Optimal Consumption/Investment Policies with Undiversifiable Income Risk and Borrowing Constraints*

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Abstract. We examine the optimal consumption and portfolio choice of an investor having an initial wealth endowment and an uncertain stream of income from non-traded assets. The income stream is not spanned by traded assets, and the investor is not allowed to borrow against future income, so the financial market is incomplete. We solve the corresponding stochastic control problem numerically with the Markov chain approximation method. In particular, we find that the implicit value, the agent attaches to an uncertain income stream, can be much smaller in this incomplete market than it is in the complete market.

Keywords. Optimal consumption and portfolio policies, undiversifiable income risk, borrowing constraints, wealth equivalent of income, Markov chain approximation

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1 Introduction

The consumption/investment choice problem of a price-taking investor is a classical problem of financial economics. In two pioneering papers, Merton (1969, 1971) introduced stochastic control techniques to analyze the continuous-time version of the problem for an investor with an additively time-separable utility function. In particular, Merton studied the case, where the investor has access to a complete financial market, in which risky asset prices follow geometric Brownian motions, and the investor's utility function for consumption is of the constant relative risk aversion type. He was able to solve analytically the Hamilton-Jacobi-Bellman (HJB) equation associated with the problem and hence to obtain closed-form expressions for the optimal control policies in feedback form, both for a finite horizon and an infinite horizon.

One of many interesting generalizations of Merton's setting appears, when the investor besides having an initial endowment of wealth also receives a stream of income throughout her planning horizon. Merton (1971, Section 7) stated that the optimal policies when the agent has a deterministic stream of income are as if the agent has no income stream but instead adds the capitalized lifetime income flow discounted at the risk-free rate to her initial wealth. However, it is easy to show, see e.g. He and Pagès (1993, Example I), that under the policies derived this way the wealth process may go below zero. Due to moral hazard and adverse selection problems, it may be impossible for the investor to borrow against future income, so that the investor can only choose her consumption/investment policy among those that keep her financial wealth non-negative.

Svensson and Werner (1993) assume that cumulative income follows an Itô process with drift and volatility depending on some one-dimensional state variable. In their set-up the income rate is not restricted to be positive. The authors find explicit results in the case of negative exponential utility, when either (1) income is not spanned by traded assets[1] and cumulative income follows an arithmetic Brownian motion, or (2) cumulative income is instantaneously riskless with drift proportional to the state variable, which follows an arithmetic Brownian motion. Like Merton, Svensson and Werner implicitly assume that wealth is not restricted to be non-negative.

He and Pagès (1993) study a model, where the income rate is spanned, such that the only source of incompleteness is that wealth has to stay non-negative. Using martingale techniques, they find that the presence of borrowing constraints has a smoothing effect on the optimal consumption across time. If the investor expects her income to rise, she will increase her consumption at a smaller rate than if she was not subjected to borrowing constraints. In a similar set-up, El Karoui and Jeanblanc-Picqué (1996) show that the optimal trading strategy is to invest part of the wealth

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[1] The income process is said to be spanned by traded assets, if it can be replicated by a self-financing dynamic trading strategy in the available assets. In that case all the risk of the income stream can be diversified away.

in the strategy, which is optimal in the corresponding unconstrained case, and the remainder in an American put option written on the optimal wealth process in the unconstrained case.

Maintaining the borrowing constraints of He and Pagès, but dropping the spanning assumption, Duffie and Zariphopoulou (1993) study an infinite horizon model with a single risky asset, whose price follows a geometric Brownian motion, and a non-negative income rate given by an Itô process driven by a Brownian motion imperfectly correlated with the risky asset price. In this general setting, we cannot be sure that the value function, also known as the indirect utility function, is a solution of the associated HJB equation. In fact, the value function may not even be smooth. Duffie and Zariphopoulou are able to show that the value function is the unique *constrained viscosity solution* in the class of concave functions of the HJB equation.

The model of Duffie and Zariphopoulou is specialized in Duffie, Fleming, Soner, and Zariphopoulou (1997), henceforth abbreviated DFSZ. For power utility of consumption, they reduce the control problem from a two-variable (wealth and income) to a one-variable (wealth divided by income) problem and show that the HJB equation for the reduced problem has a unique smooth solution. The optimal policies and the value function of the original problem can easily be restored from the solution to the reduced problem. With analytical derivations, DFSZ are able to find some characteristics of the solution, but they cannot solve the problem completely.

In this paper, we solve the reduced problem in DFSZ numerically with the so-called Markov chain approximation approach. The basic idea is to approximate the continuous time, continuous state stochastic control problem with a discrete time, discrete state stochastic control problem, which is easily solved numerically. The procedure is described in Kushner (1990) and more detailed in Kushner and Dupuis (1992). See also Fleming and Soner (1993, Chap. IX). The Markov chain approximation method has previously been applied to consumption/portfolio problems by Fitzpatrick and Fleming (1991) and Hindy, Huang, and Zhu (1997).

Using ideas of Koo (1996), we contrast the numerically computed value function and optimal controls to the complete market case, where the income rate is spanned by traded assets and the investor is not borrowing constrained. In particular, we find that the implicit value, the investor attaches to the uncertain income stream, is much smaller in the non-spanned, borrowing constrained case than in the complete market case, even for high ratios of initial wealth to initial income. We find that this implicit value of income is very insensitive to the correlation between changes in the risky asset price and changes in the income rate. Since a perfectly positive correlation corresponds to spanning, this suggests that the large difference between the complete markets case and the non-spanned, borrowing constrained case is to be attributed to the borrowing constraints. We study the sensitivity of both the optimal policies and the implicit value of the income stream with respect to various parameters. Among other things we find that, *ceteris paribus*, borrowing constraints as modeled in this paper are most restrictive for agents with a low financial wealth relative to income, an income rate positively correlated with changes in the risky asset price, and a high time preference for consumption.

The outline of the rest of the paper is as follows. The problem is formalized in Section 2, where it is also shown how the problem for power utility functions can be reduced from a two-dimensional to a one-dimensional problem. Furthermore, the analytical results obtained by DFSZ are reviewed and two measures of the value of an uncertain stream of income are introduced. In Section 3, we implement the Markov chain approximation method and prove convergence. Numerical results are presented and discussed in Section 4. In Section 5, we study the sensitivity of the results with respect to selected parameters. Finally, Section 6 briefly summarizes our results.

2 The Problem

2.1 Statement of the Problem

Let $W = (W_1, W_2)$ be a standard two-dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ be the filtration generated by W. Consider an investor who wants to maximize her expected life-time utility from consumption $\mathrm{E}\left[\int_0^\infty e^{-\beta t} u(c(t))\, dt\right]$. The investor can invest in a risky asset with a price process $S(\cdot)$ given by

$$dS(t) = S(t)\left(b\, dt + \sigma\, dW_1(t)\right), \quad S(0) > 0,$$

and in a riskless asset with a constant continuously compounded rate of return r. The investor has an initial wealth endowment $x \geq 0$ and receives income from non-traded assets at a rate given by the process $Y(\cdot)$, where

$$dY(t) = Y(t)\left(\mu\, dt + \delta\rho\, dW_1(t) + \delta\sqrt{1-\rho^2}\, dW_2(t)\right), \quad Y(0) = y > 0.$$

Here b, σ, μ, and δ are positive constants, and $\rho \in (-1, 1)$ is the correlation between changes in the risky asset price S and changes in the income rate Y. Hence, the income rate process is not spanned by traded assets, i.e. the investor faces undiversifiable income risk.

A consumption process is defined to be an $\mathbb{F}$-progressively measurable process $c : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ with $\mathrm{E}\left[\int_0^T c(t)\, dt\right] < \infty$ for all $T \geq 0$ and $\mathrm{E}\left[\int_0^\infty e^{-\beta t} u(c(t))\, dt\right] < \infty$.[2] $c(t)$ is the rate of consumption at time t. The set of consumption processes is denoted by $\mathcal{C}$. A portfolio process is an $\mathbb{F}$-progressively measurable process $\pi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ satisfying $\int_0^T \pi(t)^2\, dt < \infty$ (a.s.) for all

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[2] $\mathbb{R}_+$ denotes $[0, \infty)$.

$T \geq 0$. $\pi(t)$ is the dollar amount invested at time t in the risky asset. The set of portfolio processes is denoted by $\mathcal{P}$.

Define $X(t)$ as the time t value of the agent's portfolio of financial assets. Hence, $X(t)$ is the liquid wealth of the investor at time t and does not take into account the value of her future uncertain income stream. Given a consumption process $c \in \mathcal{C}$ and a portfolio process $\pi \in \mathcal{P}$, $X(\cdot) = X^{c,\pi}(\cdot)$ evolves as

$$dX(t) = (rX(t) + \pi(t)(b-r) - c(t) + Y(t))\, dt + \pi(t)\sigma\, dW_1(t), \quad X(0) = x. \tag{2.1}$$

The investor can choose c and π from the set $\mathcal{A}(x,y)$ of admissible controls, where

$$\mathcal{A}(x,y) = \left\{(c,\pi) \in \mathcal{C} \times \mathcal{P} : X^{c,\pi}(t) \geq 0 \ (a.s.),\ \forall t \geq 0\right\}.$$

The admissible control policies have the property that the liquid wealth is non-negative (with probability one) at all points in time. The investor is not allowed to borrow funds against her future uncertain income.

