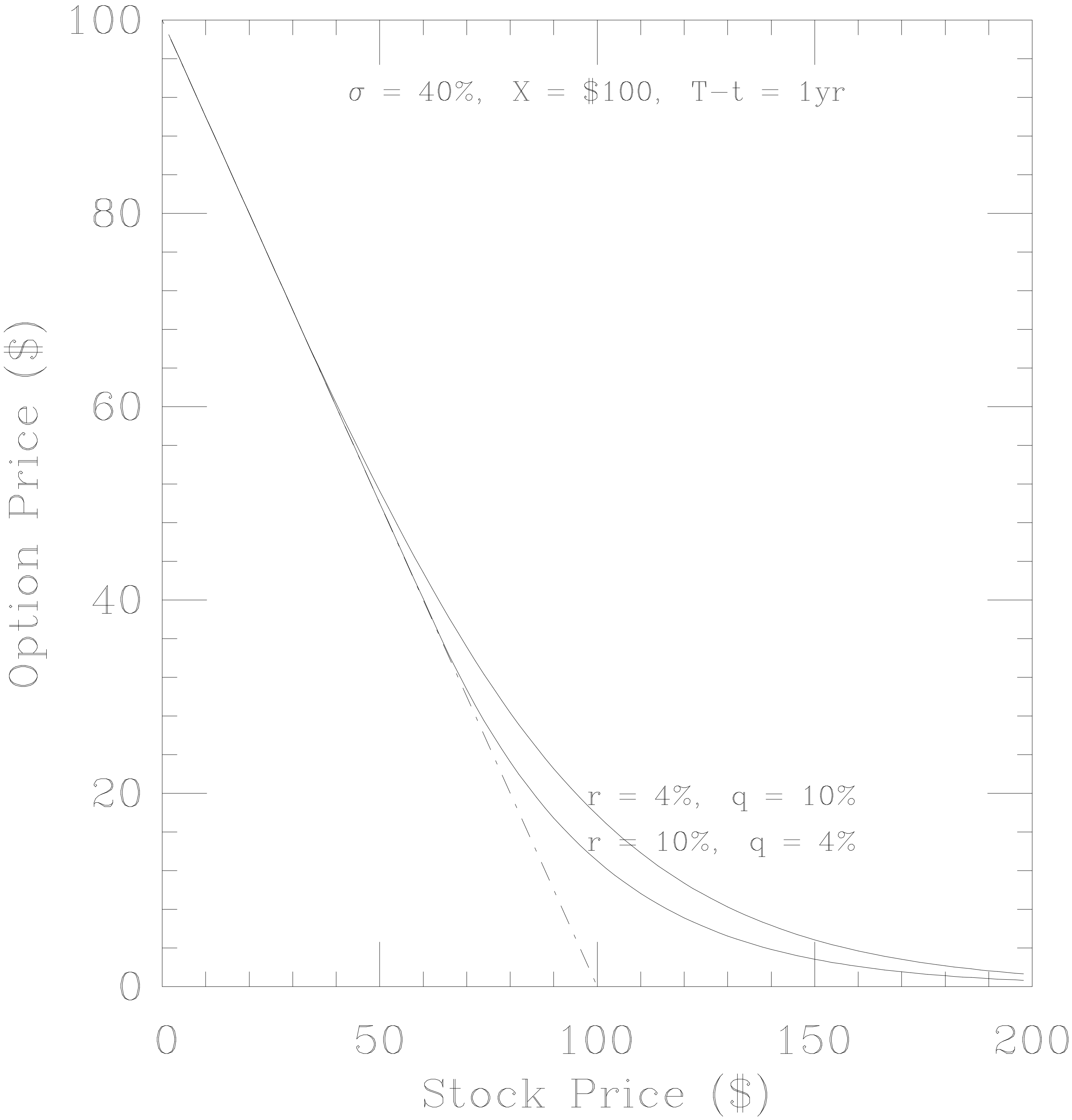

Figure 2. Option price for an American put on a stock paying dividends. Dividend yields are paid continuously at a rate q. Critical prices are \$63.2 and \$31.6 for $r = 10\%$, $q = 4\%$ and $r = 4\%$, $q = 10\%$ respectively.

