The Random–Time Binomial Model

Dietmar P.J. Leisen

Stanford University*

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Abstract

In this paper we study a binomial model with random time steps and explain how to calculate values for European and American call and put options. We prove weak convergence of the discrete processes to the Black–Scholes setup as well as convergence of the values for European *and* American put options. Computational experiments exhibit a smooth convergence structure and suggest that we can obtain an order of convergence of two via an extrapolation procedure. Approximations to jump–diffusions are straightforward.

Keywords

binomial model, option valuation, lattice–approach

JEL Classification

C63, G13

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* Stanford University, Hoover Institution, Stanford, CA 94305, U.S.A., e–mail: leisen@hoover.stanford.edu

1 Introduction

In a continuous setup where the evolution of a single stock is modelled by geometric Brownian motion, Black and Scholes (1973) derived a closed–form solution for the value of the European–style call and put option by presenting a strategy that duplicates its payoff through continuous trading in the stock and one bond. Later Harrison and Kreps (1979) and Harrison and Pliska (1981) developed the equivalent martingale measure concept, which gives an elegant technique to express and solve pricing problems in terms of expected discounted payoffs.

This paper addresses the pricing of American put options in the Black–Scholes setup. No closed–form solution is known to this problem, and prices need to be calculated numerically. The main financial tools for this purpose are binomial models, partial differential equations and the Monte–Carlo technique. Other approaches have been adopted from the literature on numerical dynamic programming; e.g., Carr and Faguet (1996) and Meyer and van der Hoek (1997) using Rothe's method of lines.

Cox, Ross, and Rubinstein (1979) (henceforth CRR) and Rendleman and Bartter (1979) independently presented the binomial model, which is a discrete process approximation of the original Black–Scholes framework. Binomial models are an easy way to explain how continuous trading takes place and how infinitely many states are spanned. Furthermore, binomial models can yield simple approximations for option values where no closed form solution is available as, for example, for the American put option. Binomial models were generalized by He (1990) to a basket of lognormally distributed assets and Nelson and Ramaswamy (1990) to general diffusion processes [1].

Monte–Carlo methods were introduced as a finance pricing tool by Boyle (1976). They rely on the equivalent martingale measure technique which calculates prices as expected discounted payoffs. After discretizing the time axis by a refinement n, a sequence of m price path trajectories is simulated according to the risk–neutral dynamics and the corresponding mean and variance of the payoff is calculated. According to the Law of Large Numbers the empirical mean converges to the true price for European–style options. The empirical variance gives an estimation of how close the mean is to the correct price in a model with refinement n of the time–axis. A great advantage of the Monte–Carlo approach is its being readily applicable to any pricing problem. For an overview on the state–of–the–art of Monte–Carlo techniques in option pricing,

[1] There also exists a different interpretation for binomial models: Black and Scholes (1973) noted that financial derivatives must fulfil a specific partial differential equation. Discretizing this by explicit finite differences is equivalent to the CRR model.

see Boyle, Broadie, and Glasserman (1997).

Monte–Carlo techniques are suggested as a solution to the "curse of dimensionality." In binomial models with a fixed refinement the computational cost exponentially increases in the number of assets, but it is independent for Monte–Carlo techniques. For American put options the exercise decision and the price are upward biased; the downward biased estimator constructed recently by Broadie and Glasserman (1997) now allows valuing these options properly, too. Their approach is closely related to the "random multigrid" of Rust (1997) for solving dynamic programming problems. Rust (1997) analyzed the complexity of his approach and proved that it is successful in breaking the "curse of dimensionality" for discrete decision processes.

This paper focuses on the one–dimensional Black Scholes setup, where the "curse of dimensionality" does not apply. Improvements of the original CRR approach have been suggested by several authors: Jarrow and Rudd (1983), Boyle (1988) and Tian (1993). Leisen and Reimer (1996) proved that the order of convergence in pricing European options for these variations is equal to one, which is also the order of convergence of CRR; thus the many improvements on the CRR model are all equivalent. Furthermore the convergence structure exhibits waves and is not monotonic. This is unsatisfactory since effective use of extrapolation would require a convergence structure as smooth as possible. Whereas this problem can be addressed in the case of European put options, no adjustments are known for the early–exercise premium of American put options.

The idea of approximating the Black–Scholes setup by a binomial model with random time–steps appeared in Föllmer and Sondermann (1986) and Sondermann (1987) in the form of a two–sided compound jump–process. Binomial models with random time–steps only recently have been brought back to consideration by Dengler and Jarrow (1997) and Rogers and Stapleton (1998). The latter studied a model in the Black–Scholes framework where the hedging portfolio is adjusted when certain prespecified barrier lines are reached. Although this is a clever way to value barrier options, it faces the drawback that the distribution of the number of jumps needs to be approximated.

Our random–time binomial model assumes that the difference between two trading dates is random. The main contribution of this paper is that such a randomized model smoothes the convergence structure and results in quadratic convergence order by extrapolation. We do not present a formal proof. However this is suggested by a detailed analysis of the deficiencies as well as simulations. This makes our approach a competitive valuation tool. We suppose that a Poisson process is driving the jumps; i.e., the time increment is exponentially distributed. We also present an easy valuation formula for European style options. In a further step we extend it to the valuation of American put options.

Related to our approach is Carr (1997), using exponentially distributed random variables to study contracts with random maturity.

A second contribution is that our model can be used in a natural way to construct approximations for jump diffusions. Such a model has been suggested as a response to the observation that market participants are well aware of sudden strong price changes ("crashes"). In fact, the assumption of continuous sample paths has also been criticized in empirical studies (see, for example, Jarrow and Rosenfeld (1984), Ball and Torous (1985) and Jorion (1988)). Here, we adopt the framework of Merton (1976) who superimposed a compound Poisson process on the Black–Scholes setup. Whereas geometric Brownian motion describes the arrival of "normal" information, the jump part models large (discontinuous) price changes due to the arrival of rare "information shocks." A discrete framework for the valuation of options in this setup was given by Amin (1993). The jump part was simply put on top of the binomial model. In contrast it is merely an adjustment to the intensity of the driving process in our model.

The remainder of the paper is organized as follows. In section 2 we review the Black–Scholes setup and the binomial model. Section 3 discusses numerical issues related to the structure of convergence. Section 4 presents our model as well as necessary and sufficient conditions for weak convergence to geometric Brownian motion. Section 5 discusses the valuation of European and American options. Section 6 discusses jump diffusions. Section 7 concludes the paper. Throughout the paper, all figures related to practical implementations use the same parameter selection of a put option with strike 110 and one year maturity, written on a stock with today's price equal to 100 and whose volatility is 0.3 when the short–term interest rate is equal to 0.1. All proofs are postponed to the appendix.

2 Binomial Models

We suppose that the stock price process can be described under the objective measure, as in Black and Scholes (1973), by

$$S_t = S_0 \cdot \exp\left\{\mu t + \sigma W_t\right\}, \tag{1}$$

where the *drift* μ and the interest rate r, as well as the *volatility* σ, are supposed to be constant. $(W_t)_t$ is a standard Wiener process on a suitable probability space $(\Omega, \mathcal{F}, Q)$. It can be seen that $E[S_t] = S_0 \cdot \exp\{\mu t + (\sigma^2/2)t\}$ for all $t \geq 0$. We assume that $\mu := r - \frac{\sigma^2}{2}$ under the measure Q, which is then the risk–neutral measure. According to Harrison and Pliska (1981), option prices are expectations w.r.t. Q.

A European call option gives its holder the right to buy the stock at some date T for a price K. If the holder rationally decides to exercise the option — i.e. if $S_T \geq K$ — by selling the stock immediately after exercising the option, he incurs a profit of $S_T - K$. The payoff is therefore determined by the function $f : x \mapsto (x - K)^+$. Similarly a European put option gives the owner the right to sell and is described by $f : x \mapsto (K - x)^+$. Black and Scholes (1973) were the first to present price formulae for these options:

$$
\begin{aligned}
C^e(S,T,K,\sigma^2,r) &= S \cdot \mathcal{N}(d_1) - K \cdot e^{-rT}\mathcal{N}(d_2) && \text{for a call,} \qquad (2)\\
P^e(S,T,K,\sigma^2,r) &= K \cdot e^{-rT}\mathcal{N}(-d_2) - S \cdot \mathcal{N}(-d_1) && \text{for a put,} \qquad (3)\\
\text{where} \quad d_{1,2} &= \frac{\ln(S/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}
\end{aligned}
$$

and $\mathcal{N}(\cdot)$ is the cumulative standard normal distribution function.

An American option gives its holder the right to exercise the claim at any date up to maturity T. Merton (1973) pointed out that an American call option on a dividend–free stock should never be exercised prior to maturity, and so the price is equal to its European counterpart. This is no longer true for an American put option, whose price is

$$
P^a = \sup_{\sigma \in \mathcal{S}} E\left[e^{-r\sigma}\left(K - S_\sigma\right)^+\right], \tag{4}
$$

where $\mathcal{S}$ denotes the set of stopping times ("policies") smaller than maturity T and adapted to the filtration generated by the stock process $(S_t)_t$ (see Myneni (1992)). No closed–form solution is known for P^a.

For an American put option with maturity date T, we know from Van Moerbeke (1976) that there is a smooth function $t \mapsto B_t$ separating the time–state space into two regions: the option should be exercised if — and only if — the stock price at time t is below or at B_t. From Carr, Jarrow, and Myneni (1992), we know that the so–called early–exercise premium $\pi = P^a - P^e$ takes the form:

$$
\pi = rK \int_0^T e^{-rt} E[I_{B,t}]dt \ . \tag{5}
$$

Here, for any $t \in [0,T]$, $I_{B,t}$ denotes the cash–or–nothing option with strike B_t and maturity date t; i.e., the option paying one unit at t if the stock price at that date is below or equal to B_t and zero, otherwise.

In the following it will be more convenient to work on the logarithmic stock price process, since it is homoscedastic. Let us define

$$
\begin{aligned}
& X_t = \ln(S_t/S_0) \\
\iff \quad & X_t = \mu t + \sigma W_t. \qquad (6)
\end{aligned}
$$

Now fix two sequences $(\kappa_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, with $\kappa_n = \mathcal{O}(\Delta t_n)$. Then, for any refinement n, a discrete setup is defined as follows. The set $\mathcal{T}^n = \{0 = t_{n,0} < t_{n,1} \ldots < t_{n,n} = T\}$ of trading dates is assumed to be equidistant $t_{n,i+1} - t_{n,i} = \Delta t_n = \frac{T}{n}$.