Define the value function of the investor's problem by

$$v(x,y) = \sup_{(c,\pi)\in\mathcal{A}(x,y)} \mathrm{E}\left[\int_0^\infty e^{-\beta t} u(c(t))\, dt\right]. \tag{2.2}$$

The HJB equation[3] associated with this control problem is

$$\begin{aligned}\beta v(x,y) = {} & \frac{1}{2}\frac{\partial^2 v}{\partial y^2}(x,y)\delta^2 y^2 + \frac{\partial v}{\partial x}(x,y)\left[rx + y\right] + \frac{\partial v}{\partial y}(x,y)\mu y + \sup_{\bar{c}\in\mathbb{R}_+}\left\{u(\bar{c}) - \frac{\partial v}{\partial x}(x,y)\bar{c}\right\} \\ & + \sup_{\bar{\pi}\in\mathbb{R}}\left\{\frac{1}{2}\frac{\partial^2 v}{\partial x^2}(x,y)\bar{\pi}^2\sigma^2 + \frac{\partial^2 v}{\partial x \partial y}(x,y)\bar{\pi}\sigma\delta\rho y + \frac{\partial v}{\partial x}(x,y)\bar{\pi}(b-r)\right\}\end{aligned} \tag{2.3}$$

2.2 Reduction of the Problem

Now assume that the investor has a CRRA (Constant Relative Risk Aversion) utility function

$$u(c) = c^\gamma, \quad 0 < \gamma < 1,$$

where $1-\gamma$ is the coefficient of relative risk aversion. For any positive constant k, it follows from the linearity of the wealth dynamics in (2.1) that $(c,\pi) \in \mathcal{A}(x,y)$ if and only if $(kc, k\pi) \in \mathcal{A}(kx, ky)$ and hence that $v(kx,ky) = k^\gamma v(x,y)$, i.e., v is homogeneous of degree γ. As demonstrated below, this allows a reduction in the dimension of the problem from two to one.[4]

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[3] It is not clear *a priori* that the value function v is twice differentiable and hence that the HJB equation (2.3) has a solution in the classical sense. In fact, the HJB equation is of the *degenerate elliptic* type, so one cannot expect v to be a smooth solution to (2.3); see, e.g., the discussion in Fleming and Soner (1993, Sec. IV.5).

[4] The reduction idea was suggested by Davis and Norman (1990). A reduction is also possible in the case of logarithmic utility, $u(c) = \log c$, and for the finite horizon case, a reduction of the problem's dimension is possible,

Define $F : \mathbb{R}_+ \to \mathbb{R}_+$ by $F(z) = v(z,1)$. Then v can be recovered from F since, for $y \neq 0$, $v(x,y) = y^\gamma v(x/y,1) = y^\gamma F(x/y)$. For $y = 0$, the value function $v(x,0)$ is known from Merton's work. Define

$$A = \frac{\beta - r\gamma}{1-\gamma} - \frac{\gamma(b-r)^2}{2(1-\gamma)^2\sigma^2}.$$

Under the assumption $A > 0$, Merton showed that $v(x,0) = A^{\gamma-1}x^\gamma$ with optimal policies $c(t) = C(X(t),0)$ and $\pi(t) = \Pi(X(t),0)$, where

$$C(x,0) = Ax, \qquad \Pi(x,0) = \frac{b-r}{\sigma^2(1-\gamma)}x. \tag{2.4}$$

Exploiting $v(x,y) = y^\gamma F(x/y)$, we get from (2.3)

$$\begin{aligned} \hat{\beta}F(z) = {} & \frac{1}{2}\delta^2 z^2 F''(z) + k_2 z F'(z) + \sup_{\zeta \geq -1} \{-\zeta F'(z) + (1+\zeta)^\gamma\} \\ & + \sup_{\psi \in \mathbb{R}} \left\{ \left(\frac{1}{2}\sigma^2\psi^2 - \sigma\delta\rho z\psi\right) F''(z) + k_1\psi F'(z) \right\}, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \hat{\beta} &= \beta - \mu\gamma + \frac{1}{2}\delta^2\gamma(1-\gamma), \\ k_1 &= b - r - (1-\gamma)\sigma\delta\rho, \\ k_2 &= \delta^2(1-\gamma) + r - \mu, \end{aligned}$$

and the variables $\zeta = \bar{c}/y - 1$ and $\psi = \bar{\pi}/y$ have been introduced. The maximizers of the two sup-terms are

$$\zeta^*(z) = (F'(z)/\gamma)^{1/(\gamma-1)} - 1, \qquad \psi^*(z) = \frac{\delta\rho z}{\sigma} - \frac{k_1}{\sigma^2}\frac{F'(z)}{F''(z)},$$

so for $x, y > 0$ the candidate optimal consumption and investment policies in feedback form are $c(t) = C(X(t),Y(t))$, $\pi(t) = \Pi(X(t),Y(t))$, where

$$C(x,y) = y\left(\frac{F'(x/y)}{\gamma}\right)^{1/(\gamma-1)}, \qquad \Pi(x,y) = \frac{\delta\rho}{\sigma}x - \frac{k_1 y}{\sigma^2}\frac{F'(x/y)}{F''(x/y)}. \tag{2.6}$$

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e.g., when the utility function for terminal wealth is identically equal to zero or if it is identical to the utility function of consumption. The homogeneity property is also sustained, if the amount invested in the risky asset is constrained to a convex cone, a specification which contains many portfolio constraints of interest, see, e.g., Cuoco (1997). There can also be multiple risky assets as long as their drift and noise terms are constant. Of course, this will increase the number of controls in the reduced problem, but not the dimension of the state variable. Indeed, the extension to multiple risky assets is a real extension of the model in the sense that portfolio separation does not obtain.

By direct computations it can be verified that

$$\begin{aligned}\sup_{\psi\in\mathbb{R}}\left\{\left(\frac{1}{2}\sigma^2\psi^2-\sigma\delta\rho z\psi\right)F''(z)+k_1\psi F'(z)\right\}&\\ =-\frac{1}{2}\frac{k_1^2}{\sigma^2}\frac{F'(z)^2}{F''(z)}-\frac{1}{2}\delta^2\rho^2z^2F''(z)+\frac{k_1\delta\rho z}{\sigma}F'(z)&\\ =\sup_{\varphi\in\mathbb{R}}\left\{\frac{1}{2}\sigma^2\varphi^2F''(z)+k_1\varphi F'(z)\right\}+\frac{\rho k_1\delta z}{\sigma}F'(z)-\frac{1}{2}\delta^2\rho^2z^2F''(z),&\end{aligned}$$

where the maximizers ψ^* and φ^* of the two sup-terms are related by the equation $\psi^*=\delta\rho z/\sigma+\varphi^*$, and hence (2.5) may be rewritten as

$$\begin{aligned}\hat{\beta}F(z)=\frac{1}{2}\delta^2(1-\rho^2)z^2F''(z)+kzF'(z)+\sup_{\zeta\geq-1}\{-\zeta F'(z)+(1+\zeta)^\gamma\}&\\ +\sup_{\varphi\in\mathbb{R}}\left\{\frac{1}{2}\sigma^2\varphi^2F''(z)+k_1\varphi F'(z)\right\},&\end{aligned}\tag{2.7}$$

where $k=k_2+\rho k_1\delta/\sigma$.