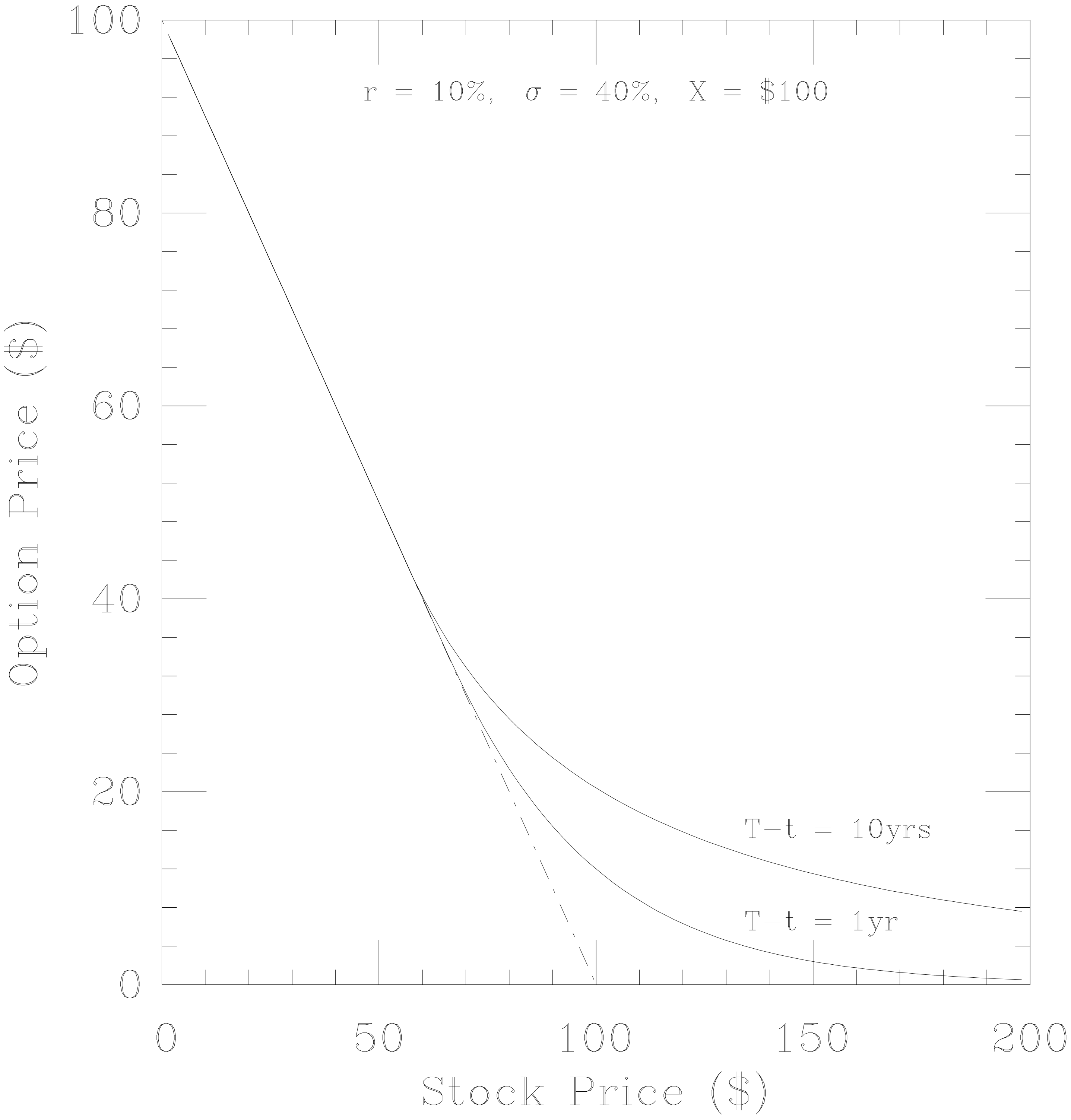

Figure 1. Option price for an American put on a stock paying no dividends. The symbols have their standard meaning. Critical prices are \$66.8 and \$56.3 for time to maturities 1yr and 10yrs respectively.

Table III
Comparison of Puts Adjusted for Cash Dividends

Our results P for the put prices are compared with $P(\mathrm{CR})$ of Cox and Rubinstein (1985). Interest rate is 0.0488 and the stock price is 40 dollars. Time to maturity is 0.0833 for the first part of the table and 0.5833 for the second part. A 50 cent dividend is paid in 0.5, 3.5 and 6.5 months. Also shown are the results $P(\mathrm{GJ})$ of Geske and Johnson (1984). The last column shows our corrected results P_+ for the put prices. We have ignored the possibility of early exercise for $X = 45$ in the second part of the table and our results are then lower as expected.

σ	X	$P(\mathrm{CR})$	$P(\mathrm{GJ})$	P	P_+
0.2	35.0	0.01	0.0116	0.0116	0.0116
0.2	40.0	1.11	1.1079	1.1084	1.1076
0.2	45.0	5.41	5.4209	5.4138	5.4132
0.3	35.0	0.11	0.1073	0.1073	0.1073
0.3	40.0	1.56	1.5590	1.5591	1.5583
0.3	45.0	5.50	5.4996	5.4985	5.4970
0.4	35.0	0.31	0.3049	0.3048	0.3048
0.4	40.0	2.01	2.0120	2.0124	2.0117
0.4	45.0	5.70	5.7015	5.7024	5.7007
0.2	35.0	0.66	0.6580	0.6569	0.6567
0.2	40.0	2.58	2.5717	2.5715	2.5711
0.2	45.0	6.02	6.0300	5.9915	5.9911
0.3	35.0	1.55	1.5454	1.5434	1.5431
0.3	40.0	3.74	3.7435	3.7433	3.7429
0.3	45.0	6.99	6.9977	6.9727	6.9722
0.4	35.0	2.52	2.5277	2.5256	2.5253
0.4	40.0	4.92	4.9116	4.9114	4.9109
0.4	45.0	8.10	8.0914	8.0733	8.0727

Table II

Comparison with Cox and Rubinstein (1985)

Our results P for the put prices are compared with $P(\mathrm{CR})$ of Cox and Rubinstein (1985). Interest rate is 0.0488 and the stock price is 40 dollars. Time to maturity is 0.0833 for the first part of the table and 0.5833 for the second part. Also shown are the results $P(\mathrm{GJ})$ of Geske and Johnson (1984). The last two columns show our corrected results P_+ for the put prices and $\overline{S}$ for the critical prices.

σ	X	$P(\mathrm{CR})$	$P(\mathrm{GJ})$	P	P_+	$\overline{S}$
0.2	35.0	0.01	0.0062	0.0062	0.0062	31.740
0.2	40.0	0.85	0.8528	0.8529	0.8522	36.274
0.2	45.0	5.00	4.9985	5.0026	5.0000	40.808
0.3	35.0	0.08	0.0774	0.0774	0.0774	29.779
0.3	40.0	1.31	1.3100	1.3107	1.3100	34.033
0.3	45.0	5.06	5.0599	5.0644	5.0597	38.287
0.4	35.0	0.25	0.2466	0.2466	0.2466	27.849
0.4	40.0	1.77	1.7679	1.7689	1.7682	31.827
0.4	45.0	5.29	5.2855	5.2900	5.2869	35.805
0.2	35.0	0.43	0.4321	0.4336	0.4329	29.085
0.2	40.0	1.99	1.9905	1.9948	1.9907	33.240
0.2	45.0	5.27	5.2719	5.2804	5.2671	37.395
0.3	35.0	1.22	1.2194	1.2216	1.2200	25.483
0.3	40.0	3.17	3.1733	3.1750	3.1699	29.124
0.3	45.0	6.24	6.2365	6.2562	6.2439	32.764
0.4	35.0	2.16	2.1568	2.1574	2.1551	22.156
0.4	40.0	4.35	4.3556	4.3586	4.3530	25.321
0.4	45.0	7.39	7.3831	7.3946	7.3833	28.486

Table I
Comparison with Parkinson (1977)

Our results P for the put prices are compared with $P(\mathrm{P})$ of Parkinson (1977). The stock and the exercise price is each one dollar with one year remaining to maturity. Also shown are the results $P(\mathrm{GJ})$ of Geske and Johnson (1984). The last two columns show our corrected results P_+ for the put prices and $\overline{S}$ for the critical prices.

σ	r	$P(\mathrm{P})$	$P(\mathrm{GJ})$	P	P_+	$\overline{S}$
0.1	0.005	0.038	0.0377	0.03771	0.03768	0.8324
0.1	0.010	0.036	0.0357	0.03576	0.03572	0.8590
0.1	0.020	0.033	0.0322	0.03232	0.03225	0.8878
0.1	0.030	0.030	0.0292	0.02932	0.02926	0.9054
0.2	0.020	0.071	0.0712	0.07120	0.07111	0.7419
0.2	0.040	0.064	0.0640	0.06419	0.06405	0.7917
0.2	0.080	0.053	0.0525	0.05282	0.05275	0.8453
0.2	0.120	0.044	0.0439	0.04394	0.04415	0.8767
0.3	0.045	0.101	0.1005	0.10068	0.10048	0.6811
0.3	0.090	0.086	0.0859	0.08638	0.08615	0.7505
0.4	0.080	0.126	0.1258	0.12634	0.12601	0.6378
0.5	0.125	0.148	0.1476	0.14846	0.14798	0.6057