The per period return in the risk–less bond is $r_n = \exp\{r\Delta t_n\}$; and the per period return in the asset is modeled by the sequence of $(\overline{R}_{n,i})_i$ of i.i.d. random variables

$$
\overline{R}_{n,i} \sim \begin{cases} \kappa_n + v_n & ; q_n \\ \kappa_n - v_n & ; 1 - q_n \end{cases} , \qquad (7)
$$

where q_n will be specified later. Denote for all $0 \le t \le T$:

$$
\begin{aligned}
& \overline{X}_t^{(n)} = \sum_{i=1}^{N_t^{(n)}} \overline{R}_{n,i} \ , \\
\text{and} \quad & \overline{S}_t^{(n)} = S_0 \exp \overline{X}_t^{(n)} \ , \\
\text{where} \quad & N_t^{(n)} = \left\lfloor \frac{t}{\Delta t_n} \right\rfloor \ .
\end{aligned}
$$

The process $(\overline{S}_t^{(n)})_{0\le t\le T}$ is called a *Binomial Model* with refinement n. An ex-

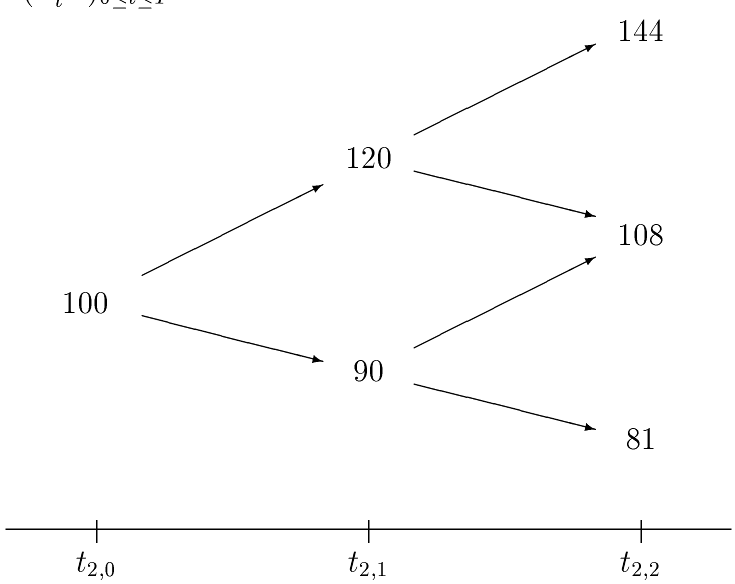

Fig. 1. Example dynamics

ample for possible dynamics is given in figure 1, where we assume a refinement of $n = 2$, today's stock price $S_0 = 100$, and $\kappa_n + v_n = \ln 1.2, \kappa_n - v_n = \ln 0.9$.

We denote by $\stackrel{d}{\Longrightarrow}$ the weak convergence (also called convergence in distribution) for stochastic processes. A necessary condition for $\overline{X}^{(n)} \stackrel{d}{\Longrightarrow} X$ is the convergence of any finite–dimensional distribution at date t. This requires

$$E\left[\frac{\overline{R}_{n,1}}{\Delta t_n}\right] \stackrel{n}{\longrightarrow} \mu \tag{8}$$

$$\frac{\operatorname{Var}\left(\overline{R}_{n,1}\right)}{\Delta t_n} \stackrel{n}{\longrightarrow} \sigma^2 \tag{9}$$

At the beginning of this section, we stated that $\mu t = (r - \frac{\sigma^2}{2})t$ and $\sigma^2 t$ are the expectation and variance of X_t. To price options as expected payoffs, we must set the probability q_n for an upward move as the risk–neutral probability; i.e., for given v_n, κ_n

$$E[\exp\{\overline{R}_{n,1}\}] = r_n = \exp\{r\Delta t_n\} \tag{10}$$

must hold. In the series expansion of the exponential function, it can be checked that, if condition (8) is fulfilled, the martingale measure condition $E[\exp\{\overline{R}_{n,1}\}] = \exp\{r\Delta t_n\}$ is matched up to second–order terms. Yet, for the logarithmic process X, condition (8) corresponds to equation (10). We therefore require the equality $E\left[\overline{R}_{n,1}\right] = \mu\Delta t_n$. It is straightforward to show that

$$q_n = \frac{\mu\Delta t_n - \kappa_n + v_n}{2v_n} \ .$$

Using (8) and the equality $\operatorname{Var}(\overline{R}_{m,1}) = E[(\overline{R}_{m,1})^2] - E[\overline{R}_{m,1}]^2 = v_n^2 - E[\overline{R}_{m,1}]^2$, we see that condition (9) requires

$$\frac{|v_n|}{\sqrt{\Delta t_n}} \stackrel{n}{\longrightarrow} \sigma \ .$$

To fulfil this condition, v_n is chosen equal to $\sigma\sqrt{\Delta t_n}$. The models proposed in the literature differ in the specification of κ_n:

(1) Jarrow and Rudd (1983):
Setting $\forall n : \kappa_n := \left(r - \frac{\sigma^2}{2}\right)\Delta t_n$, we get $q_n = \frac{1}{2}$

(2) Cox, Ross, and Rubinstein (1979):
Setting $\forall n : \kappa_n := 0$, we get $q_n = \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t_n}$

We call the vector $(u_n, d_n, r_n, q_n) = (\exp(\kappa_n + v_n), \exp(\kappa_n - v_n), \exp\{r\Delta t_n\}, q_n)$ the *characteristic terms* of the specific binomial model under consideration. Although r_n and q_n are somewhat redundant, we prefer to include both since this will make the exposition in section 5 easier. We can now state a general convergence

Theorem 1 *For a sequence of binomial models which fulfil conditions (8) and (9), we have*

$$\begin{aligned} \overline{X}^{(n)} &\overset{d}{\Longrightarrow} X \\ \textit{and} \quad \overline{S}^{(n)} &\overset{d}{\Longrightarrow} S \ . \end{aligned}$$

Using theorem 1 we deduce that $\overline{S}^{(n)} \overset{d}{\Longrightarrow} S$ for the models proposed in the literature. Since the payoff function for the European put is bounded, convergence of price follows. Put–call parity then implies the same for European call options. From Lamberton and Pagès (1990) we deduce that American put option prices converge.

The remainder of this section treats pricing in a fixed binomial model with characteristic terms (u_n, d_n, r_n, q_n). If S is the current stock price at date $t_{n,i}$, the European option price can be calculated as discounted expected prices at the next trading dates $t_{n,i+1}$:

$$\begin{aligned} P_n^e(t_{n,i}, S) &= r_n^{-1} E[P_n^e(t_{n,i+1}, S \cdot \exp \overline{R}_{n,i})] \\ &= r_n^{-1} \{q_n P_n^e(t_{n,i+1}, S \cdot u_n) + (1 - q_n) P_n^e(t_{n,i+1}, S \cdot d_n)\} \ . \end{aligned}$$

Since the conditional terminal payoff is known, backward induction yields today's option price.

The algorithm is slightly modified for an American option. We check at each point in time whether exercise might yield a higher payoff, taking the maximum with $(K - S)^+$; i.e.,

$$P_n^a(t_{n,i}, S) = \max\{(K - S)^+, r_n^{-1} E[P_n^a(t_{n,i+1}, S \cdot \exp \overline{R}_{n,i})]\} \ .$$

At each time $t_{n,i}$, $B_i^{(n)}$ will denote the highest node at which exercise occurs. A backward induction argument performed in Leisen (1998) is used to deduce a description for the discrete early–exercise premium $\pi_n = P_n^a - P_n^e$ in terms of the discrete boundary $\overline{B}^{(n)} = (\overline{B}_i^{(n)})_i$:

$$\pi_n = \pi_1^n + \pi_n^2, \tag{11}$$

$$\text{where} \quad \pi_n^1 = \sum_{i=0}^{n-1} r_n^{-i} \cdot K(1 - r_n^{-1}) \cdot E\left[I_{\overline{B}^{(n)}, t_{n,i}} \right] \tag{12}$$

$$\begin{aligned} \text{and} \quad \pi_n^2 = \sum_{i=0}^{n-1} r_n^{-i} \cdot E\Big[& 1_{\overline{S}_i^{(n)} \le \overline{B}_i^{(n)}; u_n \cdot \overline{S}_i^{(n)} > \overline{B}_{i+1}^{(n)}} \\ & \cdot \Big(f(u_n \overline{S}_i^{(n)}) - r_n^{-1} P_n^a(t_{i+1}^n, u_n \overline{S}_i^{(n)}) \Big) \Big]. \end{aligned} \tag{13}$$

Here, 1_A denotes the indicator random variable corresponding to the set A and $I_{\overline{B}^{(n)},t_{n,i}}$ is defined similarly to the continuous case.

3 The Convergence Structure in Binomial Models

This section discusses the convergence structure and ways to improve it in detail. First, we consider the European put option and later we study the American put by its premium component.

3.1 The European put option

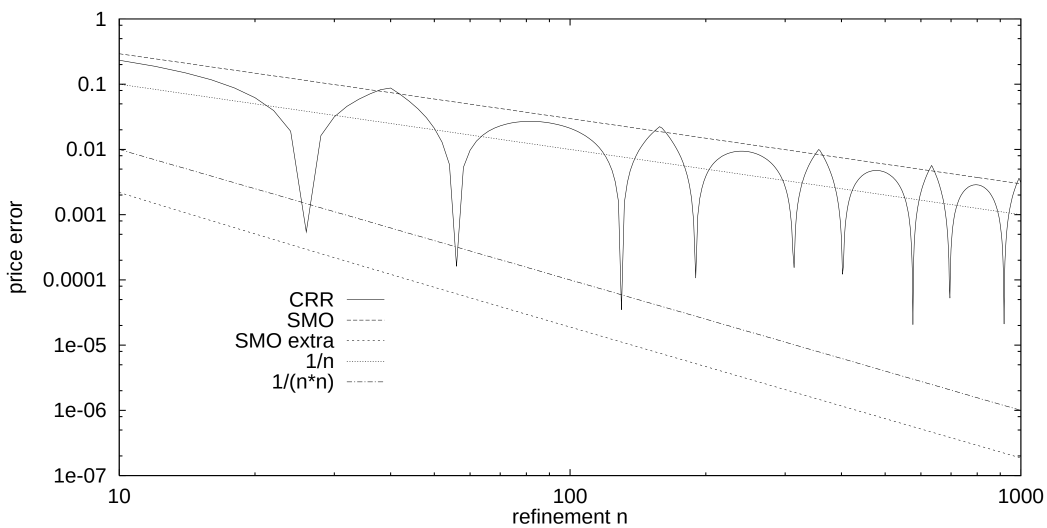

Fig. 2. Error picture for the European put option

Figure 2 shows the error $e_n = |P^e - P^e_n|$ in calculating European put prices for the parameters $S_0 = 100$, $K = 110$, $T = 1$, $r = 0.1$, $\sigma = 0.3$. The selection will be used in figures throughout this paper in order to make them comparable. In all figures the resulting errors for the different refinements are connected by straight lines to present the convergence structure more clearly.

