The CAPM-Extended Divisia Monetary Aggregate

with Exact Tracking under Risk

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> "The economic statistics that the government issues every week should come with a warning sticker: User beware. In the midst of the greatest information explosion in history, the government is pumping out a stream of statistics that are nothing but myths and misinformation."
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> Michael J. Mandel, "The Real Truth about the Economy: Are Government Statistics so much Pulp Fiction? Take a Look," *Business Week*, cover story, November 7, 1994, pp. 110-118.

1. Introduction

In the case of perfect certainty, the Divisia index exactly tracks any aggregator function. This follows from the fact that the Divisia line integral is directly derivable from the first order conditions for optimizing behavior. This result is especially well known in the case of consumer behavior, in which the Divisia index is derived directly from the total differential of the demand function, after substitution of the first order conditions for maximizing utility subject to a budget constraint. However, the exact tracking property of the Divisia index also applies to the demand for monetary services by firms and the supply of produced monetary services by financial intermediaries. See Barnett (1987).

Risk aversion is another story. The first order conditions in the case of risk aversion are Euler equations. Since those are not the first order conditions used in deriving the Divisia index under perfect certainty, the tracking ability of the unadjusted Divisia index is compromised. The degree to which the tracking ability degrades is a function of the degree of risk aversion and the amount of risk. In principle this problem could be solved by estimating the Euler equations by generalized method of moments and producing the estimated exact rational expectations monetary aggregator function. This inference procedure is in accordance with the one widely advocated as the solution to the Lucas Critique and more recently also advocated as the solution to what Chrystal and MacDonald (1994, p. 76) have called the Barnett Critique. However estimation of aggregator functions, while in strict accordance with the principles of microeconomic aggregation theory, produces results that depend upon the parametric specification of the aggregator function and the choice of econometric estimator for estimating the parameters of the aggregator.

Index number thoery exists precisely for the purpose of permitting specification-free, nonparametric tracking ability. The Divisia index is such a parameter free index number and hence depends only upon data. While the Divisia index number is known to permit exact tracking for any economic aggregator function under perfect certainty (see Hulten (1973)), that index has never been extended to a statistical index number that will track exactly under risk aversion. In fact to our knowledge, no nonparametric, statistical index numbers have ever previously been derived directly from Euler equations in a manner that retains tracking ability under risk. In this paper, we derive a statistical index number directly from the Euler equations. The resulting index number turns out to be an extension of the original Divisia index derived by Francois Divisia (1925) under perfect certainty, such that our extended Divisia index remains exact under risk aversion and reduces to the usual Divisia index in the special case of perfect certainty. The derivation is analogous to that for the usual Divisia index, but our extended Divisia index is derived from the Euler equations that are the correct first order conditions produced from rational behavior of economic agents under risk.

If additional assumptions are imposed, we find that the resulting generalized Divisia index has a direct connection with the capital asset pricing model (CAPM) in finance. In a sense our theory is a generalization of the CAPM and of economic index number theory, since our theory contains both as nested special cases. In particular, CAPM deals with a two dimensional tradeoff between mean return and risk, while the Divisia index deals with the two dimensional tradeoff between investment return and liquidity. Our generalized theory includes the three dimensional tradeoff between mean return, risk, and liquidity.

A particularly productive area of possible application of this new index number is monetary aggregation. When central banks first pruduces monetary aggregates, the components over which they aggregated yielded no interest. Hence there was perfect certainty about the rate of return on each component. In addition, since that rate of return was exactly zero for each component, the user costs were know to be the same for each component. Under those circumstances, it is well known in aggregation theory that the correct method of aggregation is simple summation. But monetary assets no longer yield the same rates of return and cannot be viewed as perfect substitutes. In addition, the interest yield is not a monetary service, so that the interest yield's capitalized value, while embedded in the value of the stock of such assets, is not part of the economy's economic monetary stock. The capitalized value of the monetary service flow, net of that interest yield, is the economy's economic monetary stock. Furthermore, since interest is not paid in advance, there is some degree of uncertaintly about that rate of return, which is needed to compute the foregone interest (user cost) of any interest-yielding monetary asset. These observations indicate that the ability to track a nonlinear aggregator function under risk is needed to be able to measure the economy's monetary service flow.

