

# Stock Price Volatility in a Multiple Security Overlapping Generations Model

by

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## Abstract

A number of empirical studies have reached the conclusion that stock price volatility cannot be fully explained within the standard dividend discount model. This paper proposes a resolution based upon a model that contains both a random supply of risky assets and finitely lived agents who trade in a multiple security environment. As the analysis shows there exist  $2^K$  equilibria when  $K$  securities trade. The low volatility equilibria have properties analogous to those found in the infinitely lived agent models of Campbell and Kyle (1991) and Wang (1993, 1994). In contrast, the high volatility equilibria have very different characteristics. Within the high volatility equilibria very large price variances can be generated with very small supply shocks. Using previously established empirical results the model can reconcile the data with supply shocks that are less than 10% as large as observed return shocks. The multiple security analysis also shows that within the economy some securities may trade under high volatility conditions, while others trade in low volatility conditions. Switching the economy from a high to a low volatility equilibrium for any single security may be very difficult. Depending upon the variance-covariance structure of the economy, an equilibrium change may require simultaneous control over the trading environment of every single security in the economy.

Shiller's (1981a) paper marks the beginning of a debate on whether the efficient markets hypothesis can be reconciled with observed price and dividend volatility. Naturally, the answer to this question depends to some degree on the statistical model one believes accurately describes the economy. Since Shiller's (1981a) analysis depends upon rather specific assumptions about corporate behavior, several articles have sought to cast doubt on his conclusions by employing alternative models. Papers by Marsh and Merton (1986) and Kleidon (1986) show that if dividends follow a random walk then one can potentially reconcile Shiller's results with an efficient markets model. In response, a new round of experiments were designed to provide accurate results within a wider set of dividend processes. These second generation volatility tests (West (1988), and Mankiw, Romer and Shapiro (1985, 1991)) basically reaffirmed that the efficient markets model cannot explain the observed dividend and price volatilities.

Since misspecification of the dividend process does not appear to drive the empirical results, theoretical models have sought other explanations. De Long *et al.* (1990) and Campbell and Kyle (CK) (1993) both suggest that noise trading may provide the additional freedom needed to reconcile the data with standard financial models. Since these traders purchase and sell securities for reasons that are beyond the model, price changes reflect movements in both dividends and aggregate noise trader demand. Because empirical studies (with the exception of CK) do not include a proxy for noise trader demand, these models suggest a plausible source for the previously unaccounted for volatility. Nevertheless, CK show that rationalizing the data will not be easy. Estimates of their most general model indicate that, relative to changes in fundamentals, stock prices are either 5.77 or 18.51 times too volatile (depending on whether one fixes or estimates the interest rate). Clearly, if the noise traders have sufficiently volatile demands then their presence can explain these results. However, this leaves open the question as to whether other models will also fit the data, and if these models can do so with smaller noise shocks. As Wang (1993, 1994) shows, introducing asymmetric information helps in this regard. Still, the volatility of the market remains proportional to the amount of noise trading. More noise trading implies more volatility which means that potentially large amounts of system noise may be needed to explain the data.

The model presented in this paper seeks to explain the observed dividend and stock price volatilities with as little unobservable noise as possible. Further, it tries to accomplish this task without introducing asymmetric information. While asymmetric information undoubtedly plays a critical role in the market for individual stocks, as Subrahmanyam (1991) indicates, one expects it to play a diminished role when pricing aggregate indices.

The model discussed here shares several features with De Long *et al.* (1990). As in their model the present analysis uses an overlapping generations framework, in which agents are born, invest, realize their returns, consume and then die. Both papers also use the standard market microstructure assumptions that all agents have exponential utilities, and risky assets have normally distributed payoffs. What distinguishes the present model is that it does away with the noise traders, and exchanges take place within a multiple security environment.<sup>1</sup>

The model produces two primary findings. First, the overlapping generations framework produces two equilibria per security, each with very different volatility characteristics. The low volatility equilibria have traits similar to those found in models with infinitely lived agents such as CK and Wang (1993, 1994). However, the high volatility equilibria generally have the opposite comparative statics. As an example, increasing the variance of the market's supply noise increases the price variance in the low volatility equilibrium but decreases it in the high volatility equilibrium. Thus, the model offers the hope that one can explain the apparently large discrepancies between dividend and price variances without resorting to large fluctuations in unobservable quantities.

The second insight derived from the paper is that different securities can trade in environments with different volatility characteristics. Roughly, the public has beliefs over a set of mutual funds. These funds are

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<sup>1</sup>In De Long *et al.* (1990) some traders receive a signal about the future price of the stock which they misinterpret. Their misinterpretation then induces them to submit demand functions that violate the usual rational expectations criteria. As a result, prices will exhibit more volatility than a rational expectations model might otherwise predict. In the present paper all agents maximize well defined utility functions and correctly carry out any calculations.

formed via a singular decomposition of a matrix derived from a combination of the variance-covariance matrices governing individual stock payoffs. Depending upon these beliefs different stocks will show varying levels of volatility. It is even possible for two securities with identical future expected dividend streams to trade at different prices and exhibit different price variances. Death prevents arbitrage. With infinitely lived agents any price discrepancies can be taken advantage of by simply waiting long enough. If agents die this strategy drops out of the feasible set. Another insight, that arises from the multiple security setting, is the potential difficulty facing a regulatory authority that tries to change the prevailing equilibrium. Since the equilibria depend upon vectors of stocks (through the eigenvectors formed in the decomposition) switching to a new equilibrium may require control over a long list of securities. Without a sufficiently wide scope of power changes will not take place.

The fact that overlapping generations models can produce multiple equilibria, with varying levels of volatility, goes back to Azariadis (1981). He examines a production economy where the young use labor to produce a consumption good desired only by the old. The analysis shows that the introduction of a random coordinating variable can result in price uncertainty for the consumption good. However, the results derive from the inability of investors to store the consumption good over time. Allowing investors to store output via capital accumulation (the purchase of a riskless bond in most financial settings) prevents the model from addressing Shiller's volatility findings. An interest bearing storage technology eliminates the extraneous market volatility, since the young will invest, rather than sell, their output if prices are below average. In contrast the model presented in this paper produces "excess" stock market volatility within a more traditional financial framework. Production occurs via the corporate sector, and investors can store the consumption good via a riskless bond if they choose to do so. Furthermore, while the selected equilibrium may cause stock prices to vary (both with respect to the consumption good and each other), unlike the Azariadis model, the

equilibrium selection does not alter corporate production.<sup>2</sup>

In order to find overlapping generations models in which stock prices will violate Shiller's inequality bounds, one must turn to the more recent models of price bubbles such as those by Tirole (1985) and Jackson (1994).<sup>3</sup> In contrast to Azariadis (1981), these papers employ traditional financial paradigms that include an interest bearing storage technology. Roughly, they conclude that a bubble can exist if traders believe that it will grow at the rate of interest. (Otherwise traders will refuse to hold the bubble.) Thus, if bubbles fully explain stock price volatility, then over time the dividend-price ratio should go towards zero. However, it is not apparent that this trend exists within the long run data. So while there may exist periods of time during which speculative bubbles arise other explanations may still prove useful. This paper helps to fill this gap by introducing volatility shocks that arise from the reluctance of risk averse traders to absorb supply shocks. As a result returns do not need to include a speculative return component, and thus the model does not necessarily predict any long term trends will exist in the dividend-price ratio data.<sup>4</sup>

In addition to the previously cited infinite horizon models there exists another literature that examines the issue of stock market volatility in a finite horizon setting. Romer (1993) shows that if some traders are unsure about the quality of information held by others small fundamental shocks can lead to large price changes. Eden and Jovanovic (1994) use asymmetric information about future production to generate their results. A recent paper by Kraus and Smith (1994) shows that the disparate beliefs are sufficient to induce

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<sup>2</sup>However, aggregate holdings of the risk free asset will vary across equilibria. Different equilibria leave investors with different trade offs from selling their holdings of the consumption good at current market prices versus storing it in the risk free bond.

<sup>3</sup>Another recent application of the overlapping generations framework can be found in Dow and Gorton (1994). They use their model to show that information may or may not flow into a stock market depending upon whether an investor believes others will discover his news in a timely manner.

<sup>4</sup>Another difference between the current model and the price bubble literature can be found in the assumptions needed to produce their respective results. For a bubble to grow at the rate of interest the overall economy must grow fast enough to support this as a rational belief. In contrast, the volatility in the current model can occur within an economy that does not grow, or even shrinks. Again these differences arise from the source of the results as explained above.

stock price volatility without fundamental value volatility. Allen and Gale (1994) show that restrictions on borrowing and costly market participation can leave the market so illiquid that small demand shocks will generate large price swings. Allen and Gorton (1993) show that agency issues between investors and mutual fund managers can also lead to potentially high volatility levels within a finite horizon model. While these papers highlight factors that may influence stock price movements, testing them will require the development of new data sets that include the requisite information. By contrast, the current approach is designed to provide an explanation that can be calibrated against previous empirical results, and tested against readily available data sets.

A number of market microstructure papers have also developed models that produce multiple equilibria with varying price patterns. These range from a continuum of equilibria in Bhattacharya and Spiegel (1991) to two equilibria in Grundy and McNichols (1989) and Hirshleifer, Subrahmanyam, and Titman (1994). These three models focus on issues relating to the transmission of asymmetric information via trading in a single stock. Furthermore, by design, it is probably more appropriate to apply these models to short run phenomena associated with individual securities, rather than annual time series data on market aggregates.<sup>5</sup> In contrast, the present paper seeks to explain the long run price-dividend variance ratio on the aggregate stock market. In light of the findings in the above papers, a primary contribution of this paper is to show that the economy can generate large price changes even in the presence of homogenous information and prior beliefs.

The paper is organized as follows. Section 1 presents the formal model. Section 2 provides empirical implications. Section 3 generalizes the dividend and supply processes to allow for mean reversion and discusses the impact on the paper's primary results. This section of the paper also relates the model to more recent empirical findings. Section 4 concludes.