In figure 2 we iterated over even refinements $n = 10, 12, \ldots, 1000$. For the CRR model we observe quite erratic convergence to the continuous time solution with waves. Iterating over all integers $n = 10, \ldots, 1000$ would result in a figure where the waves are even more pronounced. Fairly high refinements are required to achieve sufficiently high accuracy. In the example of figure 2, at least a refinement of $n = 200$ is necessary to ensure "penny–accuracy". We did not depict a price picture, because the error is of most interest. Furthermore, it would exhibit even worse behaviour, since price approximations which overestimate the continuous time price are followed by others which underestimate.

We have chosen a log–log–scale; exponential functions $x \mapsto c/n^\rho$ $(c, \rho \in \mathbb{R}^+)$ become straight lines which intersect $\log c$ at $n = 1$ and slope $-\rho$. Looking at $1/n$ the picture suggests that the order–of–convergence is one. Indeed Leisen and Reimer (1996) proved this result.

The oscillations in figure 2 are due to the fact that the payoff–function of the European put is not differentiable at the strike price. At maturity the difference between two adjacent nodes is $u_n - d_n = \mathcal{O}(v_n) = \mathcal{O}(\sigma\sqrt{\Delta t_n})$, and the probability of a single node at the centre of the tree is of order $\mathcal{O}(\sqrt{\Delta t_n})$ (see Feller (1966)). So, rounding off the strike to the next node induces distortions of order $\mathcal{O}(\Delta t_n)$ in addition to the time–space discretization error (for a detailed discussion, see Leisen and Reimer (1996)). To remove these waves the position of the grid in relation to the strike needs to be controlled: the strike K should always lie at a fixed proportional distance to the next two grid nodes, independent of the refinement; e.g., the strike should be on a node. Leisen (1998) presented the following model for even refinements n:

$$\begin{aligned} \kappa_n &= \frac{\ln K/S_0}{n} \\ \text{and} \quad v_n &= \sigma\sqrt{\Delta t_n}\ . \end{aligned}$$

It changes the drift in such a way that the centre grid point of the tree lies fixed on K for any even refinement. This model yields a very smooth convergence structure. It was therefore called *SMO* ("smooth") (see figure 2). The results of Leisen and Reimer (1996) can be used to prove that the order is one.

3.2 Extrapolation

Since European put option prices typically converge with order one, their error e_n can be represented in the form

$$\frac{\alpha_1(n)}{n} + \frac{\alpha_2(n)}{n^2}$$

for suitable real valued functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ with α_1 bounded. For two refinements n_1 and n_2 with $n_2 > n_1$ and corresponding prices $P^e_{n_1}$, $P^e_{n_2}$, let us suppose for a moment that function $\alpha_1(\cdot)$ is constant, equal to α_1 and $\alpha_2 \equiv 0$. That is, $P^e_n = \frac{\alpha_1}{n} + P^e_{(n_1,n_2)}$, where $P^e_{(n_1,n_2)}$ denotes the approximation for P^e under this assumption. We then resolve:

$$P^e_{n_1} = \frac{\alpha_1}{n_1} + P^e_{(n_1,n_2)} \tag{14}$$

$$P^e_{n_2} = \frac{\alpha_1}{n_2} + P^e_{(n_1,n_2)} \tag{15}$$

$$\Longrightarrow \qquad P^e_{(n_1,n_2)} = \frac{n_2 P^e_{n_2} - n_1 P^e_{n_1}}{n_2 - n_1} \; . \tag{16}$$

Equation (16) is called the *extrapolation rule*. We denote by $e_{(n_1,n_2)}$ the error $P^e_{(n_1,n_2)} - P^e$ resulting from extrapolation. Further analysis reveals

$$e_{(n_1,n_2)} = \frac{\alpha_1(n_2) - \alpha_1(n_1)}{n_2 - n_1} + \frac{n_1\alpha_2(n_2) - n_2\alpha_2(n_1)}{n_1 n_2 (n_2 - n_1)} \; . \tag{17}$$

Let us set $\bar{\alpha} = \max_n \alpha_1(n) - \min_n \alpha_1(n)$. If $\alpha_1(\cdot)$ is not constant, we have $e_{(n_1,n_2)} \leq \bar{\alpha}/(n_2 - n_1)$ up to higher order error terms. The term $\bar{\alpha}$ can be interpreted as a measure of smoothness or as a measure for the possible improvements by extrapolation. The important case is the one of a constant α_1. Then the first term on the right–hand side of equation (17) vanishes. Iterating the refinement n and using the sequence $(n, 2n)$ for extrapolation, equation (17) becomes $e_{(n,2n)} = (\alpha_2(2n) - \alpha_2(n))/n^2$. So, for α_2 bounded, extrapolated prices converge with order two. For models with constant α_1, the error picture looks "smooth." Therefore, we try to construct new models with this property; this is why we loosely speak of smoothing options when constructing better performing models (see Leisen (1998) for a detailed discussion).

Figure 2 shows the error $e_{(n,2n)}$ resulting from extrapolating SMO while iterating $n = 10, 12, \ldots, 1000$. A comparison with the $1/n^2$ line suggests that order two in n holds. This confirms our preceding analysis. We would like to point out that this improvement holds for any parameter selection, although we depicted only one example.

3.3 The early–exercise premium

This subsection studies the additional price component in American put options resulting from the possibility of exercising it before maturity. This is given by the early–exercise premia $\pi = P^a - P^e$ (see equation (5)) and $\pi_n = P^a_n - P^e_n$ (see equations (11)–(13)). We present the numerical problems and explain our approach to improve them. Only the errors resulting from the first component π^1_n in equation (11) are discussed; those of π^2_n can be treated similarly. Figure 3 shows the error $|\pi - \pi_n|$ resulting from the premium. True prices are calculated using a CRR model with refinement 50000 for American put options. Although all models exhibit waves, those of SMO seem to be more regular than those of CRR. Since the order appears to be one, we applied the extrapolation rule (16) iterating the refinement n and using the sequence

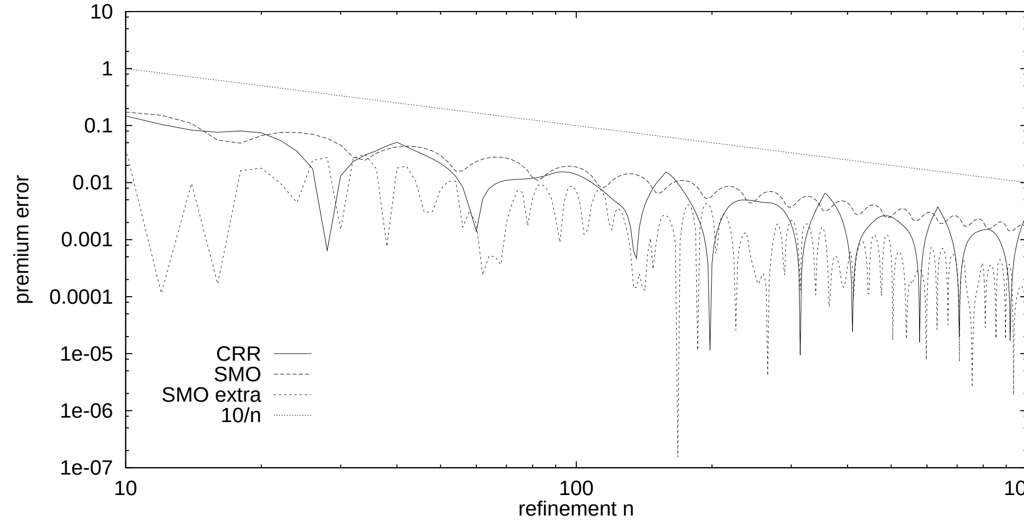

Fig. 3. Error picture for the early–exercise premium

$(n, 2n)$ as in subsection 3.1. Extrapolating SMO no longer gives significant improvements. SMO yields better results for options where the premium is small in comparison to the European put component; i.e., those where the boundary is quite far from today's stock price. Our specific parameter selection is not only of theoretical interest, but illustrates the remaining numerical problems and also represents the typical one in applications.

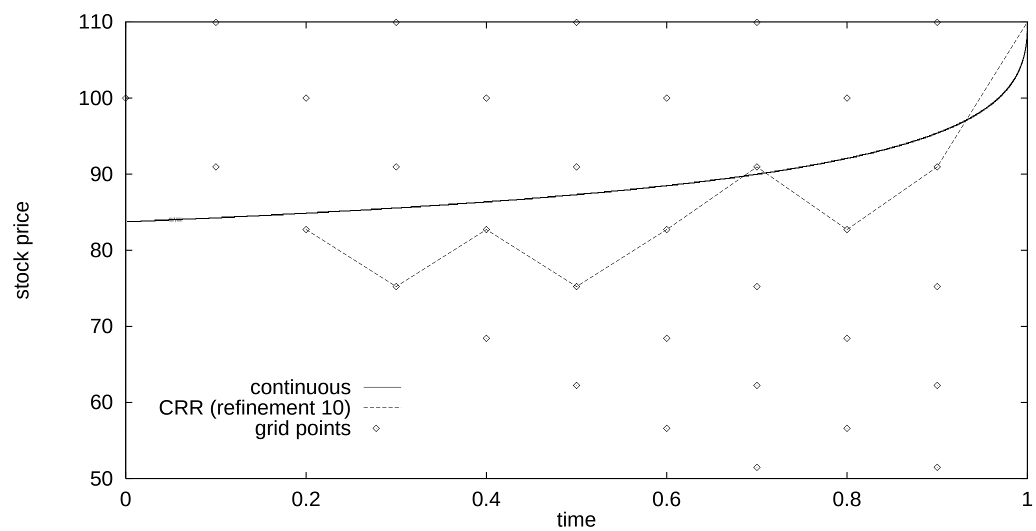

Fig. 4. The continuous boundary and its CRR approximation

Figure 4 exhibits the "continuous–time" boundary, calculated in a CRR model with a refinement of 100000, to explain the reason, why the convergence behaviour of the premium has worsened. The points represent the discrete stock grid of a CRR model with refinement $n = 10$. We connected the discrete boundary points $(t_{10,2}, \overline{B}_1^{(10)}), \ldots, (t_{10,10}, \overline{B}_{10}^{(10)})$ by straight lines; they represent an approximation to the continuous boundary. The discrete boundary seems to move in a zig–zag pattern up and down one layer of nodes. Although there are some exceptions, in general it is at the next node below the continuous boundary. Only when the distance becomes too large does it jump an entire layer following the upward trend in the continuous boundary. Near to maturity where the slope of the continuous boundary becomes infinity, the jumps become more and more important.

