Option Valuation and the Price of Risk

J. Chalupa
Box 82
Princeton, MA 01541 USA
jchalupa@ix.netcom.com

Abstract

A valuation model is presented for options on stocks for which Black-Scholes arbitrage does not entirely eliminate risk. The price dynamics of a portfolio of options and the underlying security is quantified by requiring that the excess reward-to-risk ratio of the portfolio be identical to that of the underlying stock:

$$\left.\frac{\text{excess return}}{\text{risk}}\right|_{\text{portfolio}} = \left.\frac{\text{excess return}}{\text{risk}}\right|_{\text{stock}}.$$

The nonlinear evolution equation for the portfolio value is homogeneous of degree one. A representative distribution is obtained from recent stock-history time series; numerical solutions for European calls are usually close to the Black-Scholes values, but naked and covered calls have different valuations. For infinitesimal time steps and a lognormal stock-price distribution, the evolution equation reduces to the Black-Scholes form. An analytically tractable non-lognormal distribution is analyzed near option expiration, and a formula expressing the deviation from the lognormal case is obtained for an out-of-the-money call. The present model is discussed in the context of previous work, and the effect of nonlinearity on the valuation of a portfolio of derivative securities is considered.

1 Introduction

The Black-Scholes analysis of option pricing[1] assumes that the logarithm of the stock price changes with time according to a Wiener process. The ensuing distribution is lognormal. In this case a unique option price can be determined by use of arbitrage arguments: any price other than the Black-Scholes value permits a riskless prounction of the stock price which leads to such an instantaneous return and satisfies the boundary conditions at expiration. It turns out to be independent of the first moment of the stock's lognormal distribution.

When the stock-price dynamics is not a Wiener process[2, 3], the higher moments of the distribution are not negligible. By taking an offsetting position in the stock, an option holder typically can reduce risk but may not be able to eliminate it altogether.

Thus, the issue arises of how to assign an option valuation for stocks to which the Black-Scholes arbitrage argument and risk-neutral valuation are not applicable. It is of interest for mathematical economics[4, 5, 6] and for recent calculational valuation methods [7, 8, 9, 10, 11]. For a Wiener process, the option–or any portfolio composed of the stock and its derivatives–have the same excess-reward-to-risk ratio as the stock itself[12, 13]. A natural valuation procedure is to require that this hold for the general case:

$$\frac{\text{excess portfolio return}}{\text{portfolio risk}} = \frac{\text{excess return of underlying security}}{\text{risk of underlying security}}. \quad (1)$$

In other words, the price of risk for an option portfolio is identical to the price of risk of the underlying security. The work reported here examines some consequences of this assumption. The formulation is designed to be readily impleentable in computations and calculations, and the presentation is more intuitive than deductive.

The paper is organized into five sections of which this Introduction is the first. Section 2 develops the reward-to-risk valuation model, applies it to recent time-series data for Merck and Oracle stock, and finds for thiese cases that the corresponding values of European values are close to the Black-Scholes results. Section 3 treats stock-price distributions of the Chapman-Kolmogorov class with nonnegligible higher cumulants and presents an analytically tractable example. The behavior of the option value is examined in the short-time regime and deviations from the Gaussian approximation are exhibited. Section 4 treats the relation of this work to other analyses of nongaussian stock distributions [7, 8, 9, 10, 11]. It also considers how the nonlinearity of the equation of motion affects the valuation of a portfolio of derivatives. Section 5 ends the paper with a summary and conclusions.

2 Valuation Equation

This section presents the equation of motion which determines the valuation of a portfolio Π of consisting of shares of a stock or stock index of price S and options on that stock. As discussed in the Introduction, the basic assumption is that the excess-return-to-uncertainty ratio of the portfolio equals that of the stock. Consider the portfolio at a time t. In terms of the logarithm $z = \ln S$ of the stock price, Equation 1 indicates that the portfolio's value a time increment Δt later is determined by the relation

$$\frac{\langle \Delta\Pi \rangle - r\,\Delta t\,\Pi(z,t)}{\mathcal{R}[\Pi]} = \frac{\left\langle e^{\Delta z} \right\rangle - (1 + r\ \Delta t)}{\mathcal{R}[e^{\Delta z}]} \tag{2}$$

$$\Delta\Pi \equiv \Pi(z + \Delta z, t + \Delta t) - \Pi(z,t) \tag{3}$$

where the expectation value $\langle ... \rangle$ is taken over the distribution for stock price changes. The risk metric $\mathcal{R}$ will be discussed further below. The time steps Δt are taken as finite but, for simplicity, small enough that a linearized approximation is valid for interest accrual at the riskless rate r.