In the case of the current monetary aggregates, the component assets yield rates of return having low variance and low correlation with consumption. As a result, the ordinary Divisia index produced from perfect certaintly first order conditions may be adequate to track the service flows of those collections of assets. But there is growing research interest in the possibility of incorporating into monetary aggregates assets that have substantial risk, such as common stock and bond funds. See, e.g., Barnett and Zhou (1994) and Feldstein and Stock (1994). With such potential component assets, the perfect certainty first order conditions are not suitable and hence the ordinary Divisia monetary aggregates may not track well.

1.1 The Barnett Critique

According to the Barnett Critique, as defined by Chrystal and MacDonald (1994, p. 76), an internal inconsistency exists between the microeconomics used to model private sector structure and the aggregator functions used to produce the monetary aggregate data supplied by central banks. The result can do considerable damage to inferences about private sector behavior, when central bank monetary aggregate data are used. Chrystal and MacDonald (1994, p. 76) have observed the following regarding "the problems with tests of money in the economy in recent years....Rather than a problem associated with the Lucas Critique, it could instead be a problem stemming from the 'Barnett Critique.'" In fact the Barnett Critique issues have been used to cast doubt upon many widely held views in monetary economics, as recently emphasized by Barnett, Fisher, and Serletis (1992), Belongia (1993), and Chrystal and MacDonald (1994). Based upon this rapidly growing line of research, Chrystal and MacDonald (1994, p. 108) conclude---in my opinion correctly---that: "Rejections of the role of money based upon flawed money measures are themselves easy to reject."

The Poterba and Rotemberg (1987) approach to inference about consumer behavior in the monetary sector circumvents the Barnett Critique by nesting the monetary aggregator function within the consumer's utility function and estimating the aggregator function jointly with the other parameters of the consumer's decision.

Hence Poterba and Rotemberg have extended Barnett's (1980,1987) perfect certainty theory to the case of risk. [1] Subsequently, Barnett, Hinich, and Yue (1991) have investigated the tracking abilities of various nonparametric statistical index numbers, such as the Divisia, to the Poterba and Rotemberg estimated aggregator function under risk. Under risk aversion, some small compromise in tracking ability was found for the ordinary Divisia index. The extended Divisia monetary aggregate derived in this paper produces no Lucas critique problems but in addition solves the Barnett critique problem, since the newly derived extended Divisia index exactly tracks the monetary aggregator function within the Euler equations, regardless of risk aversion.

1.2. Money in the Utility Function

A large and growing literature seeks to explain why rate-of-return-dominated money exists (i.e., has positive value) in equilibrium. This issue is important and merits much research. Nevertheless, it is well known in general equilibrium theory that if money has positive value in equilibrium, then a derived utility function containing money exists such that behavior can be described by maximizing that derived utility subject to a budget constraint. See, e.g., Quirk and Saposnik (1968), Arrow and Hahn (1971), Feenstra (1986), Phlips and Spinnewyn (1982), and Samuelson (1948). The same result is available for firms. See, e.g., Fischer (1974). In fact the resulting derived utility or production function has indeed been derived for many of the explicit mechanisms producing positive value for money in equilibrium.

The converse of the theorem is especially challenging in its implications. If a model containing an explicit motive for holding money cannot generate a derived utility function from its original utility function and constraints, then money cannot have positive value in equilibrium. Only two possible conclusions can then be reached. Either the model fails to produce a positive equilibrium quantity of money, and money is driven out of existence in equilibrium as a result of its rate dominance, or the asset being modeled as "money" is in fact not money and need not exist in equilibrium.

Empirically that well known theorem and its converse imply that behavior under explicit motives for holding money induces rational (transitive, consistent) behavior within the space of goods and monetary assets, and indeed many tests of the axioms of revealed preference have accepted those axioms in that space. See, e.g.,

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[1] See Rotemberg, Driscoll, and Poterba (1995) for a related application. The risk that matters is contemporaneous risk relative to current period interest rates that are not paid until the end of the current period. As has been proven by Barnett (1995), risk regarding future period interest rates has no effect on the contemporaneous Divisia quantity index.