The boundary behaviour has an impact on the approximation structure of the premium. Exceptional cases are those where the discrete boundary is above the continuous one; however, in order to explain the odd convergence behaviour and our improvements in a simple way, we now assume that the discrete boundary is always at the node directly below the continuous one. Then we can treat $\overline{B}^{(n)}$ as the "effective" cut–off of B to the node immediately below; i.e., $I_{\overline{B}^{(n)},t_{n,k}}$ in equation (12) can be treated as $I_{B,t_{n,k}}$. Comparing π_n^1 with the representation of the continuous boundary in equation (5), we note first that $\exp\{-rt_{n,k}\} = r_n^{-k}$. Approximating $1 - r_n^{-1} = 1 - \exp\{-r\Delta t_n\} = r\Delta t_n + \mathcal{O}(\Delta t_n^2)$, the summation can be treated essentially as a trapezoidal approximation to the integral, with expectations over indicator functions of type $I_{B,t}$ in both cases. At each date, the cut–off problem is the same as in the European put option case: with changing refinement, the difference between the boundary and the next grid point below will change. SMO placed nodes only at the critical point — the strike — at terminal time and not on the boundary at intermediate time. This explains why the convergence behaviour is no longer smooth.

Unfortunately, the functional form of the early–exercise boundary is unknown. So, it is not feasible to place nodes on it. We address the problem here from an entirely different perspective and try to smooth variations in the overall distance between the discrete and the continuous boundary over time when changing refinement. Heuristically, this is a way to minimize changes in the difference between the discrete and the continuous boundary at a fixed time for changing refinements. To do so, take two models with different refinements but the same up and down factors u, d, where the boundary does not evolve completely synchronous. If the discrete boundary approximation is a round–off of the continuous one to the next node below, this is the "standard" case (remember the zig–zag structure). A fictitious mean boundary can be calculated by averaging with equal weight one half the (linearly interpolated) boundaries resulting from both models. In those situations where both boundaries are moving up or down, there will be no change; however in those situations where they move in an asynchronous way, the new approximation will be flat. Consequently, the boundary will exhibit less variation; i.e., it will appear to be smoother.

An extension of this idea is necessary. If we reinterpret the weights as probabilities, we can think of this approach as a mix of different binomial models, drawing one randomly as the true dynamics. However, mixing randomly over two models is hardly a realistic dynamics for the evolution of the price of some underlying asset. At the first trading date, the information about the underlying binomial model will be fully revealed. The optimal exercise decision in the future will then be based on this newly revealed information, and not on an average of the two early–exercise boundaries. But this averaging is what we are trying to achieve in some market model. Therefore, we need to formalize it differently. To prevent the information about the underlying model of the actual dynamics from being fully revealed after the first time–step, we will randomize the time between two trading dates, each increment being independent of the previous ones. This will also resolve the second problem, that we can expect a good approximation only if we mix over a whole sequence of binomial models with different weights.

4 Randomization of the Binomial Model

This section extends the previous approaches, allowing the time increment between two trading dates to be random. Our discrete process is constructed such as to ensure weak–convergence to the Black–Scholes setup. The next section addresses the option valuation problem.

We start with two sequences $(\Delta x_m)_{m\in\mathbb{N}}, (\lambda_m)_{m\in\mathbb{N}} \subset \mathbb{R}$ with $\Delta x_m \xrightarrow{m} 0$ and $\lambda_m \xrightarrow{m} \infty$; for each $m \in \mathbb{N}$, a Poisson process $N^{(m)} = (N_t^{(m)})_{t\geq 0}$ with intensity λ_m. m corresponds to a refinement of the state space. Let us recall that denoting the ith interarrival time by $\tau_{m,i}$, the process $N^{(m)}$ is described by

(1) $(\tau_{m,i})_i$ are independent exponentially random variables with parameter $1/\lambda_m$
(2) $N_t^{(m)} = \max\{n \,|\sum_{i=1}^n \tau_{m,i} \leq t\}$.

For any m, take a sequence of i.i.d. random variables $(\overline{R}_{m,i})_i$, where each element is also independent of $N^{(m)}$ and distributed according to

$$\overline{R}_{m,i} \sim \begin{cases} \Delta x_m & ; p_m \\ -\Delta x_m & ; 1 - p_m \end{cases} . \tag{18}$$

Binomial models assume that the change in the process X between two trading dates $t_{n,i}, t_{n,i+1} \in \mathcal{T}^n$ is given by i.i.d. random variables $\overline{R}_{n,i}$. Similarly, here we will now approximate it by i.i.d. random variables $\overline{R}_{m,i}$ between two interarrival times $\tau_{m,i}, \tau_{m,i+1}$. In the sequel we further assume that each element of

$(\overline{R}_{m,i})_i$ is independent of $N^{(m)}$. We define the processes

$$\begin{aligned} \overline{X}_t^{(m)} &= \sum_{i=1}^{N_t^{(m)}} \overline{R}_{m,i} \\ \text{and} \quad \overline{S}_t^{(m)} &= \exp \overline{X}_t^{(m)} \end{aligned}$$

and call this the *Random–Time Binomial Model* (henceforth *RTBM*). Then we have the following

Theorem 2 *Necessary conditions for* $\overline{X}^{(m)} \overset{d}{\Longrightarrow} X$ *are that for all* $t > 0$*:*

$$E\left[\overline{R}_{m,i}\right] \lambda_m \overset{m}{\longrightarrow} \mu \tag{19}$$

$$\textit{and} \quad Var\left[\overline{R}_{m,i}\right] \lambda_m \overset{m}{\longrightarrow} \sigma^2 \tag{20}$$

Please recall from the beginning of section 2 that $\mu = r - \frac{\sigma^2}{2}$. These two conditions are also sufficient. The counterpart to theorem 1 is:

Theorem 3 *For the RTBM, suppose that conditions (19) and (20) are fulfilled. Then,*

$$\begin{aligned} \overline{X}^{(m)} &\overset{d}{\Longrightarrow} X \\ \textit{and} \quad \overline{S}^{(m)} &\overset{d}{\Longrightarrow} S \; . \end{aligned}$$

We would like to remark that $\lambda_m \overset{m}{\longrightarrow} \infty$ implies $E[\tau_{m,1}] \overset{m}{\longrightarrow} 0$. This can be interpreted in the sense that with m being sufficiently great, it can be expected that jumps will occur almost always. Indeed, theorem 3 tells us that for suitably adjusted return variables $(\exp\{\overline{R}_{m,i}\})_{m,i}$, in the limit $m \to \infty$ geometric Brownian motion is obtained. Condition (19) in theorem 2 requires

$$E[\overline{R}_{m,i}] = \lambda_m \tag{21}$$

$$\Longleftrightarrow \quad q_m = \frac{1}{2} + \frac{\mu}{2\lambda_m \Delta x_m} \; . \tag{22}$$

Although we have taken $(\Delta x_m)_m$ and $(\lambda_m)_m$ as inputs, both cannot be chosen independently of the other to ensure convergence to the Black–Scholes setup. Similar to the standard binomial case where we deduced the asymptotic form of $(v_n)_n$, here we deduce from condition (20)

Lemma 4 *A necessary condition for convergence to the Black–Scholes setup is that asymptotically we have*

$$\lambda_m \sim \left(\frac{\sigma}{\Delta x_m}\right)^2 .$$

Allowing the time increment in the binomial model to be random introduces a further risk and the market becomes incomplete. Whereas in the original binomial model framework of the previous section there was a unique equivalent martingale measure — represented by q_m — here we are losing this property. Instead we have a whole set of equivalent martingale measures, all compatible with the assumption of absence of arbitrage opportunities. We can index them by the jump–intensity λ_m. For valuation purposes we need to choose one measure among all. Yet, this cancels out in the limit according to theorem 3. The easiest way to meet the asymptotic form of Lemma 4 is obtained by setting

$$\lambda_m = \left(\frac{\sigma}{\Delta x_m}\right)^2 , \tag{23}$$

which we adopt in the sequel. Under this choice, we deduce from (22) that

$$q_m = \frac{1}{2} + \frac{\mu}{2\sigma^2}\Delta x_m .$$

For any Δx_m and $(\overline{R}_{m,i})_i$ as in (18), our RTBM is completely described by the characteristic terms $(u_m, d_m, q_m, \lambda_m) = (\exp\{\Delta x_m\}, \exp\{-\Delta x_m\}, 1/2 + \mu/(2\sigma^2)\Delta x_m, (\sigma/\Delta x_m)^2)$.

Taking fictitiously $\Delta x_m = \sigma\sqrt{\Delta t_n}$, the only difference with the CRR model consists in replacing Δt_n by random times $(\tau_{m,i})_i$. However all formulae hold with the expected time $\Delta t_n = (\Delta x_m/\sigma)^2 = 1/\lambda_m = E[\tau_{m,1}]$.

Remark 5 *Our setup could be generalized easily to allow the processes $N^{(m)}$ to be any renewal process, i.e., a process where the difference between two trading dates is i.i.d., but not necessarily exponentially distributed. Rogers and Stapleton (1998) constructed such a binomial model as follows: For any $\Delta x > 0$ they obtain the interarrival times by stopping X (see equation 6) at the grid $\Delta x \cdot \mathbb{Z}$, i.e. with $\tau_0 := 0$ by induction:*

$$\tau_{i+1} = \inf\{t \geq \tau_i |\ |X_{t+\tau_i} - X_{\tau_i}| = \Delta x\} .$$

5 Valuation

The previous section introduced the RTBM and motivated our choice of the intensity λ_m. This section presents a valuation algorithm, that gets rid of the additional randomness in an easy and straightforward manner for European and American call and put options. We further discuss convergence issues.

