Multifactor Generalization of ”Discount-Bond Derivatives on a Recombining Binomial Tree”

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Abstract

The security dynamics described by the Black-Scholes equation with price-dependent variance can be approximated as a damped discrete-time hopping process on a recombining binomial tree. In a previous working paper, such a nonuniform tree was explicitly constructed in terms of the continuous-time variance. The present note outlines how the previous procedure could be extended to multifactor Black-Scholes with price- and time-dependent coefficients. The basic idea is to derive new coordinates which give a Black-Scholes equation with all the σ’s equal to unity. In the discrete-time tree corresponding to this equation, nodes are uniformly spaced and the hopping probabilities are not constant. When the new coordinates are mapped back onto prices, the ensuing tree is nonuniform. A derivative can be valued with the new coordinates or the original prices.

A recent analysis[1] of the Bühler-Käsler discount-bond model examined the security dynamics in the discrete-time formulation. For a price-dependent variance, a recombining binomial tree with the correct continuous-time limiting behavior was explicitly constructed.

Left open in that paper was the issue of whether the tree construction procedure could be extended to time-dependent variance and to multifactor continuous-time processes. This note discusses how to go about this. A recent discussion[2] of the finite-difference method vis-à-vis the implied-tree approach[3] stresses the interest of the issue. The present version of this paper is intended to be read in conjunction with Ref. [1]. Equations for the tree structure will be derived, but only a general discussion of the expected solutions will be presented here.

In terms of the price variables z_i $(i = 1, ..., N)$, the valuation equation of interest for a derivative security f is

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{i,j} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 f}{\partial z_i \partial z_j} + \sum_i v_i \frac{\partial f}{\partial z_i} - rf = 0, \tag{1}$$

where $\rho_{ii} = 1$ holds; the quantities σ_i, ρ_{ij}, v_i and the interest rate r are functions of the price variables z_i and time t. The Black-Scholes no-arbitrage argument leads to equations of this form.

Suppose that the price variables z_i can be transformed to new variables ξ_i. In the resulting valuation equation,

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{i,j} \widetilde{\rho}_{ij} \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} + \sum_i \widetilde{v}_i \frac{\partial f}{\partial \xi_i} - rf = 0, \tag{2}$$

$\widetilde{\rho}_{ii} = 1$ holds as before, and the coefficients are functions of the ξ_i's and t. Discussion of the transformation is deferred. The expression $\frac{\partial f}{\partial t}$ has different meanings in equations (1) and (2) because the partial derivative is taken with different quantities held constant. A discrete-time multinomial-tree analog of (2) is sought. For a time increment τ, consider the expression

$$f(\xi, t) = \frac{1}{1 + \tau r(\xi, t)} \sum_{\varepsilon} p(\varepsilon) f(\{\xi_i + \varepsilon_i a\}, t + \tau) \qquad \varepsilon_i = \pm 1, \tag{3}$$

which describes a time-reversed (from $t + \tau$ to t) hopping process on a multinomial recombining tree with lattice spacing a. The notation $\xi = \{\xi_i\}$ is used. The probabilities $p(\varepsilon)$ are the likelihoods of given price changes.

As usual, equation (3) is Taylor-expanded to second order in ξ and first order in τ and matched to (2). The assignment $a = \sqrt{\tau}$ leads to a set of equations for the

$p(\xi)$'s in terms of the coefficients of (2). This set is underdetermined when the number N of price variables exceeds two.