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Appendix A

Here we outline the steps involved in obtaining Equation (5). Differentiating Q with respect to τ gives

$$\int_{-\infty}^{\infty} d\ln s \left\{ \frac{\partial P(s,\tau)}{\partial \tau} e^{-r(\tau-t)} \phi\left(\frac{s}{S}, \tau - t\right) + P(s,\tau) \frac{\partial}{\partial \tau}\left[e^{-r(\tau-t)} \phi\left(\frac{s}{S}, \tau - t\right)\right]\right\} \tag{A1}$$

where for convenience $\ln s$ is used as the variable of integration instead of s. The above can be simplified by noting that

$$f = e^{-r(\tau-t)} \phi\left(\frac{s}{S}, \tau - t\right) \tag{A2}$$

satisfies the Black-Scholes differential equation

$$\frac{\partial f}{\partial t} + \left(r - \frac{\sigma^2}{2}\right) \frac{\partial f}{\partial \ln S} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial \ln S^2} - rf = 0. \tag{A3}$$

Because f depends only on the differences $\tau - t$ and $\ln s - \ln S$, the above can be rewritten as

$$\frac{\partial f}{\partial \tau} = -\left(r - \frac{\sigma^2}{2}\right) \frac{\partial f}{\partial \ln s} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial \ln s^2} - rf. \tag{A4}$$

This can be used in Equation (A1) to replace the τ derivative of f with $\ln s$ derivatives. The $\ln s$ derivatives can now be transferred to act on P by partial integration. The result of these manipulations is Equation (5) where the Black-Scholes differential operator BS now involves s and τ variables as in Equation (6).

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just after an ex-dividend date in order to compute it. When there is only one ex-dividend date remaining, its value just after the date is simply given by the corresponding European option price. This makes it possible to obtain a direct expression for the American call price. The result agrees with the already known formula obtained by Roll (1977b), Geske (1979), Whaley (1981) and Geske (1981a).

V. Conclusion

In this article, analytic representations are derived for the American put and call option prices. Options on stocks paying dividends are also considered. The representations are all in integral form. They depend on the critical prices and lead to constraints on the critical prices that are also in integral form. The integral expressions are not easily solvable for the critical prices or the option prices. But, they lead to a very efficient approximation scheme. The approximation turns out to be a reliable one for all values of the parameters. The constraints for the critical prices are approximated and solved with simple iterative techniques. The critical prices are then used to compute the integral to obtain the option prices. The results obtained are very satisfactory and compare favorably with those known from other methods. Some understanding for the error can be obtained from the option price computed at the critical price itself. It is possible to compute corrections to the critical prices and hence the option prices if more accuracy is needed. The original constraint equation can be handled iteratively starting with the approximate solution. The corrections are not much significant beyond the first iteration. We thus have an approximation scheme that is an efficient and accurate alternative to other numerical methods.

Expressions for the various hedge parameters can be obtained easily from those derived here for the option prices. Those parameters can be valued analogously. Integral representations derived here could be useful as a basis for further discussions and investigations of American option pricing. They could be useful as a basis for developing better approximation techniques and numerical methods. The theoretical framework presented here could be helpful in handling other problems in option pricing. It is a new approach that may also have applications in some other free boundary problems.

$$D_i > X\left(e^{r(t_i - t_{i-1})} - 1\right). \tag{25}$$

This is necessary if further it can be optimal to exercise just after t_i. When the dividends are all sufficiently large, early exercise can be optimal only after the last ex-dividend date t_n. Critical prices are then undefined or effectively zero until t_n and can be approximated as in section II thereafter until maturity. Equation (10) can now be used to compute the option prices with its lower limit of integration set to t_n. In Table III, we compare our numbers with those of Cox and Rubinstein (1985) and Geske and Johnson (1984) and present our corrected results. Early exercise can be optimal before the last ex-dividend date t_n when for instance the above inequality does not hold at $i = n$. This is the case for $X = 45$ and $T - t = 0.5833$ in Table III and our results are then lower as expected. It is not easy to find good estimates for the corresponding critical prices. It may be possible to iterate the constraint equation to obtain their true values starting with a suitable ansatz that ensures convergence.

For an American call option, the option price C can have discontinuities at the ex-dividend dates. Because it is not optimal to exercise an American call in between those dates, contributions to $dQ/d\tau$ arise only from those discontinuities. Discontinuity at an ex-dividend date t_i arises from the possibility of exercise at t_{i-}, that is just prior to t_i. Exercise at t_{i-} can be optimal when the dividend D_i is sufficiently large or, more precisely, when the intrinsic value $s + I(t_{i-}) - X$ exceeds $C(s, t_{i+})$, the value of the call if no exercise is made. This forces $C(s, t_{i-})$ to take the value $s + I(t_{i-}) - X$ for s above $\overline{S}_i$ where $\overline{S}_i$ is the solution of

$$C(\overline{S}_i, t_{i+}) = \overline{S}_i + I(t_{i-}) - X. \tag{26}$$

For s below $\overline{S}_i$, it is not optimal to exercise at t_{i-} and there is no discontinuity in C at t_i. Hence, across an ex-dividend date t_i, Q jumps down by

$$\Delta_i = e^{-r(t_i - t)} \int_{\overline{S}_i}^{\infty} \frac{ds}{s}\ \left[s + I(t_{i-}) - X - C(s, t_{i+})\right] \phi\left(\frac{s}{S}, t_i - t\right). \tag{27}$$

If it is not optimal to exercise at t_{i-} for any s, Δ_i is zero. The jumps contribute to the call price as

$$C(S, t) = c(S, t) + \sum_i \Delta_i \tag{28}$$

where the summation is over the remaining ex-dividend dates and S is only the risky component of the stock price. This is a recursive expression for C, for we need to know its value dividend date. We would like to see how the present approach can be extended to handle these and other cases involving cash dividends.

To start with, let us have dividends paid continuously at a rate D that may well be time varying. Integral representations can now be derived for the option prices and the critical prices. As is usual, let us assume that it is the risky component of the stock price that follows a geometric Brownian motion. The actual stock price is obtained by adding $I(t)$, the present value of all the dividends that remains to be paid, to the risky component of the stock price. Most of the steps in our original derivation in section I can now be repeated as such if we assume that the prices S (at t) and s (at τ) that appear there are only the risky component of the stock price. The same holds for the critical price $\overline{S}(\tau)$. The difference now is that the intrinsic value of the option price $P(s,\tau)$ at τ is $X - s - I(\tau)$. This changes $BS(P)$ to

$$BS(P) = -rX + rI(\tau) - \frac{\partial I(\tau)}{\partial \tau} = -rX + D(\tau) \tag{24}$$

when s is below $\overline{S}(\tau)$. An integral representation for the American option price and an integral constraint for the critical price can now be derived as before and have $rX - D(\tau)$ in place of rX inside the integrals.