The evolution of the stock-price distribution and option value will be traced from the initial time $t_0 = 0$ and initial price z_0. The probability distribution z at time t is taken as $P(z,t;z_0,t_0)$. For times t_1 between t_0 and t, the Chapman-Kolmogorov condition[12]

$$P(z,t;z_0,t_0) = \int dz_1\, P(z,t;z_1,t_1)\, P(z_1,t_1;z_0,t_0) \tag{4}$$

is assumed. The distribution is specified to have the form

$$P(z_2,t_2;z_1,t_1) = P(z_2 - z_1, t_2 - t_1; 0, 0) \qquad (t_2 \geq t_1). \tag{5}$$

The shorthand $P(z,t)$ will be used for $P(z,t;0,0)$. Under these conditions, Equation 2 predicts that a z-independent portfolio, i.e. one consisting of cash, will compound at the riskless interest rate, and the equation admits the stock price $S = e^z$ as a solution.

The immediate task is to transform the evolution equation into a form suitable for performing computations. Consider a portfolio Π consisting of cash, the stock S, and European options depending on S. For simplicity let all the derivatives mature at the same time T. The expectation value of the change in the portfolio's value in time Δt is

$$\langle \Delta\Pi \rangle = \langle \Pi(z + \Delta z, t + \Delta t) \rangle_{P(\Delta z, \Delta t)} - \Pi(z, t). \tag{6}$$

The standard deviation

$$\mathcal{R}[\Pi] = \left\langle |\Pi(z + \Delta z, t + \Delta t) - \langle \Pi(z + \Delta z, t + \Delta t) \rangle|^2 \right\rangle^{1/2} \tag{7}$$

sugests itself as a choice of risk metric. (Actually, the sign of $\mathcal{R}$ is the sign of the derivative $\partial\Pi/\partial z$, which will be taken as positive, as will Π, unless specified otherwise.) However, more general forms like

$$\mathcal{R}[\Pi] = \langle |\Pi(z + \Delta z, t + \Delta t) - \langle \Pi(z + \Delta z, t + \Delta t) \rangle|^p \rangle^{1/p} \tag{8}$$

can also be used. The form of Equation 8 ensures that the portfolio's value scales linearly if the quantity of each asset is changed by a common multiplicative factor, i.e. that the evolution equation, although nonlinear, is homogeneous of degree one. The $p = 1$ alternative has appeal because a risk can be assigned to a δ-source (Arrow-Debreu security) whereas $p = 2$ yields the integral of the square of a δ-function in this case; moreover, the integral of $e^{p\,z} P(z, t)$ must converge at large z for the risk metric to be finite, and the $p = 1$ case accommodates distributions with stronger tails than does the $p = 2$ case. For both the $p = 1$ and $p = 2$ cases, a portfolio's risk metric is independent of its amount of cash; the cash part compounds at the riskless rate while the rest of the portfolio is governed by the risk metric.

Equation 2 allows the portfolio's value (as a function of stock price) at a given time to be determined from its value at the *subsequent* time step:

$$\Pi(z, t) = \frac{\langle \Pi(z + \Delta z, t + \Delta t) \rangle_{P(\Delta z, \Delta t)} - \omega \mathcal{R}[\Pi(z + \Delta z, t + \Delta t)]}{1 + r\ \Delta t} \tag{9}$$

where the constant ω is the Sharpe ratio for the stock:

$$\omega = \frac{\left\langle e^{\Delta z} \right\rangle - (1 + r\ \Delta t)}{\mathcal{R}[e^{\Delta z}]}. \tag{10}$$

Equation 9 can be iterated in backward time steps. Thus, one can begin at the expiration time T when the dependence of Π on the stock price is known, determine the portfolio value at the prior time step, and repeat the process until the required time to expiration is reached. If the evolution equation were linear, the Chapman-Kolmogorov condition might be used to self-convolve the distribution and price the option on the basis of the price distribution for the time to expiration. Such procedures are not possible for Equation 9 because of its nonlinearity (unless the time step is deliberately increased–even to be as large as the time to expiration–because a valuation only over longer time intervals is of interest). Thus, the numerical solution procedure can be time-consuming.

There are countervailing effects in Equation 9. For example, if the distribution is leptokurtotic, i.e. has excess weight in the tails, the first term raises the price of a well-out-of-the-money call from its Black-Scholes value. However, the risk of the call in (9) presumably also increases, which tends to decrease the option's value because of the risk discount. Moreover, it is not clear *prima facie* whether the risk of the call increases or decreases relative to the risk of the stock. In short, a complicated situation ensues and the value of an option may increase or decrease when weight is added to the underlying security's distribution tails.

In order to demonstrate the use of Equation 9, it was applied to the time series of Merck closing prices from 1 September 1988 to January 2-April 10. Under the assumption that the series is generated by a markovian process, the empirical cumulative distribution associated with the series was generated in histogram form. Results of the Kolmogorov-Smirnov and Kuiper tests [14] indicated that, at least for demonstration purposes, the time series may plausibly be taken as quasistationary. Table 1 shows the ratio of the $p = 2$ Equation 9 values to the Black-Scholes price of naked and covered European calls. The covered call value was determined by applying the expiration boundary condition to the portfolio (long stock and short call), back-stepping to compute the portfolio value at previous times, and subtracting this value from the stock price. The numerical results were checked by decreasing the histogram bin size; a stability analysis[14] of the valuation equation was not undertaken. In this particular situation the effects of nonlinearity are weak, the Central Limit Theorem gives rise to Gaussian behavior, and the deviation from the Black-Scholes result is small.

