Discount-Bond Derivatives on a Recombining Binomial Tree[1]

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Abstract

Interest-rate derivative models governed by parabolic partial differential equations (PDEs) are studied with discrete-time recombining binomial trees. For the Bühler-Käsler discount-bond model, the expiration value of the bond is a limit point of tree sites. Representative calculations give a close approximation to the continuum results. Next, situations are considered in which spatial inhomogeneity of the drift velocity can cause binomial jump probabilities to become negative. When the continuous-time boundary conditions are applied near the tree points at which this occurs, good agreement is obtained with with Hull and White's explicit-finite-difference treatment of the Cox-Ingersoll-Ross model. Finally, to mimic the effect of a drift-velocity divergence which prevents interest rates from becoming negative, Neumann boundary conditions are applied in the Vasicek model. Discrete-time computations are performed for a mean-reverting situation and for a case with constant negative short-rate drift; the ensuing bond values have nonnegative interest rates and forward rates. The results are compared with the Vasicek solution and with the leading term in a spectral expansion.

[1]Draft No. 2. Sections 3 and 4 on Neumann boundary conditions are new. Additional references have been included, and some material from the previous draft has been moved to the Appendix.

1. Introduction

This working paper applies binomial-tree models to calculate interest-rate derivative valuations. Particular attention is paid to the interplay between boundary conditions and the approach to the continuous-time limit. A natural variable for modelling a stock of price S is $\ln S$, which is unbounded ($-\infty < \ln S < \infty$). Since interest rates are in principle nonnegative[1], a discount-bond price B has an upper bound and $\ln B$ is confined to s half-space. The continuous-time valuations considered here are described by parabolic partial differential equations (PDEs); the correct behavior near the half-space boundary must be built into a tree model's passage to the continuous-time limit.

The paper is organized as follows. Section 2 considers the Bühler-Käsler discount-bond model[1]. The binomial tree has a limit point which makes the boundary unattainable in a finite number of jumps. The Cox-Ingersoll-Ross (CIR) model[2] is treated in Section 3. When CIR is mapped onto a model with constant variance, Neumann boundary conditions (vanishing slope) hold in the new variable. Although these boundary conditions are not explicitly stressed in Nelson and Ramaswamy's binomial analysis[3], they can be applied naturally in the binomial-tree formulation; it is argued that the Hull-White modification of the explicit finite-difference method[4] converges to them. In Section 4, Neumann conditions are applied to the Vasicek model[5] as a way of precluding negative interest rates. The interest-rate-dependent mean-reversion term is suppressed, and a computed example has nonnegative discount-bond rates for constant *negative* drift velocity. Section 5 presents concluding remarks. The tree structure for a nonconstant variance has been derived concisely by use of Ito's lemma[4, 3], the Appendix gives alternative derivations, one of which admits a simple multifactor generalization[6].

[1] In practice, models permitting negative interest rates should produce a negligible probability for such rates. For situations in which the predicted probability is not negligible, it is appropriate to bar negative rates outright.

2. Inaccessible Boundary: Bühler-Käsler

In the ”direct” approach to bond valuation, the bond price is the fundamental random variable. Rady and Sandmann’s article[7] reviews this line of effort, especially key contributions by Ball and Torous[8], by Briys, Crouhy and Schöbel [9], and by Bühler and Käsler[1]. The simplest lattice for a discrete-time formulation is a recombining binomial tree. The resulting picture has the same relationship to the continuous-time model as the Cox-Ross-Rubinstein binomial-tree description[10] has to Black-Scholes[11].

Rady and Sandmann[7] caution that in bond-price direct models, an absorbing boundary and nonzero probability of vanishing interest rate can arise. On a binomial tree, such behavior could be precluded if the boundary were a limit point which is inaccessible in a finite number of steps. Figure 1 depicts such a tree, on which computations can readily be performed. Representative numerical results will be presented for the Bühler-Käsler model, whose continuum limit has an economic and probabilistic ”clean bill of health”. Moreover, the tree structure can be determined analytically.

Bühler and Käsler value bond derivatives by solving the PDE

$$\frac{\partial f}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial z^2}-\frac{1}{2}\sigma^2\frac{\partial f}{\partial z}=0 \tag{1}$$

with variance

$$\sigma^2(z)=\sigma_0^2(1-e^z)^2. \tag{2}$$

Equation (1) is the Black-Scholes equation with a price-dependent variance. It is solved in the half-space $z\leq 0$ with the boundary condition

$$f(z=0,t)=f(z=0,T). \tag{3}$$

Derivative securities assume their expiration value at $z=0$.

