The Random Yield Curve and Interest Rate Options

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Abstract

This paper proposes a simple and unifying model to price the interest rate contingent claims in a complete market where trading can be made in continuous time. The underlying dynamics of the yield curve is modelled by a random string whose trajectory produces a random surface described by a Brownian sheet. Generalising Black-Scholes' PDE methodology, we derive the Kolmogorov field equation which describes the time-evolution of the contingent claims and obtain explicit pricing formulae for a large class of interest rate options including European calls, compound options, swaps, swaptions, caps and captions. This model can be thought of as an infinite-factor Gaussian model in the Heath-Jarrow-Morton framework and can be implemented without having to calibrate explicit parameters in the covariance function of the discount bond returns.

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1 Introduction

Arbitrage pricing theory for equity derivatives has been very well established by now since Black and Scholes (1973) first derived their famous option formula. A very thorough account of this theory can be found in the books by Duffie (1992) and Hull (1989). The main idea is to evaluate contingent claims as discounted payoffs by establishing the equivalence between the condition of no arbitrage and the existence of an equivalent Martingale measure. There are two techniques in evaluating the contingent claims. In the Martingale approach, a contingent claim can be priced from the expectation value of the discounted payoff under an equivalent Martingale measure as shown by Harrison and Kreps (1979) and Harrison and Pliska (1981). Alternatively, in the partial differential equation (PDE) approach, one can price the claim by solving its time-evolution equation derived from replicating the claim with a self-financing portfolio in a risk-neutral market.

However, the challenge remains to develop a satisfactory arbitrage pricing theory for interest rate contingent claims. The main difficulty lies in modelling the stochastic term structures which arise from the dependence on maturities. Earlier studies assume that the underlying dynamics is driven by a stochastic short rate process and the results do not satisfactorily capture the full term structure. More recent progress followed Heath-Jarrow-Morton (HJM) (1992) who assume that the interest rate dynamics is driven by the instantaneous forward rates. An alternative formulation based on the zero-coupon (discount) bond price process has also been suggested by Hull and White (1993). In this framework, the term structure is built in by specifying the maturity dependence in the drift and diffusion coefficients of the underlying process. Since the evaluation of contingent claims relies on the Martingale techniques, results are confined to models with a finite number of random factors. As discussed by several authors already, it is very difficult to calibrate the diffusion coefficients such that all interest rate options can be priced reasonably and simultaneously.

In this paper, we assume that the interest rate market is in a continuous time setting where bonds of maturity within a given continuous period are available to trade without friction. The market dynamics is driven by the whole yield curve (zero-rates) and can be modelled with the stochastic process of a random string, where each point on the string represents a particular maturity. This random string process can be formulated in terms of a Brownian sheet on a Hilbert space and subsequently can be described by an infinite number of Brownian particles moving randomly (similar to those studied by Ito (1978, 1983) and Funaki(1983)). In the Martingale approach, it may seem impractical to evaluate anything with an infinite number of random factors. However, there are some encouraging results from Kennedy (1994, 1995) who has obtained explicit pricing formulae for European calls and caps in the Martingale approach by formulating the forward rate process in terms of random fields.

Our approach is different from Kennedy's in that we shall follow Black-Scholes' (PDE) methodology and do not employ Martingale techniques. The idea is to model the zero-coupon bond (discount bond) return with a stochastic process of a random string and construct a self-financing portfolio in a risk-neutral market to derive the time-evolution equation for the contingent claims. This gives the infinite-dimensional analogue of Black-Scholes' PDE. We identify this equation as the Kolmogorov field equation for the underlying zero-coupon bond process whose drift term is given by the short rate. (In other words, the bond price discounted by the short rate is a Martingale.) Solving this field equation enables us to obtain explicit closed-form solutions for a wide range of interest rate options including European calls, compound options, swaptions and captions. As a confirmation of our market assumptions, we recover the same pricing formulae for calls and caps as in Kennedy (1994).

This new approach has several valuable advantages. First, the market assumptions and the derivation of Kolmogorov's field equation are easy to understand. Secondly, finite-factor HJM Gaussian models correspond to special cases of our general results by making special choices of the yield curve covariance function. This unifying feature is a direct consequence of modelling the movement of the whole yield curve with a random string. Most importantly, this model does not parametrise or impose any special functional form for the covariance function as in the finite-factor models. One can implement the results without having to calibrate explicit parameters and therefore avoid the various calibration problems in the finite-factor models.

All the results obtained in this paper rely essentially on the assumption that the interest rate market is efficient and that an investor can trade continuously to hedge away market risk so that contingent claims can be determined completely from the present term structure. Consequently, contingent claims do not depend on factors such as trading volume, information asymmetry or other econometric statistics. Because of this assumption, our model is suitable for very liquid derivatives such as caps, floors, swaptions and other European options on the government bonds of a major currency. In practice, Black-Scholes type models allow some freedom in specifying the market volatility although in theory it is given by the covariance function of interest rates. This freedom enables the users to absorb certain effects due to the lack of market efficiency by accordingly adjusting the implied volatility. Therefore, we are hopeful that a Black-Scholes type interest rate model will be useful in providing a first order approximation for the market term structure.

This paper is organised as follows. In Section 2, we formulate the stochastic yield curve as a random string and discuss its relation with an infinite-dimensional Brownian motion. Readers who are already motivated to consider an infinite-factor HJM model may wish to skip this section. In Section 3, we study this stochastic process for the zero-coupon bond price and construct a self-financing portfolio to derive the time-evolution equation of the contingent claim. This equation is identified as Kolmogorov's field equation which formally has unique solutions given by the Feynman-Kac formula. In Section 4, we solve for a large class of interest rate options explicitly. These include European calls and puts, compound options, swaps, swaptions, caps and captions. Issues regarding model implementation and comparison will be discussed in Section 5. We conclude this paper by analysing the pros and cons of this new approach to studying interest rate derivatives. A proof of the general pricing formulae given in Theorem 4.1 can be found in the Appendix.