Some deviations from the Black-Scholes values were found for Oracle (ORCL) stock. The time series from 1 September 1988 to 30 June 1996 was examined, and the valuation computations were carried out notwithstanding that evidence for stationarity was not strong. Table 3 shows the results for calls with 15 days to expiration given a stock price of 39.5. Despite the weak stationarity, the discrepancies with Black-Scholes make the results of interest. The differences between the values of naked and covered calls are due to the nonlinearity of the evolution equation: the risk of a covered call couples to the risk of the stock in the portfolio.

3 Model Stock-Price Distributions

Using the time series to value the option, as was done in the previous section, is attractive because the price is determined directly from the data rather than from a fabricated model for the stock-price dynamics. On the other hand, analyzing pretailored models offers insights not available from raw data. This section considers such a model and applies it to option behavior near expiration.

For distributions satisfying Equations 2 and 3, the Fourier transform

$$P(k,t) = \left\langle e^{-ikz} \right\rangle = \int dz\, e^{-ikz} P(z,t) \tag{11}$$

obeys the relation

$$P(k,t) = P(k,1)^t. \tag{12}$$

If $P(z,t)$ is known for a given value of t, say $t = 1$, Equation 12 allows it to be determined for any time. The lognormal distribution for geometric Brownian motion satisfies this equation. Before adopting $P(z, t < 1)$, it is necessary to check that it is a positive-semidefinite function of z.

Evaluating the option behavior near expiration requires the evolution equation for small time steps. In the limit $\Delta t \to 0$, the $p = 2$ Sharpe ratio is

$$\omega = \lim_{\Delta t \to 0} \frac{P(k=i,\Delta t) - (1+r)\Delta t}{(P(k=2i,\Delta t) - P(k=i,\Delta t)^2)^{1/2}} \tag{13}$$

$$= \Delta t^{1/2} \frac{\ln P(i,1) - r}{\ln\left[P(2i,1))/P(i,1)^2\right].} \tag{14}$$

in terms of the distribution's characteristic function.

Strike	Naked Call / Black-Scholes Call	Covered Call / Black-Scholes Call	Black-Scholes
30	.999	.999	32.44
35	.998	.999	27.67
40	.998	.998	22.94
45	.996	.998	18.36
50	.995	.997	14.08
55	.995	.996	10.32
60	.993	.995	7.22
65	.987	.997	4.84
70	.984	.997	3.11
75	.981	.997	1.93

Table 1: The January 1997 MRK call relative to the Black-Scholes value. The stock price is 62.125 and $1.02 in dividends will be paid before expiration.

Strike Price	Naked Call / Black-Scholes Call		Covered Call / Black-Scholes Call		Black-Scholes Call
	p=1	p=2	p=1	p=2	
35	.990	.990	.997	1.010	5.1116
37.5	.976	.971	.988	1.003	3.3201
40	.960	.935	.970	.983	1.9787
42.5	.942	.867	.969	.968	1.0814
45	.956	.776	.988	.956	0.5437
47.5	1.015	.630	1.153	1.060	0.2530
50	1.197	.436	1.252	1.044	0.1097

Table 2: ORCL call relative to the Black-Scholes value. The stock price is 39.5 and there are fifteen (15) days toexpiration.

The stock-price distributions under consideration have stronger tails than a gaussian, but the weight in the tails is small enough that the risk $\mathcal{R}[\mathcal{S}]$ is finite. A natural possibility is the exponential distribution

$$P(z,1) = \frac{\sqrt{2}}{2\sigma} e^{-|z|\sqrt{2}/\sigma}, \tag{15}$$

The mean $\langle e^z \rangle$ and mean square $\langle e^{2\,z} \rangle$ of the stock price are finite for sufficiently small values of the standard deviation σ. $P(z,1)$ has the Fourier transform

$$P(k,1) = \frac{1}{1+\frac{1}{2}k^2\sigma^2}. \tag{16}$$

For arbitrary time t, Equation 12 gives[15]

$$P(z,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} P(k,1)^t \tag{17}$$

$$= \frac{\sqrt{2/\pi}}{\sigma\Gamma(t)} \left(\frac{|z|}{\sigma\sqrt{2}} \right)^{t-\frac{1}{2}} K_{t-\frac{1}{2}}(|z|\sqrt{2}/\sigma), \tag{18}$$

where $K_{t-\frac{1}{2}}(|z|\sqrt{2}/\sigma)$ is a Bessel function. This distribution is an even function of z and therefore unskewed. For small times $t \to 0$, $P(z,t)$ approaches the δ-function $P(z,t=0) = \delta(z)$ in a different manner than the Gaussian associated with geometrical Brownian motion. Near the origin, the distribution diverges as $P(z,t) \sim t/|z|^{1-2t}$; its asymptotic behavior far from the origin is dominated by the exponential decay of the Bessel function, and the amplitude is proportional to $t \to 0$. (Thus, the expectation value $\overline{|x|} \sim t$ for $t \sim 0$, but the Central Limit Theorem indicates that $\overline{|x|} \sim t^{1/2}$ for $t \to \infty$. If an exponent $\widetilde{H}(t)$ is defined by

$$\widetilde{H}(t) = \frac{\partial \ln \overline{|x|}}{\partial \ln t}, \tag{19}$$