The tree structure is found by inverting

$$\xi=n\sqrt{\tau}=\int_{z_0}^{z}\frac{dz'}{\sigma(z')} \tag{4}$$

for the z-coordinate of the nth node[3, 4]. For the Bühler-Käsler model, the solution is

$$z=-\ln\left[1+e^{-n\sigma_0\sqrt{\tau}}\left(e^{-z_0}-1\right)\right]. \tag{5}$$

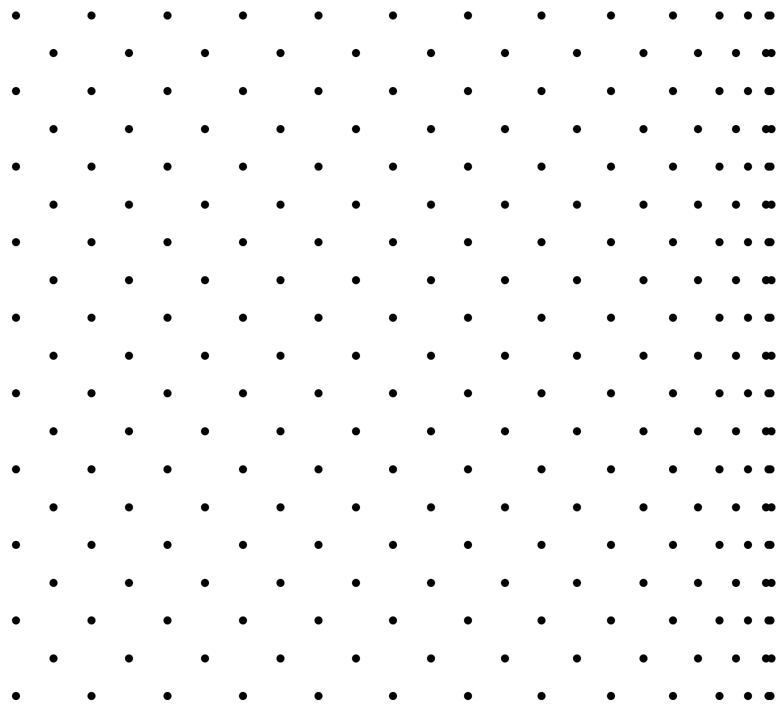

Figure 1: **Schematic diagram of a binomial tree.** Time steps occur in the vertical direction, and the horizontal direction depicts price. Hops occur between nearest neighbors on adjacent time steps. The depicted tree becomes uniform to the left and has a limit point on the right. Thus there is an infinite number of nodes both to the left and to the right of any given node.

	z=-.200000 (n=0)	z=-.214697 (n=-10)
$T-t=50\tau$	.995	.965
$T-t=100\tau$	.9975	.990
$T-t=200\tau$	.999	.997

Table 1: **Ratio of binomial calls to the Bühler-Käsler value.** The ln of the strike price is $\varepsilon=-\frac{1}{5}$, and the node index n is zero at $z_0=-\frac{1}{5}$. The standard deviation per time step is $\sigma_0=\frac{3}{20\cdot 365^{1/2}}$, and the time step is $\tau=1$.

For $n\to-\infty$, the nodes become evenly spaced, but for $n\to+\infty$, $z=0$ is an accumulation point of nodes. In random-interest-rate models like the Dothan[12] and Black-Karasinski[13] models, an accumulation point would arise at vanishing short-term rate.) The accumulation point would not exist if $1/\sigma(z)$ were integrable near $z=0$; examples of such situations will be examined in the next two sections. For constant σ, Rady and Sandmann show that the boundary conditions on the PDE can be interpreted in terms of an absorbing boundary at $z=0$ and a finite probability for a vanishing interest rate; these are absent for the Bühler-Käsler model. The absence of an absorbing barrier is consistent with the limit point at $z=0$.

Table 1 shows representative numerical results showing good agreement with the Bühler-Käsler call value [7, 1]

$$\begin{aligned} f(z,t) \;=\; & (1-e^{\varepsilon})e^{z}N\left[\frac{1}{\sigma_0\sqrt{T-t}}\left(\ln\frac{e^{z}(1-e^{\varepsilon})}{(1-e^{z})e^{\varepsilon}}+\frac{\sigma_0^2(T-t)}{2}\right)\right] \\ & -e^{\varepsilon}(1-e^{z})N\left[\frac{1}{\sigma_0\sqrt{T-t}}\left(\ln\frac{e^{z}(1-e^{\varepsilon})}{(1-e^{z})e^{\varepsilon}}-\frac{\sigma_0^2(T-t)}{2}\right)\right] \qquad (6) \end{aligned}$$

where the strike price is e^{ε} and T is the expiration time. For the given parameter values, the at-the-money Bühler-Käsler call is more than a factor of 5 less than the Black-Scholes call. The difference is due to the price dependence of the variance. This price dependence is captured by the nonuniform binomial tree.