2 Random String and Infinite-Dimensional Brownian Motion

To formulate the yield curve dynamics as the random motion of a string, let us denote the length of the string by a positive real number L (e.g. maximum maturity of the yield curve). The time evolution of this string produces a random surface, $Y(s,T)$, with string coordinate $T \in [0,L]$ and time coordinate $s \in \mathbb{R}_+ \equiv [0,\infty]$. The simplest example of such a random surface is a Brownian-sheet whose definition is given as follows.

Definition 2.1: *A Brownian sheet $B(s,T)(\omega) : ([0,\infty] \bigotimes [0,L]) \bigotimes \Omega \to \mathbb{R}$ is a Gaussian process defined on some probability space $\hat{\Omega} \equiv (\Omega, \mathcal{F}, \mathcal{P})$, such that its mean and covariance are given by*

$$
\begin{aligned}
E\left[B(s_1,T_1) - B(s_2,T_2)\right] &= 0, \\
E\left[B(s_1,T_1)B(s_2,T_2)\right] &= \min(s_1,s_2) \times \min(T_1,T_2).
\end{aligned} \tag{2.1}
$$

It is easy to see from the variance of $(B(s_1,T_1) - B(s_2,T_2))$ that the increment is given by the change in area and not in length. Since the random surface makes no distinction between the s and T coordinates, the shape of the string (along T-axis) is fairly jagged. This is not quite the type of random surface that we would expect from the trajectory of a yield curve which in general is smooth along the maturity axis.

This problem can be resolved if we impose a Hilbert space structure on the Brownian sheet as discussed in the book by Da Prato and Zabczyk (1992). The advantage of considering this structure is that the maturity coordinate T simply plays the role of a continuous label on the string. We now give a representation for such a Hilbert space.

Definition 2.2: *Let $L^2([0,L])$ denote the Hilbert space whose basis vectors are a set of orthonormal polynomials $\{e_k(T), k = 0, 1, 2, \ldots, \infty.\}$ such that the following properties are satisfied for every element $Y_s(T) \in L^2([0,L])$:*

$$(2.2)\ (i) \qquad Y_s(T) = \sum_{k=0}^{\infty} Y_s^{(k)} e_k(T), \quad Y_s^{(k)} \equiv \int_0^L du\, Y_s(u) e_k(u),$$

$$(ii) \qquad \int_0^L du\, e_j(u) e_k(u) = \delta_{jk}, \quad \sum_{k=0}^{\infty} e_k(u) e_k(T) = \delta(u - T),$$

$$(iii) \qquad |Y_s|^2 \equiv \int_0^L du Y_s(u) Y_s(u), \qquad 0 \le |Y_s| < \infty$$

$$(iv) \qquad |Y_s| = 0 \qquad \text{iff} \qquad Y_s(u) = 0, \qquad \forall u \in [0, L]$$

These properties correspond to the completeness of $\mathcal{H}$ and that every element Y_s is square-integrable in the interval $[0, L]$ and has a positive definite norm. Examples for the basis are either the shifted Jacobi polynomials or the Fourier series when L is a finite positive real number. When L is infinite, e.g. a consol bond has infinite maturity, the basis can be the Laguerre polynomials.

If the Brownian sheet $B(s,T)$ defined in (2.1) takes values in the Hilbert space, $\mathcal{H} \equiv L^2([0,L] \bigotimes \hat{\Omega})$, then we can expand it in terms of an infinite number of independent Brownian motions as follows:

$$(2.3) \qquad B(s,T)(\omega) = \sum_{k=0}^{\infty} B_s^{(k)}(\omega) \int_0^T du\, e_k(u) \qquad \in \quad \mathcal{H}$$

$$E[B_s^{(j)} - B_s^{(k)}] = 0, \quad E[dB_s^{(j)} dB_s^{(k)}] = \delta^{jk} \cdot ds, \qquad \forall j, k = 0, 1, \ldots, \infty$$

Using this construction, we consider a general stochastic process of a random string on this Hilbert space, $Y(s,T) \in \mathcal{H}$, in the following form:

$$dY(s,T) = M_y(s,T,Y) ds + \int_0^L \sigma_y(s,T,u,Y) B(ds, du),$$

Since $dY(s,T) \in \mathcal{H}$, the drift coefficient is also an element in the Hilbert space $M_y(s,T,Y) \in \mathcal{H}$ and the diffusion coefficient $\sigma_y(s,T,u,Y) \in \mathcal{L}(\mathcal{H})$ is a linear operator which maps $\mathcal{H}$ into $\mathcal{H}$. In terms of the basis expansion, this process is then specified by an infinite number of Brownian motions:

$$dY(s,T) = M_y(s,T,Y)ds + \sum_{k=0}^{\infty} \sigma_y^{(k)}(s,T,Y)dB_s^{(k)}. \tag{2.4}$$

To ensure the existence and uniqueness of a finite solution to this stochastic process equation (2.4), M_y and σ_y must satisfy the growth and Lipschitz properties (see Da Prato and Zabczyk (1992)). Bear in mind that although the individual diffusion mode $\sigma_y^{(k)}$ depends on the choice of the orthonormal basis, the covariance function of the process Y does not.

Proposition 2.1: *The covariance function of $dY(s,T)$ does not depend on the explicit choice of the basis.*

Proof: Using the properties of the orthonormal basis in (2.2.ii),

$$\mathrm{COV}\left[dY(s,T_1)dY(s,T_2)\right] = (ds) \sum_{k=0}^{\infty} \sigma_y^{(k)}(s_1,T_1,Y)\sigma_y^{(k)}(s_2,T_2,Y)$$

$$= (ds) \sum_{k=0}^{\infty} \int_0^L du_1 \sigma_y(s_1,T_1,u_1,Y)e_k(u_1) \int_0^L du_2 \sigma_y(s_1,T_2,u_2,Y)e_k(u_2)$$

$$= (ds) \int_0^L du\, \sigma_y(s_1,T_1,u,Y)\, \sigma_y(s_1,T_2,u,Y) \qquad \square$$

Notice that the covariance function is of the order ds and not dT. Thus, there is no diffusion in the maturity T direction and one can regard T as a continuous label of the string. With suitable diffusion coefficients $\sigma_y^{(k)}$, this process describes a smooth string moving randomly in the time (s) direction. Throughout this paper, we shall assume that the yield curve dynamics is described by this type of stochastic process.