It is tempting to use these results for the case of discrete cash dividends with $D(\tau)$ replaced by $\sum_i D_i \delta(\tau - t_i)$ where D_i is the dividend to be paid at an ex-dividend date t_i. This will lead to a Q that is discontinuous at the ex-dividend dates. This can not be true for an American put since a put option is continuous in time for a given value of the risky component of the stock price resulting in a continuous Q. Besides, this approach will lead to formal expressions involving critical prices at the ex-dividend dates. The critical price may not be continuous at an ex-dividend date and it is not clear what value should be used for it in such expressions.

It appears that for an American put option our earlier results in section I should still be applicable with a more complicated time dependence allowed for the critical prices. This is because $D(\tau)$ in Equation (24) is zero in between the ex-dividend dates. Besides, there are no new contributions to $dQ/d\tau$ arising at the ex-dividend dates because of continuity of Q. But it is not easy to find a suitable approximation to the constraint equation to find solutions, except of course when it is not optimal to exercise in between the ex-dividend dates for any value of the stock price. It is not optimal to exercise between t_{i-1} and the next ex-dividend date t_i if for instance that the option price agree with its intrinsic value when S is below $\overline{S}(t)$ for a put option and is above $\overline{S}(t)$ for a call. But we preferred to work with the slope requirement for reasons discussed in section II. We could still enforce duality if we agree to use the dual of Equation (19) for call options that has $X^2/\overline{S}(t)$ in place of $\overline{S}(t)$ and r and q interchanged. The dual is simply the approximation to a certain combination of the value and slope requirements. It gives us better and more reliable estimates for the critical prices for the call that are as accurate as those for the put.

The dual of Equation (19) is hence preferable as an approximation to obtain good and reliable estimates for the critical prices and the option prices for the American call. The slope requirement when approximated does give good estimates for the critical prices closer to maturity, but it becomes less accurate as we move farther from it. Similarly, for the American put option, there is an alternative to Equation (19) that gives slightly better estimates for the critical prices closer to maturity. This is obtained from Equation (19) by replacing the $-1/2$ in the end with $+1/2$ and taking its dual, that is replacing $\overline{S}(t)$ with $X^2/\overline{S}(t)$ and interchanging r and q. It too becomes less accurate and unacceptable as we move farther from maturity. Using it with q set to zero, it is possible to obtain slightly better results in Tables I and II.

The present approach is easily extended to accommodate time varying interest rates, dividend yields or volatilities. With a few more steps added to the derivation, one reaches an expression for the option price that looks very similar to those obtained here. All occurrences of r, q and σ are replaced by their averages taken over the period from t to τ, except for rX and qS that are replaced by $r(\tau)X$ and $q(\tau)S$ inside the integrals. The same is true for the constraint equations. The approximated constraints however may not be integrable without a knowledge of the time dependence.

IV. Cash Dividends

Incorporating cash dividends in an analytical framework for options is known to create some difficulties. For an American put, it is not optimal to exercise just before an ex-dividend date and for an American call it sometimes is. This makes the critical price take a complicated time dependence which appears to be at the origin of the difficulties involved. However, there are some special cases that could be handled relatively easily, as for instance when it is never optimal for an American put to be exercised any time before the last ex-

The case of the American call option is analogously handled. The results are identical to those given for the put option in Equations (17) and (18), except that now all the $N(x)$ functions that appear there are replaced by $-N(-x)$ with C and c taking the place of P and p. The constraint has no finite solution for the critical price when $q = 0$ in agreement with the fact that it is not optimal to exercise an American call prior to maturity in the absence of dividends. The approximate equation (19) still holds except for a $+1/2$ instead of the $-1/2$ in the end. The asymptotic result for $\overline{S}(t)$ is similar to (21) with $-\gamma$ replacing γ. The approximation turns out be reasonable. Compared to that for the put option, it is slightly more accurate closer to maturity, but becomes less so farther from it to the point of becoming unacceptable. To obtain better and more reliable estimates for the critical prices, we introduce in the following what may be called the dual of Equation (19).

It is known that the put-call parity that holds for European options becomes only an inequality for the American options. However, it is possible to introduce what may be called a put-call duality that holds for both the European and American options. The following is easily verified for Europeans options with identical σ, X and T:

$$p(S,t,r,q) = \frac{S}{X} c\left(\frac{X^2}{S}, t, q, r\right) \tag{22}$$

where for both p and c the third argument is the interest rate and the fourth is the dividend yield. Note that the role of r and q are reversed in the right hand side. The above relation is a duality in the sense that it holds even if we interchange p and c. It can be given a nice interpretation when the options are on currencies and the dividend yield is the foreign interest rate. For options on futures $q = r$ and the q, r interchange is automatic. It is interesting to verify that this duality holds for the American options as well provided there is a relationship between the critical price $\overline{S}_p(t)$ for a put and the critical price $\overline{S}_c(t)$ for a call given by

$$\overline{S}_p(t,r,q)\overline{S}_c(t,q,r) = X^2. \tag{23}$$

It can be shown that this follows from the constraints for the critical prices. However, the approximations we developed do not obey this relationship, at least not exactly. We do find numerically that this relationship is satisfied reasonably well. Deviations are largely due to the errors in $\overline{S}_c(t)$. It is possible to derive a constraint that can be approximated along the lines presented in this article to obtain an approximation that automatically satisfies duality. This would be the case if we had started with the value requirement demanding an extension of the previous one. Letting both S and $\overline{S}(\tau)$ tend to $\overline{S}(t)$, we perform the integrals in (18) and obtain

$$\begin{aligned} e^{-q(T-t)}\left\{N(\overline{d}_1) - N\left[(r-q+\sigma^2/2)\sqrt{T-t}/\sigma\right]\right\} &= \\ \tfrac{1}{\gamma}\left[r\tfrac{X}{\overline{S}(t)} - \tfrac{1}{2}(r+q+\sigma^2/2)\right]\left\{2N\left(\gamma\sqrt{T-t}/\sigma\right)-1\right\} - \tfrac{1}{2} & \end{aligned} \tag{19}$$

where

$$\gamma = \sqrt{(r-q+\sigma^2/2)^2 + 2q\sigma^2} \tag{20}$$

and $\overline{d}_1$ is obtained from d_1 (that has $r-q$ in place of r) by replacing S with $\overline{S}(t)$. This is again an approximation that agrees with the exact solution for the limiting values of all parameters. As $t \to -\infty$ it gives the known solution

$$\overline{S}(-\infty) = \frac{2rX}{r+q+\sigma^2/2+\gamma}. \tag{21}$$

Its limit as $t \to T$ depends on the value of q relative to r. If $\overline{S}(T)$ turns out to be less than X, it would lead to $N(\overline{d}_1)$ vanishing faster than the other terms above and becoming negligible as $t \to T$. Solving the rest of the equation, using $N(x) \approx 1/2 + x/\sqrt{2\pi}$ as $x \to 0$ and keeping terms to order $\sqrt{T-t}$, we find $\overline{S}(T)$ to be rX/q. This is the expected limit for $q > r$. For $q \leq r$, the equation is consistent only if $\overline{S}(T) = X$. We thus find that the approximation becomes exact as expected close to maturity as well.