$\widetilde{H}(t)$ will depend on the time scale. The expectation value $\overline{|x|} \sim t$ is similar to the quantity which Peters[16] analyzed to assign a Hurst exponent to economic time series, and exhibits a similar crossover downward to the Gaussian value.)

It is useful to generalize Equation 18 to a symmetric distribution which reduces to the exponential form at an arbitrary time τ. This can be done by adopting the relation

$$P(z,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} \frac{e^{-ikvt}}{(1+\frac{1}{2}k^2\sigma^2\tau)^{t/\tau}} \tag{20}$$

$$= \frac{\sqrt{2/\pi}}{\sigma\Gamma(t/\tau)} \left(\frac{|z-vt|}{\sigma\sqrt{2\tau}}\right)^{\frac{t}{\tau}-\frac{1}{2}} K_{\frac{t}{\tau}-\frac{1}{2}} \left(\frac{|z-vt|}{\sigma}\sqrt{\frac{2}{\tau}}\right). \tag{21}$$

A drift velocity v also has been introduced in Equation 21. The parameters v, σ, τ can be assigned by matching the first, second, and fourth cumulants of the time series for stock-price changes. For the Merck and Oracle series analyzed in Section 2, the values of τ are respectively $\tau_{MRK} \simeq .4$ day and $\tau_{ORCL} \simeq 5$ day. Having these characteristic time scales on the order of days is consistent with the numerical results that the Merck call values are very close to Black-Scholes prices while Oracle shows some deviation.

The overall effect of a nongaussian shape of the distribution function has long been understood[12, 17]. For the distribution of Equation 18, on general grounds one expects at- or out-of-the-money calls near maturity to be proportional to the time $T-t$ to expiration. The farther a call is out of the money, the smaller the proportionality constant will be. A Black-Scholes call, on the other hand, has the limiting behavior $\sim \sqrt{T-t}$ near the strike price and $\sim e^{-\frac{ln^2(S/X)}{2\sigma^2(T-t)}}$ far out of the money. Near expiration, the near-the-money Black-Scholes call is more valuable than that from Equation 18, and far from the money the Black-Scholes call is worth less.

In fact, limiting forms of option valuations associated with $P(z,t)$ can be derived. For example, suppose that the risk metric is approximated as the risk metric of the gaussian with the same first two moments as P:

$$\frac{R_{gauss}[\Pi]}{R_{gauss}[S]} = \frac{1}{S}\frac{\partial \Pi}{\partial z}. \tag{22}$$

In the continuous-time limit this leads to a risk-neutral form of P, denoted P_{rn}, which is identical to the original distribution except that the replacement

$$v \to r - \frac{1}{\tau}\ln(1-\frac{1}{2}\sigma^2\tau) \tag{23}$$

is made in the first moment. For an out-of-the-money call ($S << X = e^z$), the expression

$$e^{r(T-t)}C(z,T-t) = \left\langle (e^{z+\Delta z} - X)\Theta(e^{z+\Delta z} - X)\right\rangle_{P_{rn}(\Delta z,T-t)}, \qquad (24)$$

where $\Theta(x) \equiv \frac{1}{2}(1+\frac{|x|}{x})$, can be evaluated by using the asymptotic form

$$P(\Delta z, T-t) \simeq \frac{(T-t)/\tau}{|z|} e^{-|\Delta z|\sqrt{2/\tau}/\sigma} \qquad (T-t \to 0^+, |\Delta z| \to \infty) \quad (25)$$

of the Bessel function. This yields

$$C(S,T-t) \simeq \frac{\sigma^2(T-t)}{2(1-\sigma\sqrt{\frac{\tau}{2}})\ln(X/S)} \left(\frac{S}{X}\right)^{\sqrt{2/\tau}/\sigma} \left(1 + \mathcal{O}\left(\frac{1}{\ln(X/S)}\right)\right). \quad (26)$$

The effect of the exponential tail in P makes the shows up in the exponential decrease with z of the out-of-the-money call. This decrease is larger than the Gaussian decrease associated with the Black-Scholes call. Table 3 shows results for ORCL; the parameters of the model distribution were computed by matching the first, second, and fourth cumulants from the time series. This procedure is straightforward, but a cogent alternative[18, 19] is to determine the implied stock distribution from market prices of options.