3 Risk-Neutral Portfolio and Kolmogorov's Field Equation

In the previous section, we have proposed to study the yield curve dynamics in terms of an infinite-dimensional Ito process $Y(s,T)$ given in equation (2.4) by introducing the Hilbert space structure to a random string. To construct a Black-Scholes type of pricing model, we need a self-financing portfolio to replicate the derivatives. Since one can trade directly with bonds and not with yields or forward rates, it is therefore easier to formulate the model in terms of the stochastic process of the zero-coupon bond prices. Denote $P(s,T)$ as the current price of a zero-coupon bond maturing at time $T \in [0,L]$ with a principal (face value) of one unit of the currency. By definition, $P(s,T)$ is related to the yield (zero rates) $Y(s,T)$ as follows,

$$P(s,T) = e^{-Y(s,T)(T-s)}, \qquad s \leq T, \qquad P(T,T) = 1. \tag{3.1}$$

Therefore, we come to the first major assumption in this paper:

Assumption 3.1: *The zero-coupon bond price $P(s,T)$, with the face value of one unit of the currency, paid at maturity $P(T,T) = 1$, is described by the following random string process,*

$$\frac{dP(s,T)}{P(s,T)} = M(s,T,P)ds + \sum_{k=0}^{\infty} \sigma^{(k)}(s,T,P)dB_s^{(k)}. \tag{3.2}$$

The drift coefficient M and the diffusion coefficient σ can be determined from the yield curve process in equation (2.4) using Ito's Lemma. In particular, the covariance of the bond returns is related to the yield covariance by

$$\mathrm{COV}\left[\frac{dP(s,T_1)}{P(s,T_1)}\frac{dP(s,T_2)}{P(s,T_2)}\right] = (T_1 - s)(T_2 - s)\mathrm{COV}[dY(s,T_1)dY(s,T_2)]. \tag{3.3}$$

In other words, our model can be thought of as an infinite-factor HJM model in disguise. Had we chosen a special set of diffusion coefficients, say with only m nonzero $\sigma^{(k)}$, this bond process would be equivalent to the one in an m-factor HJM model. In contrast with most literature using the HJM framework, we shall use the PDE approach instead of the Martingale approach to evaluate the prices of contingent claims.

To proceed, we shall assume that the interest rate market is efficient and that there is no arbitrage opportunity. Investors can borrow and lend cash at the same rate for any amount; they can trade and hedge their portfolio continuously at any time; and there are no bid/offer spreads, no transaction/administrative costs, no taxs and no defaults. For simplicity, we will also restrict our discussion to one currency only in this paper.

Assumption 3.2: *The interest rate market is efficient; namely, there is no arbitrage opportunity and all contingent claims can be replicated by a self-financing portfolio in a risk neutral world.*

Under this assumption, an interest-rate contingent claim depends only on the present term structure of the zero-coupon bonds and can be written as a functional of these bonds. Denote the present value of the claim at time s by $C[s,\{P\}]$ where $\{P\}$ is a short-hand notation for the whole term structure $\{P(s,u), \forall\, u \in [0,L]\}$. Using Ito's Lemma for the underlying bond process given in equation (3.2), we have the following infinitesimal variation of the contingent claim:

$$dC[s,\{P\}] = \partial_s C[s,\{P\}]ds + \int_0^L du \frac{\delta C[s,\{P\}]}{\delta P(s,u)} dP(s,u) \tag{3.4}$$
$$+\frac{1}{2}\int_0^L du_1 \int_0^L du_2 \frac{\delta^2 C[s,\{P\}]}{\delta P(s,u_1)\delta P(s,u_2)} \mathrm{COV}[dP(s,u_1)dP(s,u_2)].$$

This formula is similar to those for the finite-dimensional Ito processes. The only difference is in the replacement of ordinary derivatives with functional derivatives $\delta/\delta P(s,u)$ and in summing over contributions from all bonds by integrating over all maturities u.

Suppose an investor is investing with a self-financing portfolio of value V which consists of one unit of the contingent claim and a bond portfolio with a holding of $\theta(s,u)$ units of $P(s,u)$ for all $u \in [0,L]$:

$$V[s,\{P\}] = C[s,\{P\}] + \int_0^L du\, \theta(s,u)P(s,u).$$

By the definition of a self-financing trading strategy $\theta(s,u)$, there is no external cash flow and the earning or loss from the bond portfolio comes solely from the change of the bond prices:

$$dV[s,\{P\}] = dC[s,\{P\}] + \int_0^L du\, \theta(s,u)dP(s,u). \tag{3.5}$$

Substituting in equation (3.4) for $dC[s,\{P\}]$, one can see that the diffusion terms on the right hand side of equation (3.5) cancel if we choose the holding $\theta(s,u)$ to be

$$\theta(s,u) = -\frac{\delta C[s,\{P\}]}{\delta P(s,u)}. \qquad \text{(no risk)} \tag{3.6}$$

In other words, one can hedge away the market risk provided that one can trade continuously to adjust the holding of the bond portfolio according to equation (3.6). With this trading strategy, the gain or loss of this investment can be rewritten as

$$dV[s,\{P\}] = V[s,\{P\}] \cdot r[s,\{P\},V]ds, \tag{3.7}$$

where $r[s, \{P\}, V]$ can be interpreted as the instantaneous return rate of this investment. When the market is efficient (no arbitrage), this return rate should be the same as the short rate offered in the money market. Consistency requires that the short rate depends on the bond prices via

$$r[s, \{P\}] = \lim_{T \to s} -\partial_T \ln P(s, T).$$

Thus, we have the following proposition for the time-evolution equation of the contingent claim.

