Options on a Stock with Market-Dependent Volatility[1]

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Abstract

A market is considered whose index has strongly price-dependent local volatility. A tractable parametrization of the volatility is formulated, and option valuation of a stock with two-factor dynamics is investigated. One factor is the market index; when the second factor is uncorrelated with the first, the option valuation equation can separate. A formal solution is given for a European call. The call value depends on both the stock price and the market index. Even if the prices of a set of calls were fitted with a one-factor implied volatility, the calls could not be hedged solely with an offsetting position in the stock. For example, delta-hedging involves two deltas, one corresponding to the stock and the other to the market index. In a numerical example, the magnitude of the market delta is found to be significant. The CAPM is used as an example to explore how market-dependent volatilities could be implemented in multifactor models. In the process, the Black-Scholes equation with standard boundary conditions is reduced to quadrature for volatilities of the form $\sigma^2 = \sigma_0^2(1 + a_N S_m^n)/(1 + a_D S_m^n)$; S_m is the market index, and n, σ_0, a_N and a_D are constants.

[1]Draft No. 2 of ewp-fin/9710005 on <http://econwpa.wustl.edu>. A discussion has been added concerning how local volatilities could be implemented in the CAPM and related multifactor models.

1. Introduction

Equity-option prices deviate from Black-Scholes[1] values. The leading interpretation is that the anticipated price distribution is not lognormal. Much current effort is focused on inverting actual option prices to determine the implied distribution[2]. The implied distribution can be used to value exotic options. The market's valuation of stock-index options can be interpreted in terms of price- and time-dependent volatility. For typical contemporary market conditions, the risk-neutral distribution is negatively skewed. Empirical and theoretical esearch remains active[3, 4]; for example, it has been suggested that stochastic or history-dependent volatilities may be needed to explain the temporal behavior of option prices.

This paper treats European options on individual dividendless stocks. Suppose that the market's implied volatility changes due to a sharp price movement. A typical stock's implied volatility would be affected; it is very plausible that it would change even if that particular stock's price did not participate in the sharp market move. This suggests that investigation of a two-factor (at least) description of a stock's risk-neutral dynamics is warranted. The approach in (the present draft of) this paper is to formulate a continuous-time model with a computationally tractable exact solution. European options on non-dividend-paying equities are considered throughout. The calculations will be outlined in Section 2, and results for a numerical example will be presented in the Figures and Tables.

The two-factor model of Section 2 is an *ad hoc* construct for the dynamics of a single equity; formulated for analytic convenience, it is only peripherally related to the multifactor models—the CAPM[5] and successors like ICAPM[6, 7], APT[8], Cox-Ingersoll-Ross[9], Fama-French three-factor[10]—which have explanatory power for equity-portfolio returns. (The mismatch is not with the form of the valuation equation but with the solutions; the $\beta = 1$ calculation of Section 2 can be interpreted as a special case of the CAPM with a residual factor. Issues arise for $\beta \neq 1$.) Thus, Section 3 and an Appendix have been added to discuss the application of local volatilities in factor models; the CAPM is used as a prototype one-factor model. To facilitate the interpretation of the CAPM for price-dependent volatilities, the Black-Scholes equation with standard boundary conditions is reduced to quadrature for volatilities of the form $\sigma^2 = \sigma_0^2 \left(\frac{1+a_N S_m^n}{1+a_D S_m^n}\right)$; S_m is the market index, and n, σ_0, a_N and a_D are constants. Section 4 has concluding remarks.

2. A Two-Factor Case

Let the market index be $S_m = e^{z_m}$ and the stock price be $S = e^{z_\perp + bz_m}$; b is constant. The random variables z_m and $z_\perp$ are independent ($\overline{z_m z_\perp} = \overline{z_m}\,\overline{z_\perp}$) and follow Ito processes with respective volatilities σ_m and $\sigma_\perp$.

For a flat yield curve with interest rate r, the valuation equation of a derivative security depending on S and S_m is[9]

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_m^2 \frac{\partial^2 f}{\partial z_m^2} + \frac{1}{2}\sigma_\perp^2 \frac{\partial^2 f}{\partial z_\perp^2} + \tilde{v}_m \frac{\partial f}{\partial z_m} + \tilde{v}_\perp \frac{\partial f}{\partial z_\perp} - rf = 0. \tag{1}$$

Since $S_m = e^{z_m}$ and $S = e^{z_m + bz_\perp}$ are solutions, the risk-neutral drifts are

$$\begin{aligned} \tilde{v}_m &= r - \frac{1}{2}\sigma_m^2 & (2)\\ \tilde{v}_\perp &= \frac{1}{2}\sigma_\perp^2 + (b-1)\left(r + \frac{1}{2}b\sigma_m^2\right) & (3)\\ &= \frac{1}{2}\sigma_\perp^2 \qquad\qquad (b = 1). & (4) \end{aligned}$$

The $b = 1$ case will be studied henceforth. The valuation equation is

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_m^2 \frac{\partial^2 f}{\partial z_m^2} + \frac{1}{2}\sigma_\perp^2 \frac{\partial^2 f}{\partial z_\perp^2} + (r - \frac{1}{2}\sigma_m^2)\frac{\partial f}{\partial z_m} - \frac{1}{2}\sigma_\perp^2 \frac{\partial f}{\partial z_\perp} - rf = 0. \tag{5}$$

For the dependences $\sigma_m = \sigma_m(z_m)$ and $\sigma_\perp = \sigma_\perp(z_\perp)$, the equation separates, i.e. it has solutions of the form $f(z_m, z_\perp, t) = f_m(z_m, t)\, f_\perp(z_\perp, t)$.