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<sup>5</sup>The asymmetric information assumptions in these models seem unlikely to have a major impact on volatility estimates derived from annual data on the price of the market index. For example, suppose insiders obtain information a week in advance. Then the selected equilibrium in these models determines when, during the week, the information will come out. However, irrespective of the selected equilibrium, by week's end the information will be incorporated into the stock price

# 1 Mathematical Model with Random Walk Processes.

## 1.1 Agents and Assets

The economy contains  $K+1$  assets. Asset  $K+1$ , represents a riskless bond that pays  $r$  units per period. The price of the bond constitutes the numeraire for the economy, and thus always sells for a price of 1. The remaining  $K$  assets represent stocks. In period  $t$ , the stocks pay a dividend  $D_t = \{D_{1t}, \dots, D_{Kt}\}'$  of

$$D_t = D_{t-1} + \delta_t \quad (1)$$

where  $\delta_t = \{\delta_{1t}, \dots, \delta_{Kt}\}'$  equals a period  $t$  shock. For simplicity, this formulation assumes that dividends follow a random walk, Section 3 will explore what happens under more general conditions. The  $\delta_t$  shocks are normally distributed with means of zero, and a variance-covariance matrix of  $\Sigma_\delta$ .

As in many other market microstructure models, the supply of stock  $N_t = \{N_{1t}, \dots, N_{Kt}\}$  varies over time according the following equation,<sup>6</sup>

$$N_t = N_{t-1} + \eta_t. \quad (2)$$

The  $\eta_t = \{\eta_{1t}, \dots, \eta_{Kt}\}'$  represent normally distributed shocks with means of zero, and variance-covariance matrix  $\Sigma_\eta$ .<sup>7</sup> There are several ways one can interpret equation (2). For example, the stock supply may represent a combination of traded assets, and untraded assets that are correlated with the traded assets. Assets that might produce this effect include human capital and real estate. The shocks thus derive from changes in the supply of these untraded assets. Alternatively, as firms grow one may expect that the variability of their

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<sup>6</sup>Among the many other papers that analyze economies with a random stock supply see Bray (1981), Diamond and Verrecchia (1981), Glosten (1989), Bhattacharya and Spiegel (1991), Paul (1995), and Wang (1994).

<sup>7</sup>As with the dividend process, Section 3 considers what happens under more general processes for the supply shocks.

dividend stream will also grow. In this case the shocks represent changes in variability due to the growth of established firms. Whatever the cause, the resulting analysis will remain basically unchanged, so long as market clearing requires demand to equal the supply given by (2).

While stocks in the economy live on forever, the individual investors do not. An investor born in period  $t$ , comes endowed with shares of the bond, and a personal allocation of the period  $t$  supply shock  $\eta_t$ . Since the initial endowments do not impact the equilibrium, assume for simplicity that it is spread equally among the newly born population.

Agents live for two periods.<sup>8</sup> An agent born in period  $t$  can buy and sell securities in period  $t$ , there are no short sale constraints so negative positions are possible. The model assumes that there exists an atomless continuum of investors with unit mass. This assumption implies that each trader acts as a price taker, and in fact no individual can move prices. After trade ends, the economy moves into period  $t+1$ . At this point in time agents receive their period  $t+1$  payoff from the securities they own. Next the agents sell all of their securities, and consume the single consumption good. Utility derives from an exponential utility function over final period consumption, with risk aversion parameter  $\theta$ .

## 1.2 Trading and Timing of the Dividends

At the start of period  $t$  all shares pay  $D_t$ , and bonds  $r$  to their owners. After these payouts, trading takes place. Designate  $X_t(i) = \{X_{1t}(i), \dots, X_{Kt}(i)\}'$  as the period  $t$  demands of individual  $i$ , and  $P_t = \{P_{1t}, \dots, P_{Kt}\}'$  as the market clearing prices. At the beginning of period  $t+1$  the trader receives dividends from his stocks and interest from his bonds, he then sells off his stock portfolio to fund consumption. Thus,  $i$ 's final wealth equals

$$W_{t+1}(i) = X_t(i)'(P_{t+1} + D_{t+1}) + (1+r)b_t(i) \quad (3)$$

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<sup>8</sup>None of the model's qualitative results depend upon the assumption that traders live for only two periods. High and low volatility equilibria exist even when individuals live for one unit of time and can trade continuously. The only essential element when passing from discrete to continuous time is that traders face a constant amount of volatility over their lives. See De Long *et al.* (1990) for a more detailed discussion of this issue.

where  $b_t(i)$  represents his position in the bond. To conserve on notation, the paper suppresses the variable  $i$  from here on in.

Equation (3) presents the terminal (period  $t+1$ ) wealth of an investor born in period  $t$ . This investor's primary trading decision takes place in period  $t$ , when he faces the budget constraint

$$X_t' P_t + b_t = W_t$$

where  $W_t$  equals the value of the trader's initial endowment.

Plugging the budget constraint (4) into (3), allows one to rewrite  $W_{t+1}$  as

$$W_{t+1} = X_t'(P_{t+1} - P_t + D_{t+1} - rP_t) + (1+r)W_t. \quad (5)$$

Thus, each trader born in period  $t$  seeks to maximize his expected utility over  $X_t$  given the terminal wealth relationship in the above equation.

### 1.3 Equilibrium

Following standard solution techniques, one now conjectures a form for the price function, and then verifies that if everybody believes the postulated form, it will in fact hold. In this case there are two state vectors  $N_t$  and  $D_t$  and thus one expects prices to have the linear form

$$P_t = AN_t + BD_t.$$

The  $K \times K$  matrices  $A$  and  $B$  map the state of nature into prices. Since this paper only examines the set of stationary equilibria the conjecture contained in (6) is that these matrices do not vary over time. One can now employ (6) in (5) to produce an equation for each trader's terminal wealth in terms of the matrices  $A$  and  $B$

$$W_{t+1} = X_t' \{A\eta_{t+1} + B\delta_{t+1} + D_t + \delta_{t+1} - rP_t\} + (1+r)W_t. \quad (7)$$

Standard arguments found elsewhere in the exponential-normal literature imply that given (7) each trader will

seek to

$$\max_{X_t} X_t' [D_t - rP_t] + (1+r)W_t - .5\theta X_t' \{A' \Sigma_\eta A + (I+B)' \Sigma_\delta (I+B)\} X_t. \quad (8)$$

This leads to the first order conditions,

$$X_t = \theta^{-1} [A \Sigma_\eta A' + (I+B) \Sigma_\delta (I+B)']^{-1} [D_t - rAN_t - rBD_t] \quad (9)$$

where equation (6) has been used to eliminate  $P_t$  from (9). Since the traders have unit mass the left hand side of (9) equals the aggregate level of demand. In equilibrium, prices must set supply ( $N_t$ ) equal to demand ( $X_t$ ).

Imposing this requirement and then matching terms leaves one with the equilibrium equations

$$D_t - rBD_t = 0 \quad (10)$$

and

$$N_t = -\frac{r}{\theta} [A \Sigma_\eta A' + (I+B) \Sigma_\delta (I+B)']^{-1} AN_t. \quad (11)$$

For (10) to hold for all  $D$ , it must follow that  $B = \frac{1}{r}I$ . Using this result to eliminate  $B$  from (11), produces a quadratic matrix equation describing  $A$ . Some manipulation of (11) yields the following lemma.

*Lemma 1: The matrix  $A$  mapping the supply of shares to prices is symmetric and has the following solution*

$$A = -\frac{1}{2} \frac{r}{\theta} \Sigma_\eta^{-1} + \left[ \frac{1}{4} \left( \frac{r}{\theta} \right)^2 \Sigma_\eta^{-2} - \left( \frac{1+r}{r} \right)^2 \Sigma_\eta^{-1/2} \Sigma_\delta \Sigma_\eta^{-1/2} \right]^{1/2}. \quad (12)$$

Proof: See appendix.

Equation (12) provides the heart of the model. Notice that  $\Sigma_\delta$  does not always multiply  $\Sigma_\eta$ . This implies that

stock price volatility can exist even in the absence of dividend variability. Using equation (12) and the solution for  $B$  one finds that the variance-covariance price matrix equals

$$\Sigma_p = \frac{1}{2} \left( \frac{r}{\theta} \right)^2 \Sigma_\eta^{-1} + \frac{r}{\theta} \Sigma_\eta^{-1/2} \left( \frac{1}{4} \left( \frac{r}{\theta} \right)^2 I - \left( \frac{1+r}{r} \right)^2 \Sigma_\eta^{1/2} \Sigma_\delta \Sigma_\eta^{1/2} \right)^{1/2} \Sigma_\eta^{-1/2} - (2+r) \Sigma_\delta, \quad (13)$$

after some algebra. In their study of a single security model with noise traders De Long *et al.* (1990) also find that prices can vary even when dividends do not. What makes the result presented here unique is that it occurs within an environment in which all agents maximize well defined utility functions and trade in a variety of securities.

#### 1.4 Single Stock Economy

Since single security economies have received so much attention in the literature, it is useful to analyze this case to provide a touchstone with the rest of the literature. In this case  $A$  becomes a scalar and takes on the somewhat simpler form

$$A = -\frac{1}{2} \frac{r}{\theta} \sigma_\eta^{-2} \pm \sigma_\eta^{-2} \sqrt{\frac{1}{4} \left( \frac{r}{\theta} \right)^2 - \left( \frac{1+r}{r} \right)^2 \sigma_\eta^2 \sigma_\delta^2} \quad (14)$$

where, following notational standards,  $\sigma_\eta^2$  and  $\sigma_\delta^2$  have replaced  $\Sigma_\eta$  and  $\Sigma_\delta$  to signify that they are now scalar variables.

Notice that  $A$  has two solutions corresponding to two possible equilibrium beliefs agents may hold.<sup>9</sup> To see the correspondence consider the case where  $\sigma_\delta^2$  equals zero, so that the stock pays a constant dividend every period. In this case the *positive* root of the equation sets  $A$  equal to zero. Since  $A$  and  $B$  both equal zero,

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<sup>9</sup>A footnote in De Long *et al.* (1990) makes it clear that they understood that this result might exist. However, the footnote does not indicate if they pursued this line of analysis.

stock prices do not move at all. Of course this is what occurs in most models. If dividends are constant forever, and this is common knowledge, then in general one expects the price of the stock to remain constant. However, the *negative* root of  $A$  corresponds to an economy where prices change over time even though dividends do not, a divergence from what one normally expects to see in a stock market model with fully rational agents.