5.1 The European put

Let us denote for any European option with maturity T its payoff function by f. For example for a European call with strike K, this is $f : x \mapsto (x - K)^+$. Except in places where we discuss convergence issues, we fix some RTBM $(u_m, d_m, q_m, \lambda_m)$. Due to the independence of $N^{(m)}$ and the random variables in the sequence $\overline{R}_m$, the value V_m^e of the option can be split up by first conditioning on the number of jumps $N_T^{(m)}$ and then averaging all these:

$$\begin{aligned} v_m^e &= e^{-rT} E\left[f(\overline{S}_T^{(m)}) \right] \\ &= e^{-rT} E\left[E[f(\overline{S}_T^{(m)})|N_T^{(m)}] \right] \\ &= e^{-rT} \sum_{n=0}^{\infty} Q\left[N_T^{(m)} = n \right] \cdot E\left[f(\overline{S}_T^{(m)})|N_T^{(m)} = n \right] . \end{aligned}$$

$E\left[f(\overline{S}_T^{(m)})|N_T^{(m)} = n \right]$ can be interpreted as the value calculated by backward–induction in an n–step binomial model grid with characteristic terms $(u_m, d_m, 1, q_m)$, as at the end of section 2: i.e., it is the price in an n–step binomial model with up (down) factor u_m (d_m) and probabilities q_m if we *do not* perform discounting. Denoting this value by

$$\Phi_n^{(m)} = \sum_{i=0}^{n} \binom{n}{i} q_m^i (1 - q_m)^{n-i} f(u_m^i d_m^{n-i} S_0) ,$$

we have

$$v_m^e = e^{-rT} \sum_{n=0}^{\infty} Q\left[N_T^{(m)} = n \right] \cdot \Phi_n^{(m)} . \tag{24}$$

To implement our RTBM for the specific option contract under consideration, we now specify the selection of the sequence $(\Delta x_m)_m$ to ensure good convergence. Please remember from the SMO model that nodes should always lie on critical points of the payoff function. We use the parameter m to discretize the state space equidistantly in the logarithm with an integer multiplicity of Δx_m between S_0 and K. For a given call or put option with strike K and a refinement m, we adopt

$$\Delta x_m := \frac{|\ln S_0/K|}{m} . \tag{25}$$

If $S_0 = K$, any positive constant replacing $\ln S_0/K$ is fine. This completely specifies our RTBM with the characteristic terms $(u_m, d_m, q_m, \lambda_m) = (\exp\{\Delta x_m\}, \exp\{-\Delta x_m\}, 1/2 + \mu/(2\sigma^2)\Delta x_m, (\sigma/\Delta x_m)^2)$.

Since $\Delta x_m \xrightarrow{m} 0$, we have $\lambda_m \xrightarrow{m} \infty$, and so theorem 3 implies $\overline{S}^{(m)} \overset{d}{\Longrightarrow} S$. Convergence of put option prices follows from this; convergence of call prices follows then via put–call parity. So we have the following:

Theorem 6 *For a sequence of RTBM with the above characteristic terms, European call and put option prices converge to their counterpart in the Black–Scholes setup.*

To implement our approach, we have to cut off the infinite sum in equation (24) at some appropriate γ_m. We are looking for a cut–off with $\lim_{m\to\infty} Q[N_T^{(m)} \in \{0, \dots, \gamma_m\}] = 1$. The Central Limit Theorem for renewals states

$$\frac{N_T^{(m)} - \lambda_m T}{\sqrt{\lambda_m T}} \overset{d}{\Longrightarrow} \mathcal{N}(0,1) \ .$$

We deduce that $\gamma_m = 2\lfloor \lambda_m T \rfloor$ is *one* appropriate choice. In the sequel we adopt this one and use the cut–off:

$$v_m^e \approx e^{-(r+\lambda_m)T} \sum_{n=0}^{2\lfloor \lambda_m T \rfloor} \frac{(\lambda_m T)^n}{n!} \cdot \Phi_n^{(m)} \tag{26}$$

Remark 7 *These observations hold for any renewal and include the "stopping–approach" proposed by Rogers and Stapleton (1998). However, Rogers and Stapleton (1998) can calculate the probabilities $Q[N_T^{(m)} = n]$ only through a limit theorem expansion. This procedure is quite complicated in our eyes. In contrast our approach taking a Poisson process $N^{(m)}$ allows us to calculate them in a straightforward manner.*

An important observation prevails which greatly simplifies the calculation of $(\Phi_n^{(m)})_{n=0,\dots,2\lfloor \lambda_m T \rfloor}$. For an even refinement $\tilde{n}$, the tree is recombining after exactly two periods. Since our grid is time–homogeneous for any even refinement $\tilde{n}$, the binomial model for even refinement n' with $0 \leq n' \leq \tilde{n}$ is contained as the binomial model, starting at the same level $\tilde{n} - n'$ periods later. Because we do not perform discounting, we obtain $\Phi_{n'}^{(m)}$ as intermediate calculations of $\Phi_{\tilde{n}}^{(m)}$ (see figure 5). We proceed similarly for odd refinements $0 \leq n' \leq \tilde{n}$ (see figure 6). In total, calculating $\Phi_{\tilde{n}}^{(m)}$ and $\Phi_{\tilde{n}-1}^{(m)}$ for $\tilde{n} = 2\lfloor \lambda_m T \rfloor$ in a binomial model with characteristic terms $(u_m, d_m, 1, q_m)$ gives us all the values $(\Phi_{n'}^{(m)})_{n'=0,\dots,2\lfloor \lambda_m T \rfloor}$ as intermediate calculations. Therefore computing prices in an RTBM with refinement m is comparable to a CRR with refinement $2\lfloor \lambda_m T \rfloor$ in terms of the computational cost.

Figure 7 presents pricing examples for the European put option. The im-

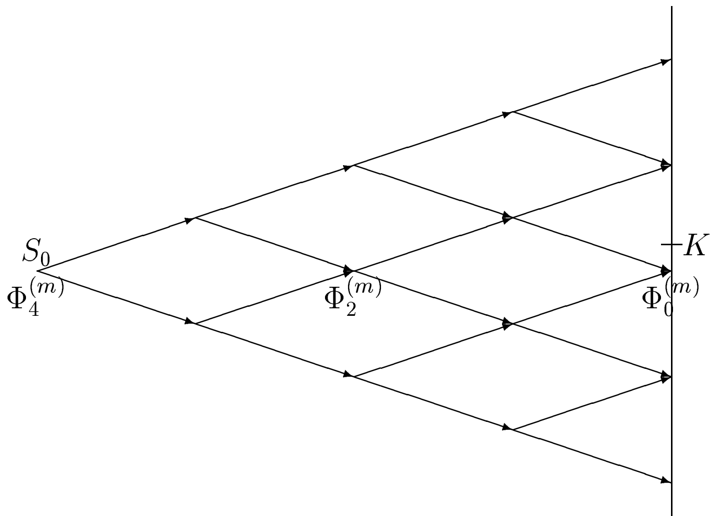

Fig. 5. Example grids for n even $(n = 0, 2, 4)$

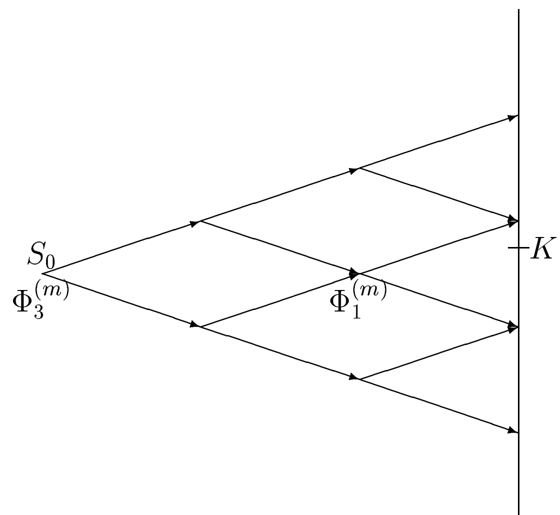

Fig. 6. Example grids for n odd $(n = 1, 3)$

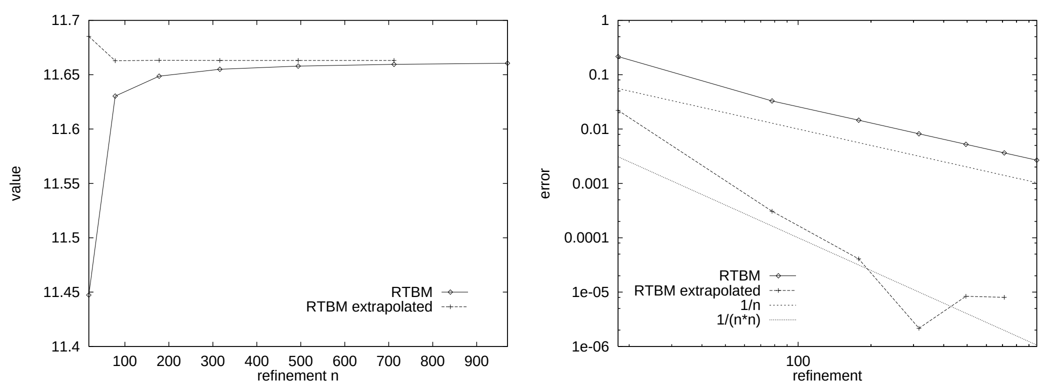

Fig. 7. Price, error and bounding error function for a European put

plementation proceeds as follows: For any refinement m we calculate Δx_m by equation (25), which results in $\lambda_m = (\sigma/\Delta x_m)^2$ by equation (23). We calculate the European option value in a $2\lfloor\lambda_m T\rfloor$ and a $2\lfloor\lambda_m T\rfloor - 1$ step binomial model with characteristics $(u_m, d_m, 1, q_m)$ and write $\Phi_1^{(m)}, \ldots, \Phi_{2\lfloor \lambda_m T \rfloor}^{(m)}$ to a separate list. Our price approximation for fixed m is then calculated by equation (26). In figure 7 we calculated European put option prices iterating over the refinements $m = 1, \ldots, 7$. We display them depending on a fictitious refinement of $2\lfloor \lambda_m T \rfloor = 18, 78, 178, 316, 494, 712, 970$, since this corresponds to the CRR model which is comparable in terms of the computational cost. On the left hand we display the values according to the refinement. The right–hand part contains the absolute difference to the true (continuous time) price.

Figure 7 for our RTBM has the same smooth structure and starts with the same small initial errors as can be observed in figure 2 for SMO. To apply extrapolation, we need to determine the order of convergence. We see that the error is bounded by the line $1/n$ for the RTBM. This suggests that it converges with order one in the fictitious refinement $2\lfloor \lambda_m T \rfloor$. The following theorem establishes this result for the RTBM:

Theorem 8 *The order of convergence in calculating European call and put option prices v_m^e using the sequence of RTBM models $(\overline{S}^{(m)})_m$ is one in λ_m:*

$$|v^e - v_m^e| = \mathcal{O}\left(\frac{1}{\lambda_m}\right) \tag{27}$$

Here v^e denotes the Black–Scholes continuous time solutions (equations (2) and (3)).

Repeating the derivation of the extrapolation rule (equation (16)) in section 3, using λ_m instead of n, we deduce the following extrapolation rule for RTBM in terms of the refinement m:

$$v_{(m_1,m_2)}^e = \frac{\lambda_{m_2} v_{m_2}^e - \lambda_{m_1} v_{m_1}^e}{\lambda_{m_2} - \lambda_{m_1}} \ . \tag{28}$$

It gives us a way to calculate a new value $v_{(m_1,m_2)}^e$ from the values $v_{m_1}^e$ and $v_{m_2}^e$ corresponding to two refinements m_1 and m_2. Equations (27) and (28) confirm our remark made in section 4 that all formulae for CRR hold with the expected time difference $1/\lambda_m$.