A functional form will be adopted to model the market volatility. For two situations in the 1995 market, Derman, Kani and Zou[11] found that roughly a 10% change in index level produces about a 50% change in local volatility. To encompass such rapid variations, the parametrization

$$\sigma_m(z_m) = \frac{\sigma_0}{\sqrt{1 + ae^{nz_m}}} \tag{6}$$

is adopted. For sufficiently large n, there is a range of market levels in which σ_m rises steeply as S_m decreases. It is convenient to examine the problem at $r = 0$ because an expansion in Bessel functions arises[12]. The valuation equation for the Arrow-Debreu market security expiring at $t = T$ is[13]

$$\frac{\sigma_0^2 S_m^2}{2(1 + aS_m^n)} \frac{\partial^2 G_m}{\partial S_m^2} + \frac{\partial G_m}{\partial t} = -\delta(S_m - S_{m0})\delta(t - T) \tag{7}$$

When this is Fourier-transformed in time and solved in the usual way[13] in terms of the discontinuity in slope at $S_m = S_{m0}$, the result is

$$G_m(S_m,t;S_{m0},T) = -\frac{2\pi i(1+aS_{m0}^n)\sqrt{S_m S_{m0}}}{n\sigma_0^2 S_{m0}^2}\times$$

$$\int \frac{d\omega}{2\pi} e^{i\omega(T-t)} J_{\frac{1}{n}\left(1+\frac{8i\omega}{\sigma_0^2}\right)^{1/2}}\left(\frac{-2i(2ia\omega)^{1/2}S_{m<}^{n/2}}{\sigma_0 n}\right)\times$$

$$H^{(2)}_{\frac{1}{n}\left(1+\frac{8i\omega}{\sigma_0^2}\right)^{1/2}}\left(\frac{-2i(2ia\omega)^{1/2}S_{m>}^{n/2}}{\sigma_0 n}\right). \tag{8}$$

$S_{m>}$ is the greater of (S_m, S_{m0}) and $S_{m<}$ is the lesser. There are two branch cuts which are taken along the imaginary axis in the upper half-plane. Deforming the contour of integration around the cuts leads to

$$G_m(S_m,t;S_{m0},T) = G_{m1}(S_m,t;S_{m0},T) + G_{m2}(S_m,t;S_{m0},T). \tag{9}$$

The first term,

$$G_{m1}(S_m,t;S_{m0},T) = \frac{2(1+aS_{m0}^n)\sqrt{S_m S_{m0}}}{n\sigma_0^2 S_{m0}^2}\times$$

$$\int_0^{\frac{\sigma_0^2}{8}} d\lambda e^{-\lambda(T-t)} J_{\frac{1}{n}\left(1-\frac{8\lambda}{\sigma_0^2}\right)^{1/2}}\left(\frac{2(2a\lambda)^{1/2}S_m^{n/2}}{\sigma_0 n}\right) J_{\frac{1}{n}\left(1-\frac{8\lambda}{\sigma_0^2}\right)^{1/2}}\left(\frac{2(2a\lambda)^{1/2}S_{m0}^{n/2}}{\sigma_0 n}\right), \tag{10}$$

corresponds to the discontinuity of the integrand over the lower cut. The second,

$$G_{m2}(S_m,t;S_{m0},T) = \frac{2(1+aS_{m0}^n)\sqrt{S_m S_{m0}}}{n\sigma_0^2 S_{m0}^2}\int_{\frac{\sigma_0^2}{8}}^{\infty} d\lambda e^{-\lambda(T-t)}\times$$

$$Re\left[J_{\frac{i}{n}\left(\frac{8\lambda}{\sigma_0^2}-1\right)^{1/2}}\left(\frac{2(2a\lambda)^{1/2}S_m^{n/2}}{\sigma_0 n}\right) H^{(2)}_{\frac{i}{n}\left(\frac{8\lambda}{\sigma_0^2}-1\right)^{1/2}}\left(\frac{2(2a\lambda)^{1/2}S_{m0}^{n/2}}{\sigma_0 n}\right)\right], \tag{11}$$

incorporates discontinuities over both cuts. Applying the identity[12]

$$H_\nu^{(2)}(z) = i\csc(\nu\pi)\left(J_{-\nu}(z) - e^{\nu\pi i}J_\nu(z)\right) \tag{12}$$

demonstrates that the integrand in G_{m2} is symmetric in S_m and S_{m0}.

For $r = 0$, $G_m(S_m, t; S_{m0}, T)$ is a probability distribution in S_{m0}[14]. The time-independence of the normalization and first moment provides a consistency check of numerical computations with G_m. Software to compute the Bessel functions is readily available[15].

An Arrow-Debreu security $G_\perp$ can be associated with $S_\perp \equiv e^{z_\perp}$. Before a model for $G_\perp$ is specified, it is helpful to examine the expression for a European call on S with strike E:

$$\begin{aligned} c(S, S_m, T-t; E) &= \int dS_{m0} dS_{\perp 0} G_m(S_m, t; S_{m0}, T) G_\perp(S_\perp, t; S_{\perp 0}, T) \times \\ & \qquad (S_{\perp 0}S_{m0} - E)\Theta(S_{\perp 0}S_{m0} - E). \end{aligned} \tag{13}$$

$\Theta()$ is the unit step function. The formula simplifies if $z_\perp$ follows a generalized Wiener process ($\sigma_\perp$ constant): $G_\perp$ is lognormal in $S_{\perp 0}$, so the $S_{\perp 0}$-integration can be done, leaving

$$\begin{aligned} c(S, S_m, T-t; E) &= \int dS_{m0} G_m(S_m, t; S_{m0}, T) S_{m0} \times \\ & \qquad c_{Black-Scholes}\left(\frac{S}{S_m}, t; \frac{E}{S_{m0}}, T; \sigma_\perp\right). \end{aligned} \tag{14}$$

This scenario will be adopted for its simplicity and to expedite numerical evaluation.

The choice of parameters in a numerical example is driven by the Derman-Kani-Zou cases, in which changing the market index by $\sim \pm 10\%$ leads to a $\sim \pm 50\%$ change in local volatility. This suggests values like

$$\sigma_0 = \frac{10}{27} \tag{15}$$

$$n = 14 \tag{16}$$

$$a = \frac{4}{900^{14}} \tag{17}$$

in equation (6) when the market index is $S_m = 900$. The ensuing market volatility is $\sim 16.5\%$. The stock price is taken as $S = 50$ and the stock

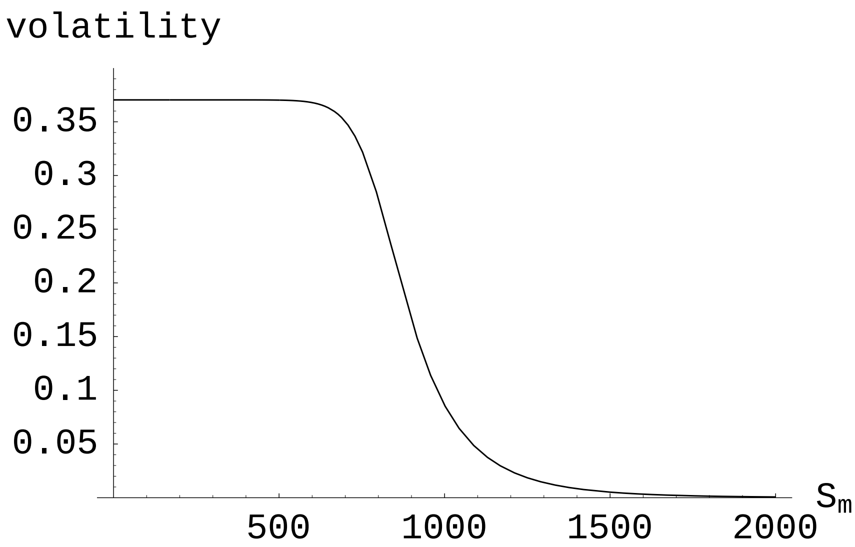

Figure 1: Market volatility $\sigma_m(S_m)$ for the parameters in the text.