The two equilibria in the model correspond to two possible self fulfilling beliefs agents may have about stock prices. Roughly, stock prices will be “excessively” volatile if people believe they will be. To see why beliefs are so important consider an agent’s decision problem when  $\sigma_\delta^2$  equals zero. If this agent lives in the positive root economy then he believes that he knows with certainty next period’s stock price. Since the investor also knows the stock’s dividend payout, he views the stock as a risk free asset. Thus, people in the economy willingly provide a perfectly elastic demand schedule for the stock at a price equal to  $D/r$ , just as in standard introductory finance texts. However, if agents think stock prices will be volatile then a different picture emerges. Because agents do not believe that they can perfectly forecast prices, they no longer voluntarily provide infinite levels of liquidity to the market. Consider what happens in the economy under the negative root for  $A$  when a large supply shock hits the market. If an agent agrees to purchase the stock he now takes on risk. The larger the supply shock investors must absorb, the larger the compensation level they will demand. In the stock market increased compensation levels derive from lower stock prices, and so large supply shocks must be associated with low stock prices. For similar reasons small supply shocks must lead to high prices. In the end, prices become random and negatively correlated with the supply of equity in the economy. Thus, if agents think that prices are volatile, then their actions will make these beliefs self fulfilling.

Within the empirical excess volatility literature the basic issue revolves around the fact that stock price volatility exceeds dividend volatility by several orders of magnitude. The next theorem shows that this model can harmonize these two contradictory facts.

*Theorem 1: As the variance of the supply noise goes to zero ( $\sigma_\eta^2 \rightarrow 0$ ), the variance of the per period price change ( $\Delta P = P_t - P_{t-1}$ ) goes to infinity under the negative root equilibrium.*

Proof: The variance of the per period price change equals  $A^2 \sigma_\eta^2 + B^2 \sigma_\delta^2$ . Plugging in the solutions from equations (14) and (10) produces,

$$\sigma_p^2 = \left( -\frac{1}{2} \frac{r}{\theta} - \sqrt{\frac{1}{4} \left( \frac{r}{\theta} \right)^2 - \left( \frac{1+r}{r} \right)^2 \sigma_\eta^2 \sigma_\delta^2} \right)^2 \sigma_\eta^{-2} + \frac{1}{r^2} \sigma_\delta^2 \quad (15)$$

after some minor algebra. To prove the theorem, let  $\sigma_\eta^2$  go to zero. Now expand the term in brackets and pull out a  $\sigma_\eta^{-2}$  to get,

$$\sigma_p^2 = \left( \frac{1}{2} \left( \frac{r}{\theta} \right)^2 - \left( \frac{1+r}{r} \right)^2 \sigma_\eta^2 \sigma_\delta^2 + \frac{r}{\theta} \sqrt{\frac{1}{4} \left( \frac{r}{\theta} \right)^2 - \left( \frac{1+r}{r} \right)^2 \sigma_\eta^2 \sigma_\delta^2} \right) \sigma_\eta^{-2} + \frac{1}{r^2} \sigma_\delta^2. \quad (16)$$

As  $\sigma_\eta^2$  goes to zero the first term on the right hand side goes to  $\frac{(r/\theta)^2}{\sigma_\eta^2}$  which goes to infinity. Since the second term on the right hand side does not depend upon  $\sigma_\eta^2$ , the price variance must go to infinity. Q.E.D.

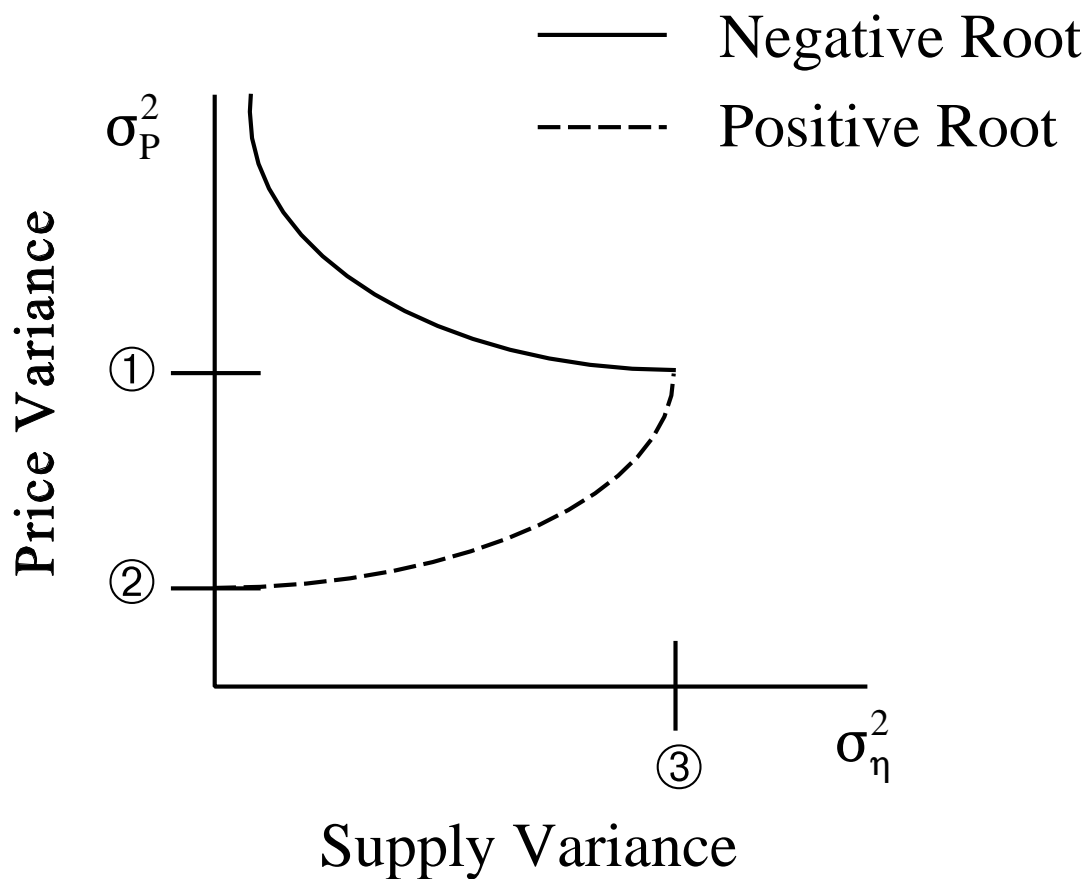
Theorem 1 shows that a very volatile price series can be compatible with an economy consisting of fully rational maximizing agents. In fact, the more volatile the price series the lower the required volatility on the supply series in the negative root equilibrium.

Theorem 1 provides a mapping from supply variance into price variance. However, econometric data does not contain supply variance or the selected root from equation (14) as observable quantities. Instead researchers have information about prices, dividends and individual levels of risk aversion. What the next proposition shows is that the observable data can pin down the unobservable information.

*Proposition 1: Assume that an econometrician knows the risk aversion of the population, the variance of the dividend series, and the variance of the stock's price. Then there exists only one value for the supply noise and one equilibrium consistent with the available data if  $\sigma_p^2 \geq \sigma_\delta^2/r^2$ . If  $\sigma_p^2 < \sigma_\delta^2/r^2$  then the model cannot explain the data.*

Proof: See the appendix.

The proof of Proposition 1 shows that within the positive root equilibrium the price variance increases in the supply variance. In contrast, for the negative root equilibrium exactly the opposite relationship holds. Furthermore, the largest possible price variance in the positive root equilibrium exactly equals the smallest possible price variance in the negative root equilibrium. **Figure 1** shows both the position and general shape of the curves describing the two equilibria.



$$: \textcircled{1} = \left[ \left( \frac{1+r}{r} \right)^2 + \frac{1}{r^2} \right] \sigma_\delta^2, \textcircled{2} = \sigma_\delta^2 / r^2, \text{ and } \textcircled{3} = .25 \left( \frac{r}{\theta} \right) \left( \frac{1+r}{r} \right)^{-2} \sigma_\delta^{-2}.$$

Proposition 1 also shows that the model can be tested empirically. While the model can, in principle, rationalize any variance for the price series that exceeds the variance of the dividend series divided by  $r^2$ , it can only do so via a particular supply noise. Thus, one can think of this model as testable in the same sense one tests models like the consumption CAPM. If estimates of the model's unobservable parameters seem unreasonable, then people conclude that the model does not generate the observed data. Section 2 provides some insight into this issue by showing that the model can produce reasonable estimates for the supply variance based upon the parameters produced in past empirical studies.

## 1.5 Variance in a Multiple Security Economy

While single security models provide numerous insights, there exist many issues that can only be addressed in a multiple security environment. This section examines the way stocks interact within the set of potential equilibria, and shows why moving from one equilibrium to another may be difficult.

While a single security economy contains two potential equilibria, the multiple security economy contains  $2^K$  potential equilibria. Naturally, they derive from the same source: the square root term in the solution to  $A$ . If a linear equilibrium exists the term under the square root in (12) is a symmetric positive definite matrix. Write

$$\Gamma' \Lambda \Gamma = \frac{1}{4} \left( \frac{r}{\theta} \right)^2 \Sigma_{\eta}^{-2} - \left( \frac{1+r}{r} \right)^2 \Sigma_{\eta}^{-1/2} \Sigma_{\delta} \Sigma_{\eta}^{-1/2} \quad (17)$$

where  $\Gamma$  represents an orthonormal matrix of eigenvectors, and  $\Lambda$  a diagonal matrix of eigenvalues. Because equation (17) appears within a square root in the solution for  $A$ , one can restate the far right term in  $A$  (from equation (12)) as  $\Gamma' A^{1/2} \Gamma$ . This implies that there exist two potential values for each diagonal element of  $\Lambda^{1/2}$  that are compatible with the equilibrium requirements. With  $K$  securities there exist  $K$  diagonal elements, and so there must be  $2^K$  possible solutions, each corresponding to a unique equilibrium. One thus has the following proposition.

*Proposition 2: With  $K$  securities in random supply there exist  $2^K$  equilibria. These equilibria correspond to the set of feasible eigenvalues for the matrix in equation (17).*

In the single security case the equilibrium in the economy depends upon the selected root for equation (14). When the market contains numerous securities the square root of the eigenvalues within  $\Lambda$  play an analogous role. As one might suspect the volatility conclusions from the single security case also generalize.

*Proposition 3: Consider an equilibrium, and the variance of any portfolio under that equilibrium. Then switching any eigenvalue in the matrix  $\Lambda^{1/2}$  from negative to positive, reduces the portfolio's variance.*

Proof: See appendix.