The critical prices and hence the option prices can now be computed in this approximation sheme as in the previous section. A plot of the option prices presented in Figure 2 for some chosen parameter values shows the expected behavior. It is found numerically that the method is both good and efficient. This can be verified by computing corrections to the approximation. Here too, the error in the option price computed at the critical price is small and is usually within half a percent. Errors in the critical prices show the oscillatory behavior about zero for $r - q > \sigma^2/2$ and turn out to be less than about a percent of the exercise price. It is not easy to see why this is the case. The reasons given earlier for the no dividend case could still be operative, at least for small q. The other reason could be that the approximation gives exact solutions for the limiting values of all the parameters. Though the original integral equation is too complicated to give an exact solution, it is satisfying that the approximate ones are remarkably close to the true results and will be sufficient for most practical purposes.

III. Puts and Calls

In section I, we derived an integral representation for the price of an American put option on a non-dividend paying stock. We also obtained an integral constraint on the critical prices. In the previous section, we developed an analytic approximation that turned out to be very satisfactory. These results should be extendible to options on dividend paying stocks. In this section, we derive integral representations for American put and call options on stocks paying dividend yields. Though they too are complicated enough to admit solutions, they lead to an equally good approximation scheme. It is also possible to compute corrections to this approximation if more accuracy is needed.

Let us first consider a put option on a stock that pays dividend yields at a rate q. This leads to a few changes to our original derivation in section I for the put option on a non-dividend paying stock. The difference now is that the definition of ϕ contains $r-q$ in place of r and the Black-Scholes differential operator contains $r-q$ in place of r multiplying $s\partial/\partial s$. $BS(P)$ is now $-rX+qs$ for $s<\overline{S}(\tau)$ and zero otherwise. Proceeding as we did in section I, we obtain

$$P(S,t) = p(S,t) + \int_t^T d\tau \left[rXe^{-r(\tau-t)}N(-d_2^\tau) - qSe^{-q(\tau-t)}N(-d_1^\tau)\right] \qquad (17)$$

where the definition of d_2^τ now contains $r-q$ in place of r and $d_1^\tau = d_2^\tau + \sigma\sqrt{\tau - t}$. It is easily verified that this expression for $P(S,t)$ does satisfy the Black-Scholes differential equation for S above $\overline{S}(t)$ and that it does so only upto $-rX+qS$ for S below. Here too, it is the limit of integration t that gives rise to a different behavior for $S<\overline{S}(t)$. The expression behaves as expected as a function of S and t. It depends on the critical prices for all τ from t to T. One may obtain a constraint on the critical prices from the value requirement $P = X - S$ or the slope requirement $\partial P/\partial S = -1$ for $S \leq \overline{S}(t)$. We expect the slope requirement to lead to better estimates. The constraint arising from the slope requirement is

$$\begin{aligned} &Se^{-q(T-t)}N(d_1) + qS\textstyle\int_t^T d\tau e^{-q(\tau-t)}N(d_1^\tau) = \\ &\qquad\qquad \textstyle\int_t^T d\tau \left(rX - q\overline{S}(\tau)\right) e^{-r(\tau-t)}\phi\left(\frac{\overline{S}(\tau)}{S}, \tau - t\right) \quad . \end{aligned} \qquad (18)$$

This is an integral constraint on the critical prices that should hold for all $S \leq \overline{S}(t)$. Its S-independence may however be verified by doing a Laplace transformation of both the sides with respect to $T-t$.

The above system of equations are more complicated than the previous ones. Since an exact solution is beyond our reach at present, we look for an approximation that is prices computed at future times from T down to t. If more accuracy is needed, one can even compute corrections to the critical prices and hence the option prices. Errors in our results for the option prices are largely due to the approximation done to the critical prices. The original constraint equation can be handled iteratively starting with the approximate solution. It is an iteration of a set of integral equations, one for every time until maturity, and is hence computationally more demanding. The corrections however are not much significant beyond the first iteration. It is convenient to iterate the constraint equation resulting from the value requirement at the critical price itself. Iterations converge to a desired accuracy when the number of grid points used to compute the integral is sufficiently large. Our corrected results for the option prices and the critical prices obtained using a grid of 100 points are shown in the last two columns of Tables I and II. Errors in our approximation to the option prices are within half a percent of the corrected results. They remain within one percent even when a grid of only 10 points is used to compute the approximate prices. Errors in the critical prices do exhibit the oscillatory behavior mentioned above. They turn out to be less than about a percent of the exercise price for all parameter values.

Recall that we had a choice in choosing the constraint equation for $\overline{S}(t)$ in the previous section. We could have started with the value requirement that demands that the put price agree with its intrinsic value when S is below $\overline{S}(t)$. But we preferred to work with the slope requirement. Though the two requirements should lead to identical critical prices when solved exactly, it is not necessarily the case with their approximations. We find numerically in agreement with our argument above that the slope requirement gives better estimates for the critical prices. It can be shown that the value requirement on the American put on a non-dividend paying stock gives only an upper bound for the critical price when approximated along the lines presented here, in contrast to the slope requirement that leads to an estimate whose error has an oscillatory behavior about zero. The bound is saturated at both the asymptotic limits $t \to -\infty$ and T. Numerical study indicates that solutions of (15) do satisfy this bound and that this bound is significantly above it. We learn from here that an agreement with the exact solution for $t = -\infty$ and T alone is not enough to lead to a good approximation.

have implications for $r \leq \sigma^2/2$ as well.