Equation (2.7) is actually the HJB equation associated with the control problem

$$F(z)=\sup_{(\zeta,\varphi)\in\hat{\mathcal{A}}(z)}\mathrm{E}\left[\int_0^\infty e^{-\hat{\beta}t}(1+\zeta(t))^\gamma\,dt\right]\tag{2.8}$$

where

$$dZ(t)=(kZ(t)+k_1\varphi(t)-\zeta(t))\,dt+\sigma\varphi(t)\,dW_1(t)+\delta Z(t)\sqrt{1-\rho^2}\,dW_2(t),\quad Z(0)=z,\tag{2.9}$$

and

$$\hat{\mathcal{A}}(z)=\{(\zeta,\varphi)\in\mathcal{C}_{-1}\times\mathcal{P}:Z(t)\geq0\text{ (a.s.)},\ \forall t\geq0\}.$$

Here $\mathcal{C}_{-1}$ is the set of progressively measurable processes ζ with $\mathrm{E}\left[\int_0^t\zeta(s)\,ds\right]<\infty$ and $\zeta(t)\geq-1$ for all $t\geq0$, which is as the set $\mathcal{C}$ except that the non-negativity constraint is replaced with the greater than or equal to -1 constraint. Note that the HJB equation (2.7) can be written as

$$\hat{\beta}F(z)=\sup_{\zeta\geq-1,\varphi\in\mathbb{R}}\{(1+\zeta)^\gamma+\mathcal{L}_{\zeta,\varphi}F(z)\},\tag{2.10}$$

where the operator $\mathcal{L}_{\zeta,\varphi}$ is given by

$$\mathcal{L}_{\zeta,\varphi}F(z)=\frac{1}{2}\left(\delta^2(1-\rho^2)z^2+\sigma^2\varphi^2\right)F''(z)+(kz+k_1\varphi-\zeta)\,F'(z).$$

2.3 Review of Analytical Results

Applying viscosity solution techniques, DFSZ are able to show that, under the parameter restrictions $\hat{\beta}>0$, $A>0$, and $r>\mu$, the reduced HJB equation (2.7) has a unique $C(\mathbb{R}_+)\cap C^2((0,\infty))$ solution F in the class of concave functions; cf. Theorem 1 of DFSZ. Furthermore, the unique optimal feedback policy (C, Π) for the original problem (2.2) is given by (2.6) for $x, y > 0$, by (2.4) for $y = 0$, and by

$$C(0,y) = \left(\frac{F'(0)}{\gamma}\right)^{1/(\gamma-1)} y, \qquad \Pi(0,y) = 0,$$

for $x = 0$ and $y > 0$, and DFSZ provide some results on the behavior of $F(\cdot)$ near zero; cf. their Proposition 1. DFSZ also show that as the ratio of wealth to income goes to infinity, the value function, respectively the optimal policies, are asymptotically equal to the value function, respectively the optimal policies, in Merton's problem.

Koo (1996) studies a model very similar to that of DFSZ. Although he allows for multiple risky assets, we shall only consider the single risky asset case. For $x > 0$ and $y > 0$, it is fairly easy to check that the homogeneity property of the value function implies that the following two mappings $B(\cdot)$ and $A(\cdot)$ are well-defined continuous functions of $z = x/y$:

$$B(x/y) = \frac{\frac{\partial v}{\partial y}(x,y)}{\frac{\partial v}{\partial x}(x,y)}, \qquad A(x/y) = \left(\frac{v(x,y)}{(x + B(x/y)y)^{\gamma}}\right)^{1/(\gamma-1)}.$$

Indeed,

$$B(z) = \gamma \frac{F(z)}{F'(z)} - z \tag{2.11}$$

and

$$A(z) = \left(\frac{\gamma}{F'(z)}\right)^{\gamma/(1-\gamma)} / F(z).$$

The value function can now be written as

$$v(x,y) = A(z)^{\gamma-1} \left(x + B(z)y\right)^{\gamma}, \quad x > 0,\ y > 0,$$

and hence

$$F(z) = A(z)^{\gamma-1}(z + B(z))^{\gamma}. \tag{2.12}$$

In the complete market, where there are no borrowing constraints and the income rate is spanned, the certainty wealth equivalent of lifetime income is $\mathrm{E}[\int_0^\infty p(t)Y(t)\,dt]$, where

$$p(t) = \exp\left\{-rt - \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 t - \frac{b-r}{\sigma} W_1(t)\right\}$$

is the unique state-price density. Define the constant $\lambda = r - \mu + \delta(b-r)/\sigma$. Then a simple computation yields that, if $\lambda > 0$, the certainty wealth equivalent of lifetime income is equal to y/λ. If the constants A and λ are both positive, we therefore have that, in the complete market, the value function is given by

$$v_{\text{com}}(x,y) = A^{\gamma-1}\left(x + \frac{y}{\lambda}\right)^{\gamma},$$

with optimal consumption policy

$$C_{\mathrm{com}}(x,y) = A\left(x + \frac{y}{\lambda}\right).$$

The optimal risky investment is[5]

$$\Pi_{\mathrm{com}}(x,y) = \frac{b-r}{\sigma^2(1-\gamma)}\left(x + \frac{y}{\lambda}\right) - \frac{\delta}{\sigma}\frac{y}{\lambda}.$$

Thus, in the complete market, $B(z) = 1/\lambda$ and $A(z) = A$ for all $z > 0$.

Koo shows that, if $\hat{\beta} > 0$, $A > 0$, $r > \mu$, and $|\rho| \neq 1$, then, for $x > 0$ and $y > 0$, optimal consumption is given by

$$C(x,y) = A(z)\left(x + B(z)y\right),$$

and the optimal amount invested in the risky asset is given by

$$\Pi(x,y) = \frac{b-r}{\sigma^2}\frac{x + B(z)y}{1-\gamma+B'(z)} + \frac{\delta\rho}{\sigma}\left(x - \frac{1-\gamma}{1-\gamma+B'(z)}(x + B(z)y)\right),$$

where $z = x/y$; cf. his Theorem 4.1. If, furthermore, $\lambda > 0$, then $B(z)$ is strictly increasing, and

$$\lim_{z\to\infty} B(z) = \frac{1}{\lambda}, \quad \lim_{z\to\infty} A(z) = A.$$

In particular, the ratio of optimal consumption to the accounting total wealth $x + y/\lambda$ is strictly increasing in z.

Due to these results, the term $B(z)y$ can be interpreted as the implicit value the agent associates with her future uncertain income stream in the presence of borrowing constraints and undiversifiable income risk. An alternative, and perhaps more natural, measure of this value is the wealth equivalent

$$\mathcal{W}(x,y) = \inf\left\{q \geq 0 : v(x+q,0) \geq v(x,y)\right\},$$

i.e. the least increase in initial wealth which the investor would accept in exchange for the entire stream of income. Since $v(x+q,0) = A^{\gamma-1}(x+q)^\gamma$, we get

$$\mathcal{W}(x,y) = A^{(1-\gamma)/\gamma}v(x,y)^{1/\gamma} - x = A^{(1-\gamma)/\gamma}yF(x/y)^{1/\gamma} - x = B^*(z)y,$$

where

$$B^*(z) = A^{(1-\gamma)/\gamma}F(z)^{1/\gamma} - z, \tag{2.13}$$

and, as before, $z = x/y$. Notice that from (2.12) we have

$$B(z) = A(z)^{(1-\gamma)/\gamma}F(z)^{1/\gamma} - z.$$

Since $\lim_{z\to\infty} A(z) = A$, we see that for large z the two income multipliers $B(z)$ and $B^*(z)$ will be approximately equal.

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[5] In the case of a certain income stream, where $\delta = 0$, we see that the value function and the optimal policies are exactly as suggested by Merton (1971, Section 7), cf. our introduction.

3 The Numerical Method

3.1 An Approximating Markov Chain

The reduced control problem (2.8) is solved with the Markov chain approximation approach. We approximate the controlled state variable process $Z = (Z(t))_{t\in\mathbb{R}_+}$ by a controlled discrete-time Markov chain $\xi^h = (\xi^h_n)_{n\in\mathbb{Z}_+}$ on the discrete state space $\mathcal{R}^h = \{0, h, 2h, \dots, Ih\}$. Here, $\bar{z} \equiv Ih$ is an artificially imposed upper boundary. A control is a sequence $(\zeta^h, \varphi^h) = (\zeta^h_n, \varphi^h_n)_{n\in\mathbb{Z}_+}$, where $\zeta^h_n = \zeta^h(\xi^h_n)$ and $\varphi^h_n = \varphi^h(\xi^h_n)$. The controls are bounded by the requirements

$$-1 \le \zeta^h(z) \le K_\zeta z \qquad \text{and} \qquad |\varphi^h(z)| \le K_\varphi z, \tag{3.1}$$

where K_ζ and K_φ are positive constants. The stochastic evolution of the controlled Markov chain ξ^h is given by the transition probabilities[6]

$$p^h(z, z-h \mid \zeta, \varphi) = \frac{\frac{1}{2}\left(\sigma^2\varphi^2 + \delta^2(1-\rho^2)z^2\right) + h\left(k^- z + (k_1\varphi)^- + \zeta^+\right)}{Q^h(z)}, \tag{3.2a}$$

$$p^h(z, z+h \mid \zeta, \varphi) = \frac{\frac{1}{2}\left(\sigma^2\varphi^2 + \delta^2(1-\rho^2)z^2\right) + h\left(k^+ z + (k_1\varphi)^+ + \zeta^-\right)}{Q^h(z)}, \tag{3.2b}$$

$$p^h(z, z \mid \zeta, \varphi) = 1 - p^h(z, z-h \mid \zeta, \varphi) - p^h(z, z+h \mid \zeta, \varphi) \tag{3.2c}$$

for $z \in \{h, 2h, \dots, (I-1)h\}$, where

$$Q^h(z) = \sigma^2 K_\varphi^2 z^2 + \delta^2(1-\rho^2)z^2 + h\left(|k|z + |k_1|K_\varphi z + \max\{1, K_\zeta z\}\right).$$