Figure 7 presents also the extrapolated values using the sequence $(m, m+1)$. We did not extrapolate the CRR model since the wavy patterns would be even more pronounced due to the non–smooth convergence behaviour. On the other hand, when we extrapolate our RTBM, we start with very small errors which are almost immediately accurate to the penny. Moreover, extrapolated prices seem to converge with a higher order of two.

A rough cost analysis to evaluate our improvements is the following: To calculate prices for the standard CRR model with refinement n, the number of calculations in the backward induction algorithm is of order n^2. The same holds for our RTBM and its extrapolation, since RTBM and CRR are subject to the same computational cost in calculating prices and extrapolation amounts to a computational cost which is larger only by a constant factor. On the other hand, extrapolation gives the order two. Thus, a rough estimate is

$$\text{error} = \mathcal{O}\left(\frac{1}{\text{cost}^{1/2}}\right) \qquad \text{for CRR}$$
$$\text{error} = \mathcal{O}\left(\frac{1}{\text{cost}}\right) \qquad \text{for RTBM} .$$

These observations make it a powerful pricing tool in the Black–Scholes setup.

5.2 The American put option

In the European option setup, the simple adjustment which led to the SMO yields the same impressive results as our RTBM. We see, however, in this subsection dealing with American options, that in contrast to SMO and CRR, our model is capable of smoothing the American put option and achieves impressive improvements for this case, too.

Similar to equation (4), we take for an RTBM with characteristic terms $(u_m, d_m, q_m, \lambda_m)$ the value V_m^a of the American put is determined by the optimal stopping problem

$$v_m^a = \sup_{\sigma \in \mathcal{S}^{(m)}} E\left[e^{-r\sigma}\left(K - \overline{S}_\sigma^{(m)}\right)^+\right] , \tag{29}$$

where here $\mathcal{S}^{(m)}$ denotes all stopping times at trading dates adapted to the filtration generated by the stock process $(\overline{S}_t^{(m)})$.

From Lamberton and Pagès (1990) we deduce, checking their condition (H) by the sufficient conditions of Mulinacci and Pratelli (1998), the following:

Theorem 9 *For a sequence of RTBM with $\Delta x_m \xrightarrow{n} 0$ and characteristics $(u_m, d_m, q_m, \lambda_m)$ given by equations (18), (22) and (23), American put option values v_m^a converge to their price in the Black–Scholes setup.*

This consistency result allows calculating continuous–time prices by our RTBM. Define $\Psi_n^{(m)}$ as the value calculated in an n–step binomial model with characteristic terms $(u_m, d_m, E[\exp\{-r\tau_{m,1}\}|N_T^{(m)} = n], q_m)$, according to the American put backward algorithm presented at the end of section 2. For European put options, equation (24) shows that the price is a mixture over different scenarios corresponding to the number of trading dates occurring in the interval

$[0, T]$, each weighted by its probability. Here we have a similar weighting result for American options using the terms $\Psi_n^{(m)}$.

Proposition 10 *For fixed m, the value of the American put option is*

$$v_m^a = e^{-\lambda_m T} \sum_{n=0}^{\infty} \frac{(\lambda_m T)^n}{n!} \cdot \Psi_n^{(m)} \ .$$

The core of the proof is that conditioning on the (random) number of jumps does not change the value (see also Chow, Robbins, and Siegmund (1991)). The implementation is exactly as in the European put case. However we calculate $\Psi_n^{(m)}$ using $\exp\{-rT/n\}$ instead of $E[\exp\{-r\tau_{m,1}\}|N_T^{(m)} = n]$. A series expansion reveals that both are equal to $E[1 - r\tau_{m,1}|N_T^{(m)}] = 1 - T/n$ up to second–order error terms.

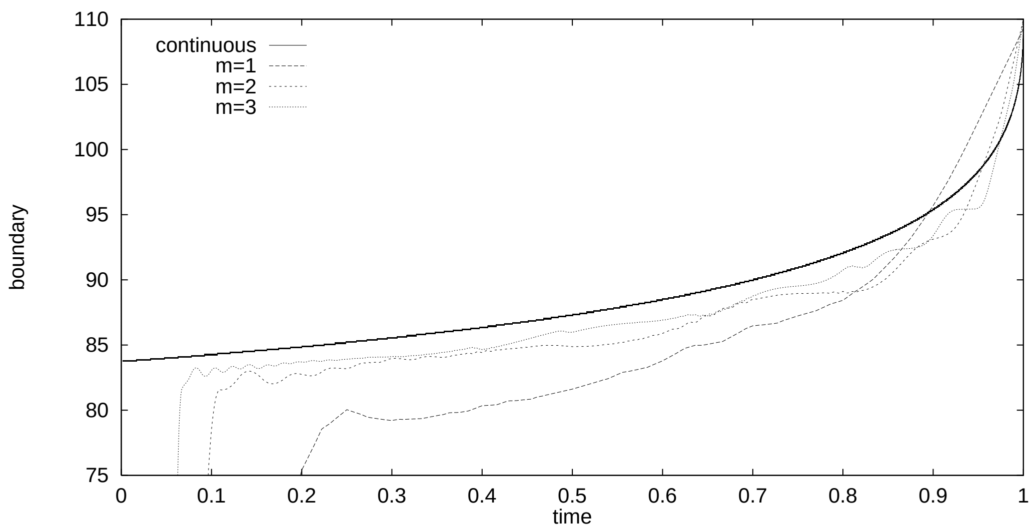

Fig. 8. Boundary approximation

From each n–step binomial model with characteristic terms $(u_m, d_m, \exp\{-rT/n\}, q_m)$, a (linearly interpolated) boundary $\overline{B}^{(m,n)}$ results. Figure 8 presents the boundary $\overline{B}^{(m)} := \sum_{n=0}^{2\lfloor\lambda_m T\rfloor} Q[N_T^{(m)} = n] \cdot \overline{B}^{(m,n)}$. The discrete boundary $\overline{B}^{(m)}$ is not completely smooth, however the overall variation to the continuous boundary over time is reduced. This result is astonishing as the discrete grid is quite large. For example, for $m = 1$ the grid is the same as that in figure 4 — except that we have taken here a smaller scale of $[75, 110]$ — and nodes lie at $75.1, 82.6, 90.9, 100, 110$. This is in line with our motivation at the end of section 3 that mixing boundaries resulting from different refinements would result in one with a reduced variation and suggests good price approximations.

For the American put option, figure 9 presents in the left–hand part price calculations and in the right–hand part the error to the continuous–time solution.

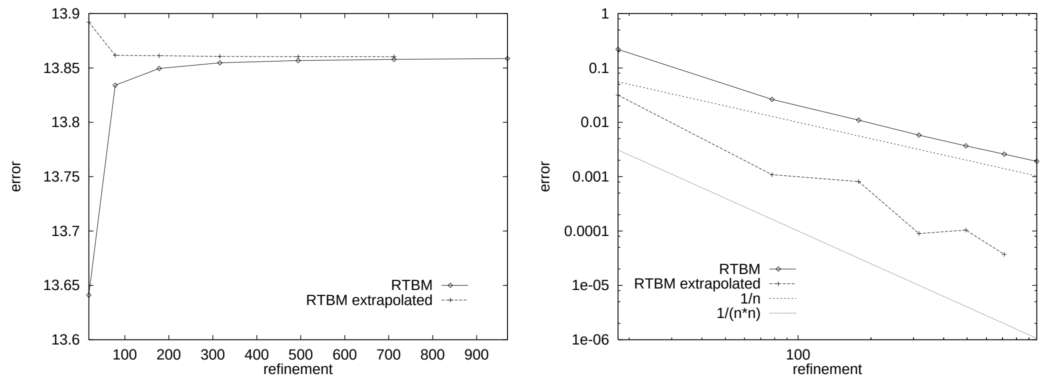

Fig. 9. Price, error and bounding error function for an American put

Different to the wavy patterns in figure 3, here a smooth convergence structure can be seen. Extrapolating our RTBM yields impressive results in terms of accuracy. We have very small initial errors which give us "penny–accuracy" almost immediately. Moreover extrapolated prices seem to converge with a higher order of two. We would like to point out that besides our specific selection these impressive results hold for any parameter selection different to SMO.

To compare CRR and RTBM in terms of the computational cost necessary to achieve a certain precision level, note that for CRR the number of calculations and the order needed remain unchanged compared to the European case (n^2 and order one). However for RTBM we get order two again, but the number of calculations needed is now of order n^3, since all intermediate calculations need to be performed separately. Thus for American put options, the rough estimate is

$$\text{error} = \mathcal{O}\left(\frac{1}{\text{cost}^{1/2}}\right) \qquad \text{for CRR} \tag{30}$$

$$\text{error} = \mathcal{O}\left(\frac{1}{\text{cost}^{2/3}}\right) \qquad \text{for RTBM.} \tag{31}$$

In this way, it gives also in the American put option case a very competitive pricing tool, increasing the order by extrapolation from 1/2 to 2/3.

6 Approximating Jump Diffusions

This section studies the jump diffusion setup of Merton (1976). Adding the jump part to the RTBM is straightforward, and we remain in the same framework. The simple and competitive valuation tools developed in section 5 immediately apply here.

In an extension of the Black–Scholes model (see equation (1)), Merton (1976) supposed that the stock–price can be described by

$$S_t = S_0 \exp\{\mu t + \sigma W_t\} \prod_{i=1}^{N_t} (1 + U_i) , \tag{32}$$

under the objective measure, where the drift $\mu \in \mathbb{R}$ is some constant, $(U_i)_{i\in\mathbb{N}} \subset]-1, \infty[$ a sequence of iid random variables and $(N_t)_t$ a Poisson process with intensity λ.

Setting $X_t = \ln(S_t/S_0)$, we have

$$X_t = \mu t + \sigma W_t + J_t , \tag{33}$$

$$\text{where} \quad J_t = \sum_{i=1}^{N_t} \tilde{U}_i$$

$$\text{and} \quad \tilde{U}_i = \ln(1 + U_i) . \tag{34}$$

This setup represents an incomplete market. In the language of Harrison and Pliska (1981) this means there is no longer a *unique* equivalent martingale measure that precludes arbitrage. Föllmer and Sondermann (1986) specified the measure that minimizes the writer's risk in some sense. The choice of a martingale measure can also be made by assuming that an exogenously fixed risk–premium is required for the jump risk. Similar to Merton (1976), we assume that jump risk can be fully diversified and is therefore not priced. The risk–neutral probability measure is then the one with $\mu := r - \frac{\sigma^2}{2} - \lambda E[U_i]$. All valuations are performed with respect to this measure.