To decide whether we have an upper or a lower bound, we need to look at ϕ inside the integral in (11). Whenever ϕ is replaced by its upper or a lower bound and the resulting equation is solved for S set to $\overline{S}(t)$, one obtains a corresponding bound for the critical price. This is because $SN(d_1)$ that appears in the left hand side of the constraint is an increasing function of S. Note that the argument of exponential in ϕ is $-(d_2^\tau)^2/2$. Letting S and $\overline{S}(\tau)$ tend to $\overline{S}(t)$ as we did above to obtain the approximation amounts to increasing d_2^τ. If d_2^τ remains negative for all values of τ from t to T during this process, this decreases its magnitude and hence increases ϕ. This is what happens for $r \leq \sigma^2/2$ and it leads to an upper bound. If d_2^τ is not negative to start with, the process increases its magnitude and decreases ϕ. This is the case for $r > \sigma^2$ and it leads to a lower bound, but it happens only when sufficiently far away from maturity. Note that for d_2^τ to be not negative for all values of τ from t to T, at least as $S \to \overline{S}(t)$, we should have

$$\overline{S}(\tau)e^{(r-\sigma^2/2)(T-\tau)} \leq \overline{S}(t)e^{(r-\sigma^2/2)(T-t)}. \tag{16}$$

Sufficiently far back in time, this inequality will indeed be satisfied for $r > \sigma^2/2$ because the expression of t that appears in the right hand side above first decreases from its value of ∞ at asymptotic past and remains finite thereafter. Numerical study reveals that for $r > \sigma^2/2$ it increases close to maturity to reach its value of X at maturity. This opens up the possibility that the lower bound of a distant past could be transforming itself to an upper bound close to maturity. Besides, we should expect an upper bound at some point in time because it is unlikely under the circumstances for an upper bound at $r = \sigma^2/2$ to become a lower bound for all times when r is just above $\sigma^2/2$. In other words, there should be a moment besides $t = -\infty$ and T when our approximation agrees with the exact solution. As one keeps increasing r, this point in time keeps moving closer to maturity. Though we do not know when this phenomenon takes place, it is enough to convince us that we have got a good approximation.

The method outlined here is very efficient. The results are fairly accurate. Besides, some estimate for the error can be obtained by computing the option price at the critical price itself and comparing it with its intrinsic value. For instance, our estimates in Tables I and II can be improved by subtracting out a percentage error that is half of that computed at the critical price. It is also possible to improve the approximate constraint (15) itself. The original constraint (11) remains integrable when $\ln S/\overline{S}(\tau)$ is set proportional to $\tau - t$ instead of zero. The proportionality constant may be chosen from a knowledge of the critical

The approximate constraint (15) is easily solved for the critical price with simple iterative techniques such as the Newton-Raphson method using an error function routine for N. One may choose for the initial guess the exercise price itself or one that satisfies $\overline{d}_1 = 0$. When computing the critical prices for all τ from t to T, it becomes unnecessary to go beyond the first iteration if the result from the previous τ is chosen as the initial guess. Given the critical prices for all τ, an approximation to the option price is obtained by computing the integral in Equation (10) numerically. The integral is computed using a grid of 100 points even though reasonable results could be obtained by as low as 10 points. The results for some chosen parameter values are plotted in Figure 1. They behave as expected. As mentioned earlier, it is possible to test how good the approximation is. The results given in Figure 1 are for all S, both above and below $\overline{S}(t)$. Note the remarkable agreement of the option prices with $X - S$ for S below $\overline{S}(t)$. The approximation appears to be not just reasonable but excellent. This appears to be the case for all values of the parameters. A test of the approximation is the error in the option price computed at the critical price itself. It is small and is usually within half a percent. We also compare our numbers with those known from a numerical method and an analytical computation. This is done in Tables I and II. The numerical results are taken from Parkinson (1977) and Cox and Rubinstein (1985) while the analytical values are from Geske and Johnson (1984). Our numbers compare very favorably with all those results.

One could have inferred directly from the original constraint (11) why the approximation approached the exact solution both in the distant past and close to maturity. The integrand in (11) is exponentially suppressed for $t \to -\infty$ unless the variable of integration τ is also sent to $-\infty$. When both t and τ approach $-\infty$, there is not much difference between $\overline{S}(t)$ and $\overline{S}(\tau)$. The same is true close to maturity when both t and τ approach T. But it is puzzling why the approximation turned out to be a good one for all t, and to gain some understanding we next analyze its nature as an upper or a lower bound. What we find is that our approximation to the critical price is an upper bound for $r \leq \sigma^2/2$. It has a more interesting behavior for $r > \sigma^2/2$. It is a lower bound at distant past and, as we move closer to maturity, becomes an upper bound. In other words, there is some finite moment in time in addition to asymptotic past and maturity when it coincides with the exact value. The error in using the approximation thus appears to have an oscillatory behavior around zero as one varies t from $-\infty$ to T. This is a strong indication that the approximation to $\overline{S}(t)$ remains a good one for a long period of time. Though this holds only for $r > \sigma^2/2$, it should value on the option price for S below $\overline{S}(t)$, rather we require that our formalism reproduce them to our satisfaction.

Before we proceed, let us verify that the integral equations do lead to the known limits and bounds. A little analysis reveals that $\overline{S}(t)$ should tend to X as $t \to T$ for the constraint equation to hold. The limit $t \to -\infty$ is more interesting to analyze. This limit is known to be

$$\begin{aligned} \overline{S}(-\infty) &= \frac{rX}{r + \sigma^2/2} \\ P(S, -\infty) &= X - S && \text{for } S \leq \overline{S}(-\infty) \\ &= \frac{\sigma^2}{2r}\overline{S}(-\infty)\left(\frac{S}{\overline{S}(-\infty)}\right)^{-2r/\sigma^2} && \text{for } S \geq \overline{S}(-\infty). \end{aligned} \tag{13}$$

It is an instructive exercise to obtain these limits from our results. It is equally instructive to verify that the constraint (11) is satisfied with $\overline{S}(t) \to X$ when $r \to \infty$ or $\sigma \to 0$. The case of $r = 0$ or $\sigma \to \infty$ with $\overline{S}(t) \to 0$ is easy to verify. We expect all known bounds to be satisfied as well. For instance, the bound

$$P(S, t) - p(S, t) \leq X\left(1 - e^{-r(T-t)}\right) \tag{14}$$

easily follows from (10) on replacing $N(-d_2^\tau)$ with its upper limit of unity. It gets saturated as $S \to 0$. It is even possible to derive new bounds. An upper bound to the option price is obtained by replacing the critical price $\overline{S}(\tau)$ with its upper limit of X inside the integral in Equation (10). It becomes a lower bound if the lower limit $\overline{S}(-\infty)$ given above in Equation (13) is used instead. A better lower bound is obtained when $\overline{S}(\tau)$ is replaced by $\overline{S}(t)$, but this bound depends on the critical price at t that is yet undetermined.