volatility as $\sqrt{\sigma_m^2 + \sigma_\perp^2}$, with the assignment

$$\sigma_\perp = \frac{2\sqrt{5}}{27} = .1656, \tag{18}$$

so the volatility of $\ln S$ is $\sim 23\%$. Plots of $\sigma_m(S_m)$ and of $G_m(S_m = 900, t = T-1; S_{m0}, T)$ are shown in Figures 1 and 2. $G_m(S_{m0})$ has a shoulder resembling those found in distributions fitted to market data[11, 16]; the negatively skewed price dependence of $\sigma_m(S_m)$ makes $S_{m0}^{maximum\ likelihood}$ greater than $S_{m0}^{mean} = 900$. Table 1 gives results for a call with one-year maturity.

The call of Table 1 can be fit to a local volatility dependent only on the stock price (and time). Because the true call depends on two factors, a one-factor fit will not correctly describe variations in the call value due to changes in the other factor. (Similarly, the value of an exotic option on the stock differs from the value based on a one-factor local volatility.) In particular, the two-factor call cannot be hedged solely with an offsetting position in the stock. For a derivative security f,

$$f = f(S_m, S_\perp), \tag{19}$$

Strike Price	25	30	35	40	45	50	55	60	65	70	75
Two-Factor	25.1	20.3	15.6	11.4	7.65	4.70	2.63	1.34	.63	.27	.11
Black-Scholes	25.0	20.0	15.3	11.0	7.37	4.66	2.79	1.90	.88	.47	.24

Table 1: **Two-factor call values.** The market index is $S_m = 900$, and the stock price is $S = 50$. Time to expiration is 1 year. For comparison, the Black-Scholes call with volatility equal to the local volatility is given.

Strike	25	30	35	40	45	50	55	60	65	70	75
Δ_S	.991	.973	.939	.873	.759	.596	.416	.258	.143	.072	.033
Δ_m	-.001	-.003	-.006	-.010	-.012	-.012	-.010	-.007	-.004	-.002	-.001

Table 2: **Deltas of the calls of Table 1.** The Δ_m's obtained by directly computing $\frac{\partial G_m}{\partial S_m}$ at $S_m = 900$ are consistent within 10% with the Δ_m's obtained by numerical differentiation at $S_m = 900 \pm 9$. The corresponding sets of Δ_S's agree within 0.2%.

Strike	25	30	35	40	45	50	55	60	65	70	75
$\Delta_S \delta S$	1.28	1.26	1.22	1.13	.98	.77	.54	.33	.185	.09	.04
$\Delta_m \delta S_m$	.01	.05	.10	.16	.20	.20	.17	.12	.07	.04	.02

Table 3: **Typical weekly fluctuations of the calls of Table 1.** The price fluctuations of the underlying equities during a period $\tau = \frac{1}{52}$ are $\delta S_m \sim \pm 16.5$ and $\delta S \sim \pm 1.3$. The call-price fluctuations are expressed as magnitudes.

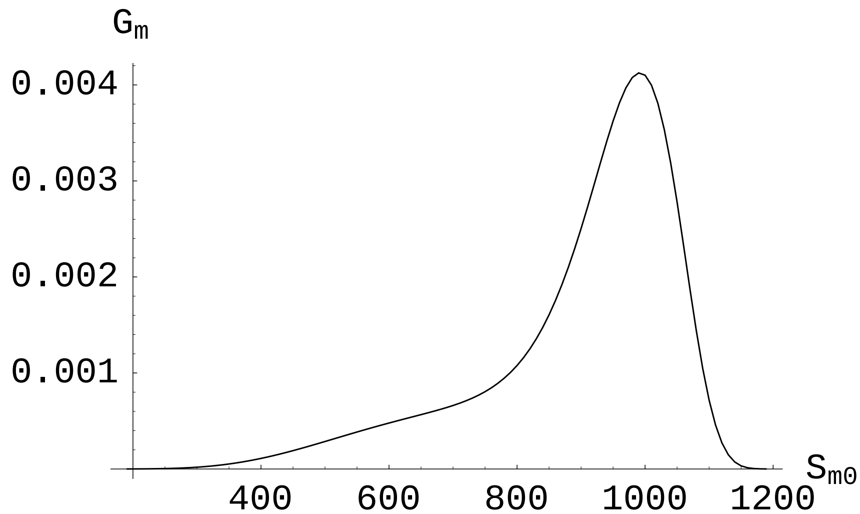

Figure 2: Probability density $G_m(S_{m0})$ for the parameters in the text.

(time dependence is suppressed), differentiation yields an expression for changes of f with respect to changes in S_m and S in terms of the changes of f with respect to changes in the independent variables S_m and $S_\perp$ ($S_\perp = S/S_m$ for $b = 1$):

$$\begin{aligned} df &= \left.\frac{\partial f}{\partial S_m}\right|_{S_\perp} dS_m + \left.\frac{\partial f}{\partial S_\perp}\right|_{S_m} dS_\perp && (20) \\ &= \frac{1}{S_m}\left.\frac{\partial f}{\partial S_\perp}\right|_{S_m} dS + \frac{1}{S_m}\left[S_m \left.\frac{\partial f}{\partial S_m}\right|_{S_\perp} - S_\perp \left.\frac{\partial f}{\partial S_\perp}\right|_{S_m}\right] dS_m && (21) \\ &\equiv \Delta_m dS_m + \Delta_S dS. && (22) \end{aligned}$$

Numerical Δ's are shown in Table 2, which was computed by analytically differentiating the call valuation formula and computing the resulting integral, and checked for consistency by numerically differentiating the call value. Δ_m is negative because, should the market drop while stock S remains unchanged, the stock's volatility increases because of the market-index factor. Δ_m is much smaller than Δ_S, but a typical fluctuation in the market index in a short time τ is

$$\delta S_m \sim \pm S_m \sigma_m \sqrt{\frac{2\tau}{\pi}} \tag{23}$$

whereas a typical fluctuation in S is

$$\delta S \sim \pm S \sigma_S \sqrt{\frac{2\tau}{\pi}}. \tag{24}$$

For an at-the-money call, the value of a S_m-hedge is about 25% of the value of an S-hedge. In the present example, a market hedge is a significant complement to a stock hedge. Table 3 displays the magnitudes of call-value fluctuations associated with the stock and market fluctuations of equations 23 and 24. It clearly shows that, in practice as well as in principle, both factors must be hedged to achieve a riskless option position.