3. Neumann Boundary Conditions: Cox-Ingersoll-Ross

It is known that artificial negative jump probabilities can arise in literal discrete-time treatments of models with large drift velocities. The Cox-Ingersoll-Ross

(CIR) model provides an example. CIR is a one-factor model driven by a random short-term interest r whose variance is proportional to r. For vanishing price of risk and a drift velocity $v = a(b-r)$ with $a, b > 0$, the valuation equation for a derivative security is[2]

$$\frac{\partial f}{\partial t} + \frac{\sigma^2}{2} r \frac{\partial^2 f}{\partial r^2} + a(b-r)\frac{\partial f}{\partial r} - rf = 0 \tag{7}$$

The derivative f can be priced in terms of (a presumably complete set of) eigenfunctions ψ generated by the substitution $f \to e^{-\lambda(T-t)}\psi(r,\lambda)$. The transformation of variables $\xi = 2\sqrt{r}/\sigma$ produces

$$\frac{1}{2}\frac{\partial^2\psi}{\partial\xi^2} + \left[\left(\frac{2ab}{\sigma^2} - \frac{1}{2}\right)\frac{1}{\xi} - \frac{1}{2}a\xi\right]\frac{\partial\psi}{\partial\xi} + \left(-\frac{1}{4}\sigma^2\xi^2 + \lambda\right)\psi = 0. \tag{8}$$

Values of λ are determined by the boundary conditions. These are that ψ is regular at $r = 0$ and vanishes at $r = \infty$. Examination of the power series solution[14] indicates that the term of order ξ is absent. Thus, the regularity requirement at $r = 0$ becomes the Neumann boundary condition

$$\frac{\partial\psi}{\partial\xi} = 0 \tag{9}$$

Since $r \propto \xi^2$, this is satisfied by the CIR discount-bond value $A(t)e^{-B(t)r}$.

How is the Neumann boundary condition to be implemented on the binomial tree associated with the time-dependent version of equation 8? The nodes are at $\xi = n\sqrt{\tau} \quad (n = 0, 1, 2, \ldots)$. For $n = 0$ and perhaps some neighboring sites, and for sufficiently large values of n, one hopping probability is negative (and the other is greater than one). A natural procedure is to apply the boundary condition at the values n^* of n at which the probabilistic interpretation breaks down[2]; using the condition $\partial f(\xi = 0, t)/\partial\xi = 0$ rather than $f(\xi = 0, t) = 0$ permits f to remain nonzero at finite r. Table 2 shows a comparison with the exact CIR results and Hull and White's explicit-finite-difference solution[4]. The Neumann conditions are given the form $f(n^*\sigma\sqrt{\tau}, t) = f((n^*+1)\sigma\sqrt{\tau}, t)$. This leads to two interleaving binomial trees which interact at their boundaries. The time step τ was chosen identical to Hull and White's, and the ξ-step size is $1/\sqrt{3}$ smaller than that on their trinomial tree; this is partially offset by the fact that computing a trinomial jump requires fewer multiplications than computing a binomial one.

[2] By no means is this the only choice. For example, v could be regularized, e.g. by $\frac{v\sqrt{\tau}}{\sigma} \to$

	r=.06	r=.08	r=.10	r=.12	r=.14
$T-t=5$ yrs	.6631 (.6631) ((.6624))	.6353 (.6353) ((.6345))	.6086 (.6086) ((.6078))	.5830 (.5830) ((.5823))	.5585 (.5585) ((.5578))
$T-t=10$ yrs	.4091 (.4092) ((.4083))	.3898 (.3898) ((.3889))	.3714 (.3714) ((.3705))	.3538 (.3538) ((.3529))	.3371 (.3371) ((.3362))
$T-t=15$ yrs	.2502 (.2502) ((.2494))	.2382 (.2382) ((.2374))	.2268 (.2268) ((.2259))	.2159 (.2159) ((.2151))	.2055 (.2055) ((.2147))
$T-t=20$ yrs	.1528 (.1528) ((.1521))	.1454 (.1454) ((.1448))	.1385 (.1385) ((.1378))	.1318 (.1318) ((.1311))	.1255 (.1255) ((.1248))

Table 2: **CIR Results.** Hull and White's parameters are adopted: $a = .4$, $b = .1$, $\sigma = .06$, and $\tau = .05$. The top entry in each box is the present calculation; linear interpolation was used on the tree values. The second (in parentheses) is Hull-White's explicit-finite-difference result, and the third ((in double parentheses)) is the exact CIR result quoted by Hull and White.