Clearly, society would like to operate in an economy where all the selected roots from  $\Lambda^{1/2}$  are positive. If some of the roots are negative, then there exists an incentive for some organization to try and change the equilibrium. Potentially, one can view a stock exchange as playing this role. A market maker with deep enough pockets, and a sufficiently long horizon may be able to credibly announce that he will make a market in accordance with a less volatile equilibrium. Since investors prefer lower volatility they should go along with the change. While this scenario seems inviting, it has one serious drawback. The equilibria do not, in general, associate each eigenvalue with a single security. Rather an eigenvalue determines the volatility of a mutual fund with portfolio weights determined by the associated eigenvector. To change the equilibrium a specialist must somehow alter the behavior of an entire mutual fund and not just a single security. Unless he can credibly accomplish this task, investors will not believe that they are in a new equilibrium and so the old equilibrium will prevail. It seems that once a high volatility equilibrium gets started, it may be very difficult to alter it.

Equation (17) also shows how two securities with identical dividend streams can trade at different prices, and exhibit different price volatilities. A simple examination makes it clear that  $\Sigma_\delta$ , the dividend variance-covariance matrix, can be singular. Only the supply shock matrix,  $\Sigma_\eta$ , needs to be of full rank. Thus, depending upon the supply shocks, one can create an equilibrium in which two stocks with perfectly correlated dividend streams have different moments associated with their prices. Initially, it might seem that if the two stocks have different supply levels then in some sense they are not identical. However, this is only true to a limited degree. In principle, since both stocks have identical dividend streams one can divide the supply shocks up in any manner one wishes, without changing the fundamentals of the economy. Thus, in this case the correlation between the supply shocks is in some sense arbitrary. Nevertheless, if the dividend streams are

identical and the prices are not, does this open up an arbitrage opportunity? No, providing investors have finite lives. An infinitely lived individual can buy the cheap stock, sell the expensive stock and make a risk free profit. However, in any finite length of time an investor that tries this risks having the prices move further in the wrong direction. Thus, stocks that have identical dividend streams but different pricing patterns do provide people with a profit opportunity, but one that contains risk. The equilibrium equations tell us the degree to which prices can diverge and still produce demands that balance out the supplies. Can a long lived mutual fund make an arbitrage profit? Again the answer remains no. Consider an open ended mutual fund. If the prices of the underlying stocks vary so will the value of the mutual fund itself. Thus, if the shareholders of the mutual fund are finitely lived they will take on, through the mutual fund, the same risks as they would buying the shares directly. The mutual fund only acts as a shell and cannot change the equilibrium simply via its existence.

There already exist several models that can potentially reconcile the empirical volatility results through the introduction of “noise” from various sources. One contribution of this paper was to show, in Theorem 1, that a model with fully rational agents can produce the requisite variances with small supply noise levels. That theorem used equation (15) to derive its results, and was based on the assumption that the market contains only one risky security. As the next set of results show, in a multiple security environment, equation (15) can vastly overstate the required aggregate supply noise. Propositions 4 and 5 show that under fairly mild assumptions, as the number of securities grows to infinity the supply variance of the market portfolio needed to reconcile any particular aggregate price and dividend variance goes to zero. Lemma 2 then shows, under the assumption that all securities have independent identically distributed payoffs and supplies, that if there exist  $K$  securities then the value of  $\sigma_{\eta}^2$  derived via (15) will overstate the required aggregate supply variance by a factor of  $K^2$ .

There are two routes that lead to the conclusion that as the number of securities goes to infinity the supply noise needed to produce any particular variance in the price innovations goes to zero. One can either place some restrictions on the form of the variance-covariance matrix describing the supply noise, or the

dividend shocks. Proposition 4 takes the former tack, and Proposition 5 the latter. For the limiting results that follow let “ $K$ ” both index a particular economy and represent the number of securities. Further, assume that at a particular time  $t$ :

*Assumption 1:*  $N'N = 1$  for all  $K$ .

*Assumption 2:* Let  $n_j$  represent then  $j$ th element of  $N$  at time  $t$ . Then

$$\lim_{K \rightarrow \infty} \max\{n_1, \dots, n_K\} \rightarrow 0. \quad (18)$$

*Assumption 3:*  $N'\Sigma_\delta N = k_\delta$ , where  $k_\delta$  is a constant independent of  $K$ .

*Assumption 4:*  $N'\Sigma_p N = k_p$ , where  $k_p$  is a constant independent of  $K$ .

*Assumption 5:*  $k_p \geq \frac{1}{r^2}k_\delta$ .<sup>10</sup>

Assumption 1 holds the size of the economy constant as the number of securities increases. Setting  $N'N = 1$  just normalizes the currency so that the economy has an initial value of one unit. Assumption 2 states that as the number of securities goes to infinity any one firm becomes only a small part of the economy. Assumptions 3 and 4 hold the aggregate dividend and price variance constant. Finally, Assumption 5 states that stock prices exhibit “excess” volatility.

An econometrician cannot observe the supply noise directly. Thus, a reasonable starting point for empirical research may be to assume that the variance-covariance matrix of the security supply is a diagonal matrix proportional to  $I$ . If this assumption describes the economy then the next proposition shows that under Assumptions 1 through 5 the aggregate supply noise goes to zero as  $K$  goes to infinity.

*Proposition 4:* Assume that one can write  $\Sigma_\eta$  as  $\sigma_\eta^2 I$ , where  $\sigma_\eta^2$  is a scalar. Then under assumptions 1 through 5  $N'\Sigma_\eta N = \sigma_\eta^2$  goes to zero as  $K$  goes to infinity.

Proof: From equation (12) an equilibrium exists if and only if

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<sup>10</sup>In Assumptions 1 through 5 the date index  $t$  has been suppressed for notational simplicity.

$$\frac{1}{4} \left( \frac{r}{\theta} \right)^2 I - \left( \frac{1+r}{r} \right)^2 \Sigma_{\eta}^{1/2} \Sigma_{\delta} \Sigma_{\eta}^{1/2} \quad (19)$$

is a positive semi-definite matrix. Under the assumption that  $\Sigma_{\eta} = \sigma_{\eta}^2 I$ , equation (19) simplifies to

$$\frac{1}{4} \left( \frac{r}{\theta} \right)^2 I - \left( \frac{1+r}{r} \right)^2 \sigma_{\eta}^2 \Sigma_{\delta}. \quad (20)$$

If (20) is a positive semi-definite matrix then pre and post multiplying by  $N$  must produce a nonnegative number. Using Assumptions 1 and 2 one can prove that  $N'N$  goes to zero as  $K$  goes to infinity. Since  $\Sigma_{\delta}$  is a positive definite matrix, and by Assumption 3  $N' \Sigma_{\delta} N$  does not depend on  $K$  one must have that  $\sigma_{\eta}^2$  goes to zero. Since  $N' \Sigma_{\eta} N = \sigma_{\eta}^2$  this proves that the aggregate supply variance goes to zero if an equilibrium exists.

Finally, given the restrictions imposed by Assumption 5 one can always select a value for  $\sigma_{\eta}^2$  that satisfies Assumption 4, and causes an equilibrium to exist by solving equation (13) and picking positive or negative eigenvalues as needed. Since the proof follows along the same lines as the proof for Proposition 1 it is not repeated here. Q.E.D.

While Proposition 4 allows for a general dividend covariance matrix it imposes assumptions on the supply's covariance structure. Since the supply cannot be observed directly it may be possible to construct a more robust statistical test by restricting  $\Sigma_{\delta}$  instead as in the next proposition.

*Proposition 5: Assume that as  $K$  goes to infinity the smallest eigenvalue of the matrix  $\Sigma_{\delta}$  approaches a number strictly greater than zero. Then under Assumptions 1 through 5 the supply variance of the market portfolio  $N' \Sigma_{\eta} N$  goes to zero.*

Proof: An equilibrium exists if and only if the matrix (19) is positive semi-definite, and thus

$$\lim_{K \rightarrow \infty} N' \Sigma_{\eta}^{1/2} \Sigma_{\delta} \Sigma_{\eta}^{1/2} N \rightarrow 0 \quad (21)$$

since Assumptions 1 and 2 imply  $N'N$  goes to zero as  $K$  goes to infinity. Let  $Y = \Sigma_{\eta}^{1/2} N$ . Then using (21) one has

$$N' \Sigma_{\eta}^{1/2} \Sigma_{\delta} \Sigma_{\eta}^{1/2} N = Y' \Sigma_{\delta} Y \geq v_{\min} Y' Y \quad (22)$$

where  $v_{\min}$  represents the smallest eigenvalue of  $\Sigma_{\delta}$ . By assumption  $v_{\min}$  approaches a strictly positive number as  $K$  goes to infinity. Thus, (21) can only hold if  $Y'Y$  goes to zero which proves the proposition since  $Y'Y = N' \Sigma_{\eta} N$ . Q.E.D.

Propositions 4 and 5 show that within a multiple security environment very small shocks can produce large price changes. To obtain some intuition as to just how quickly the supply shock declines consider what happens as one divides the economy's risky capital asset into  $K$  identically sized independent pieces. As the following lemma shows this divides the required supply variance by  $K^2$ .

*Lemma 2: Assume  $\Sigma_{\eta} = \sigma_{\eta}^2 I$ ,  $\Sigma_{\delta} = \sigma_{\delta}^2 I$ , and  $n_i = 1/K$  in addition to Assumptions 1 through 5. As  $K$  increases assume that the equilibrium only uses either the positive or negative roots from the matrix  $\Lambda^{1/2}$ . Then  $\sigma_{\delta}^2 = K k_{\delta}$  and if  $\sigma_{\eta}^2 = \hat{\sigma}_{\eta}^2$  equals the solution in the 1 security case then  $\sigma_{\eta}^2 = \hat{\sigma}_{\eta}^2 / K$  in the  $K$  security case. The latter result implies that the supply variance of the market portfolio equals  $\hat{\sigma}_{\eta}^2 / K^2$ .*

Proof: Since  $\Sigma_{\delta} = \sigma_{\delta}^2 I$ , and  $n_i = 1/K$  the dividend variance will equal  $k_{\delta}$  if  $\sigma_{\delta}^2 = K k_{\delta}$ . Use this to substitute out  $\sigma_{\delta}^2$  in equation (13). Next replace  $\sigma_{\eta}^2$  with  $\hat{\sigma}_{\eta}^2 / K$  to verify that this keeps  $N' \Sigma_{\eta} N$  constant. Finally, with  $n_i = 1/K$ , and  $\sigma_{\eta}^2 = \hat{\sigma}_{\eta}^2 / K$  it follows that the supply variance of the market portfolio equals  $\hat{\sigma}_{\eta}^2 / K^2$ . Q.E.D.

Lemma 2 shows that in practice a very small aggregate supply variance can easily produce a very large price variance. Empirically the lemma indicates that estimates of the model based only on the aggregate market index will probably overestimate the supply variance by a factor of perhaps several hundred.