3. Factor Models

The previous section has considered only a single stock. Models like the CAPM *et seq* describe the dynamics of a variety of stocks and portfolios with a small number of factors (and residual factors). How would local volatility work in such models? This issue is beyond the scope of Section 2. Exploratory comments will be made below.

A one- (plus residual) factor model is the obvious place to start. In the Fama-French three-factor model, the size and book-to-market factors are uncorrelated with each other but are correlated with the market factor[17]. Thus, the prototypical CAPM will be used as an example. The presumed market factor is the S&P 500, but, for some stocks, an alternate choice—e.g. Value Line, NASDAQ, a sector index—might provide a heuristically better single-factor description.

For present purposes, the key point about the CAPM is that, if there are no residual factors and $S_m = e^{z_m}$ can assume all nonnegative values, the identity

$$f(S_m, t) = \int dS_{m0} G_m(S_m, t; S_{m0}, T) f(S_{m0}, T), \tag{25}$$

where G_m is the Arrow-Debreu security corresponding to a δ-sink at $(S_m, t) = (S_{m0}, T)$, is satisfied by all regular CAPM securities. In fact, the foregoing is a working definition of "regular" security. It excludes securities like barrier options, which require a G_m with tailored boundary conditions. In some situations, like a portfolio of options with different expiration dates, (25) must be utilized attentively. The overall intention is to use (25) to model CAPM stocks and their vanilla options; G_m, the f's, and concomitant quantities—presumably determined from option prices at a given t—can be applied to formulate hedges, as inputs to the valuation of exotics, etc. Until further notice, the residual factor(s) associated with each stock will be neglected.

For constant σ_m and r, consider the Black-Scholes equation[1]

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_m^2 \frac{\partial^2 f}{\partial z_m^2} + (r - \frac{1}{2}\sigma_m^2)\frac{\partial f}{\partial z_m} - rf = 0. \tag{26}$$

Because they have constant beta, eigenfunctions of the form

$$f_\nu = e^{bz_m + \nu t} = S_m^b e^{\nu t} \tag{27}$$

$$\nu = -\left(\frac{1}{2}\sigma_m^2 b + r\right)(b-1) \tag{28}$$

are taken to represent potential CAPM stocks[2]. (The $S = e^{z_\perp + bz_m}$ of Section 2 is time-independent because of the extra factor $z_\perp$ and its b-dependent drift.) There is a value of ν below which b becomes complex. It can be seen from the original Black-Scholes derivation[1], from Ito's lemma, or by changing variables in (26) that options on the equities (27) are described by a Black-Scholes equation with volatility $b\sigma_m$. Moreover, the betas of linear combinations of eigensolutions (27) can be price- and time-dependent even if σ_m is constant; such combinations, not just single eigenfunctions, can be used to model CAPM equities.

What happens when a time-independent σ_m depends on z_m? By analogy with the constant-σ_m *Ansatz*, it is natural to represent CAPM stocks with eigenfunctions

$$f(S_m, t) = f_\nu(S_m)e^{\nu t}. \tag{29}$$

[1]For constant σ and $\widetilde{v}$, the one-factor equation $\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial \xi^2} + \widetilde{v}\frac{\partial f}{\partial \xi} - rf = 0$ can be reduced to the form (26) by a transformation $\xi = \text{constant} \times \xi'$.

[2]For constant S_m, f_ν will decrease with time when $b > 1$ and $b < -\frac{2r}{\sigma_m^2}$; for $-\frac{2r}{\sigma_m^2} < b < 1$, it will increase. In a multifactor model for which (26) generalizes to a multivariate equation with constant coefficients, a b_i could be associated with each factor z_i, leading to structure in $\nu(\{b_i\})$.

satisfying

$$\frac{1}{2}\sigma_m^2\frac{\partial^2 f_\nu}{\partial z_m^2} + (r - \frac{1}{2}\sigma_m^2)\frac{\partial f_\nu}{\partial z_m} + (-r+\nu)f_\nu = 0 \tag{30}$$

or

$$\frac{1}{2}\sigma_m^2 S_m^2\frac{\partial^2 f_\nu}{\partial S_m^2} + rS_m\frac{\partial f_\nu}{\partial S_m} + (-r+\nu)f_\nu = 0. \tag{31}$$

As has been mentioned, combinations of eigenfunctions could also be put to use. Option values can be expressed in terms of the Arrow-Debreu security and the expiration boundary conditions. For the $r = 0$ market model of Section 2, f is a Bessel function which has oscillatory behavior in S_m when ν is small and positive, corresponding to b slightly less than 1 in the constant-σ_m case. This happens at $r = 0$ because $S_m^2\sigma_m(S_m) \to 0$ as $S_m \to \infty$, and is a consequence of the assumptions adopted to make the valuation equation as tractable as possible; the oscillatory behavior is indicated by the asymptotic structure of the differential equation. It is counterintuitive, especially for $\nu \simeq 0$ i.e. $\beta \simeq 1$, that a CAPM stock's value not depend monotonically in the market index. The vanishing of σ_m can be avoided with a choice like

$$\sigma_m(S_m) = \sigma_0\sqrt{\frac{1 + a_N S_m^n}{1 + a_D S_m^n}}. \tag{32}$$

The eigenfunctions and G_m are worked out in the Appendix. The numerical example considered there has relatively small deviations from exponential behavior, so perhaps analytic approximations exist which are adequate for real-time use.