The approximation presented below is the simplest to explain but turns out to be a good one to work with. We just replace $\overline{S}(\tau)$ inside the integral in Equation (11) with its lowest value $\overline{S}(t)$. The integral can now be handled as $S \to \overline{S}(t)$ giving

$$\overline{S}(t)N(\overline{d}_1) = \frac{rX}{r + \sigma^2/2}\left\{2N\left[(r + \sigma^2/2)\sqrt{T - t}/\sigma\right] - 1\right\} \tag{15}$$

where $\overline{d}_1$ is obtained from d_1 by replacing S with $\overline{S}(t)$. The original constraint is now replaced by this approximate one that is far simpler to handle. The solution has the expected behavior. It tends to the limits mentioned above for the limiting cases of r and σ. It tends to X as $t \to T$. Further, it tends to $\overline{S}(-\infty)$ given above as $t \to -\infty$. In other words, it is an approximation that becomes exact on both the asymptotic limits $t \to -\infty$ and T. Hence, it could be a reasonable approximation for finite values of t.

should give identical solutions to $\overline{S}(t)$ when solved exactly. The slope requirement gives the following constraint on $\overline{S}(t)$ using $\partial p/\partial S = -N(-d_1)$:

$$SN(d_1) = rX \int_t^T d\tau e^{-r(\tau-t)} \phi\left(\frac{\overline{S}(\tau)}{S}, \tau - t\right) \tag{11}$$

where, as usual, d_1 denotes

$$d_1 = \frac{\ln S/X + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \tag{12}$$

This is an integral equation for $\overline{S}(t)$ that should be satisfied for all $S \leq \overline{S}(t)$. One may explicitly see how the S-dependence drops off from the equation and does so for S less than the critical prices by doing a Laplace transformation of both the sides with respect to $T - t$. Those details are not presented here as they are not relevant for the discussion to follow.

It is not a miracle that the integral representations exist for the free boundary problem at hand. A well-known fact in electromagnetics, a theory based on linear differential equations, is that the electromagnetic potential can be obtained by summing up or integrating the contributions from the charges. The charges can be obtained by the action of the relevant differential operators on the potential. Equation (10) simply represents this fact. Equations for options on stocks paying dividend yields are analogously derived. They are presented later in section III where we discuss similar equations for call options. The present equations provide us with a simpler framework to illustrate our approach and to discuss an interesting approximation scheme which is our next topic.

II. Analytic Approximation

Equations (10) and (11) determine the option price for an American put on a non-dividend paying stock. But they are complicated integral equations. They do not appear to be easily solvable. One may try various approximations to understand the solution. Our Strategy will be to find an approximation to Equation (11) that is easily handled with iterative techniques. This will give an approximate solution to the critical price. The critical prices computed for all τ from t to T are then used to evaluate the integral in Equation (10) numerically to obtain the option price. This can be done for all S, both above and below $\overline{S}(t)$. Though we do not have to compute the option price for S below $\overline{S}(t)$ in practice, as it is known to take its intrinsic value of $X - S$ then, we do so to verify that our approximation to the critical prices is a reasonable one. In other words, we do not impose the intrinsic due to the fact that $P(s, \tau)$ is then $X - s$. There are no delta function contributions since P and $\partial P/\partial s$ are continuous at $\overline{S}(\tau)$. Thus $BS(P) = -rX$ if $s < \overline{S}(\tau)$ and zero otherwise. This gives

$$\frac{dQ(\tau)}{d\tau} = -rXe^{-r(\tau-t)} \int_0^{\overline{S}(\tau)} \frac{ds}{s}\, \phi\left(\frac{s}{S}, \tau - t\right) = -rXe^{-r(\tau-t)} N(-d_2^\tau) \tag{7}$$

where N is the cummulative normal distribution function given by

$$N(x) = \int_{-\infty}^{x} \frac{dy}{\sqrt{2\pi}} e^{-y^2/2} \tag{8}$$

and

$$d_2^\tau = \frac{\ln S/\overline{S}(\tau) + (r - \sigma^2/2)(\tau - t)}{\sigma\sqrt{\tau - t}}. \tag{9}$$

As expected, $dQ/d\tau$ has turned out to be nonzero for the American put. We know from (3) what Q is when τ is either t or T. Integrating $dQ/d\tau$ from $\tau = t$ to T and using those boundary values, we get

$$P(S,t) = p(S,t) + rX \int_t^T d\tau e^{-r(\tau-t)} N(-d_2^\tau). \tag{10}$$

This is the expression for P that we are after. It is an integral representation for the American put price on a non-dividend paying stock. We may now verify that it does indeed satisfy the Black-Scholes differential equation for S above $\overline{S}(t)$ and does so only upto $-rX$ for S below. Note that the integrand is a solution of the Black-Scholes differential equation. It is the limit of integration t that gives rise to a different behavior for S below $\overline{S}(t)$. Differentiation with respect to this limit contributes $-rXN(-d_2^t)$ that evaluates to $-rX$ for $S < \overline{S}(t)$ but zero otherwise. Note also that the integral vanishes as $t \to T$ and the payoff Max$(X - S, 0)$ comes from $p(S,t)$. It is also worth noting that the integral decreases with increasing S and vanishes as $S \to \infty$.

The above expression for P depends on the critical prices $\overline{S}(\tau)$ for all τ from t to T. Those critical prices are yet undetermined. This is because satisfying the Black-Scholes differential equation for S above $\overline{S}(t)$ and doing so upto $-rX$ for S below is not a sufficient condition to obtain the American option price. It is still necessary to require $\overline{S}(t)$ to be such that the price $P(S,t)$ agrees with its intrinsic value $X - S$ for $S \leq \overline{S}(t)$. This may be called the value requirement. Or, equivalently, it is necessary to demand $\partial P/\partial S = -1$ for $S < \overline{S}(t)$ as follows from $P = X - S$. We prefer the latter, what may be called the slope requirement, for reasons to be discussed in the next section. The two requirements

$$\phi\left(\frac{s}{S}, \tau - t\right) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau - t}} \exp\left\{-\frac{1}{2\sigma^2(\tau - t)}\left(\ln\frac{S}{s} + (r - \frac{\sigma^2}{2})(\tau - t)\right)^2\right\} \tag{2}$$

describes $\ln s$ as a normal distribution with mean $\ln S + (r - \sigma^2/2)(\tau - t)$ and variance $\sigma^2(\tau - t)$. Q depends on S and t in addition to τ. It can be interpreted as the price at time t of an European option maturing at τ with the American put price as the payoff. Or, equivalently, it can be interpreted as the price at time t of a hybrid option maturing at T with early exercise allowed only after time τ. These interpretations help us understand the values at the τ-boundaries t and T

$$Q(t) = P(S, t), \qquad Q(T) = p(S, t). \tag{3}$$

For $\tau \to t$, early exercise is allowed at all future times till maturity in the latter interpretation and Q is then no different from the American put price. For $\tau \to T$, exercise is allowed only at maturity making Q agree with the European put price. The above also follows from the limits

$$\begin{aligned} \phi\left(\frac{s}{S}, \tau - t\right) &\to \delta(\ln s - \ln S) \qquad \text{as } \tau \to t \\ P(s, \tau) &\to \mathrm{Max}(X - s, 0) \qquad \text{as } \tau \to T. \end{aligned} \tag{4}$$