## 2 Empirical Implications

### 2.1 Calibrating the Model

Normal-exponential models define payoffs in terms of units of consumption. In a purely theoretical analysis this does not present a problem. By altering the means, variances, and the population's risk aversion parameter scaling problems can be eliminated. Unfortunately, calibrating the model with previously reported empirical results is not trivial. Empirical researchers often rescale variables to eliminate growth trends, heteroskedasticity, and other miscellaneous factors. As a result, the reported figures are no longer in dollar terms that can be plugged directly into the model from the previous section. However, with some adjustment these problems can be overcome by employing combinations of variables that produce scale free numbers.

Intuitively, one would like the model to explain the data with “relatively small” supply shocks. However, providing a useful definition for “relatively small” turns out to be fairly complex. Since the present analysis seeks to explain market volatility it seems intuitive to compare the supply volatility to the wealth volatility induced from one share of stock. For notational consistency define  $\pi_t$  as the wealth from holding one share of stock this period (i.e.,  $\pi_t = P_t$ ), and  $\pi_{t+1}$  as the wealth from selling the share next period and cashing in its dividend (i.e.,  $\pi_{t+1} = P_{t+1} + D_{t+1}$ ). Further define  $\sigma_\pi^2$  as the per period variance in  $\pi$ , so that  $\sigma_\pi^2 = E(\pi_{t+1} - \pi_t)^2$  given  $\pi_t$ . At this point one may be tempted to compare  $\sigma_\eta$  directly with  $\sigma_\pi$ . Unfortunately, a direct comparison is not productive since  $\sigma_\eta$  and  $\sigma_\pi$  are in inverse units to each other. To see this consider a unit of account such that there exists only one share of stock and fix both  $\sigma_\eta$  and  $\sigma_\pi$ . Now consider the same economy but split the stock into two shares. This doubles the number of shares which doubles  $\sigma_\eta$  and simultaneously cuts the dividend per share in half thereby dividing  $\sigma_\pi$  in half. Thus, depending on the normalization selected one can set  $\sigma_\eta/\sigma_\pi$  to any value. Fortunately, one can easily circumvent this problem by normalizing with the appropriate units thereby producing a scale free test statistic

$$S = \frac{\sigma_{\eta} \pi}{N \sigma_{\pi}} = \frac{\sigma_{\eta} P}{N \sigma_{\pi}}. \quad (23)$$

If  $S$  equals one the supply shocks and wealth shocks have similar magnitudes. Larger values indicate that  $\sigma_{\eta}$  predominates, and smaller values that  $\sigma_{\pi}$  predominates. Since it seems unlikely that the aggregate supply shocks exceed the stock price shocks a “good” model should produce fairly small values for  $S$ .

While (23) takes care of the scaling problem it requires one to know both  $N$  and  $P$  which for fairly obvious reasons do not appear among the statistics generally reported. However, this problem can be circumvented by requiring  $N$  and  $P$  to take on values that will induce the model to produce statistics that correspond with well accepted properties associated with market returns. Under a standard CAPM like model, with dividends that follow a random walk,  $P_t = D/r^*$  where  $r^* > r$  represents the dividend discount rate. One can then write the return variance ( $\sigma_r^2$ ) as

$$\sigma_r^2 = \frac{(1+r^*)^2 \sigma_{\delta}^2}{D_t^2}. \quad (24)$$

Well known estimates from Ibbotson Associates (1989) place  $\sigma_r^2$  at .209 for annual data.<sup>11</sup> Thus, filling in (24) with  $\sigma_r^2$  at .209 and a particular study’s reported value for  $\sigma_{\delta}^2$  and  $r^*$  one obtains the implicit price scale that will induce the model to produce a sensible return series. The value of  $N$  in (23) is somewhat simpler to find. Let  $W$  represent the dollar value of the wealth held by each trader in the stock market. Then by definition  $W_t = P_t N$ . Substituting this into (23) eliminates  $N$  from the equation.

To obtain  $S$  in terms of observable variables one now only needs an estimate of  $\sigma_{\pi}$ . Since  $\pi_{t+1} = P_{t+1} + D_{t+1}$  and  $\pi_t = P_t$  it follows that

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<sup>11</sup>However, as the text will shortly explain,  $S$ ’s value does not depend upon  $\sigma_r^2$ .

$$\sigma_{\pi} = \left( \frac{1+r^*}{r^*} \right) \sigma_{\delta} \quad (25)$$

since  $P_t = D_t/r^*$ .

Using (24), and (25) along with the substitutions  $N=W/P_t$  and  $P_t=D_t/r^*$  the statistic  $S$  reduces to

$$S = \frac{\sigma_{\eta} \sigma_{\delta} (1+r^*)}{W_t r^* \sigma_r^2}. \quad (26)$$

Even with the values of  $\sigma_{\delta}$  and  $\varphi$  reported by a study the model cannot pin down  $\sigma$  without knowledge of  $\theta$ . Under a mean-variance analysis a standard CAPM type model will produce the first order condition

$$r^* - r - \theta \sigma_{\delta}^2 (1+r^*)^2 \frac{X_t}{r^* D_t} = 0. \quad (27)$$

Solving this for  $\theta$  one obtains

$$\theta = \frac{r^* (r^* - r) D_t}{\sigma_{\delta}^2 (1+r^*)^2 X_t}. \quad (28)$$

Now use (24) to eliminate  $\sigma_{\delta}$  from the above equation, along with  $P_t = D_t/r^*$  and  $W_t = P_t X_t$  to produce

$$\theta = \frac{r^* - r}{\sigma_r^2 W_t}. \quad (29)$$

Using the above value for  $\theta$ , along with values from a study for  $\sigma_{\delta}$  and  $\sigma_p$ , equation (15) will yield the value of  $\sigma_{\eta}$  needed to reconcile the model with the reported empirical results. Notice that the implied risk aversion

parameter is *not* scale free. Moving from dollars to cents will multiply  $W_t$  by 100 while leaving all the other variables on the right hand side of (29) unchanged. Thus, the value of  $\theta$  depends inversely on the units of account selected. Normally, people think of preference parameters as exogenous to the model and independent of the currency unit. But, the parameter  $\theta$  represents risk aversion per unit of consumption as measured in the unit of account. Changing the unit of account changes the risk aversion through the change in translation from currency to consumption units. This means that one cannot assign a risk aversion parameter within the model unless the unit of account has been specified

At first it may appear that the estimate of  $S$  will depend heavily upon the values used for  $\sigma_r^2$  and  $W_t$ . In fact, however,  $S$  is invariant to changes in either of these parameters. To see why notice that  $\theta$  depends inversely upon both  $\sigma_r^2$  and  $W_t$  and so does  $S$ . At the same time one can see from equation (15) that  $\sigma$  depends inversely upon  $\theta$ . Thus, the changes induced to the model through both  $\sigma_r^2$  and  $W_t$  leave  $S$  unchanged. Intuitively, the reason this occurs is that  $S$  has been made into a scale free statistic, and the values of  $\sigma_r^2$  and  $W_t$  depend arbitrarily upon the selected scale.

## 2.2 Applying the Model to Past Empirical Studies

Shiller's (1981a) paper brought out the point that long term stock price and dividend series exhibit very different volatilities. This leaves stock price volatility models with the challenge of explaining these differences within plausible parameter values. The first explanations (Marsh and Merton (1986) and Kleidon (1986)) attacked the problem by criticizing the robustness of the initial econometric estimates. They show that if dividends follow a random walk then Shiller's (1981a) estimates may be compatible with the standard dividend discount model. In response several papers tried to develop tests that were robust in such an environment. Among the many variance inequalities that have been tried one of the earliest is

$$\sigma(\Delta P) \leq \sigma(\Delta D) / \sqrt{2r^3 / (1+2r)} \quad (30)$$

which can be found in Shiller (1981b) with  $\Delta D$  defined as  $D_t - D_{t-1}$ . Using one year intervals, the historical

data does not violate equation (30). However, when the interval increases to ten years Shiller (1981b) finds that  $\sigma(\Delta P)=69.4$ , and  $\sigma(\Delta D)=16.5$ . Thus, in his test, markets can only be efficient if the annual interest rate remains below 3.2%. Since actual discounts rates appear to lie above 4%, the model's predictions do not conform with the data.

Since the model presented in Section 1 predicts that stock price volatility has both a dividend and supply component it can potentially reconcile Shiller's (1981b) estimates with an environment populated by rational traders. By adjusting  $\sigma_\eta$  equation (15) can produce an expected price volatility equal to that reported in past empirical work given any previously estimated dividend variance. However, in order to calculate  $\sigma_\eta$  from (15) and  $S$  from (26) estimates of the dividend discount factor and the return variance are needed. Based upon the values used in West (1989), Mankiw, Romer, and Shapiro (1991) and CK an appropriate discount factor seems to lie between 4% and 7%. Thus for completeness the next table presents results for discount rates within this range.

Table I provides values of the model's parameters that will produce data consistent with Shiller's (1981b) estimates in a one security environment. However, as Lemma 2 shows the resulting estimates of the aggregate supply standard deviation will exceed the supply standard deviation in a multiple security economy by a factor of  $K$ . Thus, the single risky security aggregate supply variance figures calculated in all of the tables that follow should be viewed as extremely conservative upper bounds. To provide a more realistic scenario figures are also provided under a 30 security economy based on the assumptions leading to Lemma 2.

<b>Table I</b>				
Model Statistics Implied by				
Shiller's (1981b) Equation (I-3) Empirical Estimates				
(In all cases the estimates imply that the negative root equilibrium holds.)				
r	.04	.05	.06	.07
$1-\sigma_\delta/r\sigma_p$	0.505	0.622	0.699	0.754
<i>S</i>				
K = 1	0.552	0.502	0.458	0.422
K = 30	0.018	0.017	0.015	0.026
Notes: Row labeled r: interest rate used in each column. Row labeled $1-\sigma_\delta/r\sigma_p$ : fraction of the price variance that cannot be accounted for in the standard fixed supply dividend discount model. West (1988 equation (17)) uses an identical measure multiplied by 100. Rows labeled S: numerical value of <i>S</i> under the given interest rate, the return standard deviation, and the number of traded securities under the assumptions of Lemma 2.				

Employing the empirical estimates within the model, requires that the economy operate under the negative root equilibrium. Thus, it appears that the additional equilibrium found in this model provides a useful degree of freedom with which to explain the data.