Proposition 3.1: *Under the market assumptions (3.1) and (3.2), the present price of the contingent claim, denoted by $C[s, \{P\}]$, satisfies the following field equation, given the initial term structure of the zero-coupon bond $\{P\}$ and the final payoff function $C[T, \{P\}] = \Phi(T)$:*

$$\partial_s C[s, \{P\}] - r[s, \{P\}] \left\{ C[s, \{P\}] - \int_0^L du P(s, u) \frac{\delta C[s, \{P\}]}{\delta P(s, u)} \right\} + \frac{1}{2} \int_0^L du_1 \int_0^L du_2 \, Z(s, u_1, u_2) P(s, u_1) P(s, u_2) \frac{\delta^2 C[s, \{P\}]}{\delta P(s, u_1) \delta P(s, u_2)} = 0. \tag{3.8}$$

where $Z(s, u_1, u_2)$ is the covariance function for the bond return:

$$\text{COV}\,[dP(s, u_1) dP(s, u_2)] \equiv Z(s, u_1, u_2) P(s, u_1) P(s, u_2) ds, \tag{3.9}$$

$$Z(s, u_1, u_2) = \sum_{k=0}^{\infty} \sigma^{(k)}(s, u_1) \sigma^{(k)} k(s, u_2).$$

Proof: Combine equation (3.5) and equation (3.7) and substitute equation (3.6) for the self-financing trading strategy θ.

□

This equation describes how the value of a contingent claim evolves in time and is determined by the initial term structure and the correlation function of the bond returns Z defined in equation (3.9). Since this covariance function does not depend on which orthonormal basis is used, it is not necessary to specify each individual diffusion coefficient $\sigma^{(k)}$ in order to evaluate the contingent claim. *What is observed and really matters is the total covariance function* $Z(s, T_1, T_2)$. We shall discuss in more details in Section 5 on how to calibrate this function empirically.

Similarly to Black-Scholes' PDE, equation (3.8) can be interpreted as the Kolmogorov field equation for an underlying bond process whose drift term is given by the short rate $r[s,\{P\}]$. (We have called this a field equation to reflect the infinite number of degrees of freedom involved.) Hence the question of how to evaluate an interest-rate contingent claim can be answered by solving this field equation with a boundary condition given by the final payoff function. Formally, one can prove that there exists a unique solution given by the Feynman-Kac formula (see Da Prato and Zabczyk (1992)):

$$C[s,\{P\}] = E\Big[C[T_0,\{P\}]e^{-\int_s^{T_0} r[t,\{P\}]dt}\Big]$$

In other words, the current price of a contingent claim is the expected value of its final payoff discounted by the short rate. In practice, it is not easy to evaluate this expectation nor to solve the field equation. Luckily, a large class of interest rate options belongs to the class of degree-one homogeneous functions of the zero-coupon bonds. This homogeneous property of contingent claims were first considered by Merton (1973). For these options, the field equation (3.8) becomes much simpler because the two terms multiplied by the short rate cancel each other. As shown in the next section, we can solve for these pricing formulae explicitly when the covariance function Z is deterministic.

4 Pricing Formulae of Interest Rate Options

Let us consider European (path-independent) interest rate options whose final payoff is of the following type.

Definition 4.1: *Let the pay-off (boundary) function Φ be a class of homogeneous functions of degree 1 which satisfy*

$$\sum_{j=0}^{N} y_j \frac{\partial}{\partial y_j}\Phi[y_0,y_1,\ldots,y_N] = \Phi[y_0,y_1,\ldots,y_N]. \tag{4.1}$$

For the examples considered in this paper, such as European calls and compound options, the payoff function Φ belongs to one or a combination of the following two types:

(4.2) (*i*)
$$\Phi[y_0,y_1,\ldots,y_N] = \sum_{j=0}^{N} a_j y_j,$$

(*ii*)
$$\Phi[y_1,y_2] = y_2\,\mathcal{N}\left(\frac{\ln(\frac{y_2}{Ky_1}) + \frac{1}{2}a^2}{a}\right) - K\cdot y_1\,\mathcal{N}\left(\frac{\ln(\frac{y_2}{Ky_1}) - \frac{1}{2}a^2}{a}\right),$$

where a is deterministic, a_j and K are constant, and $\mathcal{N}(h)$ is the accumulated normal distribution function, $\mathcal{N}(h) \equiv \int_{-\infty}^{h} \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

Given a boundary condition of this type, we can proceed easily to solve for Kolmogorov's field equation when the covariance function of the bond return $Z(s, u_1, u_2)$ is deterministic. The solutions can be written in terms of integrals similar to those of Harrison-Pliska (1981). We state the result as follows (the details of the proof can be found in the appendix):

Theorem 4.1: *If, under the market assumptions in (3.1) and (3.2),*

(a) *the final payoff of the contingent claim at time T_0 is given by the homogeneous degree-1 payoff function Φ defined in equation (4.1),*

$$C_+[T_0, \{P\}] = \Big(\Phi\left[P(T_0,T_0), P(T_0,T_1), \ldots, P(T_0,T_N)\right]\Big)_+ \tag{4.3}$$

where $0 \leq s \leq T_0 < T_1 \ldots < T_N$ and $(A)_+$ denotes $\max(A, 0)$, and

(b) *The following correlation matrix W is deterministic and not singular*

$$W_{ij}(s) = \frac{1}{b_i(s) b_j(s)} \int_s^{T_0} dt\, Z(t, T_i, T_j), \tag{4.4}$$
$$b_i^2(s) = \int_s^{T_0} dt\, Z(t, T_i, T_i), \qquad i = 0, 1, \ldots, N.$$

then the pricing formula for the contingent claim is given by

$$C_+[s, \{P\}] = \int_{-\infty}^{\infty} d^{N+1}x\ g(x_0, x_1, \ldots, x_N, W) \times \tag{4.5}$$
$$\left(\Phi\left[P(s,T_0)e^{b_0 x_0 - \frac{1}{2}b_0^2}, P(s,T_1)e^{b_1 x_1 - \frac{1}{2}b_1^2}, \ldots, P(s,T_N)e^{b_N x_N - \frac{1}{2}b_N^2}\right]\right)_+$$

where $g(x_0, x_1, \ldots, x_N, W)$ is the multivariate normal distribution density

$$g(x_0, x_1, \ldots, x_N, W) = \frac{1}{\sqrt{(2\pi)^{N+1} \det W}} \exp\left(-\frac{1}{2}\sum_{i,j=0}^{N} x_i (W^{-1})_{ij} x_j\right) \tag{4.6}$$

Proof: See Appendix.