Call option values are a mixture of Black–Scholes prices (see Merton (1976)):

$$e^{-\lambda T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} E[C^e(S_0 Z_n e^{-\lambda k T}, T, K, \sigma^2, r)] , \tag{35}$$

where $C^e(S_0, T, K, \sigma^2, r)$ is the Black–Scholes formula for call options (equation (2)), $k = E[U_i]$ and $Z_n \stackrel{d}{=} \prod_{i=1}^{n}(1 + U_i) = \exp\{\sum_{i=1}^{n} \tilde{U}_i\}$.

Amin (1993) presents the following extension of the binomial model to deal with jump diffusions. As N is a Poisson process with intensity λ, the probability of one jump equals $\lambda\Delta t_n \cdot \exp\{-\lambda\Delta t_n\}$ on a discrete interval Δt_n. The probability of more than one jump is small in comparison to this, and $\exp\{-\lambda\Delta t_n\}$ is approximately equal to one. Therefore Amin (1993) assumes that between two dates at the most one jump occurs and that the probability of this event equals $\lambda\Delta t_n$. Using a binomial model with grid $v_n = \sigma\sqrt{\Delta t_n}$, $\kappa_n = \mu\Delta t_n$, he assumes that jumps occur only to points of this grid. Furthermore, he explains how to suitably approximate $\tilde{U}_i$ by a random variable $\tilde{U}_{n,i}$

which takes values at the grid–points at time $t_{n,i}$ only. The return $\overline{R}_{n,i}$ is then modelled by

$$\overline{R}_{n,i} \sim \begin{cases} \kappa_n + \upsilon_n & ; (1 - \lambda \Delta t_n) q_n \\ \kappa_n - \upsilon_n & ; (1 - \lambda \Delta t_n)(1 - q_n) \\ U_{n,i} & ; \lambda \Delta t_n \end{cases} .$$

The probability q_n is set according to equation (2) with $\mu := r - \frac{\sigma^2}{2} - \lambda E[U_i]$. Amin (1993) proved weak convergence for the processes and convergence for American option prices. In this approach the jump part is simply put on top of the binomial model and it inherits its poor convergence properties from this. Moreover, although the model is computationally correct, it does not match the idea of a rare event at random time.

For the RTBM corresponding in the jump diffusion setup let us first construct along the lines of section 4 a sequence $(N^{(m)}, \overline{R}_m)_m$, each element independent of the other and independent of N such that

$$\prod_{i=1}^{N^{(m)}} \overline{R}_{m,i} \stackrel{d}{\Longrightarrow} \mu t + \sigma W_t \ ,$$

where $\mu := r - \frac{\sigma^2}{2} - \lambda E[U_i]$. Define approximations $\tilde{U}_{m,i}$ of $\tilde{U}_i$ along the lines of Amin (1993) and the process

$$\overline{N}^{(m)} = N + N^{(m)} \ ,$$

which is a Poisson process with intensity $\bar{\lambda}_m = \lambda + \lambda_m$; the sequence of random variables

$$Z_{m,i} \sim \begin{cases} \tilde{U}_{m,i} & ; \frac{\lambda}{\lambda + \lambda_m} \\ \overline{R}_{m,i} & ; \frac{\lambda_m}{\lambda + \lambda_m} \end{cases} ;$$

and the processes

$$\overline{X}_t^{(m)} := \sum_{i=1}^{\overline{N}_t^{(m)}} Z_{m,i}$$
$$\text{and} \quad \overline{S}_t^{(m)} := \exp \overline{X}_t^{(m)} \ .$$

This has the same structure as the model we constructed in section 4, and therefore, we call this model also *Random–Time Binomial Model (RTBM)*. We have the following:

Theorem 11 *For the RTBM:*

$$\overline{X}^{(m)} \stackrel{d}{\Longrightarrow} X$$
$$\text{and} \quad \overline{S}^{(m)} \stackrel{d}{\Longrightarrow} S \ .$$

The put payoff function is continuous and bounded. This fact, together with the above theorem and put–call parity, imply:

Proposition 12 *The value of European put and call option converges to their corresponding continuous time solution.*

The methods and proofs presented in the previous section immediately carry over:

Proposition 13 *The European call and put option value in the RTBM is*

$$v_m^e = e^{-(r+\bar{\lambda}_m)T} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}_m T)^n}{n!} \Phi_n^{(m)} \ ,$$

where $\Phi_n^{(m)}$ is its price in an n–step tree with characteristic terms $(u_m, d_m, 1, p_m)$. The American put option value is

$$v_m^a = e^{-\bar{\lambda}_m T} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}_m T)^n}{n!} \Psi_n^{(m)} \ ,$$

where $\Psi_n^{(m)}$ is its price in an n–step tree with characteristic terms $(u_m, d_m, E[\exp\{-r\tau_{m,1}\}|N_T^{(m)} = n], p_m)$.

Theorem 14 *The value of American put options converges to their continuous time solution.*

The construction is very easy and straightforward to perform. We believe that it also puts more emphasis on the didactical advantages of the original CRR binomial model. Moreover we observe in simulations that the remarkable convergence properties of the RTBM carry over to the approximation of jump–diffusions.

7 Conclusion

In this paper we studied a binomial model with random time steps. We described the difficulties with standard binomial models. We present a way to easily calculate price approximations for European and American put and call options in the Black–Scholes setup and proved convergence to the continuous–time solution. For European put options, we proved that the order of convergence is equal to one. The major contribution lies in a smoothing of the convergence structure for American put options, which allows speed–ups by extrapolation: Simulations suggest a much smaller initial constant and order of convergence two. The same holds for American put options. Thus this model can serve as an efficient tool in the Black–Scholes setup. A second contribution is that our model gives intuitive and straightforward approximations to jump–diffusions which seem to preserve the outstanding properties.

A Proofs

In contrast to the common literature, this paper defined the CRR model on the whole interval $[0, T]$ instead of only at dates $t_{n,i} \in \mathcal{T}^n$. This is for technical convenience and as long as we restrict trading dates to the discrete set $\mathcal{T}^n$, this makes no difference. The processes we study are processes whose paths are continuous to the right, but have left–side limits (called *càdlàg* processes). We will further suppose that the space $\mathcal{D}$ of càdlàg processes is equipped with the Skorohod topology and denote by $\stackrel{d}{\Longrightarrow}$ the weak convergence $\mathcal{D}$.

Proof of Theorem 1 The proof of the first assertion is an application of Donsker's theorem in a suitable form for our case (see corollary VII.3.11 in Jacod and Shiryaev (1987)). Please note that we only need to check the first two moments. The second assertion follows from the observation that the exponential function is continuous.

Proof of Theorem 2 Necessary conditions include convergence of the first moment to its continuous value $E\left[X_t\right] = \left(r - \frac{\sigma^2}{2}\right) t$ and of the second moment to $\mathrm{Var}(X_t) = \sigma^2 t$. For the first moment, using the Wald equality, we have

$$\begin{aligned} E\left[\overline{X}_t^{(m)}\right] &= E\left[\sum_{i=1}^{N_t^{(m)}} \overline{R}_{m,1}\right] \\ &= E\left[N_t^{(m)}\right] E\left[\overline{R}_{m,1}\right] \\ &= \mu_m(t) E\left[\overline{R}_{m,i}\right] \ . \end{aligned}$$

We have

$$
\begin{aligned}
& E\left[\left(\overline{X}_t^{(m)}\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{N_t^{(m)}} \overline{R}_{m,i}\right)^2\right] \\
&= E\left[\sum_{i=1}^{N_t^{(m)}} \left(\overline{R}_{m,i}\right)^2 + \sum_{i=1}^{N_t^{(m)}} \sum_{j=1, j\neq i}^{N_t^{(m)}} \overline{R}_{m,i}\,\overline{R}_{m,j}\right] \\
&= E\left[E\left[\sum_{i=1}^{N_t^{(m)}} \left(\overline{R}_{m,i}\right)^2 + \sum_{i=1}^{N_t^{(m)}} \sum_{j=1, j\neq i}^{N_t^{(m)}} \overline{R}_{m,i}\,\overline{R}_{m,j}\,\middle|\, N_t^{(m)}\right]\right] .
\end{aligned}
$$

Due to the independence of $(N^{(m)})_m$ and $(\overline{R}_{m,i})_{m,i}$, this can be simplified to

$$
\begin{aligned}
& \sum_{n=0}^{\infty} P[N_t^{(m)} = n] E\left[\sum_{i=1}^{n} \left(\overline{R}_{m,i}\right)^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \overline{R}_{m,i}\,\overline{R}_{m,j}\right] \\
&= \sum_{n=0}^{\infty} P[N_t^{(m)} = n] \left(nE\left[\left(\overline{R}_{m,i}\right)^2\right] + n(n-1)E\left[\overline{R}_{m,i}\right]^2\right) .
\end{aligned}
$$

If we denote for $t > 0 : \mu_m(t) := E[N_t^{(m)}] = \lambda_m t$, the previous result implies

$$
\begin{aligned}
& \mathrm{Var}\left(\overline{X}_t^{(m)}\right) \\
&= E\left[\left(\overline{X}_t^{(m)}\right)^2\right] - E\left[\overline{X}_t^{(m)}\right]^2 \\
&= E\left[\left(\overline{X}_t^{(m)}\right)^2\right] - \left(\mu_m(t) E\left[\overline{R}_{m,1}\right]\right)^2 \\
&= \mu_m(t) E\left[\left(\overline{R}_{m,1}\right)^2\right] + E\left[\left(N_t^{(m)}\right)^2\right] E\left[\overline{R}_{m,1}\right]^2 \\
&\qquad - \mu_m(t) E\left[\overline{R}_{m,1}\right]^2 - (\mu_m(t))^2 E\left[\overline{R}_{m,1}\right]^2 \\
&= \mu_m(t) \mathrm{Var}\left(\overline{R}_{m,1}\right) + E\left[\overline{R}_{m,1}\right]^2 \underbrace{\left(E\left[\left(N_t^{(m)}\right)^2\right] - (\mu_m(t))^2\right)}_{=\mathrm{Var}(N_t^{(m)})} .
\end{aligned}
$$

As $\mathrm{Var}(N_t^{(m)}) = \mu_m(t) = \lambda_m t$, we conclude with the result on the first moment (equation (19)).