Notice that column labeled  $1-\sigma_\delta/r\sigma_p$  shows that dividend shocks explain less than half of the price volatility. Nevertheless, the calculated value of *S* lies well below 1. The model, therefore, explains over half the price volatility with relatively small supply shocks. Recall that supply shocks are generally taken to represent changes in human capital and other illiquid assets. As such it seems likely that the variance of these supply shocks should be small relative to the stock return shocks. Table I indicates that the model can produce estimates which conform to this intuition.

The inequality used by Shiller (1981b) was probably the first that did not depend upon a dividend process without a unit root. However, it has since been supplanted by more powerful second generation tests. Perhaps the strongest results appear in West (1988).

The dividend process estimated in West (1988) is more general than the one used in the model

presented here, making it impossible to produce a direct comparison. However, with this caveat in mind one can draw some inferences. West provides a number that he refers to as the excess price variance in Table II column (8) of his paper. Under the assumptions leading to the model presented here West's measure can be written as

$$WS = 100 \frac{\sigma_p^2 - \sigma_\delta^2 / r^2}{\sigma_p^2}. \quad (31)$$

Table II shows the parameter values needed to produce values for WS equal to West's reported results.

<b>Table II</b>							
Model Statistics Implied by West's (1988) Table II Empirical Estimates (In all cases the estimates imply that the negative root equilibrium holds.)							
Data	S&P				Modified Dow-Jones		
Model	1	2	3	4	5	6	7
Estimated r	0.074	0.062	0.074	0.058	0.064	0.066	0.079
$1-\sigma_s/r\sigma_p$	0.929	0.800	0.931	0.786	0.969	0.898	0.973
S							
K=1	0.29226	0.45079	0.28852	0.46301	0.19864	0.34439	0.18455
K=30	0.00974	0.01503	0.00962	0.01543	0.00662	0.01148	0.00615
Notes: Descriptions of the data can be found in West (1988). Estimated r comes from column (3) in West's (1988) Table II. The entries in the $1-\sigma_s/r\sigma_p$ row equal the values reported in West's (1988) Table II column (8) divided by 100.							

Based on Theorem 1 the figures presented in Table II are not too surprising. Compared to the results in Shiller (1981b) the statistical results in West (1988) lead to an even stronger rejection of the fixed supply dividend discount model. Theorem 1 shows that under the negative root equilibrium the higher the price variance, holding constant the dividend variance, the smaller the supply shock needed in equilibrium. Thus, West's (1988) stronger rejection of the fixed supply model leads the model in Section 1 to rationalize his results with a smaller supply noise and thus the values of  $S$  in Table I exceed those for Table II. As with the earlier results the model explains a large fraction of the price variance (over 75%) with a relatively small supply shock.

The results to this point show that the model developed in Section 1 can reconcile a number of observed statistical results with a rational agent model. One underlying assumption has been that dividends follow a random walk. This assumption has been used to both simplify the analysis and to provide the greatest degree of congruence possible with the empirical literature. However, as a number of authors have pointed out (Marsh and Merton (1986), and Kleidon (1986)) a random walk dividend process provides the standard dividend discount model with its greatest opportunity to explain the data. In fact, the second generation volatility tests were designed to overcome this particular difficulty. But there does exist evidence in both

DeJong and Whiteman (1991) and CK that dividends may have a trend stationary component. If this is in fact true then the case against the standard fixed supply dividend discount model becomes much stronger. The next section of the paper explores the model's implications under a more general dividend structure. It shows that neither a trend stationary dividend nor a trend stationary supply process reduces the model's ability to reconcile the data.

### 3 A Trend Stationary Model

#### 3.1 Modified Model and its Solution

Section 1 assumes that both dividends and the supply of equity follow a random walk. Now relax that assumption and assume that dividends are generated via the following formula

$$D_t = D_{t-1} - \beta(D_{t-1} - \bar{D}) + \delta_t, \quad (32)$$

and the stock supply by

$$N_t = N_{t-1} - \alpha(N_{t-1} - \bar{N}) + \eta_t. \quad (33)$$

Here  $\bar{D}$  represents a vector of constants towards which dividends move over time, while  $\bar{N}$  represents a similar vector for the stock supply. The terms  $\beta$  and  $\alpha$  are scalar constants that determine the speed at which the processes revert towards their stationary values. Setting  $\beta$  or  $\alpha$  to zero causes equations (32) and (33) to revert back to their respective form in Section 1.

Since the analysis remains basically identical to that found in Section 1, only the results are presented here. The equation relating prices to the underlying state variables now includes a constant component ( $K \times 1$ ) vector  $C$ , and thus prices equal

$$P_t = AN_t + BD_t + C. \quad (34)$$

Repeating the steps found earlier one finds that  $A$ ,  $B$  and  $C$  satisfy the following equations

$$A = -\frac{1}{2}\left(\frac{\alpha+r}{\theta}\right)\Sigma_{\eta}^{-1} + \left(\frac{1}{4}\left(\frac{\alpha+r}{\theta}\right)^2\Sigma_{\eta}^{-2} - \left(\frac{1+r}{\beta+r}\right)^2\Sigma_{\eta}^{-1/2}\Sigma_{\delta}\Sigma_{\eta}^{-1/2}\right)^{1/2}, \quad (35)$$

$$B = \frac{1-\beta}{\beta+r}I, \quad (36)$$

and

$$C = \frac{(1+r)\beta}{(\beta+r)r}\bar{D} + \frac{\alpha}{r}\left[-\frac{1}{2}\left(\frac{\alpha+r}{\theta}\right)\Sigma_{\eta}^{-1/2} + \left(\frac{1}{4}\left(\frac{\alpha+r}{\theta}\right)^2\Sigma_{\eta}^{-2} - \left(\frac{1+r}{\beta+r}\right)^2\Sigma_{\eta}^{-1/2}\Sigma_{\delta}\Sigma_{\eta}^{-1/2}\right)^{1/2}\right]\bar{N}. \quad (37)$$

While  $C$  seems to add a great deal of complexity to the solution, it drops out of the variance-covariance calculations. Thus, as the next subsection will show a trend stationary dividend does not manifestly change the model's qualitative properties.

### 3.2 Implied Price Variance

The variance-covariance matrix of prices can be calculated as

$$E(P_{t+1}-P_t)(P_{t+1}-P_t)' = A\left[\Sigma_{\eta} + \alpha^2(N_t-\bar{N})(N_t-\bar{N})'\right]A' + B\left[\Sigma_{\delta} + \beta^2(D_t-\bar{D})(D_t-\bar{D})'\right]. \quad (38)$$

To obtain the formula only in terms of endogenous variables one can use equations (35) and (36) to eliminate  $A$  and  $B$  from (38).

Notice that increasing  $\alpha$ , and  $\beta$  from 0 does not materially alter the equilibrium's properties. This becomes even more apparent in the scalar case. When there exists only one risky stock,  $A$  reduces to

$$A = -\frac{1}{2}\left(\frac{\alpha+r}{\theta}\right)\sigma_{\eta}^{-2} \pm \sigma_{\eta}^{-2}\sqrt{\frac{1}{4}\left(\frac{\alpha+r}{\theta}\right)^2 - \left(\frac{1+r}{\beta+r}\right)^2\sigma_{\delta}^2\sigma_{\eta}^2}. \quad (39)$$

Typically models find that larger values of  $\beta$  reduce volatility by driving  $A$  towards zero. This result also holds here but only for the positive root equilibrium where  $\partial A/\partial\beta > 0$  (since  $A < 0$ ). Under the negative root equilibrium  $\partial A/\partial\beta < 0$ , thus *increasing* price volatility. Nevertheless, because the basic structure of (39) is identical to (14) the conclusions reached in earlier sections remain unchanged.

### 3.3 Past Empirical Results and their Implications for the Model

#### 3.3.1 Mankiw, Romer and Shapiro (1991)

The empirical test developed by Mankiw, Romer and Shapiro (1985, 1991) has a fundamentally different structure from the other excess volatility tests considered so far. While most tests use variance bounds the Mankiw-Romer-Shapiro (MRS) test uses a pure volatility bound. The difference is that one can calculate the MRS test statistic without demeaning the data. Some algebra shows that applying the random walk model from Section 1 to the MRS model will produce implied estimates for the supply variance that depend upon the aggregate stock supply. This appears to make it impossible to properly apply that model to the MRS results. However, the stationary model does not suffer this difficulty if one assumes that the supply equals a constant plus a shock that does not persist ( $\alpha = 1$ ).

MRS define  $P_t^0$  as a “naive” forecast based upon period  $t$  information. Their paper then sets  $P_t^0 = D/r^*$  under the assumption that dividends follow a random walk with zero drift which corresponds to setting  $\beta = 0$  in Section 3.1. Defining  $P_t^*$  as the perfect foresight value of  $P$  at time  $t$  MRS test whether

$$E\left(\frac{P_t^* - P_t^0}{P_t}\right)^2 = E\left(\frac{P_t^* - P_t}{P_t}\right)^2 + E\left(\frac{P_t - P_t^0}{P_t}\right)^2 \quad (40)$$

holds. Excess volatility exists to the extent that the right hand side of the equation exceeds the left hand side. The test does not presume that  $E(P_t^* - P_t^0)$ ,  $E(P_t^* - P_t)$ , or  $E(P_t - P_t^0)$  equal zero (which is why the MRS test is not a variance bound test).

Using the model from Section 3.1

$$P_t^* = \frac{A\eta_{t+1} + \left(\frac{1+r}{\beta+r}\right)D_{t+1} + C}{1+r} \quad (41)$$

where the value of  $P_{t+1}$  has been derived by plugging  $\alpha = 1$  into equations (35), (36), and (37). These same values for  $\alpha$  and  $\beta$  also imply that

$$P_t = A\eta_t + \frac{1-\beta}{\beta+r}D_t + C \quad (42)$$

where A can be found from (35). Table 1 in MRS provides three columns of estimates, one for each squared term in (40). To fit all three columns of numbers the model requires at least three free parameters. However, unless  $\beta$  equals 1 the volatilities will depend upon  $\bar{D}$ ,  $D_t$ , and the initial period dividend  $D_0$ . To avoid this problem the subsequent analysis assumes that the dividend process begins infinitely far in the past. This allows the model to produce unconditional expected values for the MRS statistics without reference to the exogenously specified initial value for the dividend process. The details can be found in the Appendix.