In the rest of this section, we apply this result to obtain explicit option formulae for European calls, puts; compound options; swaps, swaptions; caps and captions.

Example 4.1: European Call and Put

For a European call on a zero-coupon bond maturing at time T with a strike price K, the final payoff at the option expiry T_0 is

$$C_{call}(T_0) = \Big(P(T_0,T) - K \cdot P(T_0,T_0)\Big)_+ .$$

This corresponds to the boundary condition in (4.2.i) with $N = 1$. Therefore, we can integrate equation (4.5) correspondingly and obtain the call price at the present time s as

$$\text{(4.7)} \quad C_{call}[s, P(s,T_0), P(s,T)] = P(s,T)\mathcal{N}\left(\frac{\ln\frac{P(s,T)}{K\cdot P(s,T_0)} + \frac{1}{2}\zeta^2(s,T_0,T)}{\zeta(s,T_0,T)}\right)$$
$$-K \cdot P(s,T_0)\mathcal{N}\left(\frac{\ln\frac{P(s,T)}{K\cdot P(s,T_0)} - \frac{1}{2}\zeta^2(s,T_0,T)}{\zeta(s,T_0,T)}\right)$$

and the volatility function $\zeta(s,T_0,T)$ is given by

$$\text{(4.8)} \qquad \zeta^2(s,T_0,T) = \int_s^{T_0} dt\Big(Z(t,T,T) + Z(t,T_0,T_0) - 2Z(t,T_0,T)\Big).$$

On the other hand, an European put on a zero-coupon bond $P(s,T)$ has a final payoff at the option expiry T_0 given by

$$C_{put}(T_0) = \Big(K \cdot P(T_0,T_0) - P(T_0,T)\Big)_+ .$$

A similar integration shows that the put price satisfies the put-call parity,

$$\text{(4.9)} \qquad C_{call}(s) - C_{put}(s) = P(s,T) - K \cdot P(s,T_0).$$

Equation (4.7) is very similar to Black-Scholes' formula except for the different volatility function ζ in equation (4.8). This supports the rough approximation given by Black-Scholes' formula which has been used frequently among practitioners. The merit here is that the volatility function $\zeta(s, T_1, T_2)$ in the pricing formulae should take into account the maturity dependence in the covariance function of the bond returns $Z(s, T_1, T_2)$ via the relation given in equation (4.8). Different interest-rate models in the HJM framework correspond to different choices of Z's. We shall return for more discussion on the calibration of the correlation function in section 5.

Example 4.2: Compound Option (Call on Call)

Consider a compound option which is an European call maturing at time T_0 on another European call maturing at a later time T_1. Let the underlying zero-coupon bond of the call be $P(s, T_2)$ and the strike price of the call be K, while the strike price for the compound option is K_c. Then, the final payoff of the compound option at expiry T_0 is

$$C_{cmp}(T_0) = \Big(C_{call}\left[T_0, P(T_0, T_1), P(T_0, T_2)\right] - K_c \cdot P(T_0, T_0)\Big)_+$$

where the call price C_{call} is given in equation (4.7). This corresponds to solving for the boundary condition in (4.2.ii) with $\Phi[y_1, y_2] = C_{call}[T_0, y_1, y_2]$. Substituting this Φ into equation (4.5) and evaluating the integral with $N = 2$, we obtain the following pricing formula for the compound option:

$$\text{(4.10)} \qquad C_{cmp}(s) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{G(x_1,x_2)} dx_0 \quad g(\{x\}, W)$$
$$\left(C_{call}\left[T_0, P(s, T_1)e^{b_1 x_1 - \frac{1}{2}b_1^2}, P(s, T_2)e^{b_2 x_2 - \frac{1}{2}b_2^2}\right] - K_c P(s, T_0) e^{b_0 x_0 - \frac{1}{2}b_0^2}\right),$$

where the x_0 integration limit is given by

$$\text{(4.11)} \quad G(x_1, x_2) = \frac{1}{b_0} \ln\left(\frac{C_{call}\left[T_0, P(s, T_1)e^{b_1 x_1 - \frac{1}{2}b_1^2}, P(s, T_2)e^{b_2 x_2 - \frac{1}{2}b_2^2}\right]}{K_c P(s, T_0)}\right) + \frac{b_0}{2},$$

and C_{call} is given in equation (4.7). In general, this integral can only be evaluated numerically.

Example 4.3: Swap and Swaption

A payer forward swap is an agreement to pay a fixed-rate interest and receive a floating-rate interest in return. Suppose the swap is settled and paid in advance for N periods starting from T_0 and ending at T_N. Take the float-rate to be the LIBOR rate $L(T_j)$ during the period $[T_j, T_{j+1}]$ and the fixed-rate to be some constant K. Then the payment at the beginning of each period is

$$C^{(j)}_{swap}(T_j) = \frac{(T_{j+1} - T_j)\,(L(T_j) - K)}{1 + L(T_j)(T_{j+1} - T_j)} \qquad \forall \quad j = 0, 1, \ldots, N-1.$$

Here, we have taken account of discounting the payment from T_{j+1} to T_j. This payment can be written in terms of zero-coupon bond prices because the LIBOR rate is related to the zero-coupon bond price as follows:

$$P(T_j, T_{j+1}) = \frac{1}{1 + L(T_j)(T_{j+1} - T_j)}. \tag{4.12}$$

The swap payment due at the beginning of the j-th period is therefore

$$C^{(j)}_{swap}(T_j) = P(T_j, T_j) - (1 + K(T_{j+1} - T_j))\, P(T_j, T_{j+1}).$$

Since this is a linear functional of the bonds, the corresponding solution of the Kolmogorov equation is simply

$$C^{(j)}_{swap}(s) = P(s, T_j) - \Big(1 + K(T_{j+1} - T_j)\Big) P(s, T_{j+1}).$$

It is straightforward to verify this solution using the property of the functional derivative given in the appendix. Therefore, the total forward swap price at the present time s is simply the sum of all the contributions from N periods:

$$C_{swap}(s) = P(s, T_0) - P(s, T_N) - \sum_{j=0}^{N-1} K(T_{j+1} - T_j) P(s, T_{j+1}). \tag{4.13}$$