Here, $p^h(z, z' \mid \zeta, \varphi)$ denotes the probability of the state changing from z to z' in one "time step", when the control (ζ, φ) is currently applied. At the upper boundary, we can take

$$p^h(\bar{z}, \bar{z}-h \mid \zeta, \varphi) = \frac{\frac{1}{2}\left(\sigma^2\varphi^2 + \delta^2(1-\rho^2)\bar{z}^2\right) + h\left(k^- \bar{z} + (k_1\varphi)^- + \zeta^+\right)}{Q^h(\bar{z})}, \tag{3.2d}$$

$$p^h(\bar{z}, \bar{z} \mid \zeta, \varphi) = 1 - p^h(\bar{z}, \bar{z}-h \mid \zeta, \varphi). \tag{3.2e}$$

Since Z must stay non-negative, we conclude from (2.9) that φ at $Z = 0$ must be zero, and subsequently that ζ has to be non-positive at $Z = 0$. Therefore, we take

$$p^h(0, h \mid \zeta, \varphi) = \frac{h\zeta^-}{Q^h(0)} = \zeta^-, \tag{3.2f}$$

$$p^h(0, 0 \mid \zeta, \varphi) = 1 - p^h(0, h \mid \zeta, \varphi) = 1 - \zeta^-. \tag{3.2g}$$

All other transition probabilities are zero.

The control $(\zeta^h, \varphi^h) = (\zeta^h_n, \varphi^h_n)_{n\in\mathbb{Z}_+}$ is called an admissible control for ξ^h, if (3.1) is satisfied for all $z \in \mathcal{R}^h$, and ξ^h is a Markov chain, when it is controlled by (ζ^h, φ^h). The set of admissible controls for ξ^h given $\xi^h_0 = z$ is denoted $\mathcal{A}^h(z)$.

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[6] The plus and minus superscripts indicate the positive and negative part, respectively, i.e. $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

Define the interpolation interval function $\Delta t^h(z) = h^2/Q^h(z)$, and let $\Delta t_n^h = \Delta t^h(\xi_n^h)$ and $t_n^h = \sum_{m=0}^{n-1} \Delta t_m^h$. Define

$$F^h(z) = \sup_{(\zeta^h,\varphi^h)\in\mathcal{A}^h(z)} \mathrm{E}\left[\sum_{n=0}^{\infty} e^{-\hat{\beta} t_n^h} (1+\zeta_n^h)^\gamma \Delta t_n^h \,\middle|\, \xi_0^h = z \right], \quad z \in \mathcal{R}^h. \tag{3.3}$$

The dynamic programming equation for the discrete-time Markov chain control problem (3.3) is

$$F^h(z) = \sup_{-1\le\zeta\le K_\zeta z, |\varphi|\le K_\varphi z} \left\{ \Delta t^h(z)(1+\zeta)^\gamma + e^{-\hat{\beta}\Delta t^h(z)} \sum_{z'\in\mathcal{R}^h} p^h(z,z' \mid \zeta,\varphi) F^h(z') \right\}. \tag{3.4}$$

We note that the Markov chain ξ^h is *locally consistent* with the process $Z(\cdot)$, since

$$\mathrm{E}\left[\xi_{n+1}^h - \xi_n^h \mid \xi_n^h = z, \zeta_n^h = \zeta, \varphi_n^h = \varphi\right] = \Delta t^h(z)(kz + k_1\varphi - \zeta), \tag{3.5}$$

and

$$\mathrm{Var}\left[\xi_{n+1}^h - \xi_n^h \mid \xi_n^h = z, \zeta_n^h = \zeta, \varphi_n^h = \varphi\right] = \Delta t^h(z)(\sigma^2\varphi^2 + \delta^2(1-\rho^2)z^2) + o(h\Delta t^h(z)), \tag{3.6}$$

for every $z \in \mathbb{R}^h$, $-1 \le \zeta \le K_\zeta z$, and $|\varphi| \le K_\varphi z$. Equation (3.5) says that the expected change in the Markov chain ξ^h divided by the length of the time step is equal to the drift of the process $Z(\cdot)$, cf. (2.9). Similarly, Equation (3.6) says that the variance of the change in ξ^h divided by the length of the time step is approximately equal to the squared volatility of $Z(\cdot)$.

3.2 Convergence of the Numerical Method

We shall argue that the discrete-time value function $F^h(\cdot)$ converges to the continuous-time value function $F(\cdot)$ as $h \to 0$, $K_\zeta, K_\varphi \to \infty$, and $\bar{z} = Ih \to \infty$. The convergence of the Markov chain approximation approach has been proved for various stochastic control problems by, e.g., Kushner and Dupuis (1992, Chap. 14), Fleming and Soner (1993, Chap. IX), Fitzpatrick and Fleming (1991), and Hindy, Huang, and Zhu (1997) using viscosity solution techniques. While these proofs only require that the value function of the continuous-time control problem is a viscosity solution of the associated HJB equation, we know from DFSZ that the value function for the problem (2.8) is actually a classical solution of the HJB equation (2.7). Therefore, convergence will follow immediately from the stability and consistency of the Markov chain approximation approach, which we shall prove below.

Define the operator $\mathcal{T}^h$ on the space $\mathcal{B}$ of real functions on $\mathcal{R}^h$ by

$$\mathcal{T}^h F(z) = e^{-\hat{\beta}\Delta t^h(z)} \sum_{z'\in\mathcal{R}^h} p^h(z,z' \mid \zeta,\varphi) F(z').$$

Obviously,

$$\begin{aligned} \left|\mathcal{T}^h F_1(z) - \mathcal{T}^h F_2(z)\right| &\le e^{-\hat{\beta}\Delta t^h(z)} \sum_{z'\in\mathcal{R}^h} p^h(z,z' \mid \zeta,\varphi) \left|F_1(z') - F_2(z')\right| \\ &\le e^{-\hat{\beta}\Delta_h} \max_{z'\in\mathcal{R}^h} \left|F_1(z') - F_2(z')\right|, \end{aligned}$$

where $\Delta_h = \min_{z \in \mathcal{R}^h} \Delta t^h(z) > 0$. Hence, $\mathcal{T}^h$ is a strict contraction in terms of the operator norm $\|\mathcal{T}\| = \sup_F \|\mathcal{T}F\|_\infty / \|F\|_\infty$, where $\|\cdot\|_\infty$ is the sup-norm on $\mathcal{B}$. This implies the stability of the method.

Note that with the transition probabilities given by (3.2), we have

$$\sum_{z' \in \mathcal{R}^h} p^h(z, z' \mid \zeta, \varphi) F^h(z') = \Delta t^h(z) \mathcal{L}^h_{\zeta,\varphi} F^h(z) + F^h(z),$$

where the operator $\mathcal{L}^h_{\zeta,\varphi}$ is given by

$$\begin{aligned}\mathcal{L}^h_{\zeta,\varphi} F^h(z) = \frac{1}{2}\left(\delta^2(1-\rho^2)z^2 + \sigma^2\varphi^2\right) D^2 F^h(z) \\ + \left(k^+ z + (k_1\varphi)^+ + \zeta^-\right) D^+ F^h(z) - \left(k^- z + (k_1\varphi)^- + \zeta^+\right) D^- F^h(z),\end{aligned}$$

and

$$D^+ F^h(z) = \frac{F^h(z+h) - F^h(z)}{h}, \qquad D^- F^h(z) = \frac{F^h(z) - F^h(z-h)}{h},$$

and

$$D^2 F^h(z) = \frac{F^h(z+h) - 2F^h(z) + F^h(z-h)}{h^2}.$$

Hence, the discrete dynamic programming equation (3.4) can be rewritten as

$$\frac{1 - e^{-\hat{\beta}\Delta t^h(z)}}{\Delta t^h(z)} F^h(z) = \sup_{-1 \le \zeta \le K_\zeta z, |\varphi| \le K_\varphi z} \left\{ (1+\zeta)^\gamma + e^{-\hat{\beta}\Delta t^h(z)} \mathcal{L}^h_{\zeta,\varphi} F^h(z) \right\}. \tag{3.7}$$

Comparing (3.7) with the HJB equation (2.10), we see that consistency of the method follows from the facts that $\left(1 - e^{-\hat{\beta}\Delta t^h(z)}\right)/\Delta t^h(z) \to \hat{\beta}$, $e^{-\hat{\beta}\Delta t^h(z)} \to 1$, and $\mathcal{L}^h_{\zeta,\varphi} \to \mathcal{L}_{\zeta,\varphi}$ as $h \to 0$ and $K_\zeta, K_\varphi, \bar{z} \to \infty$. Note the link between the consistency of the method and the local consistency property of the approximating Markov chain.