The Gaussian analysis motivates examination of the corresponding $p = 1$ case. On the basis of the Gaussian-risk approximation, a solution of the form

$$C(z,T-t) = C(z,T-t=0) + (T-t)f(z) \qquad (27)$$

is assumed in the limit $T-t \to 0$. Substituting this into the evolution equation and matching leading terms in Δt and $T-t$ gives

$$C(z,T-t) \simeq (1-2\omega)\frac{\sigma^2(T-t)}{2(1-\sigma\sqrt{\frac{\tau}{2}})\ln(X/S)}\left(\frac{S}{X}\right)^{\sqrt{2/\tau}/\sigma} \qquad (28)$$

$$\omega = \frac{v - r - \frac{1}{\tau}\ln(1-\frac{1}{2}\sigma^2\tau)}{2\ln(1+\sigma\sqrt{\frac{\tau}{2}})}\tau \qquad (29)$$

for $S-X \ll 0$. This is essentially the Gaussian form reduced by an additional price-of-risk discount for the higher moments of the distribution.

If the stock growth rate v is high enough, the calculated value of a naked call becomes negative. A potential buyer of a naked call who prices with the $p=1$ risk metric concludes that any positive call value gives an insufficient return-to-risk ratio. Such a buyer will refuse to pay even an infinitesimal price for the call. This decision is a consequence of the choice of risk metric. In particular, the uncertainty of the return at time $t+\Delta t$ is defined with respect to the expectation value of the return at $t+\Delta t$. Section 4 will indicate how this convention is necessary for the methodology to reduce to the arbitrage results for binomial discrete-step stock motions. Conveniently, use of the expectation value at $(t+\Delta t)$ greatly simplifies the numerical solution of the evolution equation by enabling backward-stepping in time. However, careful interpretation is needed of the unexpected consequences, particularly in asymptotic limiting cases, which can arise from the nonlinearity of the option valuation method. Legitimate reward-to-risk criteria may not invariably lead an investor to place a positive value on a call.

For the $p=2$ risk metric, the situation can be even more extreme. For an out-of-the-money naked European call near expiration at strike price X, one finds

$$\left\langle \Delta C(Se^{\Delta z}, \Delta t)\right\rangle \simeq \frac{\Delta t}{2\ln(X/S)} \left(\frac{S}{X}\right)^{\sqrt{2/\tau}/\sigma} \frac{\sigma^2}{1-\sigma\sqrt{\tau/2}} \tag{30}$$

and

$$\mathcal{R}\left[\left(C(Se^{\Delta z}, \Delta t)\right)^2\right] \simeq \frac{(\Delta t)X^2}{\ln(X/S)} \left(\frac{S}{X}\right)^{\sqrt{2/\tau}/\sigma} \frac{\sigma^3\sqrt{\frac{\tau}{2}}}{(1-\sigma\sqrt{2\tau})(1-\sigma\sqrt{\frac{\tau}{2}})}. \tag{31}$$

Use of these two expressions and Equation 27 gives

$$C(S,X,T-t) \simeq (T-t)X\left[\frac{\sigma^2(S/X)^{\sqrt{2/\tau}/\sigma}}{2(1-\sigma\sqrt{\frac{\tau}{2}})\ln(\frac{X}{S})} - \omega\sqrt{\frac{(S/X)^{\sqrt{2/\tau}/\sigma}\sigma^3\sqrt{\frac{\tau}{2}}}{(1-\sigma\sqrt{2\tau})(1-\sigma\sqrt{\frac{\tau}{2}})\ln(\frac{X}{S})}}\right]. \tag{32}$$

The (negative) risk term above is asymptotically of the form $\mathcal{R}[C] \approx (S/X)^{\sqrt{2/\tau}/(2\sigma)}$, whereas the first term is of the form $\langle \Delta C\rangle \approx (S/X)^{\sqrt{2/\tau}/\sigma}$. For sufficiently large X (and positive ω), there ensues

$$\frac{\mathcal{R}[C]}{\Delta C} \gg 1, \tag{33}$$

and the expression for $C(S, X, T-t)$ is negative in this limit. No matter how small the excess return, the uncertainty in C drives the far-out-of-the-money valuation negative.

The distribution studied above is an even function of z whereas empirical stock returns are skewed. By considering a distribution of the form

$$P_{skew}(z,\tau) = \begin{cases} \dfrac{\kappa_p \kappa_n}{\kappa_p + \kappa_n} e^{-\kappa_p (z - v\tau)} & (z \geq v\tau) \\ \\ \dfrac{\kappa_p \kappa_n}{\kappa_p + \kappa_n} e^{\kappa_n (z - v\tau)} & (z \leq v\tau) \end{cases} \tag{34}$$

the model can be generalized to incorporate skewness. A drift term v and a timescale factor τ have also been included. The Fourier transform is[15]

$$P_{skew}(z,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(z - v\,t)} (\kappa_p \kappa_n)^{t/\tau}}{\left(\frac{1}{4}(\kappa_p + \kappa_n)^2 + (k - \frac{i}{2}(\kappa_p - \kappa_n))^2\right)^{t/\tau}} \tag{35}$$

$$= \sqrt{\tfrac{\kappa_p + \kappa_n}{\pi |z - v\,t|}} \left(\tfrac{\kappa_p \kappa_n |z - v\,t|}{\kappa_p + \kappa_n}\right)^{t/\tau} \tfrac{e^{-(\kappa_p - \kappa_n)(z - v\,t)/2}}{\Gamma(t/\tau)} K_{\frac{t}{\tau} - \frac{1}{2}}\left(\tfrac{1}{2}(\kappa_p + \kappa_n)\,|z - vt|\right). \tag{36}$$

