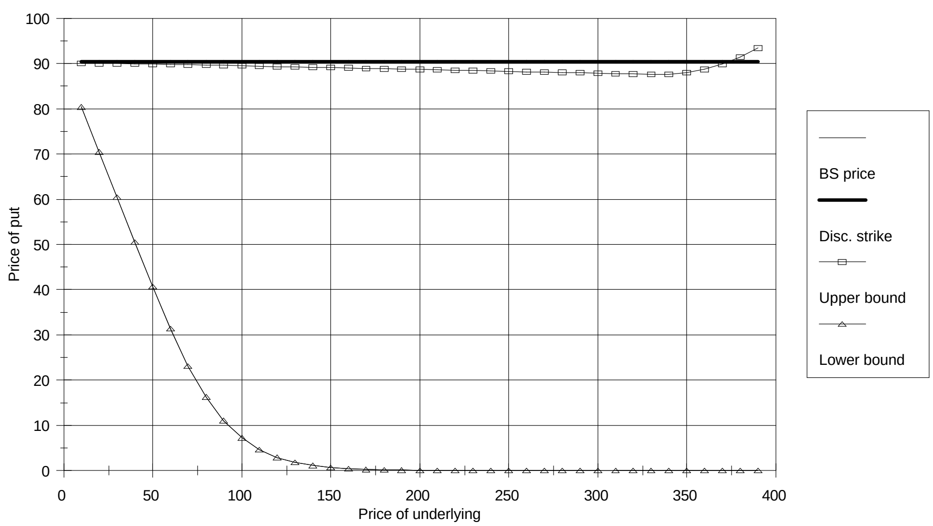

Figure 3: The arbitrage-based bounds for a European put option, when short sales are prohibited. $I = 1000, N = 50$. The lines corresponding to the lower bound and the Black-Scholes price are indistinguishable.

Numerically computed bounds are depicted in Figure 3. As expected, the lower put price bound is very close to the Black-Scholes put price, and the upper put price bound is close to the discounted strike price.

5.3 Hedging a Straddle under Short-Sale Prohibition

To illustrate the non-linearity of the no-arbitrage bounds, we consider hedging a bottom straddle under short-selling prohibition. A bottom straddle is a combination of a long European call and a long European put on the same underlying asset, with the same maturity date and the same exercise price. A straddle with exercise price χ and time to maturity T on the risky asset with price process P has the terminal payoff

$$\varphi(P(T)) = (P(T) - \chi)^{+} + (\chi - P(T))^{-} = |P(T) - \chi|.$$

Since the terminal payoff is not a monotonic function of the price of the underlying asset, the replicating portfolio may or may not involve short-selling the underlying asset, depending on it's price. If the price of the underlying asset is high compared to the exercise price, the straddle is very much like a call option. Therefore, for high values of P, the upper bound of the straddle price should be close to the Black-Scholes price of the straddle and the lower bound should be close to zero. If the price of the underlying asset is low compared to the exercise price, the straddle resembles a put option. Hence, for low values of P, the upper bound of the straddle price should be close to the discounted exercise price and the lower bound should be close to the Black-Scholes price of the straddle. Figure 4 fully confirms these expectations. Note that the upper bound on the bottom straddle is much smaller than the sum of the upper bounds for its two components.

6 Concluding Remarks

We have shown that with convex constraints on the dollar investments in the primary assets, the price of a contingent claim must be in some interval, if arbitrage opportunities are ruled out. We demonstrated how the bounds of this interval can be computed numerically in an efficient way.

The price bounds studied in this paper are arbitrage-based in the sense that all risk-averse investors should agree that the equilibrium price of the contingent claim cannot lie outside the interval between the bounds. Note however that this applies only when all investors face the same constraints on their *hedge* portfolio. It seems more reasonable to consider constraints on the *total* portfolio of the investor. Since the investors can have different preferences and endowments, and hence different optimal portfolios without considering engaging in contingent claims trading, they will typically face different constraints on their hedge portfolio. Consequently, only investor-specific

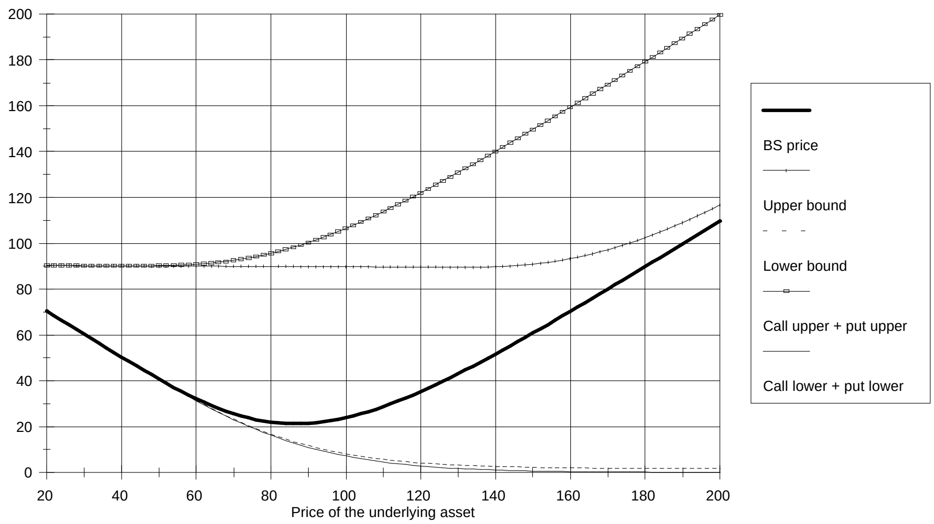

Figure 4: The arbitrage-based bounds for a bottom straddle under short-sale prohibition of the underlying asset. Results with $I = 10000$ and $N = 1000$.

(or to be more precise: preference- and endowment-specific) bounds on the price of the contingent claim can be given.

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