The forward swap rate is chosen to be the fixed rate K_f such that the total float-rate payment equals the total fixed-rate payment at the beginning of the swap. Namely, the forward swap price is zero at the present time s,

$$K_f = \frac{P(s,T_0) - P(s,T_N)}{\sum_{j=0}^{N-1}(T_{j+1}-T_j)P(s,T_{j+1})}. \tag{4.14}$$

These swap formulae can be found in the work of Brace and Musiela (1994). Next, we consider a payer swaption where the owner buys the right to swap the float rate payments with the fixed rate payments (say a strike rate K) for N periods in $[T_0, T_N]$ starting at some future time $T_0 > s$. This is equivalent to a European call on a payer swap of N periods starting at T_0. At the expiry of the swaption, T_0, the value of the contract is

$$C_{swaption}(T_0) = \Big(C_{swap}(T_0)\Big)_+$$
$$= \left(P(T_0,T_0) - P(T_0,T_N) - K\sum_{j=0}^{N-1}(T_{j+1}-T_j)P(T_0,T_{j+1})\right)_+ .$$

This corresponds to the boundary condition given in (4.2.i). Thus, the present price of the swaption can be evaluated by substituting the payoff function $\Phi = C_{swap}[T_0, \{P(T_0,T_j)\}]$ into equation (4.5):

$$C_{swaption}(s) = \int_{-\infty}^{\infty} d^N x \int_{G_N}^{\infty} dx_0\, g(\{x\},W)\times \tag{4.15}$$
$$\left(P(s,T_0)e^{b_0x_0-\frac{1}{2}b_0^2} - \sum_{j=1}^{N}\Big(\delta_{j,N} + K(T_j - T_{j-1})\Big)P(s,T_j)e^{b_jx_j-\frac{1}{2}b_j^2}\right),$$

with the x_0 integration limit G_N given by

$$G_N = \frac{1}{b_0}\left(\ln\left(\frac{\sum_{j=1}^N a_jP(s,T_j)e^{b_jx_j-\frac{1}{2}b_j^2}}{P(s,T_0)}\right) + \frac{1}{2}b_0^2\right),$$

and $a_j \equiv \delta_{j,N} + K(T_{j+1} - T_j)$. The b's and W are given as in equation (4.4). For an one-period swap, $N = 1$, this integration gives rise to an accumulated bivariate normal distribution function. For $N > 1$, it is in general not possible to integrate this analytically. Numerical methods will be needed to evaluate this integral. This swaption formula differs from those of Brace and Musiela

(1994,1995) because the covariance function of bond returns (or the forward rates) does not factorise in general as assumed in their papers.

Example 4.4: Cap and Caption

Let us consider a forward start cap consisting of N caplets starting at T_0 and ending at T_N. This can be rewritten as N European calls on the LIBOR rates $L(T_j)$ for the period $[T_j, T_{j+1}]$ where each call has a cap-rate K_j and the payment due at the beginning of the period T_j is

$$C_{cap}^{(j)}(T_j) = \left(\frac{(T_{j+1} - T_j)(L(T_j) - K_j)}{1 + L(T_j)(T_{j+1} - T_j)} \right)_+ .$$

In terms of the zero-coupon bonds, the j-th payment at the beginning of the period is

$$C_{cap}^{(j)}(T_j) = \Big(P(T_j, T_j) - (1 + K_j(T_{j+1} - T_j))\, P(T_j, T_{j+1}) \Big)_+ .$$

Thus, each individual caplet is equivalent to a European put on a bond maturing at the end of the period T_{j+1} where the bond principal is $(1+K_j(T_{j+1}-T_j))$ and the strike price is the inverse of the principal. The present value of each caplet is then

$$C_{cap}^{(j)}(s) = P(s, T_j)\mathcal{N}(h_j^+) - (1 + K_j(T_{j+1} - T_j))P(s, T_{j+1})\mathcal{N}(h_j^-), \tag{4.16}$$

where

$$h_j^{\pm} \equiv \left(\frac{-\ln\left(\frac{(1+K_j(T_{j+1}-T_j))P(s,T_{j+1})}{P(s,T_j)}\right) \pm \frac{1}{2}\zeta^2(s, T_j, T_{j+1})}{\zeta(s, T_j, T_{j+1})} \right), \tag{4.17}$$

and the volatility function is given by

$$\zeta^2(s, T_j, T_{j+1}) \equiv \int_s^{T_j} dt \Big(Z(t, T_j, T_j) + Z(t, T_{j+1}, T_{j+1}) - 2Z(t, T_j, T_{j+1}) \Big). \tag{4.18}$$

With N such caplets, the total cap price at the present time s is simply the sum of all caplets

$$C_{cap}(s) = \sum_{j=0}^{N-1} C_{cap}^{(j)}(s). \tag{4.19}$$

Next, we consider a caption which is an European call on a cap starting at time T_0 with cap rate as in equation (4.16), namely the interest rate is capped at K_j for each j-th period. The final payoff of the caption at expiry T_0 is the value of the underlying cap if the cap is positive or else the payoff will be zero:

$$C_{cptn}(T_0) = \Big(C_{cap}[T_0, P(T_0, T_0), P(T_0, T_1), \ldots, P(T_0, T_N)] \Big)_+ .$$

Hence the caption's present price can be evaluated as a compound option on the cap given in equation (4.19), namely

$$C_{cptn}(s) = \int_{-\infty}^{\infty} d^N x \int_{G(x_1,\ldots,x_N)}^{\infty} dx_0\, g(\{x\}, W) \times \tag{4.20}$$

$$\left(\sum_{j=0}^{N-1} C_{cap}^{(j)} \left[T_0, P(s, T_j) e^{b_j x_j - \frac{1}{2} b_j^2}, P(s, T_{j+1}) e^{b_{j+1} x_{j+1} - \frac{1}{2} b_{j+1}^2} \right] \right),$$

where the x_0 integration limit $G(x_1, \ldots, x_N)$ is the solution to x_0 in the following equation:

$$\sum_{j=0}^{N-1} C_{cap}^{(j)} \left[T_0, P(s, T_j) e^{b_j x_j - \frac{1}{2} b_j^2}, P(s, T_{j+1}) e^{b_{j+1} x_{j+1} - \frac{1}{2} b_{j+1}^2} \right] = 0$$

and the caplet $C_{cap}^{(j)}$ can be found in equation (4.16)-(4.19). Again, for $N > 1$, this integration has to be carried out numerically.