Swofford and Whitney (1987). Knowledge of the induced preference preordering over that Cartesian product space is not sufficient to produce a unique inverse mapping back to the original decision. Hence, the properties of the derived utility function do not uniquely reveal the explicit motive for holding money. The nonuniqueness of the inverse mapping is the constructive reason for interest in models containing an explicit motive for holdng money. But for some purposes we do not need to know that motive.[2]

Macroeconomists, monetary economists, and central bankers have many reasons for wanting an explanation of the fact that positive quantities of money are held in equililbrium, despite the fact that many monetary assets are rate dominated as investments. However, in aggregation theory we have no need for such an explanation. Monetary assets are indeed held in positive quantities, and not all monetary assets yield the same investment rate of return. Within the derived utility function over the Cartesian product space of goods and monetary assets, we know that monetary assets having different own rates of return are most certainly not perfect substitutes. That fact alone is sufficient to permit us to deduce that simple sum aggregation over currency and any interest yielding monetary asset makes no sense whatsoever, and any empirical research using such a simple sum aggregate over such assets is contaminated by that data. We need no further nformation about the explicit motive for holding money to deduce that result in microeconomic aggregation theory.

In this paper, we display the derivation of the CAPM-extended Divisia monetary index based upon the derived utility function containing money. However, it should be observed that exactly the same result would be produced from any explicit motive for holding money, such as a model having transactions technology constraint, or from a production model.[3] Even the case of perfect substitution between components is a nested special case of the index derived below. However, it should be observed that if the components are perfect substitutes but yield different rates of return, there will be a corner solution at the highest rate of return, with all other components

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[2] Manufacturers of roller skates may need to know why people buy roller skates, despite the fact that roller skates are dominated by subway trains as means of transportation. Hence manufacturers of roller skates may wish to know about the basic-wants (elementary consumption characteristics) and preferences over those characteristics applicable to roller skate and subway train service consumption. However, we know that since roller skates have positive value in equilibrium, a preference preordering exists over the space of subway trains and roller skates, and we know that the two goods are most certainly not perfect substitutes. That fact alone is sufficient to permit us to deduce that simple sum aggregation over subway trains and roller skates makes no sense whatsoever, and any empirical research using such a simple sum aggregate over those two goods is contaminated by that data. We need no information about the preference preordering over the space of elementary consumption characteristics to deduce that result in microeconomic aggregation theory.

[3] See Barnett (1987) for the result that production or consumption modeling produce the same monetary aggregation index number formula.

going out of existence. Hence for the results in this paper to be nontrivial, it should be assumed that the component monetary assets are not perfect substitutes, so that a nonlinear aggregator function is nested within the structure of the economy. The use of a utility function as that structure is arbitrary and for expository convenience. However, the existence of a nonlinear aggregator function nested within the structure is necessary for our results to be useful. Hence it is the existence of the aggregator function, and not the nature of the structure within which the aggregator is nested, that is central to our results.

2. Consumer Demand for Monetary Assets

2.1 The Decision

In this section we formulate a representative consumer's stochastic decision problem over consumer goods and monetary assets. The consumer's decisions are made in discrete time over an infinite planning horizon for the time intervals, t, t+1, ..., s, ..., where t is the current time period. The variables used in defining the consumer's decision are as follows: $\mathbf{x}_s$ = n dimensional vector of real consumption of goods and services during period s, $\mathbf{p}_s$ = n dimensional vector of goods and services prices and of durable goods rental prices during period s, $\mathbf{a}_s$ = k dimensional vector of real balances of monetary assets during period s, $\boldsymbol{\rho}_s$ = k dimensional vector of nominal holding period yields of monetary assets, A_s = holdings of the benchmark asset during period s, R_s = the one-period holding yield on the benchmark asset during period s, I_s= the sum of all other sources of income during period s, and $p^*_s = p^*_s(\mathbf{p}_s)$ = the true cost of living index.