The four quantities $(v, \kappa_p, \kappa_n, \tau)$ can be assigned to characterize a data set, e.g. so that the first four moments of P_{skew} match the first four moments of an empirical distribution. Whether the matching procedure yields a solution and the quality of the fit must be determined on a case-by-case basis.

Finally, it should be noted that the infinite value of $P(z = 0, t < 1/2)$ is due to the fact that the Fourier transform $(1 + \frac{1}{2}k^2\sigma^2)^{-t}$ is not integrable for $t < \frac{1}{2}$. An alternative like $P_{alt}(k, t = 1) = \mathrm{sech}(k/\kappa)$ gives[15]

$$P_{alt}(z,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} \mathrm{sech}^t(k/\kappa) \tag{37}$$

$$= \frac{2^{t-2}\kappa}{\pi \Gamma(t)} \left| \Gamma\left(\frac{1}{2}(t + i\kappa x)\right) \right|^2, \tag{38}$$

which is manifestly positive for all finite x and t, and is therefore an acceptable model distribution. For small t, $P_{alt}(z = 0, t \to 0^+) \approx 1/t$. Thus, in the $t \to 0$ limit, $P_{alt}(z,t)$ is larger for $z \approx 0$ and $z \to \infty$ than a Gaussian of the

Strike	$\frac{\text{Risk-Neutral Call}}{\text{Black-Scholes Call}}$	Black-Scholes
30	1.002	9.6321
32.5	1.002	7.2633
35	.997	5.1116
37.5	.981	3.3201
40	.962	1.9787
42.5	.966	1.0814
45	1.022	.5437
47.5	1.155	.2530
50	1.399	.1097

Table 3: Risk-neutral ORCL call relative to the Black-Scholes value. The call value was computed with a time-evolved exponential distribution. The stock price is 39.5 and there are fifteen (15) days to expiration.

same mean and variance. Because Equation 38 is a special case of an integral representation of a hypergeometric function, generalizations of $P_{alt}(z,t)$ can be essayed to incorporate skewness.

To illustrate the use of the valuation equation in the continuum limit, this section has presented calculations utilizing well-characterized special functions. The exponential form of the distribution (5) at $t = \tau$ was chosen for its analytical tractability. However, application of the Chapman-Kolmogorov condition to many distributions, e.g. the exponentially modulated Lévy form considered by Montegna and Stanley, will be a computational task. Because a distribution which is positive-semidefinite for $t = \tau$ may have negative values for $t < \tau$, candidate distributions must be analyzed on a case-by-case basis.

4 Discussion

This section covers three topics. First, it will be indicated that the present formalism reduces to the Black-Scholes case for geometrical Brownian motion. Second, the relationship of the present work to the papers of Bouchaud and Sornette and subsequent workers will be discussed. Finally, some issues arising from adoption of a nonlinear valuation equation will be briefly explored.

The reduction to Black-Scholes will be demonstrated in a concise but indirect way. Consider a portfolio Π such that $\Pi, \frac{\partial \Pi}{\partial z} \geq 0$. The stock price changes at discrete time intervals Δt. The distribution of changes z is taken to be binary:

$$P(z) = q\delta(z-a) + (1-q)\delta(z-b) \qquad (b > a), \tag{39}$$

where $\delta(z)$ denotes a Dirac delta-function. For this distribution the risk metric

$$\begin{aligned} \mathcal{R}[\Pi] &= \langle |\Pi(z+\Delta z, t+\Delta t) - \langle \Pi(z+\Delta z, t+\Delta t)\rangle|^p\rangle^{1/p} & (40) \\ &= \left((q^p + (1-q)^p)^{1/p} \left(\Pi(z+b, t+\Delta t) - \Pi(z+a, t+\Delta t)\right)\right. & (41) \end{aligned}$$

is linear in Π. Thus a perfect hedge for a call can be constructed in the standard way by taking a compensating short position in the stock, and the standard binomial option valuation[20] is recovered. Since geometrical Brownian motion and Poisson[21] hopping processes can be extracted from the binomial distribution by appropriately taking the continuum limit $\Delta t \to 0$, the riskless-hedge option pricing results for these cases are recovered as well. Note that the risk metric

$$\tilde{\mathcal{R}}[\Pi] = \langle |\Delta \Pi|\rangle^{1/p} = \langle |\Pi(z+\Delta z, t+\Delta t) - \langle \Pi(z,t)\rangle|^p\rangle^{1/p}_{P(\Delta z, \Delta t)} \tag{42}$$

is *not* linear in Π for the binomial distribution and therefore does not lead to consistency with the arbitrage valuations.