5 Model Implementation and Comparison

In order to implement the results in this paper properly, it is crucial to have a good estimate of the covariance function of bond returns, $Z(s, T_1, T_2)$, from empirical data. Since our pricing formulae are derived for the case when Z is a deterministic function of time, while in reality Z is most likely to be stochastic. We need to find an approximation function for Z which will fit the market prices. Different assumptions on how Z behaves produce different interest rate models. Here, we discuss two types of covariance functions suggested in the literature.

5.1 The Random Field Model (Kennedy):

Let us rewrite Z in terms of the covariance function of the instantaneous forward rates, $F(s,u) \equiv -\partial_u \ln P(s,u)$,

$$Z(s,T_i,T_j)\,ds = \int_s^{T_i} dv_1 \int_s^{T_j} dv_2 \,\mathrm{COV}[dF(s,v_1)dF(s,v_2)].$$

Then the result in Example 4.1 recovers the European call price (and the caplet in Example 4.3) obtained in the random field model by Kennedy (1994). Furthermore, if we substitute the forward rate covariance function suggested by Kennedy (1995), assuming that the forward rate is a time-stationary Markov process, Z is then specified by three constant parameters, σ, λ and μ ($\mu > \lambda$):

$$Z(s,T_i,T_j) = \int_s^{T_i} du_1 \int_s^{T_j} du_2 \,\sigma^2\, \lambda e^{\lambda s}\, e^{((2\mu-\lambda)\min(u_1,u_2)-\mu(u_1+u_2))}.$$

Empirical testing will then require estimating these three constant parameters and their stability. However, one may not expect a good fitting if the instantaneous forward rate does not resemble a Markov process.

5.2 The m-Factor HJM Gaussian Models (Brace and Musiela):

By choosing a finite number of nonzero deterministic diffusion coefficients $\sigma^{(k)}$, we can easily recover the results in multi-factor Gaussian models. For example, to compare with the results of Brace and Musiela (1994a), let the forward rate diffusions $\sigma_f^{(k)} = 0$ for $k \leq m$ and the covariance function of bond-returns becomes

$$Z(s,T_i,T_j) = \int_s^{T_i} dv_1 \int_s^{T_j} dv_2 \sum_{k=0}^{m-1} \sigma_f^{(k)}(s,v_1)\sigma_f^{(k)}(s,v_2).$$

Notice that the following covariance function is related to our correlation matrix W as follows:

$$\Delta_{ij} \equiv \mathrm{COV}\,[\ln\, P(T_0,T_i)\, \ln\, P(T_0,T_j)] = W_{ij}\, b_i\, b_j.$$

If Δ_{ij} can be factorised into a product of two m-dimensional vectors $\vec{\gamma}_i \cdot \vec{\gamma}_j$, then, for contingent claims with a linear payoff-function as in (4.1.i), our pricing formula in Theorem 4.1 agrees with Theorem 3.1 in their paper. For implementation, these authors further assume that the forward rate diffusions are functions of time to maturity only, $\sigma_f^{(k)}(s,v) = \tau^{(k)}(v-s)$.

5.3 Nonparametric Calibration for the Covariance Function Z

In order to capture the full term structure accurately, it is better to calibrate the whole covariance function of the bond returns $Z(s, T_1, T_2)$ and not the forward rate diffusions $\sigma_f^{(k)}(s,T)$. Here, we propose how one may proceed with a non-parametric calibration.

First, from the discount bonds and their correlation matrix including the variances at a set of discrete maturities, say monthly or quarterly (for either implied or historical volatility), we can obtain a table of data which define (e.g. generate numerically from Mathematica) an interpolating function $F(T_i - s, T_j - s) = Z(s, T_i, T_j)$ for a given date s. By studying a series of such interpolating functions, we can see how they vary in time s. If the discount bond covariance is time-stationary,

$$Z(s+h, T_i+h, T_j+h) = Z(s, T_i, T_j)$$

we can integrate the interpolation function $F(T_i - s, T_j - s)$ over time s to obtain the correlation matrix W in the pricing formulae. If the discount bond covariance is not time-stationary, the covariance function can be approximated by a time-dependent scaling factor $H(s)$ which is determined empirically.

$$Z(s, T_i, T_j) = H(s) \times F(T_i - s, T_j - s).$$

In contrast with Brace and Musiela (1994), we do not assume that the correlation matrix factorises into some finite-dimensional vectors $\Delta_{ij} = \vec{\gamma}_i \cdot \vec{\gamma}_j$. More empirical studies are needed in order to have a better understanding of how the covariance function vary in time.

6 Conclusions

In this paper we generalise Black-Scholes' methodology (PDE approach) to formulate the arbitrage pricing theory of the interest rate contingent claims in a complete market. The underlying assumptions are that the yield-curve dynamics is driven by a stochastic process of a random string and that an investor can trade continuously in an arbitrage-free market where the bonds mature within a continuous period $T \in [0, L]$. The first assumption leads us to specify the stochastic process of the discount bond returns in terms of an infinite number of Brownian motions and the second enables us to derive the Kolmogorov equation which describes the time evolution of a contingent claim. We obtain explicit pricing formulae for a large class of interest rate options by solving this equation when the covariance function of the bond returns $Z(s, T_1, T_2)$ is deterministic. Although these are obviously idealised descriptions of the real interest rate market, the advantage of obtaining simple analytic results is well worth pursuing. Our results are also very general in the sense that they include finite-factor HJM Gaussian models by choosing special functional forms of the covariance function of the bond return.

Since we formulate the interest rate dynamics in terms of the bond returns directly rather than the instantaneous forward rates as in the usual HJM models, the option pricing formulae depend explicitly on the covariance function of the bond returns Z (or the yield curve differentials as in (3.3)). This covariance function is not derived theoretically but requires input from empirical data. So far, much effort has been put in the implementation of finite-factor HJM models by calibrating the parameters in the forward rate volatility function. Our results suggest that it is better to calibrate the whole covariance function of the bond-returns $Z(s, T_1, T_2)$ directly or equivalently the implied volatility function ζ's defined in equation (4.8).