If σ_m depends on S_m *and* t, ν is not an eigenvalue and cannot be directly used to characterize a CAPM stock $S(S_m, t)$. To do so, one might take refuge in an expansion about a well-characterized case. Writing S by use of an Arrow-Debreu security, i.e.

$$S(S_m, t) = \int dS_{m0} G_m(S_m, t; S_{m0}, T) S(S_{m0}, T) \tag{33}$$

indicates that the functional form of $S(S_m, t)$ is predicated on an assumption about future values of $\sigma_m(S_m, t)$. Let σ_m be time-independent for $t > T^*$ and consider some $S_\nu(S_m, t)$ in this interval since ν *is* an eigenvalue for $t > T^*$. For $t \leq T^*$, (33) determines the value of S for a given S_ν: G_m incorporates the functional dependence of σ_m on S_m and t. (Another simple case is $\sigma_m =$

$\sigma_m(z_m - c\,t)$ with c constant.) If the stock-*qua*-eigenfunction interpretation holds beyond some time horizon, the functional dependence of a CAPM stock on the market index S_m depends on the assessment of future volatility for times before the horizon is reached.

In contrast to the previous paragraph, suppose that no attempt is made to associate a stock $S = S(S_m, t)$ with a single eigenfunction. Let option values be known at all strike prices E. G_m has been determined from market-index option prices. A call c on S satisfies

$$\begin{aligned} c(S(S_m,t),t;E,T) &= \int dS_{m0} G_m(S_m,t;S_{m0},T) \times \\ &\qquad (S(S_{m0},T) - E)\ \Theta\left(S(S_{m0},T) - E\right) \qquad (34) \\ &= \int \frac{dS_0}{|dS_0/dS_{m0}|} G_m(S_m,t;S_{m0}(S_0),T) \times \\ &\qquad (S_0 - E)\,\Theta(S_0 - E), \qquad (35) \end{aligned}$$

where $S_0 \equiv S(S_{m0}, T)$. Differentiating twice with respect to E leads to

$$\frac{\partial^2 c}{\partial E^2} = G_m(S_m,t;S_{m0}(S_0 = E),T) / \left|\frac{dS_0}{dS_{m0}}\right|\Bigg|_{S_0=E} \qquad (36)$$

For $\frac{dS_0}{dS_{m0}} > 0$, this becomes

$$\frac{dS_0}{dS_{m0}} = G_m(S_m,t;S_{m0}(S_0 = E,T),T) / \left.\frac{\partial^2 c}{\partial E^2}\right|_{E=S_0}. \qquad (37)$$

For a given $\sigma_m(S_m, t)$, equations 33 and 37 relate $c(S, t; E, T)$ and $S(S_m, t)$. The $S = S_m$ case of (37) is equivalent to Breeden and Litzenberger's relation[18] for the risk-neutral distribution in terms of the call value.

Although this paper focuses on analyzing valuation models rather than on inverting empirical data in terms of the models, some possibilities about the latter will be briefly presented. Only two factors will be considered; without experience in fitting two factors, a hypothetical discussion about more than two will not be attempted. At a given time t, let call values on a stock S be known at all strikes E and expiration times T. Suppose that (34), together with any necessary criterion to ensure uniqueness, can be inverted to determine $S(S_m, t)$ as a function $\check{\mathcal{S}}(S_m, t)$. If the CAPM holds perfectly for the stock S, $\check{\mathcal{S}}(S_m, t)$ will satisfy (31) and (33).

If deviations from the CAPM, i.e. from the baseline one-factor model, are due to short-term effects and the CAPM holds for times $t > T^*$, (33) can be adapted to define

$$S_{CAPM}(S_m, t) \equiv \int dS_{m0} G_m(S_m, t; S_{m0}, T^*) S(S_{m0}, T^*) \quad t \leq T^* \tag{38}$$

The longest expiration time for which options exist is a natural initial choice for the time horizon T^*. A residual factor Ξ could be introduced to fit the option data in terms of a stock function $S(S_m, \Xi, t)$ constrained to deviate as little as possible from $S_{CAPM}(S_m, t)$. This should work well when the factor model drives most of the stock's dynamics, but probably not as well when much of the dynamics is due to the residual factor. An alternative two-factor description based on Fama-French is to take $\beta = 1$ and lump the two non-market factors together with the residual factor.

Discussion

This paper has presented a soluble two-factor model of a stock for which one of the factors has a nonuniform volatility; for a call, each factor must be hedged separately. A discussion, whose implementation remains for the future, was added concerning how local volatilities might work for a spectrum of multifactor equities. In the process, a four-parameter model of the local volatility was reduced to quadratures. Such solutions may support the development of tractable descriptions of empirical equity-option dynamics.

Appendix

This Appendix displays the eigenfunctions and Arrow-Debreu security associated with the volatility of (32). After the replacement $Y = -a_N S_m^n$ and some rearrangement of terms, (31) can be reduced to the canonical form

$$\begin{aligned} \frac{d^2 f_\nu}{dY^2} \quad &- \quad \left[-\left(\frac{2r}{n\sigma_0^2} - \frac{1}{n} + 1 \right) \frac{1}{Y} + \frac{2r}{n\sigma_0^2} \left(1 - \frac{a_D}{a_N} \right) \frac{1}{Y-1} \right] \frac{df_\nu}{dY} \\ &- \quad \left[\frac{2(\nu - r)}{n^2\sigma_0^2} \frac{1}{Y} - \frac{2(\nu - r)a_D}{n^2\sigma_0^2 a_N} \right] \frac{f_\nu}{Y(Y-1)} = 0. \end{aligned} \tag{39}$$

This is a case of the Papperitz equation[13, Eq. 5.2.38], and so the two linearly independent eigenfunctions f_ν are related to hypergeometric functions:

$$\begin{aligned} f^+_\nu(S_m) &= S_m^{n\lambda_+} F\left(\lambda_+ + \tilde{\nu}_+, 1 - \mu - \tilde{\nu}_+ - \lambda_-; \lambda_+ - \lambda_- + 1; -a_N S_m^n\right) \\ &= S_m^{n\lambda_+} F\left(\lambda_+ + \tilde{\nu}_+, \lambda_+ + \tilde{\nu}_-; \lambda_+ - \lambda_- + 1; -a_N S_m^n\right) \end{aligned} \quad (40)$$

$$f^-_\nu(S_m) = S_m^{n\lambda_-} F\left(\lambda_- + \tilde{\nu}_-, \lambda_+ + \tilde{\nu}_+; \lambda_- - \lambda_+ + 1; -a_N S_m^n\right) \quad (41)$$

$$\lambda_\pm = \frac{1}{2n}\left[1 - \frac{2r}{\sigma_0^2} \pm \sqrt{\left(1 + \frac{2r}{\sigma_0^2}\right)^2 - \frac{8\nu}{\sigma_0^2}}\right] \quad (42)$$

$$\mu = 1 + \frac{2r}{n\sigma_0^2}\left(1 - \frac{a_D}{a_N}\right) \quad (43)$$

$$\tilde{\nu}_\pm = \frac{1}{2n}\left[1 - \frac{2ra_D}{\sigma_0^2 a_N} \pm \sqrt{\left(1 + \frac{2ra_D}{\sigma_0^2 a_N}\right)^2 - \frac{8\nu a_D}{\sigma_0^2 a_N}}\right] \quad (44)$$

The eigenfunction of present interest is f^+_ν, which reduces to S_m for $\nu = 0$.