The methodology in this paper was complete when prior studies by Bouchaud and Sornette[7, 8] and subsequent workers[11] came to my attention in the financial literature[9]. While full details of their approach will be described in a forthcoming monograph[10], my understanding of it is as follows. Bouchaud and Sornette's time-integrated formalism takes the viewpoint of an option seller who minimizes risk by taking an offsetting position in the underlying stock. The trading strategy, i.e. the time dependence of the number of shares owned, is determined by minimizing the uncertainty in the seller's wealth at option expiration. The option offering price is determined by the risk premium which the seller requires. Aurell and Życkowski have built on this formalism and developed a trading strategy to optimize the gain of an investor willing to accept a given amount of risk. (They use the term "value" to characterize portfolios whose uncertainty vanishes; in this paper, an option's value to an investor is determined from the expiration boundary condition on the portfolio in which the option resides.) On the other hand, the starting point of this paper is a risk-to-reward balance for the upcoming time step ("equation of motion"), not an integrated trading history. Moreover, the price of risk is the cornerstone of the present approach and is contained in all hedging strategies based on it; in Ref. [9] an independent parameter is introduced to characterize the option seller's risk aversion.

This paper focuses on valuation of options embedded in a given portfolio, but the methodology has also been used to derive formal results about hedging. These can facilitate comparison with other work. A portfolio consisting of N shares of stock S and short one call C has the $p = 2$ risk metric

$$\mathcal{R}(\Pi) = \left\langle (\Pi(z+\Delta z, t+\Delta t) - \langle \Pi(z+\Delta z, t+\Delta t)\rangle)^2 \right\rangle^{1/2}. \tag{43}$$

The portfolio with minimal absolute risk is found from the variational condition $\partial\mathcal{R}[\Pi]/\partial N = 0$, which has the solution

$$N_{abs} = \frac{\langle \delta C \delta S\rangle}{\langle \delta S^2\rangle} \tag{44}$$

$$\delta\Pi \equiv \Pi(z+\Delta z, t+\Delta t) - \langle \Pi(z+\Delta z, t+\Delta t)\rangle\,, \tag{45}$$

and, in terms of the risk $\langle \delta S^2\rangle$ of the stock, the residual risk of the portfolio is

$$\frac{\mathcal{R}_{abs}[\Pi]}{\langle \delta S^2\rangle^{1/2}} = \sqrt{\frac{\langle \delta C^2\rangle}{\langle \delta S^2\rangle} - \left(\frac{\langle \delta C \delta S\rangle}{\langle \delta S^2\rangle}\right)^2} \tag{46}$$

If this risk is neglected, the evolution equation for $C(z,t)$ becomes linear. For this approximation, it is convenient to treat the call as a linear combination of options whose values are bounded in z so that the Fourier transform exists. If $f(z,t)$ is such a constituent option and $f(k,t)$ is its Fourier transform, the linearized evolution equation for $f(k,t)$ is

$$\frac{\partial f}{\partial t} + \left(\ln P(-k,1) + \frac{r - \ln P(i,1)}{\ln \frac{P(2i,1)}{P(i,1)^2}} \ln \frac{P(-k+i,1)}{P(-k,1)P(i,1)}\right) f = rf. \tag{47}$$

The problem has been reduced to a first-order differential equation whose solution is straightforward. Section 3 has noted that for $p = 2$ the risk metric may be so large that positive option values are not always compatible with the risk-to-reward tradeoff.

Minimizing the *fractional* uncertainty by $\frac{\partial \mathcal{R}[\Pi]/\Pi}{\partial N} = 0$ yields the solution

$$N_{frac} = \frac{C \langle \delta C \delta S \rangle - S \langle \delta S^2 \rangle}{C \langle \delta S^2 \rangle - S \langle \delta C \delta S \rangle} \tag{48}$$

$N_{frac}(t + \Delta t)$, which is computed from the option value at time $t + \Delta t$, is used to solve for Π at time t; $N_{frac}(t)$ can be calculated from $\Pi(t)$ and the solution process iterated backward in time. In the Black-Scholes analysis, $N = \partial C/\partial S$; N_{abs} and N_{frac} are homogeneous of order 1 in C and of order -1 in S. After some rearrangement, the residual risk is found to be

$$\left.\frac{\langle \delta \Pi^2 \rangle / \Pi^2}{\langle \delta S^2 \rangle / S^2}\right|_{frac} = S^2 \frac{\frac{\frac{\langle \delta C^2 \rangle}{\langle \delta S^2 \rangle} - \left(\frac{\langle \delta C \delta S \rangle}{\langle \delta S^2 \rangle}\right)^2}{\left(S \frac{\langle \delta C \delta S \rangle}{\langle \delta S^2 \rangle} - C\right)^2}}{1 + S^2 \frac{\frac{\langle \delta C^2 \rangle}{\langle \delta S^2 \rangle} - \left(\frac{\langle \delta C \delta S \rangle}{\langle \delta S^2 \rangle}\right)^2}{\left(S \frac{\langle \delta C \delta S \rangle}{\langle \delta S^2 \rangle} - C\right)^2}} \tag{49}$$