Q thus interpolates between the American and European put prices. If we had the European put price p in place of P in Equation (1), no early exercise is allowed in the latter interpretation. This makes Q independent of τ and agree with the European put price at t. Q is dependent on τ for the American put because of the possibility of early exercise. This suggests that we consider its derivative with respect to τ to learn something specific about the American option. As outlined in the appendix, differentiating the expression for Q with respect to τ gives

$$\frac{dQ(\tau)}{d\tau} = e^{-r(\tau - t)} \int_0^\infty \frac{ds}{s}\, BS\left[P(s, \tau)\right] \phi\left(\frac{s}{S}, \tau - t\right) \tag{5}$$

where BS is the Black-Scholes differential operator given by

$$BS = \frac{\partial}{\partial \tau} + rs\frac{\partial}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} - r. \tag{6}$$

What is $BS(P)$? If we had the European option price p in place of P, it would identically vanish proving our claim that Q is then τ-independent. For the American put option, it is still zero when s is above the critical value $\overline{S}(\tau)$. But, when s is below $\overline{S}(\tau)$, its value is $-rX$

time as t, expiration time as T, interest rate as r, volatility as σ and dividend yield as q. It is assumed that r, σ and q are constants, but the approach is generalizable to cover their time variations. American put and call option prices are denoted as P and C while those of the European options as p and c. All options are assumed to mature at time T. At any time $t < T$, the American put price $P(S,t)$ is a decreasing function of S for S above the critical price $\overline{S}(t)$ satisfying the Black-Scholes differential equation and takes its intrinsic value $X-S$ for $S \leq \overline{S}(t)$. P is continuous at $\overline{S}(t)$ and it has been argued that the slope $\partial P/\partial S$ should also be continuous at $\overline{S}(t)$. Similar results hold for the American call price on a stock paying dividend yields. Here $C(S,t)$ is an increasing function of S for $S < \overline{S}(t)$ satisfying the Black-Scholes differential equation and takes its intrinsic value $S-X$ for $S \geq \overline{S}(t)$. Here too, C and $\partial C/\partial S$ are continuous at $\overline{S}(t)$. The critical price $\overline{S}(t)$ increases for a put option and decreases for a call option to reach a known value as we approach maturity, with some exceptions when discrete dividends are involved. The value known is the exercise price X, or rX/q if that is in the money. $\overline{S}(t)$ tends to a finite value asymptotically back in time. This value is also known and is calculable from the Black-Scholes differential equation. Details on options can be found in Hull (1993) and Wilmott, Dewynne and Howison (1993a).

I. Integral Representation

In this section, an integral representation is derived for the American put price on a non-dividend paying stock. To start with, a function $Q(\tau)$ is constructed in such a way that it is independent of τ for European options. The specifics of the American options are hence embodied in its τ dependence. As τ runs from t to T, Q evolves from the American put price to the European put price. Studying its τ derivative, we obtain an integral representation for the difference between the American and European put prices. This representation is then used to derive a constraint on the critical price. Though this constraint is also in integral form, it lets us obtain the approximation mentioned earlier and argue why it is expected to be a good one to work with.

The function $Q(\tau)$ is defined as follows:

$$Q(\tau) = e^{-r(\tau-t)} \int_0^\infty \frac{ds}{s} \; P(s,\tau)\phi\left(\frac{s}{S}, \tau - t\right) \tag{1}$$

where τ is some time in the future before maturity, $t < \tau < T$. $P(s,\tau)$ is the option price at τ for an American put on a stock that is for the time being assumed to pay no dividends and

Rubinstein (1985) and Brennan and Schwartz (1977a). Geske and Shastri (1985a) have done a comparison of the various option valuation techniques. Besides providing us with a means to compare numerical results, analytical methods give us a better understanding of American option pricing and can potentially lead to more efficient and accurate approximations. In this article, an exact integral representation is derived for the American option price. It too is not solvable, but it leads to a very efficient approximation scheme. The approximation turns out to be more than satisfactory. It is even possible to compute corrections to this approximation if more accuracy is needed. In other words, this is more than an approximation scheme; it is an efficient and accurate alternative to other numerical methods.

Our results are first discussed in detail for the American put option on a non-dividend paying stock and later extended to cover puts and calls on stocks paying dividends. An integral representation is derived for the difference between the American and European option prices. It is a single integral over the remaining life of the option. From this representation, a constraint on the critical stock price, also known as the optimal exercise boundary, that determines exercise prior to maturity is deduced. An approximation scheme is then developed to obtain the critical prices. The critical prices are used to compute the integral and obtain the option price. The results are very satisfactory and comparable to those available from other methods. Corrections to these approximate results are also computed. The approximation is tested from various angles to make sure that it behaves as expected. It is observed that there should be a moment during the life of the option when the estimate for the critical price becomes exact, in addition to asymptotic past and maturity. This lets us believe that it should be a good approximation to work with. Numerical results are presented to support these claims.

The integral representation for the price of the American put option on a non-dividend paying stock is derived in section I. It is given by Equation (10). The constraint on the critical price is given by Equation (11). Analytic approximation is developed in section II with Equation (15) replacing the constraint. The approximation is used to obtain numerical results that are compared with those computed by other methods in Tables I and II. Integral representations for puts and calls on stocks paying dividend yields (Equations (17) and (18)) are presented in section III. Analytic approximation (Equation (19)) is developed to handle them. Section IV attempts to incorporate cash dividends and compares our results with those of other methods in Table III. Finally, section V concludes with some remarks.

Notations are as follows. The stock price is denoted as S, the exercise price as X, current

February 1996

Analytic Representations and Approximations to American Option Pricing

B. S. BALAKRISHNA*

ABSTRACT

An exact integral representation is derived for the American option price. It is not easily solvable, but it leads to an efficient approximation scheme. The results obtained are very satisfactory and comparable to those available from other methods. In this method, critical stock prices can be computed with simple iterative techniques. The critical prices can then be used to compute the integral to obtain the option price. It is possible to compute corrections to this approximation if more accuracy is needed. The method is applied to puts and calls on stocks paying dividends.

Black and Scholes (1973) derived analytic expressions for European option prices in a model where the stock price follows geometric Brownian motion. Merton (1973a) showed that American options are difficult to value because of the possibility of early exercise. There have been attempts since then to obtain solutions to American options in the model of Black and Scholes. Though such attempts have not been fully successful, they have helped develop approximation techniques that are useful as alternatives to numerical computations. Geske and Johnson (1984) have succeeded in valuing the American options analytically, but are still far from reaching a satisfactory solution. Analytic approximation methods have been developed by Johnson (1983), MacMillan (1986), Barone-Adesi and Whaley (1987) and others. Numerical methods have been developed among others by Parkinson (1977), Cox and

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*Department of Physics, CB 390, University of Colorado, Boulder Colorado 80309.
Email: bala@szikra.colorado.edu. Programs written in C can be obtained by contacting the author. I thank an anonymous referee for pointing out to me that the integral representation presented here was derived earlier differently by Kim (1990) .