The choice of parameters in a numerical example is again influenced by the DKZ cases, in which changing the market index by $\sim \pm 10\%$ leads to a $\sim \pm 50\%$ change in local volatility, which is typically between .10 and .25 for longer maturities. This suggests values like

$$\sigma_0 = \frac{1}{4} \quad (45)$$

$$n = 14 \quad (46)$$

$$a_N = \frac{9/25}{900^{14}} \quad (47)$$

$$a_D = \frac{75/25}{900^{14}} \quad (48)$$

The volatility is plotted in Figure 3. To compare a constant-volatility eigenfunction to a hypergeometric one with the same local volatility, first consider a CAPM with constant volatility $\sigma_c = \sigma_m(S_m = 900, n, a_N, a_D)$; the eigenfunctions are given by (27). Compute the $\nu(b, r, \sigma_c)$ for a given b in (28). Figure 4 shows the hypergeometric eigenfunction $f^+_\nu(S_m)$ for the $\nu(b, r, \sigma_c)$ associated with $b = \frac{2}{3}$; the ratio of this eigenfunction to the constant-σ eigenfunction is plotted in Figure 5; and Figure 6 shows

$$\frac{1}{b}\frac{d \ln f^+_\nu}{d \ln S_m}, \quad (49)$$

which characterizes the F eigenfunction's covariance with the market index relative to the market covariance of the corresponding exponential eigenfunction. Figures 7-9 show these quantities for $b = \frac{4}{3}$. The qualitative form of Figures 4-9 can be interpreted in terms of the asymptotics of the f_ν's. The relatively small deviations (of order 10%) from the exponential eigenfunctions displayed in Figures 5 and 8 raises the possibility that simple, accurate analytic approximations exist for option values.

Finally, it remains to calculate the Arrow-Debreu security, from which all vanilla options can be determined. It can be expressed as the Fourier transform[13]

$$G_m(S_m, t; S_{m0}, T) = \frac{-2}{\sigma(S_{m0})^2 S_{m0}^2} \int \frac{d\omega}{2\pi} e^{i\omega(T-t)} \frac{\psi_<(S_<)\psi_>(S_>)}{W[\psi_<(S_<), \psi_>(S_>)]} \quad (50)$$

$$S_< = \min(S_m, S_{m0}); \qquad S_> = \max(S_m, S_{m0}) \quad (51)$$

$$W[\psi_1, \psi_2] = \psi_1 \psi_2' - \psi_2 \psi_1'. \quad (52)$$

$\psi_<(S_m)$ and $\psi_>(S_m)$ are respectively chosen to be regular at $S_m = 0$ and to vanish at $S_m = \infty$. The first condition is satisfied by $\psi_<(S_m) = f^+_{\nu=-i\omega}(S_m)$. As was true for the Bessel functions of Section 2, satisfying the second condition requires a linear combination of f^+ and f^- because each function separately diverges for large argument. The asymptotic form[12] of F dictates that this combination is

$$\psi_>(S_m) = f^-_{-i\omega}(S_m) - \frac{\Gamma(\lambda_- - \lambda_+ + 1)\Gamma(\lambda_+ + \tilde{\nu}_+)\Gamma(1 - \tilde{\nu}_- - \lambda_-)}{\Gamma(\lambda_+ - \lambda_- + 1)\Gamma(\lambda_- + \tilde{\nu}_+)\Gamma(1 - \tilde{\nu}_- - \lambda_+)} f^+_{-i\omega}(S_m). \quad (53)$$

The Wronskian W satisfies the first-order differential equation[13],

$$\frac{dW}{dS_m} + \frac{2r}{S_m\,\sigma(S_m)^2} W = 0; \quad (54)$$

the small-argument behavior of the eigenfunctions determines the boundary condition, and

$$W[\psi_<(S_m), \psi_>(S_m)] = -\frac{\left(\left(1 + \frac{2r}{\sigma_0^2}\right)^2 + \frac{8i\omega}{\sigma_0^2}\right)^{1/2}}{\left(S_m (1 + a_N S_m^n)^{\frac{a_D}{a_N} - 1}\right)^{2r/\sigma_0^2}}. \quad (55)$$

is the solution. Equation (50) can now be computed numerically. However, $e^{i\omega(T-t)}$ oscillates trigonometrically for real ω of large magnitude, and it is desirable to improve convergence by deforming the integration path around the $Im\ \omega$ axis. There are two branch cuts as happened in Section 2; the location of the branch points depends on the parameter values. Moreover, the denominators and gamma functions arising in the solution could give rise to poles associated with discrete ν-eigenvalues. Such discrete eigenvalues do not arise for the parameters currently adopted, and the formula for the present G_m becomes

$$G_m(S_m, t; S_{m0}, T) = \frac{2(1 + a_D S_{m0}^n)(1 + a_N S_{m0}^n)^{-\mu}}{\pi\sigma_0^2 S_{m0}^{2(1-\frac{r}{2\sigma_0^2})}} \times \quad (56)$$

$$\int_{\Lambda_0}^{\infty} \frac{d\Lambda e^{-\Lambda(T-t)}}{\sqrt{\frac{8\Lambda}{\sigma_0^2} - \left(1 + \frac{2r}{\sigma_0^2}\right)^2}} Re\left[\psi_\Lambda^<(S_<)\psi_\Lambda^>(S_>)\right]$$

$$\Lambda_0 = \frac{\sigma_0^2}{8}\left(1 + \frac{2r}{\sigma_0^2}\right)^2. \quad (57)$$

Λ_0 is the location of branch point nearer the origin, and hence is parameter-dependent. Figure 10 shows the Arrow-Debreu security $G_m(S_m = 900, t; S_{m0}, T)$ for $T - t = 1$. The normalization condition, which corresponds to the discounting of a unit of cash, is

$$\int dS_{m0} G_m(S_m, t; S_{m0}, T) = e^{-r(T-t)} \simeq .9512 \quad ; \quad (58)$$

numerical integration of the G_m depicted in Figure 10 gives $\simeq .9505$.