In view of the correspondence of the functional form above to the minimized absolute risk, the approximation

$$\left.\frac{\langle \delta \Pi^2 \rangle / \Pi^2}{\langle \delta S^2 \rangle / S^2}\right|_{frac} \simeq \frac{\left.\frac{\langle \delta \Pi^2 \rangle / \Pi^2}{\langle \delta S^2 \rangle / S^2}\right|_{abs}}{1 + \left.\frac{\langle \delta \Pi^2 \rangle / \Pi^2}{\langle \delta S^2 \rangle / S^2}\right|_{abs}} \tag{50}$$

suggests itself. This equation is *not* exact. The reason is that the call values in Equation 49 are the solutions to the Π valuation equation with $N = N_{frac}$ whereas the valuation of calls with minimal absolute risk is determined with $N = N_{frac}$. Still, because the call value appears in the numerator and denominator of N_{frac} but only in the numerator of N_{abs}, the minimal-risk valuation equation is structurally simpler for absolute risk than for fractional risk. Equation 50 suggests that $N_{frac} \simeq N_{abs}$ may be a good approximation when the portfolio uncertainty is small compared to the uncertainty of the stock. The simpler valuation equation can be used under these conditions, although the difference may not be critical if the portfolio dynamics is being determined numerically.

However, the distinction illustrates the effect of nonlinearity, which is the final issue to be discussed in this section. The equation of motion is a first-degree homogeneous nonlinear equation. The solution depends on the boundary conditions at option expiration, and the use of those boundary conditions for option valuation must be done with deliberation. For example, investors with different amounts of the underlying stock will not value a single call identically even if they use the same risk metric; ensuing differences in option valuations are typically small but important in principle. Because the equation of motion is nonlinear, the portfolio's risk cannot be expressed as the sum of the risks of the components. (In fact, the equation can be linearized for a portfolio consisting of stock and an infinitesimal number of options.)

There is a situation in which the nonlinearity of the risk metric is clearly important. The risk term has the sign of $\partial\Pi/\partial z$ [13], and it has been assumed in this paper that the sign of $\partial\Pi/\partial z$ does not change. However, portfolios are commonly constructed for which this assumption is invalid. For example, a straddle is a long call and short put with the same strike price. Because the risk changes sign and does not vanish, it must be discontinuous as a function of stock price. This discontinuity finds its way into the equation of motion and, for finite time steps, into the portfolio value.

For the model Bessel distribution studied in this paper, the discontinuity generated by a quadratic minimum in Π is of order $\Delta t^{1/p}$ and vanishes in the continuum limit $\Delta t \to 0$. Moreover, the discontinuity is "smeared" when the equation of motion is solved by backward propagation (although a new discontinuity arises at each time step). After the continuum limit is taken, some feature–kink or twist or cusp–may well remain as the remnant of the discontinuity.

Discontinuities persist if Equation (2) is formulated for discrete time steps. Coarse-grained valuations for portfolios with extrema can be obtained heuristically, but a fully satisfactory treatment will probably be grounded in an understanding of the continuum limit [22].

5 Summary and Conclusions

This paper has examined the problem of option valuation when the underlying stock does not execute geometrical Brownian motion. The central assumption has been that option portfolios should be valued so that their excess return per risk equals that of the stock; the methodology can be used to construct optimally hedged portfolios but is not restricted to such portfolios.

The ensuing equation of motion is nonlinear and first-degree homogeneous; a solution has been displayed which appears well suited for numerical implmentation. The method has been applied to an empirical distribution obtained from time series for Merck and Oracle stock. An analytically tractable model of the stock motion has been utilized to determine the asymptotic form of option prices near expiration. Equivalence to the Black-Scholes, Poisson and binary-tree results has been noted. The relationship to the related work of Bouchaud, Sornette and others has been discussed.

Stock-price distributions deviate from the lognormal form, and option prices are in only partial agreement with the Black-Scholes value. It is believed that these two effects may be linked, and the present work supports this position. The model distribution analyzed in this paper was generated by beginning with an exponential distribution and, in effect, applying Central-Limit-Theorem arguments in reverse; as expected, deviations from the Gaussian form were most pronounced for shortest times. Sample results were presented for discrete- and continuous-time formulations of the valuation methodology. Despite the utility of discrete-time calculations, understanding the short-time continuum limit is highly desirable.

The paper has explored how nonlinearity in option valuation models increases their level of complexity. The nonlinearity is important in principle even when its effects are small or negligible. Because of its intuitive basis, the present methodology is plausible, and its consequences appear coherent. The intuitive foundation provides guidance when literal solutions have anomalous features, and it will provide guidance for modifications or generalizations.

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