Thus, the state of the art in pricing interest rate derivatives lies in the knowledge of the covariance function of the bond returns Z (or equivalently the yield curve differentials). The more we know about this function, the better we can capture the full term structure. The disadvantage of the finite-factor HJM models is that they impose restricted functional forms on Z and therefore reduce the model's capability to describe the full term structure. Our model does not impose such restrictions on Z and allows us to price the interest rate options as long as the covariance function is deterministic. As discussed in the last section, it is possible to implement the model in a non-parametric way. Nonetheless, it will be very illuminating to carry out further theoretical studies of the covariance function based on more general considerations such as those discussed by Kennedy (1995) and to test it against empirical studies.

A Appendix

Here we sketch the proof of Theorem 4.1 when the payoff function Φ is monotonically decreasing in x_0. (For other types of Φ, either monotonically increasing in x_0 or x_j, the proof can be deduced in a similar fashion.) Let us first rewrite the solution with explicit integration boundary as follows:

$$(A.1) \qquad C_+(s) = \int_{-\infty}^{\infty} d^N x \int_{-\infty}^{G_N} dx_0\, g(\{x\}, W)\Phi[y_0, y_1, \ldots, y_N],$$

where $y_j \equiv P(s, T_j)e^{b_j x_j - \frac{1}{2}b_j^2}$ and the x_0 integration limit is given by

$$(A.2) \qquad G_N(s) \equiv \frac{1}{b_0(s)}\left(\ln\frac{y_*}{P(s,T_0)} + \frac{1}{2}b_0^2(s)\right), \qquad \Phi[y_*, y_1, \ldots, y_N] = 0.$$

From equation (4.4), we have $b_j(s = T_0) = 0$ and $y_j(T_0) = P(T_0, T_j)$ for all j. Therefore, at the option expiry T_0, the integration boundary becomes

$$G_N(T_0) = \pm\infty \qquad \text{for} \qquad \pm\,\Phi[P(T_0, T_0), \ldots, P(T_0, T_N)] > 0,$$

and the dependence on x_j in the integrand drops out. Thus equation (A.1) gives the correct boundary condition.

Next, we substitute C_+ into Kolmogorov's field equation and use the following property of the functional derivative:

$$\frac{\delta\Phi[P(s,T)]}{\delta P(s,u)} = \delta(u - T)\frac{\partial\Phi}{\partial P(s,T)},$$

where the delta function is defined by $\int_0^L du H(u)\delta(u - T) = H(T)$ for $T \in [0, L]$. Then, it is easy to check that the two terms which depend on the short rate in Kolmogorov's field equation cancel each other because

$$(A.3) \qquad r[s, \{P\}]\int_0^L du\, P(s,u)\frac{\delta C_+}{\delta P(s,u)}$$

$$= r[s, \{P\}]\int_{-\infty}^{\infty} d^N x \int_{-\infty}^{G_N} dx_0\, g(\{x\}, W)\sum_{j=0}^{N} y_j \frac{\partial}{\partial y_j}\Phi[y_0, y_1, \ldots, y_N]$$

$$= r[s, \{P\}]C_+(s).$$

In the last step, we have used the homogeneous property of Φ defined in equation (4.1). The second-derivative term in Kolmogorov's equation can be evaluated similarly

$$\text{(A.4)}\qquad \frac{1}{2}\int_0^L du_1 \int_0^L du_2 Z(s,u_1,u_2)P(s,u_1)P(s,u_2)\frac{\delta^2 C_+}{\delta P(s,u_1)\delta P(s,u_2)}$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} d^N x \int_{-\infty}^{G_N} dx_0\, g(\{x\},W) \sum_{i,j=0}^{N} Z(s,T_i,T_j) y_i y_j \partial_{y_i}\partial_{y_j}\Phi$$
$$+\frac{1}{2}\int_{-\infty}^{\infty} d^N x \left\{ \frac{g(G_N, x_1, \ldots, x_N, W)}{b_0(\sum_{i=1}^N y_i \partial_{y_i}\Phi)} \sum_{j,k=1}^{N} \left(y_j \partial_{y_j}\Phi\right)(y_k \partial_{y_k}\Phi) \right.$$
$$\left. \times \Big(Z(s,T_j,T_k) - Z(s,T_j,T_0) - Z(s,T_k,T_0) + Z(s,T_0,T_0) \Big) \right\}_{x_0=G_N}.$$

On the other hand, to calculate $\partial_s C_+$, it is easier to diagonalise W first by changing the basis of the variables and rewrite the density function in terms of uncorrelated Gaussian densities, $d^{N+1}x g(\{x\},W) = d^{N+1}x' \hat{g}(x'_0)\ldots\hat{g}(x'_N)$, where $\hat{g}(x') \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x'^2}$. After calculating the time-derivative of C_+, we then transform it back to the original variables.

$$\text{(A.5)}\qquad \partial_s C_+ = \int_{-\infty}^{\infty} d^N x \int_{-\infty}^{G_N} dx_0\, g(\{x\},W) \sum_{i,j=0}^{N} \frac{1}{2}\partial_s(b_i b_j W_{ij}) y_i y_j \partial_{y_i}\partial_{y_j}\Phi$$
$$+ \int_{-\infty}^{\infty} d^N x \left\{ \frac{g(G_N, x_1, \ldots, x_N, W(s))}{2b_0(\sum_{i=1}^N y_i \partial_{y_i}\Phi)} \sum_{j,k=1}^{N} \left(y_j \partial_{y_j}\Phi\right)(y_k \partial_{y_k}\Phi) \right.$$
$$\left. \times \partial_s \Big(b_0^2 + b_j b_k W_{jk} - b_0 b_j W_{0j} - b_0 b_k W_{0k} \Big) \right\}_{x_0=G_N}.$$

Substituting in the coefficients b's and W from equation (4.4), we prove that equations (A.4) and (A.5) cancel.

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