Proof of Theorem 3 Let us define the two sequences of processes $\left(M_t^{(m)}\right)_t$ and $\left(A_t^{(m)}\right)_t$ by

$$M_t^{(m)} := \sum_{i=1}^{N_t^{(m)}} \ln \overline{R}_{m,i} - \left(r - \frac{\sigma^2}{2}\right) t$$
$$A_t^{(m)} := \sigma^2 t \ .$$

Then, for each m, the processes $\left(M_t^{(m)}\right)_t$ and $\left(\left(M_t^{(m)}\right)^2 - A_t^{(m)}\right)_t$ are martingales. As the jump sizes are of order v_n and vanish in the limit, we deduce from the Martingale Central Limit Theorem as stated in Ethier and Kurtz (1986) that $M \overset{d}{\Longrightarrow} \sigma W$.

Proof of Theorem 8 Similar to Leisen and Reimer (1996), we define the (pseudo–) moments

$$\mathfrak{m}^1_{m,n} := E\left[\exp\{\overline{R}_{m,1}\} - \frac{S_{\tau_{m,1}}}{S_0}\middle| N_T^{(m)} = n\right]$$
$$\mathfrak{m}^2_{m,n} := E\left[\left(\exp\{\overline{R}_{m,1}\}\right)^2 - \left(\frac{S_{\tau_{m,1}}}{S_0}\right)^2\middle| N_T^{(m)} = n\right]$$
$$\mathfrak{m}^3_{m,n} := E\left[\left(\exp\{\overline{R}_{m,1}\}\right)^3 - \left(\frac{S_{\tau_{m,1}}}{S_0}\right)^3\middle| N_T^{(m)} = n\right]$$
$$\mathfrak{p}_m := E\left[\left(\overline{R}_{m,1}\right)\left(\exp\{\overline{R}_{m,1}\} - 1\right)^3\right] \ ,$$

which are the structural properties of the RTBM under consideration. Now, the proof is just the application of a general technique to derive error bounds (see Kloeden and Platen (1992)). For binomial models in the Black–Scholes setup, it was applied in the proof derived by Leisen and Reimer (1996). Decomposing $v^e - v^e_m = \sum_{n=0}^{\infty} P[N_T^{(m)} = n] \cdot \left(v^e - e^{-rT}\Phi_n^{(m)}\right)$, the proof proceeds on $v^e - e^{-rT}\Phi_n^{(m)}$ exactly as in their paper. We sketch the main ideas: A Taylor–series expansion up to order three yields terms of structure $(l = 1, 2, 3)$:

$$\sum_{n=0}^{\infty} P[N_T^{(m)} = n]\, \mathfrak{m}^l_{m,n} \sum_{k=0}^{n-2} e^{-r\tau_{k+1}} E\left[\left(\overline{S}^{(m)}_{\tau_k}\right)^l \frac{\partial^l v^e}{\partial S^l}(\tau^n_{k+1}, \overline{S}^{(m)}_{\tau_k})\middle| N_T^{(m)} = n\right]$$
$$\sum_{n=0}^{\infty} P[N_T^{(m)} = n] \sum_{k=0}^{n-2} e^{-r\tau_{k+1}} E\left[\mathcal{R}_3(\tau^n_{k+1}, \overline{S}^{(m)}_{\tau_{k+1}}, \overline{S}^{(m)}_{\tau_k})\middle| N_T^{(m)} = n\right] \ ,$$

where $\mathcal{R}_3$ is a remainder term.

It remains to prove that the terms $\sum_{k=0}^{n-2} \ldots$ $(l = 1, 2, 3)$ are of order $\mathcal{O}(n)$, and the summation over the remainder is of order $\mathcal{O}(n\mathfrak{p}_m)$. The proof concludes by checking that the (pseudo–)moments have the right order: It is easy to see that $\mathfrak{p}_m = \mathcal{O}(1/\lambda_m^2)$. Furthermore for the $l = 1$ term this can be calculated in a straightforward way using $E[\overline{R}_{m,1}] = 1 + rT/\lambda_m + \mathcal{O}(1/\lambda_m^2)$ and $E[S_{\tau_{m,1}}/S_0|N_T^{(m)} = n] = 1 + rT/n + \mathcal{O}(1/n^2)$. All others can be derived immediately from this by applying the method in Appendix B of Leisen and Reimer (1996).

Proof of Proposition 10 The proof is independent of any specific choice on m. To ease exposition we will therefore omit here any dependence on m. For any $t \in [0, T]$, denote by $\mathcal{F}_t = \sigma(\overline{S}_{t'}|t' \leq t)$ the filtration corresponding to the information observing the discrete process $\overline{S}$, and by $\mathcal{S}$ the stopping times which are adapted to the filtration $(\mathcal{F}_t)_t$ and take values at trading dates before T. We refer the current time after the i–th jump has occurred by $\bar{\tau}_i = \sum_{j=1}^{i} \tau_j$.

We first prove the theorem in the case where the number N_T of jumps on the interval $[0, T]$ is bounded by some $M \in \mathbb{N}$. In a second step we will prove it in its full generality through a limit argument $M \to \infty$. Fix one $M \in \mathbb{N}$. We study

$$\begin{aligned} \text{the driving process} \qquad & N^M = \min\{N, M\} \\ \text{and the process} \qquad & \overline{S}^M = \prod_{i=1}^{N^M} \exp\{\overline{R}_i\} \ , \end{aligned}$$

as well as the filtration (set of stopping times) $\mathcal{F}_t^M = \sigma(\overline{S}_{t'}^M|t' \leq t)$ $(\mathcal{S}^M)$ corresponding to the information observing $\overline{S}$, $\mathcal{F}_t^{N,M} = \sigma(\overline{S}_{t'}^M, N_T^M|t' \leq t)$ $(\mathcal{S}^{N,M})$ corresponding to the information observing $\overline{S}$ and knowing N_T, and $\mathcal{F}_t^{n,M} = \mathcal{F}_t^M \vee \{N_T = n\}$ $(\mathcal{S}^{n,M})$ corresponding to the information that results from the observation of $\overline{S}$, and when it is known that n jumps will occur in total on the interval $[0, T]$.

For fixed M, we define a correspondence $\sigma \leftrightarrow \sigma^N$ between the stopping times $\sigma \in \mathcal{S}^M$ and $\sigma^N \in \mathcal{S}^{N,M}$ such that both yield the same expected payoff as exercise policy; i.e., for any $\sigma^N \in \mathcal{S}^{N,M}$ we assign a $\sigma \in \mathcal{S}^M$ such that $\sigma \wedge N_T^M$ is equivalent to σ^N in the sense that they yield the same expected payoff. This proceeds as follows: First, on the part where $\sigma^N = i$ and $N_T^M \geq i$ we can take $\sigma = i$ since there the exercise decision σ^N depends only on $\overline{S}_i^M$ and on the actual time $\bar{\tau}_i$ (for $i = 1, \ldots, M$). Second, the part where $\sigma^N > N_T^M$ is the complementary part to the union of the previously mentioned sets and hence depends on $\overline{S}_1^M, \ldots, \overline{S}_M^M$ only. In this part, we adopt $\sigma = M$. From the construction $\sigma \in \mathcal{S}^M$ and $\sigma^N \in \mathcal{S}^{N,M}$ yield the same payoff.

Now take any $\sigma \in \mathcal{S}^M$ and its corresponding $\sigma^N \in \mathcal{S}^{N,M}$. Then, we have

$$
\begin{aligned}
& e^{-r\sigma}(K-\overline{S}^M_\sigma)^+ \\
&= \sum_{i=1}^M I_{\sigma=\bar\tau_i} e^{-r\bar\tau_i}\left(K-\overline{S}^M_{\bar\tau_i}\right)^+ \\
&= \sum_{i=1}^M \sum_{j=i}^M I_{N_T^M=j} \underbrace{I_{\sigma=\bar\tau_i}}_{=I_{\sigma^N=\bar\tau_i}} e^{-r\bar\tau_i}\left(K-\overline{S}^M_{\bar\tau_i}\right)^+ \\
&= \sum_{j=1}^M I_{N_T^M=j} \sum_{i=1}^j I_{\sigma^N=\bar\tau_i} e^{-r\bar\tau_i}\left(K-\overline{S}^M_{\bar\tau_i}\right)^+ \ .
\end{aligned}
$$

Thus,

$$
E\left[e^{-r\sigma}(K-\overline{S}^M_\sigma)^+\right] = \sum_{j=1}^M P[N_T^M=j]\cdot E\left[e^{-r\sigma^N}(K-\overline{S}^M_{\sigma^N})^+ | N_T^M=j\right] \ .
$$

For $n \leq M$ now define,

$$
\bar\Psi_n^M := \sup_{\sigma\in\mathcal{S}^{N,M}} E\left[e^{-r\sigma}\left(K-\overline{S}^M_\sigma\right)^+\right] \ .
$$

Due to our correspondence ($\sigma \leftrightarrow \sigma^N$) taking the supremum over σ is equivalent to taking the supremum over σ^N and vice versa. Therefore,

$$
\sup_{\sigma\in\mathcal{S}^M} E\left[e^{-r\sigma}(K-S_\sigma)^+\right] = \sum_{j=1}^M P[N_T^M=j]\cdot\bar\Psi_j \ .
$$

From Theorem 4.3 in Chow, Robbins, and Siegmund (1991) follows

$$
\sup_{\sigma\in\mathcal{S}^M} E\left[e^{-r\sigma}\left(K-S_\sigma\right)^+\right] \xrightarrow{M} \sup_{\sigma\in\mathcal{S}} E\left[e^{-r\sigma}\left(K-S_\sigma\right)^+\right] \ .
$$

Thus, it remains to prove that for any $n \leq M$

$$
\Psi_n = \bar\Psi_n^M \ .
$$

Knowing the number of jumps n having occurred up to time T, calculating $\bar\Psi_n^M$ is a simple backward induction argument at dates $T-\tau_n, T-\tau_n-\tau_{n-1}, \ldots, T-\tau_n-\ldots-\tau_1=0$ using the discount factors $E[\exp\{-r\tau_{m,1}\}|N_T^{(m)}=n]$. This is however exactly how Ψ_n is defined.

Proof of Theorem 11 Denote by Y, the process $Y_t := \mu t + \sigma W_t$; and by h, the function $h : x \mapsto x + \sum_{i=1}^N U_i$ on $\mathcal{D}$.
Since

$$
\sum_{i=1}^{N^{(m)}} \overline{R}_{m,i} \overset{d}{\Longrightarrow} Y,
$$

and the latter is continuous, we have according to VI.1.23 and VI.3.8 (ii) in Jacod and Shiryaev (1987)

$$\overline{X}^{(m)} = h\left(\sum_{i=1}^{N^{(m)}} \overline{R}_{m,i}\right) \overset{d}{\Longrightarrow} h\left(Y\right) = X \ .$$

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