The optimal control policies can be expressed in terms of the value function and it's derivatives (for the continuous-time, continuous-state problem) or it's differences (for the approximating problem). Given the convergence of the value function and the convergence of finite differences to derivatives, it follows that the optimal policies for the approximating Markov control problem converges to the optimal control policies of the continuous-time, continuous-state problem. For details, the reader is referred to Hindy, Huang, and Zhu (1997, Thm. 5) and Fitzpatrick and Fleming (1991, Sec. 4).

As noted by, e.g., Fleming and Soner (1993, Chap. IX) and Hindy, Huang, and Zhu (1997), the specification of the Markov chain at the artificial upper bound is not important for the convergence of the method, but will, of course, affect the quality of the numerical results. Also note the importance of the reduction of the problem form a two-variable to a one-variable problem. Due to the control-dependence of the variance-covariance matrix of the original two-dimensional state-variable (X, Y), it is not possible to approximate it by a locally consistent Markov chain.

3.3 Solving the Approximating Markov Chain Control Problem

The DPE (3.4) is solved with the *policy iteration* algorithm. Given an arbitrary admissible control policy (ζ_0^h, φ_0^h), i.e. given $(\zeta_0^h(z), \varphi_0^h(z))$ for all $z \in \mathcal{R}^h$, perform a *policy evaluation* by solving the linear system of equations

$$F_0^h(z) = \Delta t^h(z)(1+\zeta_0^h(z))^\gamma + e^{-\hat{\beta}\Delta t^h(z)} \sum_{z' \in \mathcal{R}^h} p^h(z, z' \mid \zeta_0^h(z), \varphi_0^h(z)) F_0^h(z'), \quad z \in \mathcal{R}^h. \tag{3.8}$$

Next, a *policy improvement* is computed by

$$(\zeta_1^h(z), \varphi_1^h(z)) = \operatorname*{arg\,max}_{-1 \le \zeta \le K_\zeta z, |\varphi| \le K_\varphi z} \left\{ \Delta t^h(z)(1+\zeta)^\gamma + e^{-\hat{\beta}\Delta t^h(z)} \sum_{z' \in \mathcal{R}^h} p^h(z, z' \mid \zeta, \varphi) F_0^h(z') \right\}.$$

Then a new guess F_1^h on the value function is computed on the basis of (ζ_1^h, φ_1^h) similarly to the computation of F_0^h, etc. Note that each policy iteration step always generates an improvement, so that F_m^h converges to F^h from below. A simple criterion for stopping the algorithm is to stop the first time $\sup_{z \in \mathcal{R}^h} |F_m^h(z) - F_{m-1}^h(z)| < \varepsilon$ for some tolerance $\varepsilon > 0$.

Since the transitions of the approximating Markov chain are to only "nearest neighbors", the equation system (3.8) has a tri-diagonal matrix structure, which makes it very fast to solve. For interior states z, the j'th policy improvement step amounts to computing

$$\zeta_j^h(z) = \operatorname*{arg\,max}_{-1 \le \zeta \le K_\zeta z} \left\{ (1+\zeta)^\gamma + e^{-\hat{\beta}\Delta t^h(z)} \left(-\zeta^+ D^- F_{j-1}^h(z) + \zeta^- D^+ F_{j-1}^h(z) \right) \right\}$$

and

$$\varphi_j^h(z) = \operatorname*{arg\,max}_{|\varphi| \le K_\varphi z} \left\{ \frac{1}{2}\sigma^2 \varphi^2 D^2 F_{j-1}^h(z) - (k_1\varphi)^- D^- F_{j-1}^h(z) + (k_1\varphi)^+ D^+ F_{j-1}^h(z) \right\}.$$

At the upper bound $\bar{z}$, the corresponding equations are

$$\zeta_j^h(\bar{z}) = \operatorname*{arg\,max}_{-1 \le \zeta \le K_\zeta \bar{z}} \left\{ (1+\zeta)^\gamma - e^{-\hat{\beta}\Delta t^h(\bar{z})} \zeta^+ D^- F_{j-1}^h(\bar{z}) \right\}$$

and

$$\varphi_j^h(\bar{z}) = \operatorname*{arg\,max}_{|\varphi| \le K_\varphi \bar{z}} \left\{ -D^- F_{j-1}^h(\bar{z}) \left(\frac{1}{2}\sigma^2\varphi^2 + h(k_1\varphi)^- \right) \right\},$$

which implies $\varphi_j^h(\bar{z}) = 0$ as $D^- F_{j-1}^h(\bar{z}) \ge 0$.

Convergence of the policy space algorithm will, other things equal, be faster, the faster the probability mass "spreads". A locally consistent Markov chain with $p^h(z, z \mid \zeta, \varphi) = 0$ is obtained by replacing the denominator $Q^h(z)$ of $p^h(z, z \pm h \mid \zeta, \varphi)$ with the control-dependent denominator

$$Q^h(z, \zeta, \varphi) = \sigma^2\varphi^2 + \delta^2(1-\rho^2)z^2 + h(|k|z + |k_1\varphi| + |\zeta|).$$

This procedure does not require bounding of the controls, but the policy improvement step is now significantly more complicated, since the controls then enter both the numerator and the denominator of the probabilities. Therefore, we prefer the scheme with bounded controls.[7]

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[7] Note that $Q^h(z) = \sup\{Q^h(z, \zeta, \varphi) : -1 \le \zeta \le K_\zeta z, |\varphi| \le K_\varphi z\}$. Fitzpatrick and Fleming (1991) also impose

4 Numerical Results

In this section, we present and discuss results from an implementation of the numerical method outlined in Section 3. The basic economic parameter values are taken to be

$$\begin{array}{lllll} \text{(4.1)} & \gamma = 0.5 & r = 0.1 & b = 0.15 & \mu = 0.05 \\ & \beta = 0.2 & \sigma = 0.3 & \delta = 0.1 & \rho = 0.0 \end{array}$$

unless otherwise indicated. The auxiliary parameters are then

$$\hat{\beta} = 0.17625, \quad k_1 = 0.05, \quad k_2 = 0.055, \quad k = 0.055, \quad A \approx 0.27222.$$

K_ζ and K_φ must be determined experimentally and are, of course, dependent on the other parameters. They are chosen as small as possible under the condition that their values do not affect the optimal controls. Unnecessarily high values slow down the method. We found that $K_\zeta = K_\varphi = 3$ is an acceptable compromise, unless when the focus is on wealth/income ratios very close to zero, where the constants have to be higher. The value to be chosen for $\bar{z}$, the artificial upper bound on z, depends on the objective. If one's primary interest is small values of z, then a relatively small $\bar{z}$ is justifiable, whereas a high $\bar{z}$, of course, is required, if one wishes to study the solution for high wealth/income ratios.

4.1 Properties of the Numerical Method

The Markov chain approximation approach does not provide any measures for the precision of the numerical results. The reduced problem has a structure similar to Merton's no-income consumption/investment problem for which the solution is as shown in Section 2.2. An indication of the numerical properties of our proposed scheme can, therefore, be obtained by studying the performance of the Markov chain approximation approach on Merton's problem. Such an analysis is contained in Munk (1997). The numerically computed value function was found to be very precise over the entire range of the wealth level, which is the sole state variable of that problem. The numerically computed optimal controls were rather imprecise near the artificially imposed upper bound on the state variable, but otherwise also very precise. For our problem, there is also no reason to believe that the numerically computed optimal controls at the upper bound are anywhere near the true optimal controls. The error introduced this way will propagate to lower values of the

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bounds on the controls to avoid control dependent denominators. Their scheme corresponds to replacing $Q^h(z,\zeta,\varphi)$ with a constant Q^h given by $Q^h = \sup\{Q^h(z,\zeta,\varphi) : z \in \mathcal{R}^h, -1 \leq \zeta \leq K, -K \leq \varphi \leq K\}$ for some constant K. Of course, this will increase the probability of staying at a state, which tends to slow down convergence. Since the advantages of having a constant Q^h and, hence, a constant Δt^h, are very limited, the $Q^h(z)$ probability scheme is adopted here.

state variable. The range of states in which the numerically computed controls can be considered reliable is highly dependent on the value of the contraction parameter of the DPE, which in our case is the parameter $\hat{\beta}$. The higher the contraction parameter, the wider the reliable range.