Actually, the Neumann boundary condition at $r = 0$ is virtually implicit in the binomial formulation of CIR. Consider the continuous-time limit $\tau \to 0$; in this limit, $v(\xi = n\sqrt{\tau})\sqrt{\tau}$ approaches a constant for fixed n and so do the corresponding jump probabilities. Ignoring the $1/(1 + r\tau)$ term which is of higher order in ξ, one sees that for adjacent tree sites

$$\begin{aligned} f(n\sqrt{\tau}, t) &= p(n) f((n+1)\sqrt{\tau}, t+\tau) \\ &\quad + (1 - p(n)), \end{aligned} \tag{10}$$

where $p(n)$ is the jump probability at site n. The point is that $f(r \propto \xi^2, t)$ is assumed regular and smooth near $r = 0$, but this cannot be true for $p(n)$ because of the $1/\xi$ term in the drift velocity. Expanding about $\xi = 0$ to leading order in $\sqrt{\tau}$, one finds

$$\frac{\partial f(\xi = 0, t)}{\partial \xi} \simeq (2p(n) - 1) \frac{\partial f(\xi = 0, t+\tau)}{\partial \xi}. \tag{11}$$

For positive drift velocity, $2p - 1 > 0$ holds, and, for $p(n) < 1$, under repeated time steps a finite $\partial f(\xi \sim \sqrt{\tau}, t)/\partial \xi$ is rapidly driven to zero.

Similar considerations hold for the Hull-White explicit-finite-difference treatment of CIR. For example, consider the relation

$$\begin{aligned} f(n\Delta\phi, t) &= p_n^+ f((n+1)\Delta\phi, t+\tau) \\ &\quad + (1 - p_n^+ - p_n^-) f(n\Delta\phi, t+\tau) \\ &\quad + p_n^- f((n-1)\Delta\phi, t+\tau), \end{aligned} \tag{12}$$

in which the $p_n^\pm$'s are the probabilities of up or down jumps at site n and $\Delta\phi$ is the ξ-step size. Since $\Delta\phi = O(\sqrt{\tau})$, for $\tau \to 0$ at fixed n, this equation can be expanded about $\xi = 0$. At first order in $\Delta\phi$, there ensues

$$n\Delta\phi \frac{\partial f(\xi = 0, t)}{\partial \xi} \simeq (n + p_n^+ - p_n^-)\Delta\phi \frac{\partial f((n+1)\Delta\phi, t+\tau)}{\partial \xi}(0, t+\tau) \tag{13}$$

$$\simeq (n + p_n^+ - p_n^-)\Delta\phi \frac{\partial f(\xi = 0, t)}{\partial \xi}. \tag{14}$$

Since $p_n^\pm$ approaches a constant in the limit considered, Neumann boundary conditions hold if the Taylor expansion is valid[3]—not surprisingly, given the quanti-

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$\frac{v\sqrt{\tau}}{\sqrt{\sigma^2 + v^2\tau}}$.

[3]Equation (14) leaves open the possibility that $\partial f(\xi \to 0, t)/\partial \xi$ grows under iteration. For the functional form of Hull and White's jump probabilities and their bounds on the variables in the functions, it can be shown that this is not the case at $n = n^*$.

tative success of Hull-White reproduced in Table 2.

4. Neumann Boundary Conditions: Vasicek

The power-series solution[14] of equations of the form (8) indicates that the Neumann boundary condition describes the solution which is regular at the origin when the drift velocity has a pole there. In effect, a pole with positive residue prevents ξ from becoming negative. It is natural to consider such a divergent drift velocity in a constant-variance interest rate model, i.e. in the Vasicek model, whose variants and generalizations[15, 16, 13] are used to infer derivative valuations from empirical term structure data. The robustness of the results is of particular interest: when the residue of the pole is tuned toward zero, the limiting case is a "regular" Vasicek model with Neumann boundary conditions. Also, for constant σ, if Neumann conditions are adopted for the forward equation $\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial r^2} + v(r)\frac{\partial f}{\partial r} = 0$, the adjoint boundary condition is $\frac{1}{2}\frac{\partial f}{\partial r} - vf = 0$, which indicates the conservation of probability—time-independent normalization—in the backward (adjoint) equation $-\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial r^2} - \frac{\partial v(r)f}{\partial r} = 0$.

This section presents binomial-tree computations on the Vasicek model; the results will be interpreted with the continuous-time theory. First, a standard scenario with strong mean reversion will be considered; then, to highlight the role of boundary conditions, a situation without mean reversion and with *negative* interest-rate drift will be analyzed. The intention (at least in the present draft) is to demonstrate the procedure rather than to present a survey of the (σ, b, a, τ) parameter space.