The S statistic developed earlier needs to be modified in order to accommodate the mean reversion in the dividend process. As shown in the Appendix S now becomes

$$S = \frac{\sigma_\eta \sigma_\delta (1+r^*)}{W_t (r^* + \beta) \sigma_r^2} \quad (43)$$

which is identical to (26) except for the additional  $\beta$  in the denominator.

Table III presents statistics produced by the model when fitted to the MRS results.

<b>Table III</b>			
Model Statistics Implied by Mankiw, Romer, and Shapiro's (1991) Table 1 (In all cases the estimates imply that the <i>positive</i> root equilibrium holds.)			
r	.05	.06	.07
2 Year	0.239	0.087	0.060
	0.218	0.266	0.282
	0.035	0.037	0.070
5 Year	0.231	0.126	0.120
	0.267	0.325	0.358
	0.028	0.058	0.139
10 Year	0.139	0.093	0.115
	0.430	0.551	0.683
	-0.012	0.052	0.173

Notes: # Year refers to the number of years in the holding period as defined in MRS (1991). Each cell contains  $1 - \sigma_d / r \sigma_p$  as the top entry,  $S$  as the middle entry, and  $\beta$  as the bottom entry.

The estimates indicate that while dividends are mean reverting the tendency is very weak. This result corresponds with other studies such as DeJong and Whiteman (1991). Contrary to most other volatility studies MRS found only weak evidence against the fixed supply dividend discount model. Once the model has been fitted to their data it mimics this result by indicating that dividend shocks can explain all but 8% to 24% of the price variance. The model adjusts for their low price volatility findings by switching from the high volatility negative root equilibrium to the low volatility positive root equilibrium. By doing so the model also keeps the required supply shocks at a relatively low level. For the most part, the figures in the table indicate that the requisite supply shocks are only about one-third the size of the return shocks. Of course, if one were to apply any of the previous noisy rational expectations models to the MRS results they would undoubtedly produce similar results. Basically, if there is very little excess volatility in the world nearly any explanation will require relatively small nondividend noise shocks to reconcile the data.

### 3.3.2 Campbell and Kyle (1993)

At first it would appear that since CK estimate a normal-exponential model it should be possible to fit the present model to their results with minimal modification. Unfortunately, the two models are not fully compatible since CK estimate a more general dividend process and allow for a covariance term between their noise traders and the dividend flow. While one can easily modify the present model to fit their assumptions such an analysis seems unnecessary since the primary goal of this section is to estimate the magnitude of the supply noise needed to fit their results.

While the present model cannot fit all of the CK results one can ask what supply noise will reconcile the model with the major statistical findings in their paper. Table 8 of their paper reports a “normalized” variance-covariance matrix. The first element of this matrix reports the excess variance in prices relative to dividend news shocks. In the present context this measure is equivalent in spirit to dividing the price variance by  $\sigma_0^2$  and then with  $B^2$  from equation (36). CK also report estimates for parameters representing the degree of mean reversion for the stationary component of both their dividend and noise processes. Applying their mean reversion parameter estimate to the present model causes it to mimic their stationary component while ignoring their random walk component. Conversely, using a zero parameter value allows the model to mimic their random walk component while ignoring their stationary component. Recall from Section 3.2 that the required supply noise *shrinks* with increases in  $\alpha$  or  $\beta$  in the negative root equilibrium. Thus, if one considers all values of  $\alpha$  and  $\beta$  from 0 to the estimates in Campbell-Kyle this should completely outline the required supply variances that their estimates might imply for the present model.

CK assume that the process generating dividends can be decomposed into two components. Adapting their notation to a discrete time setting, the random walk piece equals

$$D_{0,t+1} = D_{0,t} + z_0 \tag{44}$$

where the  $t$  subscript represents time and  $z_0$  a normally distributed variable with zero mean and variance  $\sigma_0^2$ .

The second component equals

$$D_{1,t+1} = D_{1,t} - \alpha_1 D_{1,t} + z_1, \quad (45)$$

where  $\alpha_j$  is a constant that induces mean reversion, and  $z_j$  is a normally distributed error term with zero mean and variance  $\sigma_1^2$ . Thus, dividends  $D_t$  equal  $D_{0,t} + D_{1,t}$ . At this point one can show that the price variance induced by these shocks ( $\sigma_v^2$ ) in a full information normal-exponential model equals

$$\sigma_v^2 = \frac{\sigma_0^2}{r^2} + \frac{\sigma_1^2}{(r+\alpha_1)^2}. \quad (46)$$

While the model from Section 3.1 cannot reproduce (46) exactly it can mimic the influence of  $z_0$  and  $z_1$  shocks by setting

$$\left( \frac{1-\beta}{r+\beta} \right)^2 \sigma_\delta^2 = \frac{\sigma_0^2}{r^2} + \frac{\sigma_1^2}{(r+\alpha_1)^2} \quad (47)$$

and

$$\sigma_\delta^2 = \sigma_0^2 + \sigma_1^2. \quad (48)$$

Equation (47) sets the total influence from the dividend shocks to the same value in each model while (48) equalizes the total variance of the dividend innovations. Using the CK estimates from their Table 8 model C(2) one finds that the value of  $S$  ranges from a low of 0.069 to a high of 0.139. One obtains the low value by setting  $\alpha$  and  $\beta$  to zero, and the high value by using the parameter values estimated by CK.<sup>12</sup> As with the

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<sup>12</sup> Since the relationship between  $S$  and  $\alpha$  and  $\beta$  is basically linear, in the estimated range, a table showing all of the intermediate values has not been included.

other empirical models, the theoretical model in this paper can produce test statistics that mimic many of the reported results with a minimal supply shock. This conclusion becomes even stronger under the multiple security version of the model since it will likely divide the required supply standard deviation (and thus  $S$ ) by 30 or more.

## 4 Conclusion

The relative volatility of stock prices to the dividend process has produced a long string of papers. No doubt this will continue into the distant future. This paper shows that one can produce large price to dividend standard deviation ratios in a model with fully rational agents. More importantly one can produce these statistics with a rather modest variance in the unobservable supply of stock. Using results from several empirical studies the model fits the reported relative price-dividend standard deviation ratios with supply shocks that are generally less than 20% as large as the price changes being explained. Casual observation indicates that this is a useful trait for any model to possess. Changes in the aggregate supply of securities are often attributed to changes in illiquid assets like human capital. If so, then there is every reason to believe the capital asset supply variance will be much smaller than the variance in stock prices.

Another feature brought out by the analysis is the potential importance a multiple security environment can play in the market's aggregate volatility. Simply put, the greater the number of securities the smaller the required supply variance to generate any particular variance for the aggregate price index.

Finally, the model indicates that once the economy gets "stuck" in a high volatility equilibrium switching to one with lower volatility may prove very difficult. Any organization attempting the switch must control every security within a particular eigenvector formed from a decomposition of a matrix formed via a complex interaction between the variance-covariance matrices controlling the economy.

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## 6 Appendix

6.1 Proofs for claims in the main body of the text.

*Lemma 1: The matrix  $A$  mapping the supply of shares to prices is symmetric and has the following solution*

$$A = -\frac{1}{2} \frac{r}{\theta} \Sigma_{\eta}^{-1} + \left[ \frac{1}{4} \left( \frac{r}{\theta} \right)^2 \Sigma_{\eta}^{-2} - \left( \frac{1+r}{r} \right)^2 \Sigma_{\eta}^{-1/2} \Sigma_{\delta} \Sigma_{\eta}^{-1/2} \right]^{1/2}. \quad (49)$$

Proof: To prove  $A$  is symmetric, note that from (11)  $A$  solves

$$A \Sigma_{\eta} A' + \frac{r}{\theta} A + \left( \frac{1+r}{r} \right)^2 \Sigma_{\delta} = 0. \quad (50)$$

The first and third terms are symmetric since  $\Sigma_{\eta}$  and  $\Sigma_{\delta}$  are symmetric variance-covariance matrices. Since  $r$  and  $\theta$  are scalars,  $A$  must be symmetric. This allows one to replace the  $A'$  term with  $A$  in equation (50). Now pre and post multiply by  $\Sigma_{\eta}^{1/2}$ , and let  $Y = \Sigma_{\eta}^{1/2} A \Sigma_{\eta}^{1/2}$  thereby producing

$$Y^2 + \frac{r}{\theta} Y + \left( \frac{1+r}{r} \right)^2 \Sigma_{\eta}^{1/2} \Sigma_{\delta} \Sigma_{\eta}^{1/2} = 0 \quad (51)$$

which is a quadratic matrix equation. To solve for  $Y$ , complete the square by adding and subtracting  $\frac{1}{4} \left( \frac{r}{\theta} \right)^2 I$ .

After some minor algebra this generates

$$\left( Y + .5 \frac{r}{\theta} I \right)^2 = .25 \left( \frac{r}{\theta} \right)^2 I - \left( \frac{1+r}{r} \right)^2 \Sigma_{\eta}^{1/2} \Sigma_{\delta} \Sigma_{\eta}^{1/2}, \quad (52)$$

and allows one to conclude

$$Y = -\frac{1}{2} \frac{r}{\theta} I + \left( \frac{1}{4} \left( \frac{r}{\theta} \right)^2 I - \left( \frac{1+r}{r} \right)^2 \Sigma_{\eta}^{1/2} \Sigma_{\delta} \Sigma_{\eta}^{1/2} \right)^{1/2}. \quad (53)$$

Finally, substitute out for  $Y$  by using the relationship  $Y = \Sigma_{\eta}^{1/2} A \Sigma_{\eta}^{1/2}$ , and then pre and post multiply by  $\Sigma_{\eta}^{-1/2}$  to finish the proof. Q.E.D.

*Proposition 1: Assume that an econometrician knows the risk aversion of the population, the variance of the dividend series, and the variance of the stock's price. Then there exists only one value for the supply noise and one equilibrium consistent with the available data if  $\sigma_p^2 \geq \sigma_{\delta}^2/r^2$ . If  $\sigma_p^2 < \sigma_{\delta}^2/r^2$  then the model cannot explain the data.*

Proof: The proposition can be proved in three steps. The first step shows that in the positive root equilibrium  $\sigma_p^2$  increases in  $\sigma_{\eta}^2$ . The second step shows that in the negative root equilibrium  $\sigma_p^2$  decreases in  $\sigma_{\eta}^2$ . Finally, the third step shows that the largest value  $\sigma_p^2$  takes on in the positive root equilibrium, equals the smallest value it takes on in the negative root equilibrium.