For practical applications, the particular values of the controls might not be that important, as long as they induce a utility level very close to the true, but unknown value function. But since we want to study the economic properties of the optimal control policies, we cannot ignore this feature of the numerical scheme. Therefore, the numerical results discussed below do not rely on values near the artificial upper bound on z.[8]

The numerical values presented below are computed with a grid refinement of $I = 10000$ (the corresponding h-values depend, of course, on $\bar{z}$). The policy iteration algorithm is implemented with tolerance $\varepsilon = 10^{-5}$. With these specifications, running the computer program on an HP9000/E35 128MB computer takes about 5–20 seconds, depending on $\bar{z}$ and the parameter values.[9]

Our numerical results confirm the analytical findings of DFSZ concerning the behavior of $F(\cdot)$ near zero and that the value function and optimal policies of the unreduced problem asymptotically are equal to the value function and optimal policies of Merton's no-income problem as the ratio of wealth to income goes to infinity.[10]

4.2 The Computed Value Function and Optimal Controls

Next, we present the computed value function and optimal controls for non-extreme values of the wealth/income ratio. Results discussed in this subsection are obtained from an implementation with $\bar{z} = 100$. Figure 1 shows the value function and Figures 2 and 3 depict the optimal controls. All three surfaces are smooth and increasing in both coordinate directions as one would expect

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[8] A standard way to assess the speed and smoothness of the convergence of a numerical method is to look at the so-called experimental order of convergence, which can be estimated by examining the variations in the numerically computed quantity, when the grid refinement h (or, equivalently, I) is varied. With the default parameter values in (4.1), the experimental order of convergence is typically very unstable as the grid refinement h is varied. For higher values of the contraction parameter $\hat{\beta}$, the convergence is much smoother, and the convergence order is roughly one for both the value function and the controls. (For high levels of z relative to $\bar{z}$, the value function converges nicely, but for both controls, most pronouncedly for φ, the computed experimental order indicates a lack of convergence.) With such a smooth convergence, the Richardson extrapolation technique can be succesfully superimposed, but since our focus is on lower and economically more appealing β-values, we cannot benefit from extrapolation methods.

[9] The main determinant of the running time is the number of iterations the policy iteration algorithm requires for convergence. With the stated value of ε, the policy space iteration algorithm typically converges in 5–10 iterations. A smaller ε will result in additional time-consuming and essentially futile iterations.

[10] Details can be obtained from the author.

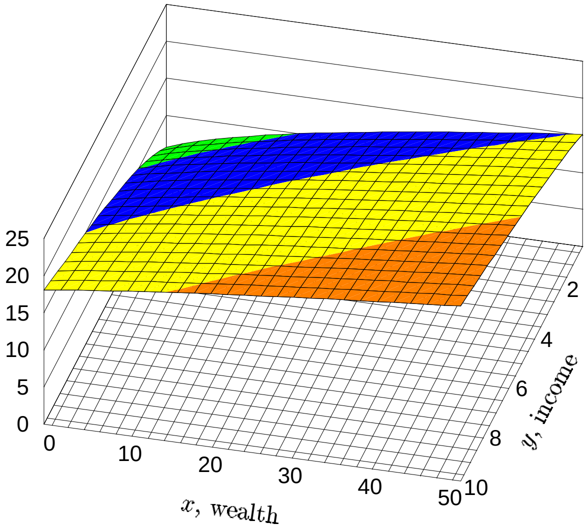

Figure 1: The value function, $v(x, y)$.

them to be.

Recall that in Merton's no-income problem, both the optimal consumption rate and the optimal risky investment are constant fractions of wealth. With undiversifiable income risk and borrowing constraints, these two fractions depend on wealth and income. For relatively high levels of wealth, the fractions of consumption and risky investment to wealth are nearly constant across income rates, but they are rapidly increasing with the initial income rate for small levels of wealth. When measured relative to the initial income rate, both controls are rather insensitive to the level of wealth as long as income is large relative to wealth, but for small levels of the income rate both the consumption/income ratio and the investment/income ratio increase rapidly as wealth increases.

4.3 The Implicit Value of an Uncertain Income Stream

In this subsection, we consider the implicit value that the investor associates with her stochastic income stream given the initial income rate. As discussed in Section 2.3, we can measure this value by $B(z)y$, where $B(z)$ is given by (2.11), or alternatively by $B^*(z)y$, where $B^*(z)$ is given by (2.13). Figure 4 shows the two multipliers $B(z)$ and $B^*(z)$ of the implicit value of the income stream as a function of z. The multipliers are strictly increasing in z, so the value associated with a given uncertain stream of income increases with wealth. Intuitively, this is because the importance of the borrowing constraint, i.e. the non-negative wealth constraint, decreases as wealth increases. Recall that for the complete markets case the multiplier is $1/\lambda$. With the parameters in (4.1), $1/\lambda = 15$, so, obviously, the implicit value attached to the income stream is much smaller in the presence of borrowing constraints and non-spanning. Since the magnitude of $B(z)$ and $B^*(z)$ is

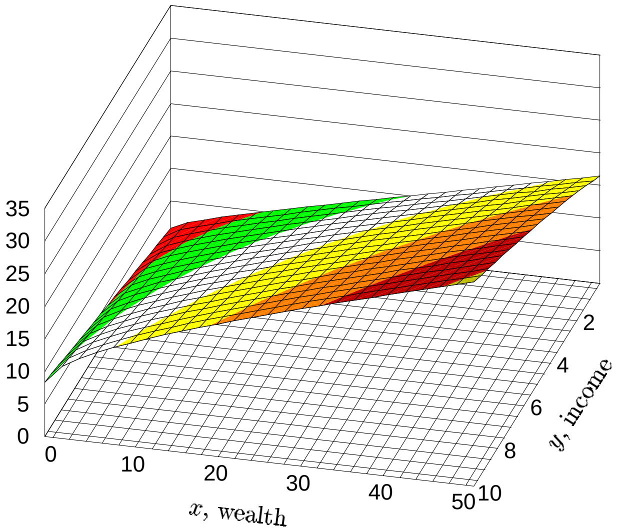

Figure 2: The optimal consumption rate, $C(x, y)$.

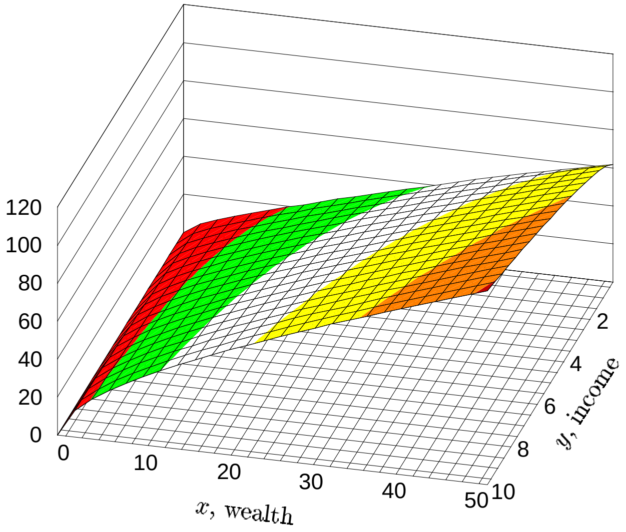

Figure 3: The amount optimally invested in the risky asset, $\Pi(x, y)$.

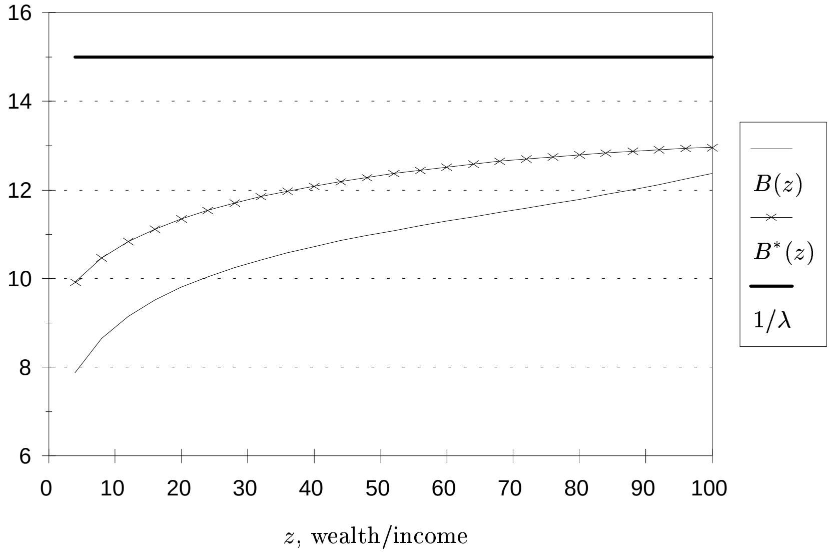

Figure 4: Convergence of the implicit value of income multipliers to the complete markets income multiplier. The results are from an implementation with $\bar{z} = 200$.

relatively insensitive to the value of the correlation parameter ρ, as will be discussed in the next subsection, the large difference between the complete and the incomplete market income valuation must be imputed mainly to the borrowing constraints.

5 Parameter Sensitivity

In this subsection, we examine the sensitivity of results to the income process parameters ρ, δ, and μ, and to the time preference rate β.