The set of constraints for the 2^N p's must be closed. A possible starting point is the observation that the multinomial tree can be viewed as a particular type of finite-difference scheme: the nodes and time increments are chosen so that, if all works out, the weights of all the $f(\xi, t+\tau)$'s are nonnegative and a probabilistic interpretation exists. The $p(\xi)$'s might be constructed directly from a finite-differencing scheme. In fact, consider a finite-difference solution of (2) in which the partial derivatives are computed from the f-values at the corners of a hypercube of side $2\sqrt{\tau}$. This corresponds to the probability

$$p(\varepsilon) = 2^{-N}\left(\sum_{m\leq n}^{N} \varepsilon_m \varepsilon_n \widetilde{\rho}_{mn} + \sum_{m=1}^{N} \varepsilon_m \widetilde{v}_m \sqrt{\tau}\right) \tag{4}$$

proposed by Boyle, Evnine and Gibbs[4]. They caution that the non-negativity of the $p(\varepsilon)$'s must be checked in each given case. Ideally, the domain of $\widetilde{\rho}$ in which the correlation matrix has no negative eigenvalues would coincide with the domain of $\widetilde{\rho}$ in which all the $p(\varepsilon)$'s are nonnegative[1].

It remains to present a scheme for determining $\xi(z)$ or $z(\xi)$. Let the notation $\partial f/\partial z_i|_{z',t}$ denote the partial derivative of f with respect to z_i with t and $z_{j\neq i}$ fixed. When the chain rule

$$\left.\frac{\partial f}{\partial z_i}\right|_{z',t} = \sum_m \left.\frac{\partial \xi_m}{\partial z_i}\right|_{z',t} \left.\frac{\partial f}{\partial \xi_m}\right|_{\xi',t} \tag{5}$$

$$\left.\frac{\partial f}{\partial t}\right|_{z} = \left.\frac{\partial f}{\partial t}\right|_{\xi} + \sum_m \left.\frac{\partial \xi_m}{\partial t}\right|_{z} \left.\frac{\partial f}{\partial \xi_m}\right|_{\xi',t} \tag{6}$$

is substituted into equation (1), the second-order term becomes

$$\sum_{i,j} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 f}{\partial z_i \partial z_j} = \sum_{m,n}\left\{\sum_{i,j}\rho_{ij}\sigma_i\sigma_j \left.\frac{\partial \xi_m}{\partial z_i}\right|_{z',t} \left.\frac{\partial \xi_n}{\partial z_j}\right|_{z',t}\right\} \frac{\partial^2 f}{\partial \xi_m \partial \xi_n} + \dots \tag{7}$$

Comparing to equation (2) gives

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[1] If the ξ_i's are strongly correlated, the p's differ strongly in magnitude; the short-time continuum dynamics might help indicate the tree paths with significant statistical weight.

$$\sum_{i,j} \rho_{ij}\sigma_i\sigma_j \left.\frac{\partial \xi_m}{\partial z_i}\right|_{z',t} \left.\frac{\partial \xi_m}{\partial z_j}\right|_{z',t} = 1 \qquad\qquad m = 1,\ldots,N. \tag{8}$$

The boundary condition can be taken, for example, as $\xi_i\left(\{z_{n\neq i}\}, z_i = 0\right) = 0$, or it can be fine-tuned to the expiration boundary conditions on the derivatives of interest. The ξ-lattice has the points

$$\xi_i = n_i\sqrt{\tau} \qquad i = 1,\ldots,N; \quad n_i = 0,\pm 1,\pm 2,\ldots \tag{9}$$

The corresponding points of the z-lattice are the intersections of the N z-hypersurfaces $\xi_i(z) = n_i\sqrt{\tau}$. Alternatively, Jacobian matrices could be used to express (8) in terms of partial derivatives at constant (ξ', t), and the ensuing expressions solved for $z(\xi)$. Also, (8) could be solved for the σ's in terms of a posited z-tree structure and the ρ's or p's.

For the tree description to be used as an economic model and not only as an approximant to the continuous-time limit, no-arbitrage conditions for the z-jumps must hold on the portion of the tree used to value derivatives. The existence of a path to the continuous-time limit enhances confidence in heuristic utilizations of nonuniform multinomial trees.

References

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[4] P.P. Boyle, J. Evnine and S. Gibbs, Rev. Financial Studies 2, 241 (1989).