The valuation equation is

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial r^2} + a(b-r)\frac{\partial f}{\partial r} - rf = 0. \tag{15}$$

This equation is usually analyzed in the domain $-\infty < r < \infty$, where its behavior simplifies[5, 17]. The mean reversion implied by the convention $a, b > 0$ typically keeps the contribution of negative rates small, but this need not be the case for sufficiently volatile assets or for nonstandard values of a and b. The solution can be expressed as an eigenfunction expansion:

$$f(r,t) = \sum_{n=0}^{\infty} \psi(r, \lambda_n) e^{-\lambda_n (T-t)}. \tag{16}$$

The eigenfunction terms are the solutions of equation (15) which vanish at $r = \infty$;

	r=.04	r=.06	r=.08	r=.10	r=.12	r=.14
$T-t=5$ yrs	.8048 (.8163) ((.8047))	.7555 (.7390) ((.7554))	.7093 (.6688) ((.7091))	.6658 (.6052) ((.6657))	.6251 (.5477) ((.6249))	.5868 (.4956) ((5866))
$T-t=10$ yrs	.6365 (.6395) ((.6364))	.5839 (.5789) ((.5836))	.5356 (.5239) ((.5353))	.4913 (.4742) ((.4910))	.4507 (.4291) ((.4503))	.4134 (.3883) ((.4130))
$T-t=15$ yrs	.5003 (.5010) ((.5003))	.4552 (.4535) ((.4550))	.4141 (.4105) ((.4134))	.3766 (.3715) ((.3762))	.3425 (.3361) ((.3421))	3115 ((.3042)) ((.3111))
$T-t=20$ yrs	.3925 (.3925) ((.3926))	.3560 (.3553) ((.3559))	.3229 (.3216) ((.3226))	.2928 (.2910) ((.2924))	.2654 (.2633) ((.2651))	.2407 (.2383) ((.2403))

Table 3: **Results for Mean-Reverting Vasicek.** The following parameters are adopted: $a = .2$, $b = .05$, $\sigma = .01$, and $\tau = .04$. The top entry in each box is the binomial calculation; linear interpolation was used on the tree values. The second (in parentheses) is the one-mode approximation, and the third ((in double parentheses)) is the Vasicek solution for $-\infty < r < \infty$.

the eigenvalues are determined by the boundary condition at $r = 0$. The ψ_λ's can be expressed in terms of confluent hypergeometric (parabolic cylinder) functions or Airy functions[18]. Because (15) is not a Sturm-Liouville equation, a weight factor must be used to orthonormalize the ψ_λ's. Interest rates will be negative if the smallest λ appearing in the expansion is negative.

Table 3 presents a canonical case with strong mean reversion. Neumann conditions are imposed at $r = 0$ and at the value of r at which a jump probability becomes negative. The numerical results are in good agreement with the Vasicek solution; the dynamics is driven by the mean reversion and boundary effects are minor. The results are also consistent with a one-term truncation of the eigenfunction expansion; as expected, the agreement improves when time to maturity increases and the leading transient dominates.

Next, a nonstandard situation will be considered in which the boundary condition is the only factor working against negative interest rates. Equation (15) will be considered in the limit $(a \to 0, ab \to v)$, and the interest-rate drift v will be taken as negative. Negative interest rates are not apparent in Table 4. In the set of

	r=.04	r=.06	r=.08	r=.10	r=.12	r=.14
$T-t=5$ yrs	.9059 (.9039) ((.9297))	.8356 (.8202) ((.8412))	.7600 (.7168) ((.7612))	.6883 (.6036) ((.6887))	.6231 (.4902) ((.6232))	.5640 (.3840) ((.5639))
$T-t=10$ yrs	.8736 (.8817) ((1.1237))	.7974 (.8001) ((.9200))	.7007 (.6992) ((.7537))	.5978 (.5888) ((.6167))	.4992 (.4781) ((.5049))	.4120 (3746) ((.4134))
$T-t=15$ yrs	.8505 (.8600) ((1.7883))	.7732 (.7804) ((1.3248))	.6771 (.6820) ((.9814))	.5718 (.5743) ((.7271))	.4666 (.4637) ((.5386))	.3691 (.3654) ((.3990))
$T-t=20$ yrs	.8254 (.8389) ((3.7934))	.7504 (.7612) ((2.5430))	.6570 (.6652) ((1.7046))	.5544 (.5602) ((1.1426))	.4511 (.4549) ((.7659))	.3543 (.3564) ((.5134))

Table 4: **Vasicek Results for Constant Drift.** The following parameters are adopted: $a = 0$, $v = ab = -.01$, $\sigma = .01$, and $\tau = .04$. The top entry in each box is the binomial calculation; linear interpolation was used on the tree values. The second (in parentheses) is the one-mode approximation, and the third ((in double parentheses)) is the Vasicek solution for $-\infty < r < \infty$.

eigenfunctions

$$\psi(r,\lambda) = e^{-rv/\sigma^2} Ai\left(\frac{r-\lambda+\frac{v^2}{2\sigma^2}}{(\frac{1}{2}\sigma^2)^{1/3}}\right), \tag{17}$$

the asymptotic properties of the Airy function in the limit $\frac{v}{\sigma^{4/3}} \to -\infty$, lead to the estimate $\lambda \approx \frac{\sigma^2}{2|v|}$, in good agreement with the Newton's-method result $\lambda \simeq .00497$. (For $v=0$, the leading λ is proportional to the Airy function's largest root, which is negative; for $v \to +\infty$, $\frac{v^2}{2\sigma^2} - \lambda$ is proportional to the zero of $Ai'(x)$, which is also negative.)