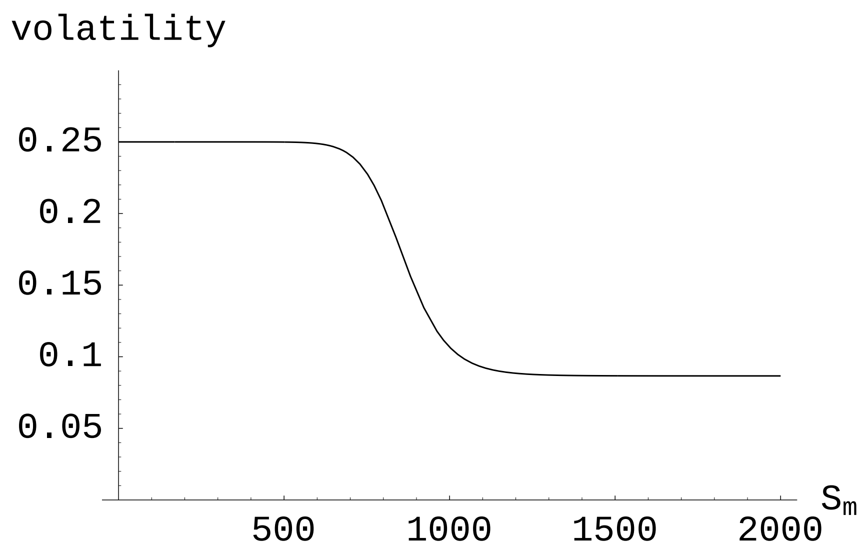

Figure 3: Market volatility $\sigma_m(S_m)$ for the parameters in the Appendix.

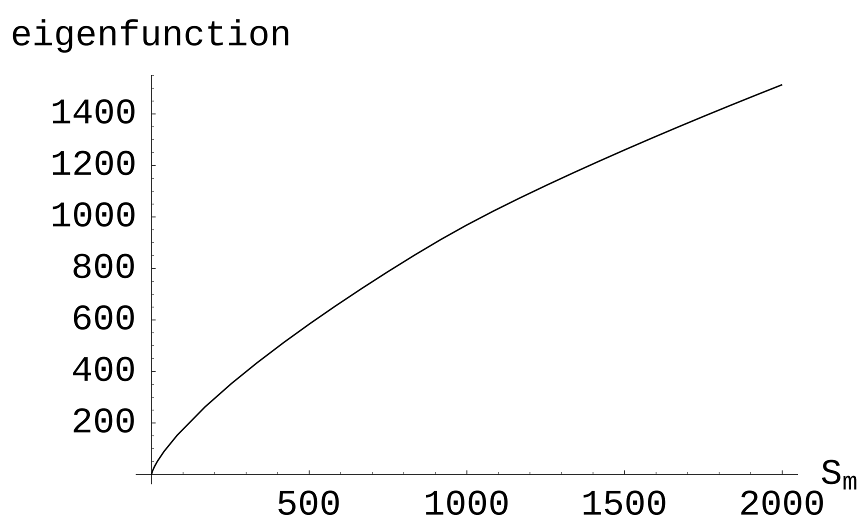

Figure 4: Eigenfunction associated with $b = \frac{2}{3}$ as described in the text. The function is normalized to equal S_m at $S_m = 900$.

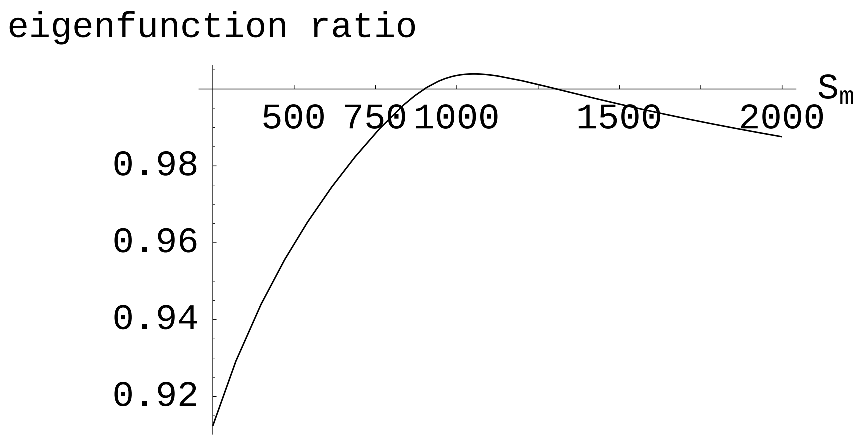

Figure 5: Ratio of the Fig. 4 eigenfunction to the exponential eigenfunction associated with b. Both functions are normalized to equal S_m at $S_m = 900$.

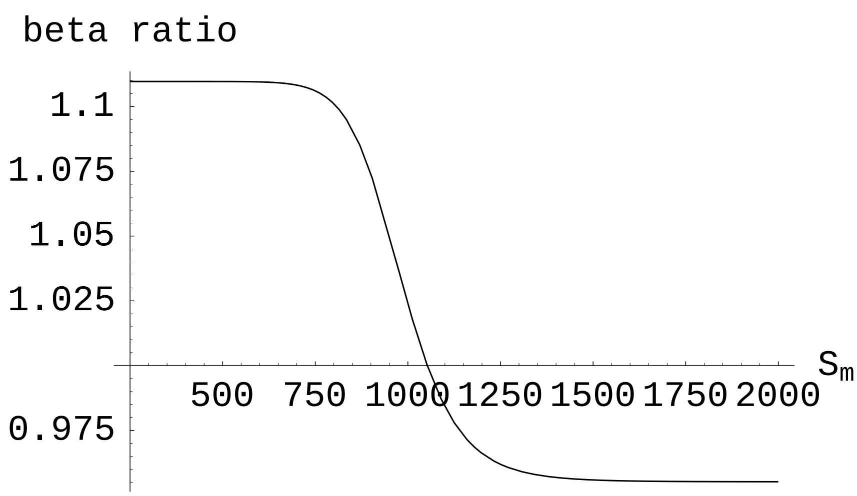

Figure 6: As described in the text, the $b = \frac{2}{3}$ eigenfunction's relative covariance with the market index.