Define Y to be the consumer's survival set, assumed to be a compact subset of the n+k+2 dimensional nonnegative orthant. The consumer's consumption possibility set, S(s) for period s is S(s) = { ($\mathbf{a}_s$, $\mathbf{x}_{s,}$, A_s) ∈ Y:

$$\sum_{i=1}^{n} p_{is}x_{is} = \sum_{i=1}^{k} [(1+\rho_{i,s-1})\, p^*_{s-1}\, a_{i,s-1} - p^*_s\, a_{is}] + (1+R_{s-1})p^*_{s-1}\, A_{s-1} - p^*_s\, A_s + I_s\}.$$

The benchmark asset A_s provides no services other than its yield R_s. As a result, the benchmark asset does not enter the consumer's contemporaneous utility function. The asset is held only as a means of accumulating wealth. The consumer's subjective rate of time preference, ξ , is assumed to be constant.[4] The single period utility function, $u(\mathbf{a}_t, \mathbf{x}_t)$, is assumed to be increasing and strictly quasiconcave.

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[4] Although money may not exist in the elementary utility function, there exists a derived utility function that contains money, so long as money has positive value in equilibrium. See, e.g., Arrow and Hahn (1971), Phlips and Spinnewyn (1982), Quirk and Saposnik (1968), Samuelson (1984), and Feenstra (1986). We implicitly are using that derived utility function.

The consumer's decision problem is the following.[5]

Problem 1: Choose the deterministic point $(\mathbf{a}_t,\mathbf{x}_t,A_t)$ and the stochastic process $(\mathbf{a}_s,\mathbf{x}_s,A_s)$, $s = t+1, ..., \infty$, to maximize

$$u(\mathbf{a}_t, \mathbf{x}_t) + E_t \left[\sum_{s=t+1}^{\infty} \left(\frac{1}{1+\xi}\right)^{s-t} u(\mathbf{a}_s, \mathbf{x}_s)\right] \tag{2.1}$$

subject to $(\mathbf{a}_s, \mathbf{x}_s, A_s) \in S(s)$ for s ³ t, and also subject to the transversality condition

$$\lim_{s\to\infty} E_t \left(\frac{1}{1+\xi}\right)^{s-t} A_s = 0.$$

The transversality condition rules out perpetual borrowing at the benchmark rate, R_t.

2.2 Existence of a Monetary Aggregate for the Consumer

In order to assure the existence of a monetary aggregate for the consumer, we partition the vector of monetary asset quantities, $\mathbf{a}_s$, such that $\mathbf{a}_s = (\mathbf{m}_s, \mathbf{h}_s)$. We correspondingly partition the vector of interest rates of those assets, ρ_s, such that $\rho_s=(\mathbf{r}_s,\mathbf{i}_s)$. We then assume that the utility function, u, is blockwise weakly separable in $\mathbf{m}_s$ and in $\mathbf{x}_s$ for some such partition of $\mathbf{a}_s$ and blockwise strongly separable in $\mathbf{h}_s$.[6] Hence there exist a monetary aggregator ("category utility") function, M, a consumer goods aggregator function, X, and utility functions, F and H, such that

$$u(\mathbf{m}_s,\mathbf{h}_s, \mathbf{x}_s) = F(M(\mathbf{m}_s), X(\mathbf{x}_s)) + H(\mathbf{h}_s), \tag{2.2}$$

and we define the implied utility function $V(\mathbf{m}_s, c_s)$ by $V(\mathbf{m}_s, c_s)=F(M(\mathbf{m}_s), c_s)$, where aggregate consumptions of goods is defined by $c_s=X(\mathbf{x}_s)$. Then it follows that the exact monetary aggregate is

$$M_s = M(\mathbf{m}_s). \tag{2.3}$$

We define the dimension of $\mathbf{m}_s$ to be k_1, and the dimension of $\mathbf{h}_s$ to be k_2, so that $k=k_1+k_2$. The fact that blockwise weak separability is a necessary condition for exact aggregation is well known in the perfect certainty

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[5] We do not consider aggregation over economic agents in this paper. But some results that may be relevant can be found in Blackorby and Schworm (1984).