The absence of negative eigenvalues for constant drift raises the issue for $v = a(b-r)$ and more general cases. For example, when $a > 0$, the appropriate eigenfunction is

$$\begin{aligned} f(r,\lambda) &= \pi e^{-r/a}\left[\frac{M\left(c,\frac{1}{2},\frac{a}{\sigma^2}(r+\frac{\sigma^2}{a^2}-b)^2\right)}{\Gamma(\frac{1}{2}+c)\Gamma(\frac{1}{2})}\right. \\ &\quad \left. - \frac{\sqrt{a}}{\sigma}(r+\frac{\sigma^2}{a^2}-b)\frac{M\left(\frac{1}{2}+c,\frac{3}{2},\frac{a}{\sigma^2}(r+\frac{\sigma^2}{a^2}-b)^2\right)}{\Gamma(c)\Gamma(\frac{3}{2})}\right], \end{aligned} \tag{18}$$

where

$$c = -\frac{1}{2a}\left[\frac{\sigma^2}{2a^2} - b + \lambda\right]. \tag{19}$$

and $f \to 0$ as $r \to \infty$; a similar solution exists for $a < 0$. The leading eigenvalue $\lambda = b - \frac{\sigma^2}{2a^2}$ in the Vasicek solution satisfies the equation $c = 0$, which can be shown to determine an eigenvalue, i.e. $\frac{1}{\Gamma(c)} = 0$, in the limit $\sigma \to 0$. For $b = \frac{\sigma^2}{a^2}$, examination of the Γ-functions indicates that the solutions to $\frac{\partial f(r=0,\lambda)}{\partial r} = 0$ must have negative c and positive λ. At present, the spectrum associated with an arbitrary value of (a,b) must be determined on an individual basis, although so far all prospective negative roots of $\frac{\partial f(r=0,\lambda)}{\partial r} = 0$ have turned out to be weakly positive or asymptotes.

Nevertheless, a qualititive argument about the spectrum can be made. Consider the forward equation $-\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial r^2} - \frac{\partial v(r) f}{\partial r} = 0$ mentioned at the beginning of this section. Since the (adjoint) boundary conditions give conservation of probability, the equation is assumed to have well-behaved solutions with a nonnegative spectrum. Therefore, the backward equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial r^2} + v(r)\frac{\partial f}{\partial r} = 0, \tag{20}$$

with Neuman boundary conditions, is also well-behaved.

Consider a binomial-tree treatment of this equation. If the initial value of the solution is nonnegative, the properly bounded jump probabilities assure that it will be nonnegative at all prior times. Introducing the damping factor $\frac{1}{1+r\tau} \leq 1$ at each jump in equation (20) produces a binomial-tree formulation of

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial r^2} + v(r)\frac{\partial f}{\partial r} - rf = 0, \tag{21}$$

which is the equation of interest. If the temporal divergences associated with negative eigenvalues are not present in (20), they will not arise in (21). Note that this rationale breaks down if r is allowed to be negative.

5. Summary and Conclusion

Three things have been done in this paper. First, the discrete-time Bühler=Käsler discont-bond model has been studied on a binomial tree which turned out to have a limit point. Second, it was noted that when the Cox-Ingersoll-Ross model is mapped onto a constant-variance model, the appropriate boundary condition in the new price variable is the Neumann condition; binomial-tree calculations are consistent with, and clarify the success of, the Hull-White explicit-finite-difference method. Third, as a heuristic way of modelling a short-range divergent drift which keeps interest rates nonnegative, Neumann conditions were imposed in the Vasicek model. In a typical mean-reverting situation, the effects are minor. When mean reversion is turned off and a constant negative interest-rate drift imposed, the consequences in a sample case are dramatic: the discount-bond valuation appears well-conditioned, whereas the Vasicek solution is dominated by negative rates. If mean reversion is not essential to obtain *prima facie* acceptable results from Vasicek, the model and its variants may accommodate a larger category of fitting functions, and be applicable to securities with greater volatility, than would be the case otherwise.

Appendix

The tree structure has been concisely derived[3, 4] by use Ito's lemma. This Appendix derives it by formulating the appropriate continuous-time limit of a binomial process. A check using the PDE commences after equation (44); it is the starting point for a simple generalization to the multifactor case.