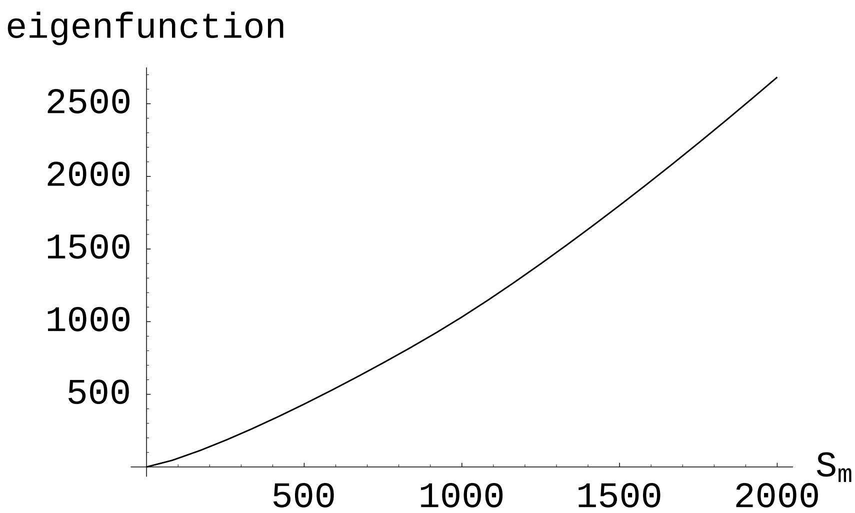

Figure 7: Eigenfunction associated with $b = \frac{4}{3}$ as described in the text. The function is normalized to equal S_m at $S_m = 900$.

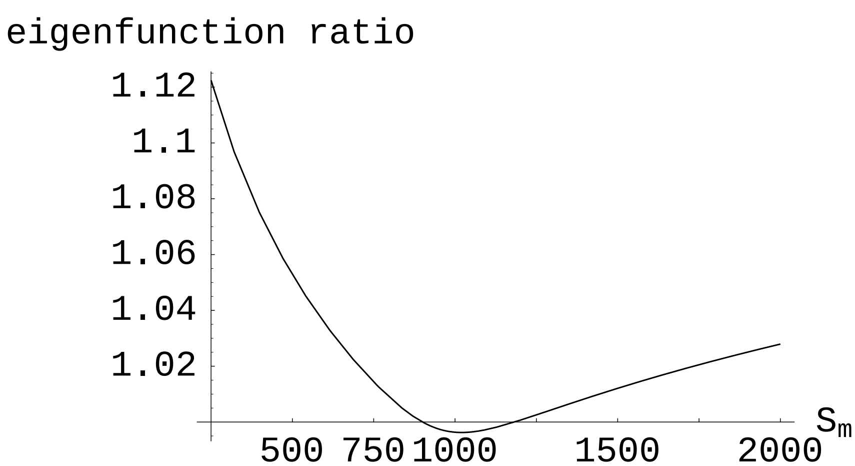

Figure 8: Ratio of the Fig. 7 eigenfunction to the exponential eigenfunction associated with b. Both functions are normalized to equal S_m at $S_m = 900$.

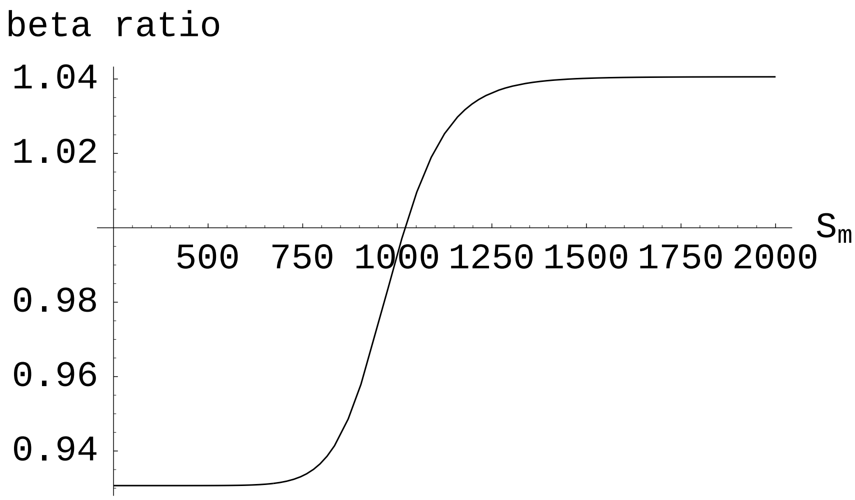

Figure 9: As described in the text, the $b = \frac{4}{3}$ eigenfunction's relative covariance with the market index.

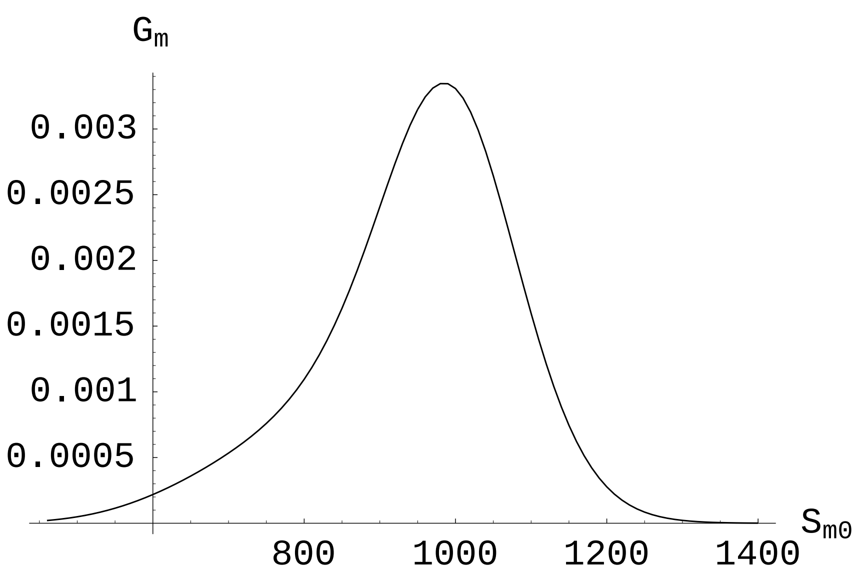

Figure 10: Arrow-Debreu security $G_m(S_{m0})$ for the parameters in the text.

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