Step 1, begin with the positive root equilibrium. In this case the price variance can be written as,

$$\sigma_p^2 = \left( \frac{1}{2} \left( \frac{r}{\theta} \right)^2 - \left( \frac{1+r}{r} \right)^2 \sigma_{\eta}^2 \sigma_{\delta}^2 - \frac{r}{\theta} \sqrt{ \frac{1}{4} \left( \frac{r}{\theta} \right)^2 - \left( \frac{1+r}{r} \right)^2 \sigma_{\eta}^2 \sigma_{\delta}^2 } \right) \sigma_{\eta}^{-2} + \frac{1}{r^2} \sigma_{\delta}^2. \quad (54)$$

which is identical to the negative root equation (16) except for the sign in front of the square root symbol. One can simplify the algebra by using the precision of  $\eta$  rather than its variance. Thus, let  $\rho_{\eta} = \sigma_{\eta}^{-2}$  and substitute into (54), and then differentiate (54) with respect to  $\rho_{\eta}$  producing

$$\frac{\partial \sigma_p^2}{\partial \rho_{\eta}} = \frac{1}{2} \left( \frac{r}{\theta} \right)^2 - \frac{\frac{1}{2} \frac{r}{\theta} \left[ \frac{1}{4} \left( \frac{r}{\theta} \right)^2 \rho_{\eta} - \left( \frac{1+r}{r} \right)^2 \sigma_{\delta}^2 \right]}{\sqrt{ \frac{1}{4} \left( \frac{r}{\theta} \right)^2 \rho_{\eta} - \left( \frac{1+r}{r} \right)^2 \sigma_{\delta}^2 \rho_{\eta} }}. \quad (55)$$

While equation (55) cannot immediately be signed for all  $\rho_\eta$  it can be signed at the lower bound of  $\rho_\eta$ . To generate a real solution to (14) the term inside the radical sign must be positive. Therefore,

$$\rho_\eta \in \left[ 4 \left( \frac{r}{\theta} \right)^{-2} \left( \frac{1+r}{r} \right)^2 \sigma_{\delta, \infty}^2 \right]. \quad (56)$$

Now divide the top and bottom of the last term in (55) by  $\rho_\eta$  and then evaluate as  $\rho_\eta$  goes to infinity. After some algebra, this produces

$$\lim_{\rho_\eta \rightarrow \infty} \frac{\partial \sigma_p^2}{\partial \rho_\eta} \propto -\frac{r}{\theta} < 0. \quad (57)$$

Thus, at  $\rho_\eta$ 's upper bound  $\partial \sigma_p^2 / \partial \rho_\eta$  is negative. To prove this holds for all  $\rho_\eta$ , consider the second derivative,

$$\frac{\partial^2 \sigma_p^2}{\partial \rho_\eta^2} \propto \left( \frac{1+r}{r} \right)^4 \sigma_\delta^4 > 0, \quad (58)$$

which arises after some extensive but straightforward algebra. Since  $\partial \sigma_p^2 / \partial \rho_\eta < 0$  at the upper bound for  $\rho_\eta$  and since  $\partial^2 \sigma_p^2 / \partial \rho_\eta^2 > 0$  for all  $\rho_\eta$ , it follows that  $\partial \sigma_p^2 / \partial \rho_\eta < 0$  for all  $\rho_\eta$ . Translating back from  $\rho_\eta$  to  $\sigma_\eta^2$ , one has now shown that  $\partial \sigma_p^2 / \partial \sigma_\eta^2 > 0$  for all  $\sigma_\eta^2$ .

Step 2, consider the negative root equilibrium. Again use  $\rho_\eta$  and differentiate (14) yielding,

$$\frac{\partial \sigma_p^2}{\partial \rho_\eta} = \frac{1}{2} \left( \frac{r}{\theta} \right)^2 + \frac{\frac{r}{\theta} \left[ \frac{1}{2} \left( \frac{r}{\theta} \right)^2 - \left( \frac{1+r}{r} \right)^2 \sigma_\delta^2 \right]}{\sqrt{\frac{1}{4} \left( \frac{r}{\theta} \right)^2 \rho_\eta^2 - \left( \frac{1+r}{r} \right)^2 \sigma_\delta^2 \rho_\eta}}. \quad (59)$$

Since the term inside the square root must be positive, and the bracketed term in the numerator is bigger than the term inside the square root, one immediately has that  $\partial \sigma_p^2 / \partial \rho_\eta > 0$ . This in turn implies that  $\partial \sigma_p^2 / \partial \sigma_\eta^2 < 0$  for

all permissible  $\sigma_\eta^2$ .

Step 3, consider the value  $\sigma_p^2$  in the positive root equilibrium. Since its value increases in  $\sigma_\eta^2$  the maximal value of  $\sigma_p^2$  must occur at the maximal value for  $\sigma_\eta^2$ . Plugging this in, and using

$$\bar{\sigma}_\eta^2 = \frac{1}{4} \left( \frac{r}{\theta} \right)^2 \left( \frac{1+r}{r} \right)^{-2} \sigma_\delta^{-2}, \text{ produces}$$

$$\sigma_p^2 \Big|_{\sigma_\eta^2 = \bar{\sigma}_\eta^2} = \left( \frac{1+r}{r} \right)^2 \sigma_\delta^2 + \frac{\sigma_\delta^2}{r^2}. \quad (60)$$

Under the negative root equilibrium, the smallest value of  $\sigma_p^2$  must occur at the maximal value  $\sigma_\eta^2$  since in the negative root equilibrium  $\sigma_p^2$  declines in  $\sigma_\eta^2$ . Plugging this into (14) produces an equation identical to (60).

Thus, each equilibrium covers a disjoint set of values for  $\sigma_p^2$ . Q.E.D.

*Proposition 3: Consider an equilibrium, and the variance of any portfolio under that equilibrium. Then switching any eigenvalue in the matrix  $\Lambda^{1/2}$  from negative to positive, reduces the portfolio's variance.*

Proof: Let  $X$  represent an arbitrary portfolio. Then the variance of that portfolio can be written as

$$\sigma_X^2 = X' \left[ A \Sigma_\eta A' + \frac{1}{r^2} \Sigma_\delta \right] X \quad (61)$$

Substituting in the solution for  $A$  from (12) plus extensive manipulation produces

$$\sigma_X^2 = X' \left[ \frac{1}{2} \left( \frac{r}{\theta} \right)^2 \Sigma_\eta^{-1} + \left[ \frac{1}{r^2} - \left( \frac{1+r}{r} \right)^2 \right] \Sigma_\delta \right] X - \frac{r}{\theta} X' \Gamma' \Lambda \Gamma X \quad (62)$$

whose size needs to be determined. The first term on the right hand side does not depend upon the selected equilibrium. Thus, one only needs to consider the far right term. Let  $\gamma_i$  represent the  $i$ th eigenvector in the matrix  $\Gamma$ . Then one can write any  $X$  as

$$X = \sum_{i=1}^K w_i Y_i \quad (63)$$

where the  $w_i$  represent weights. Substituting in for  $X$  in equation (62), the far right hand term becomes

$$\frac{r}{\theta} X' \Gamma' \Lambda \Gamma X = \sum_{i=1}^K \lambda_i w_i^2 \quad (64)$$

after making use of the fact that the columns of  $\Gamma$  are orthonormal vectors. Note that the  $w_i$  in (64) are squared, and (64) enters (62) with a negative sign. Thus, switching any  $\lambda_i$  from *negative* to *positive* must reduce the variance of the portfolio  $X$ . Q.E.D.

## 6.2 Fitting the model to the MRS test statistics.

The primary difficulty lies with the calculation of  $E(D_t - \bar{D})^2$ . Using equation (32) one can repeatedly substitute out for past values of  $D$  to get

$$D_t = (1-\beta)^t D_0 + \sum_{j=1}^t (1-\beta)^{t-j} (\beta \bar{D} + \delta_j). \quad (65)$$

Now subtract off  $\bar{D}$  and take the limit as  $t$  goes to infinity to find

$$\lim_{t \rightarrow \infty} E(D_t - \bar{D})^2 = \frac{\sigma_\delta^2}{1 - (1-\beta)^2}. \quad (66)$$

Using (66) straightforward, but very long, algebraic calculations lead to the following equalities

$$E(P_t^* - P_t^0)^2 = \frac{A \sigma_\eta^2}{(1+r)^2} + \frac{2r^2 + \beta + 2r\beta}{r^2(\beta+r)^2(2-\beta)} \sigma_\delta^2, \quad (67)$$

$$E(P_t^* - P_t)^2 = \frac{A^2 \sigma_\eta^2}{(1+r)^2} + A^2 \sigma_\eta^2 + \frac{\sigma_\delta^2}{(r+\beta)^2}, \quad (68)$$

and

$$E(P_t - P_t^0)^2 = A \sigma_\eta^2 + \frac{\beta(1+r)^2 \sigma_\delta^2}{r^2(\beta+r)^2(2-\beta)}. \quad (69)$$

where each of the expectations is understood to be the limiting expectation as period 0 recedes into the distant past. Simultaneously solving the above three equations so that they fit the information supplied by MRS after using the conversions suggested in Section 2.1 allows one to produce the results in Table III.

### 6.3 Deriving $S$ when dividends exhibit mean reversion ( $\beta > 0$ ).

The derivation follows along the lines for  $S$  when  $\beta = 0$ . First,  $P_t$  can no longer be written as  $D/r^*$ . Some algebra shows that under the dividend discount model with a dividend process that follows (32) the price of a stock becomes

$$P_t = \frac{1-\beta}{r^*+\beta} D_t + \frac{1+r^*}{r^*+\beta} \frac{\beta}{r^*} \bar{D}. \quad (70)$$

Given (70) the formula for  $\sigma_\pi$  changes from (25) to

$$\sigma_\pi = \left( \frac{1+r^*}{r^*+\beta} \right) \sigma_\delta. \quad (71)$$

With a mean reverting dividend process it becomes somewhat more convenient to work with  $P_t$  rather than  $D_t$ ,

when solving for  $\sigma_r$ . For the formulation of  $S$  without mean reversion one simply replaces  $D_t$  in equation (24) with  $r^*P_t$ . Making the appropriate changes in the derivation of (24) to allow for  $\beta > 0$ , produces

$$\sigma_r = \frac{(1+r^*)\sigma_\delta}{(r^* + \beta)P_t} \quad (72)$$

which can now be used to eliminate  $P_t$  from  $S$  after  $N$  is replaced by  $W_t/P_t$ . Following the substitutions that lead to (26) the above set of equalities yield the modified  $S$  statistic given by (43).