In the binomial processes of interest, the price of interest-rate derivatives depends on the value of a random variable z which can be, for instance, an interest rate or a bond price. Changes in z occur in discrete time steps τ. At time t and $z = z_0$, z changes to $z_0 + a_+$ with probability p and to $z_0 - a_-$ with probability $1 - p$. Derivative securities can be valued with price-of-risk methods[19, 20]. It is convenient to adapt the terminology of Ref. [21]. The value of a derivative security $f(z,t)$ is determined from its values one time step closer to expiration by

$$\frac{\langle f(z+\Delta z, t+\tau\rangle_{\Delta z} - (1+R)\, f(z,t)}{\mathcal{R}\,[f]} = \lambda \tau^{1/2}, \tag{22}$$

where λ is the price of risk and R is the interest. In principle, these quantities can depend on both z and t. The expectation value is

$$\langle f(z_0)\rangle \equiv p f(z+a_+) + (1-p)\, f(z-a_-) \qquad (a_+ + a_- > 0), \tag{23}$$

and the risk metric $\mathcal{R}$ is taken as quadratic:

$$\begin{aligned} |\mathcal{R}\,[f]| &= \left\langle (f(z+\Delta z) - \langle f\rangle)^2 \right\rangle^{1/2} && (24)\\ &= \sqrt{p(1-p)}\, |f(z+a_+) - f(z-a_-)| && (25) \end{aligned}$$

The Black-Scholes $\mathcal{R}$ has the sign of $\partial f/\partial z$, and in the present situation, $\mathcal{R}$ is assigned the sign of $f(z+a_+) - f(z-a_-)$. The solution to equation (22) is

$$\begin{aligned} f(z,t) &= \frac{1}{1+R}\left[p(1-\lambda\sqrt{\frac{\tau(1-p)}{p}})\, f(z+a_+, t+\tau)) + \right.\\ &\qquad \left. (1-p)(1+\lambda\sqrt{\frac{\tau p}{1-p}})\, f(z-a_-, t+\tau)\right] && (26)\\ &\equiv \frac{1}{1+R}\left[\tilde{p} f(z+a_+, t+\tau)) + (1-\tilde{p})\, f(z-a_-, t+\tau)\right]. && (27) \end{aligned}$$

The functional form (27) remains valid irrespective of the sign of R . If the underlying security is a bond of price B and the random variable z is $z = \ln B$, $\tilde{p}$ can be eliminated by requiring that e^z be a solution of the equation, which transforms to

$$f(z,t) = \frac{(1+R-e^{-a_-})\,f(z+a_+,t+\tau)-(1+R-e^{a_+})\,f(z-a_-,t+\tau)}{(1+R)(e^{a_+}-e^{-a_-})}. \tag{28}$$

For a market consisting of the bond B, its derivatives, and time-independent cash, the above expression further simplifies to

$$f(z,t) = \frac{(1-e^{-a_-})\,f(z+a_+,t+\tau)+(e^{a_+}-1)\,f(z-a_-,t+\tau)}{(e^{a_+}-e^{-a_-})}. \tag{29}$$

after the interest rate is dropped. Equation (29) describes a binomial hopping process with

$$\frac{1-e^{-a_-}}{e^{a_+}-e^{-a_-}}, \qquad \frac{e^{a_+}-1}{e^{a_+}-e^{-a_-}} \tag{30}$$

as transition probabilities.

As noted in Section 2, Bühler and Käsler value bond derivatives by solving the PDE

$$\frac{\partial f}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial z^2}-\frac{1}{2}\sigma^2\frac{\partial f}{\partial z}=0 \tag{31}$$

subject to

$$\sigma^2(z) = \sigma_0^2(1-e^z)^2 \tag{32}$$

$$f(z=0,t) = f(z=0,T). \tag{33}$$

Derivative securities assume their expiration value at $z=0$.

Can a binomial process be constructed which reduces to Bühler-Käsler in the $\tau \to 0$ limit? A recombining tree is sought. If σ depends on price but not on time, so will the lattice geometry. Let the price-dependent nodes of the tree be indexed by an integer n. At a given time step, all values of n are even or odd; hops take place from site n to sites $n \pm 1$. Accordingly, the quantities $a_\pm$ depend on the index n. The constraint

$$a_+(n) = a_-(n+1). \tag{34}$$

holds because the tree structure is recombining and time-independent.

A convenient feature of the Bühler-Käsler model is that the variance approaches a constant values for large negative z. The tree becomes uniform in this domain with $a_+(n) \simeq a_-(n) \simeq a \simeq \sigma_0 \tau^{1/2}$ as $n \to -\infty$ and $\tau \to 0$. For small τ, the replacement

$$a_\pm(n) = \alpha_\pm \tau^{1/2} \tag{35}$$

is reasonable; the uniform-tree results suggest that the hopping probabilities should be expanded to $O(\tau^{1/2})$. Thus the drift velocity is

$$v_{drift}\tau \simeq -\frac{1}{2}\sigma^2(z(n))\tau \simeq -\frac{1}{2}a_+(n)a_-(n) = -\frac{1}{2}a_+(n)a_+(n-1). \tag{36}$$

The first equality above comes from the PDE; the second is a small-$a_\pm$ expansion of the tree model. If the variance rather than the drift velocity is computed, to leading order in τ the same relationship,

$$\sigma^2(z(n))\tau = a_+(n)a_+(n-1), \tag{37}$$

ensues. Specifying one of the a's determines all the rest.