5.1 The Correlation Parameter, ρ

First, we focus on ρ, the correlation between changes in the price of the risky asset and changes in the income rate. Figure 5 shows the ratio of optimal risky investment to wealth, $\Pi(x,y)/x$, as a function of ρ for four different values of $z = x/y$.[11] This ratio, the portfolio weight of the risky asset, is obviously an almost linearly decreasing function of ρ for all four values of z. Intuitively, this is because a negative correlation between changes in the income rate and changes in the risky asset price provides insurance against negative wealth: If the risky asset price and therefore the value of the investment decreases, the income rate will increase and *vice versa*. For high values of z, the portfolio weight is nearly constant, whereas for low z, it is steeply decreasing in ρ. The intuition for this property is straightforward: When the financial wealth is small relative to the

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[11] Since $\varphi(z) = \Pi(x,y)/y - \delta\rho z/\sigma$, we have $\Pi(x,y)/x = \varphi(z)/z + \delta\rho/\sigma$, where, as usual, $z = x/y$.

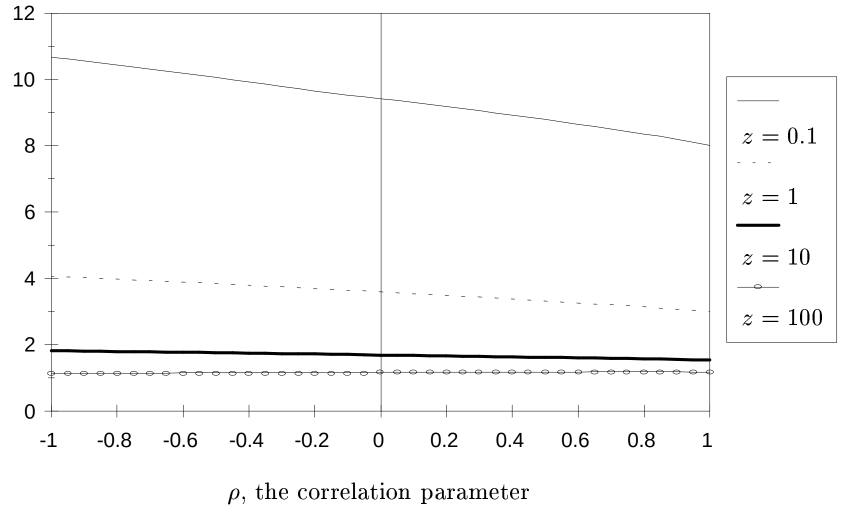

Figure 5: The dependence of the relative optimal risky investment $\Pi(x, y)/x$ on the correlation parameter ρ. The results are from an implementation with $\bar{z} = 200$.

income rate, it is very important to hedge against future changes in the income rate, and therefore the portfolio weight is more sensitive to the correlation parameter.

Figure 6 shows the ratio of the optimal consumption rate to wealth, $C(x, y)/x$, as a function of ρ.[12] The ratio is increasing in ρ, but, again, nearly constant for high wealth/income ratios. The value function is decreasing in ρ, although nearly constant.[13]

As can be seen from Figure 7, the income multiplier $B^*(z)$ decreases almost linearly in ρ. For low and moderate wealth/income ratios, the income multiplier is very insensitive to ρ, and hence the implicit value of the uncertain income stream is substantially lower than in the complete markets case for any value of ρ. For high wealth/income ratios, the implicit value of the income stream decreases significantly with ρ.

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[12] Since $\zeta(z) = C(x, y)/y - 1$, we have $C(x, y)/x = (1 + \zeta(z))/z$.

[13] Any linearly increasing transformation of the utility function of consumption will give the same optimal policies, but may change the resulting value function drastically. Therefore, one should generally avoid assessing the magnitude of changes in the value function. Instead, one can translate the value function into a measure, which is independent of the specific representation of preferences. One such measure is the *constant consumption equivalent* $\tilde{C}(x, y)$ defined as

$$\tilde{C}(x, y) = \inf \left\{ c \geq 0 \; : \; \int_0^\infty e^{-\beta t} u(c) \, dt = v(x, y) \right\}.$$

Whenever we address changes in the value function in the text, it is to be understood as changes in the constant consumption equivalent.

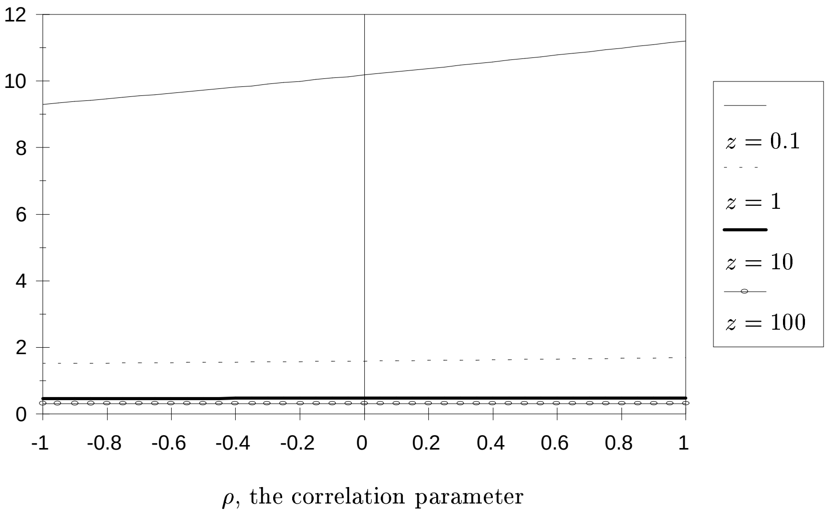

Figure 6: The dependence of the relative optimal consumption rate $C(x, y)/x$ on the correlation parameter ρ. The results are from an implementation with $\bar{z} = 200$.

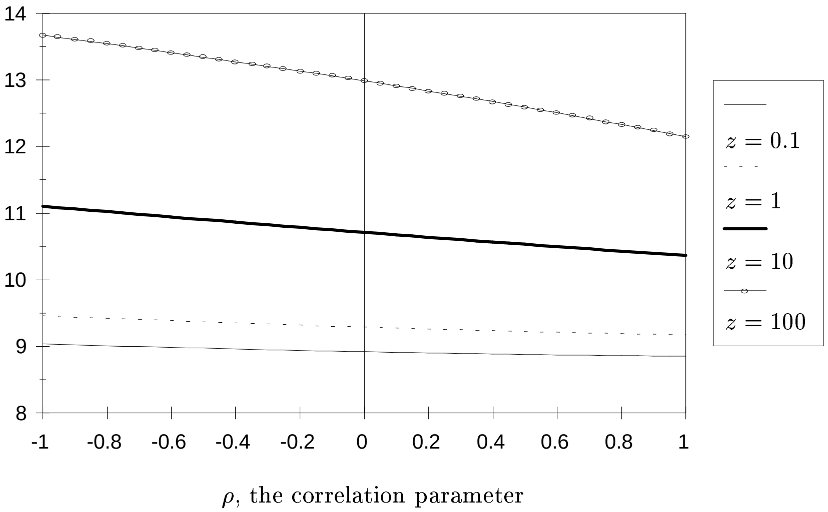

Figure 7: The dependence of the implicit value of income multiplier $B^*(z)$ on the correlation parameter ρ. The results are from an implementation with $\bar{z} = 200$.

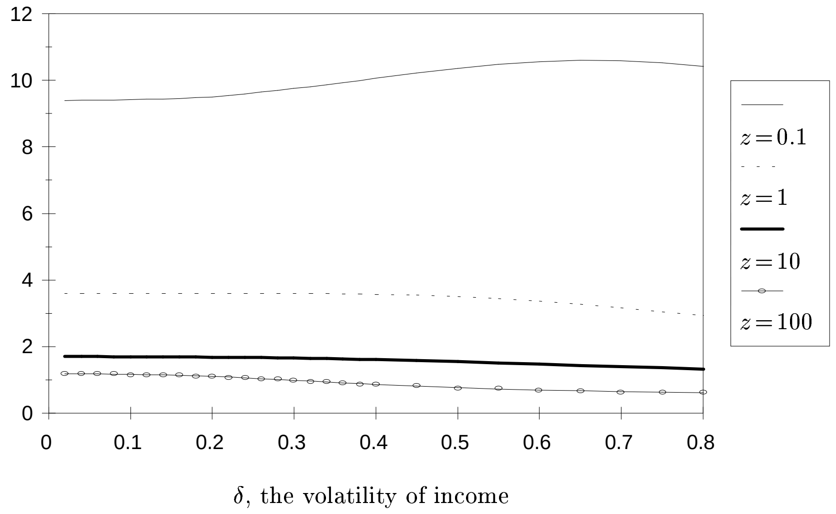

Figure 8: The dependence of the relative optimal risky investment $\Pi(x, y)/x$ on the income volatility parameter δ. The results are from an implementation with $\bar{z} = 200$.

5.2 The Income Rate Volatility, δ

Next, we study the sensitivity of results to δ, the volatility of the income rate. From Figure 8, we see that the risky asset portfolio weight is an increasing function of δ for small wealth/income ratios, but decreasing for large wealth/income ratios. Similarly, Figure 9 shows that the ratio of optimal consumption to initial wealth is a decreasing function of δ for small values of the wealth/income fraction and an increasing function of δ for very high wealth/income ratios. The intuition is that for small wealth relative to income, a higher income volatility will induce the agent to consume less and invest more to protect herself against a decline in the income rate and possibly a negative future wealth. The value function is a decreasing, concave function of δ. However, the changes of both the controls and the value function are small, when δ is varied from 0 up to about 0.3. Hence, for investors with rather steady income, the precise value of δ is not that important. The income multipliers $B(z)$ and $B^*(z)$ decrease as δ increases, so – as in the complete markets case – the value of an uncertain income stream decreases with the income rate volatility.

5.3 The Income Rate Drift, μ

Here we examine the influence of μ, the drift of the income rate, on the results. Figure 10 unveils that the risky asset weight is a decreasing function of μ for small values of the wealth/income ratios, but an increasing function of μ for large wealth/income ratios. For intermediate wealth/income ratios and relatively small μ, the optimal risky investment is almost constant. The optimal consump-

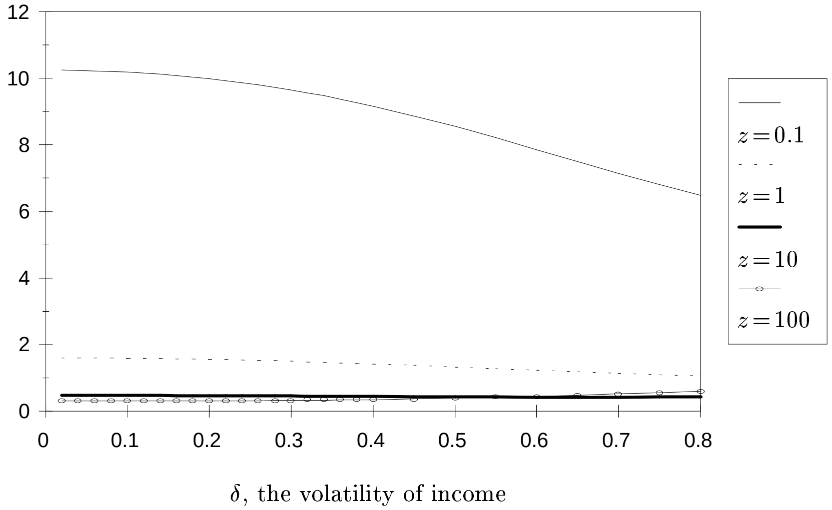

Figure 9: The dependence of the relative optimal consumption rate $C(x, y)/x$ on the income volatility parameter δ. The results are from an implementation with $\bar{z} = 200$.

tion rate is an increasing function of μ for all levels of the wealth/income ratio, as can be seen from Figure 11. The higher the wealth/income ratio, the flatter the curve. For low wealth/income ratios, the consumption policy is highly limited due to the non-negativity constraint on the liquid wealth. With a higher μ, a significantly higher consumption rate is possible without leading the investor into bankruptcy. As expected, the value function increases convexely with μ. The income multiplier $B(z)$ increases significantly, also in a convex manner, with μ.

5.4 The Time Preference Rate, β

The value of the time preference rate, β, has a great impact on the optimal policies and the implicit value of the income stream. With a high time preference rate, the investor is very eager to transform some of the future income to current consumption. Due to the borrowing constraints, this is not possible, and therefore the implicit value of the income stream is much smaller than in the unconstrained case. Figure 12 depicts the value of the income multiplier $B(z)$ for different values of β. For relatively high values of β, the income multiplier is much smaller than in the complete markets case, even for very high values of the wealth/income ratio z.

6 Concluding Remarks

We have numerically solved the optimal consumption/investment problem of a power utility investor endowed with some initial wealth and an uncertain non-spanned stream of income, when

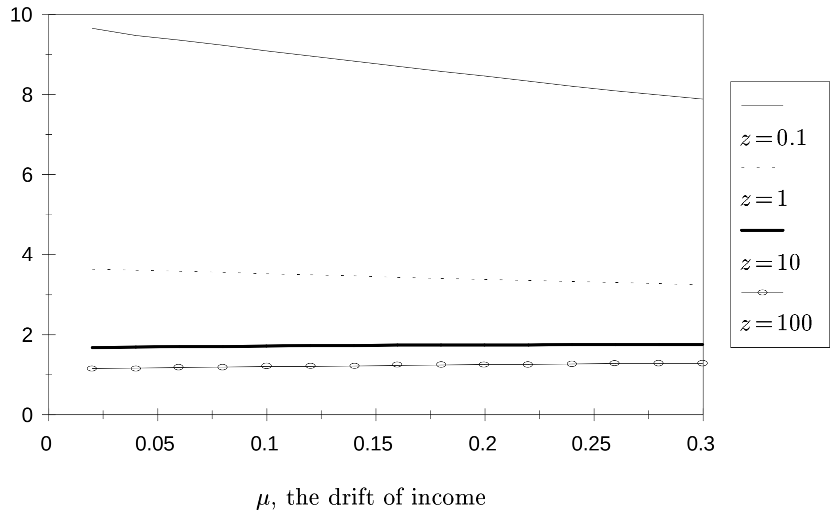

Figure 10: The dependence of the relative optimal risky investment $\Pi(x,y)/x$ on the income drift parameter μ. The results are from an implementation with $\bar{z} = 200$.

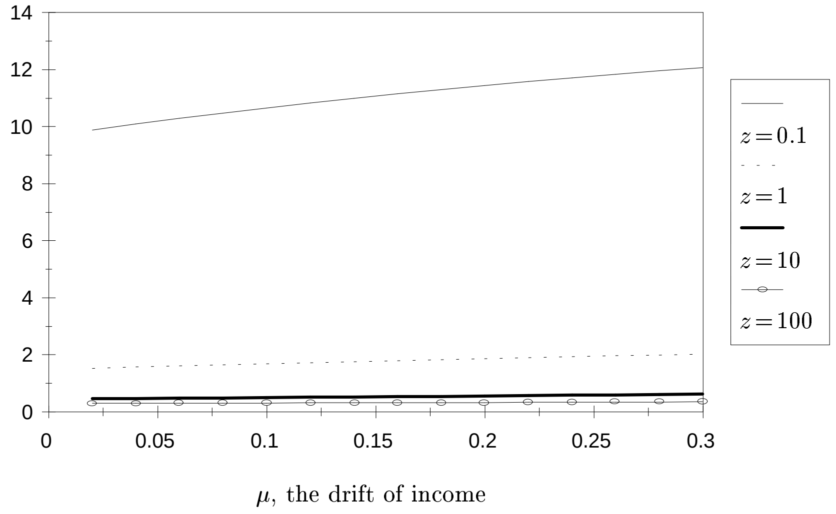

Figure 11: The dependence of the relative optimal consumption rate $C(x,y)/x$ on the income drift parameter μ. The results are from an implementation with $\bar{z} = 200$.

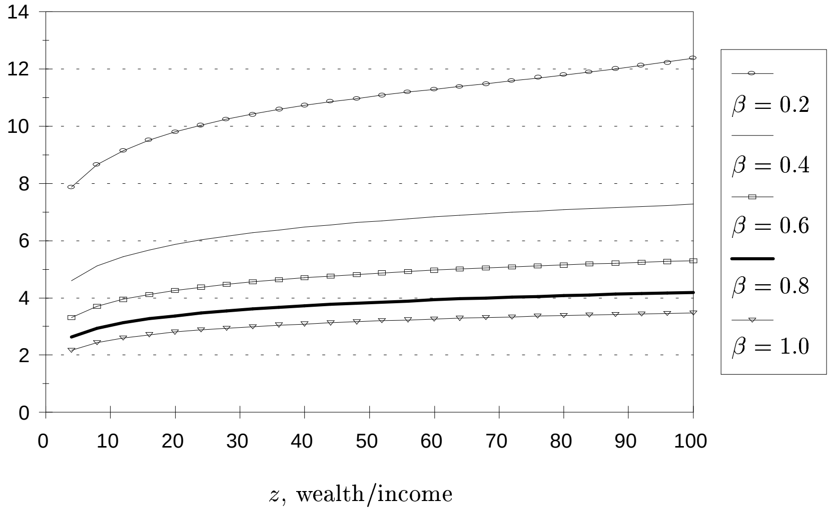

Figure 12: The implicit value of income multipliers for different values of the time preference rate, β. The results are from an implementation with $\bar{z} = 200$.

the investor is not allowed to borrow against future income. We have found that only when initial wealth is very, very large compared to the initial income rate, is it justifiable to neglect the presence of the income stream. Furthermore, the implicit value the investor associates with the entire income process is much smaller in the presence of borrowing constraints than without such restrictions. Other things equal, borrowing constraints are most restrictive for agents with a low financial wealth relative to income, an income rate positively correlated with changes in the risky asset price, and a high time preference for consumption.

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