Position on the tree can be described in terms of (z,t) or (n,t). A natural question is whether a continuum model can be developed in the $\tau \to 0$ limit. An obvious variable to try is

$$\xi \equiv n\sqrt{\tau}. \tag{38}$$

Summations over n are thus Riemann sums:

$$\sqrt{\tau}\sum_n \psi(n\sqrt{\tau}) \longrightarrow \int d\xi \psi(\xi). \tag{39}$$

Equation (37) for the a_+'s can be simplified by noting that

$$a_+(n+1) \simeq a_+(n) \equiv a(n) \tag{40}$$

to leading order in τ because $a_+(\xi+\tau) \simeq a_+(\xi)$. The $\alpha_\pm$'s defined previously can be written as

$$\alpha_\pm \simeq \alpha(\xi). \tag{41}$$

If the index n is set to zero at z_0, i.e. $n(z_0) = 0$, a z-position on the tree is determined by the number of a-steps from z_0:

$$z(\xi) - z_0 \simeq \sum_{n=0}^{n(z)} a_+(n) \simeq \int_0^{\xi} d\xi' \alpha(\xi'). \tag{42}$$

Accordingly, to leading order in τ, equation (37) can be written as

$$\frac{dz}{d\xi} = \sigma(z), \tag{43}$$

and inverting

$$\xi = n\sqrt{\tau} = \int_{z_0}^{z} \frac{dz'}{\sigma(z')} \tag{44}$$

yields the z-coordinate of the nth node.

The foregoing discussion has been intuitive, and a consistency check is desirable. In fact, there is another interpretation which naturally encompasses models other than Bühler-Käsler. The starting point is the continuum form $\xi = \int \frac{dz}{\sigma}$ for ξ given in equation (4). If its coefficients are time-independent, a PDE of the form

$$\frac{\partial f}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial z^2} + v_{eff}\frac{\partial f}{\partial z} - rf = 0 \tag{45}$$

can be rewritten as

$$\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial \xi^2}+(\frac{v_{eff}}{\sigma}-\frac{1}{2\sigma}\frac{\partial \sigma}{\partial \xi})\frac{\partial f}{\partial \xi}-rf=0 \tag{46}$$

When $\int \frac{dz}{\sigma}$ diverges at zero and infinity, the relevant binomial process associated with equation (46) involves price-dependent hopping of the form (27) on a uniform ξ-tree. The internode spacing is $\sqrt{\tau}$, and the hopping probability corresponding to the drift velocity of ξ is

$$\begin{aligned} 2\tilde{p}-1 &= (\frac{v_{eff}}{\sigma}-\frac{1}{2\sigma}\frac{\partial \sigma}{\partial \xi})\tau^{1/2} && (47)\\ &= (\frac{v_{eff}}{\sigma}-\frac{1}{2}\frac{\partial \sigma}{\partial z})\tau^{1/2}. && (48)\end{aligned}$$

Consistency of the z-tree and ξ-tree must be demonstrated. To order $\sqrt{\tau}$, the spacings are related by

$$\begin{aligned} \tau^{1/2} &= \int_z^{z+a_+(z)} \frac{dz'}{\sigma(z')} \simeq \frac{1}{\sigma(z)}\int_z^{z+a_+(z)} dz' \left(1-\frac{(z'-z)}{\sigma(z)}\frac{d\sigma(z)}{dz}\right) && (49)\\ a_+(z) &\simeq \sigma(z)\tau^{1/2}+\frac{1}{2}\sigma(z)\tau\frac{d\sigma(z)}{dz}. && (50)\end{aligned}$$

The corresponding value of a_- is

$$a_-(z) \simeq \sigma(z)\tau^{1/2}-\frac{1}{2}\sigma(z)\tau\frac{d\sigma(z)}{dz}. \tag{51}$$

Because the relationships

$$\begin{aligned} \sigma(z)^2\tau &\simeq a_+(z)a_-(z) && (52)\\ v_{eff}(z)\tau &\simeq \tilde{p}(z)a_+(z)-(1-\tilde{p}(z))a_-(z) && (53)\end{aligned}$$

hold to $O(\tau)$, the two trees are consistent. Discrete Bühler-Käsler is recovered for the appropriate choice of $\tilde{p}$.

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