[6] The strong separability assumption is largely for expository convenience. Weak separability would be sufficient. It is also possible that some of our results could be aquired from duality theory, using results such as those in Blackorby and Russell (1994) and Blackorby, Davidson, and Schworm (1991). However, the applicability of duality theory under risk is not currently highly developed, and we have based all of our results on functional structure imposed directly on the primal decision.

case.[7] In fact, if the resulting aggregator function also is linearly homogeneous, the theory of two stage budgeting can be used to prove that the consumer behaves as if the exact aggregate were an elementary good. Since two stage budgeting theory is not applicable under risk, we provide in the appendix an aggregation theorem proving that $M(\mathbf{m}_s)$ can be treated as a quantity aggregate, in a well defined sense, even under risk.

The Euler equations which will be of the most use to us below are those for monetary assets. Those Euler equations are:

$$E_s\left[\frac{\partial V}{\partial m_{is}} - \rho\frac{p_s^*(R_s - r_{is})}{p_{s+1}^*}\frac{\partial V}{\partial c_{s+1}}\right] = 0 \qquad (2.4a)$$

for s^3t and $i = 1, \dots, k_1$, where $\rho = \frac{1}{1+\xi}$ and where p_s^* is the exact price aggregate that is dual to the consumer goods quantity aggregate c_s.[8] Similarly we can acquire the Euler equation for the consumer goods aggregate c_s, rather than for each of its components. The resulting Euler equation for c_s is

$$E_s\left[\frac{\partial V}{\partial c_s} - \rho\frac{p_s^*(1+R_s)}{p_{s+1}^*}\frac{\partial V}{\partial c_{s+1}}\right] = 0 \qquad (2.4b)$$

3. The Perfect Certainty Case

In the perfect certainty case, nonparametric index number theory is highly developed and is applicable to monetary aggregation. In the perfect certainty case, Barnett (1978,1980) proved that the contemporaneous real user cost of the services of m_{it} is $^1{}_{it}$, where

$$\pi_{it} = \frac{R_t - r_{it}}{1 + R_t} \qquad (3.1)$$

The corresponding nominal user cost is $p_t^*\pi_{it}$. It can be shown that the solution value of the exact monetary aggregate $M(\mathbf{m}_t)$ can be tracked without error in continuous time (see, e.g., Barnett (1980)) by the Divisia index:

$$d\log M_t = \sum_{i=1}^{k_1} s_{it}\ d\log m_{it}, \qquad (3.2)$$

where the user cost evaluated expenditure shares are $s_{it} = \pi_{it}m_{it} / \Sigma_{j=1}^{k_1}\pi_{jt}\, m_{jt}$. The flawless tracking ability of the index in the perfect certainty case holds regardless of the form of the unknown aggregator function, M. However, under risk the ability of equation (3.2) to track $M(\mathbf{m}_t)$ is compromised.

4. The New Generalized Divisia Index

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[7] Regarding the highly developed theory of aggregation over goods under perfect certainty, see, e.g., Fisher and Shell (1972) and Blackorby, Primont, and Russell (1978)..

[8] Assuming that X is linearly homogeneous, the exact price aggregator function is the unit cost function.

4.1 The User Cost of Money Under Risk Aversion

We now return to the Euler equations for optimal behavior of consumers under risk. Those Euler equations are displayed in equation (2.4a) for monetary assets and equation (2.4b) for consumer goods. Our objective is to find the formula for the user cost of monetary services in a form that is applicable to our model of decision under risk. The following definition for the contemporaneous user cost simply states that the real user cost price of a monetary asset is the marginal rate of substitution between those asset and consumer goods.[9]

Definition 1: The contemporaneous risk adjusted real user cost price of the services of monetary asset i is Π_{it}, defined such that $\Pi_{it} = \frac{\partial V}{\partial m_{it}} \Big/ \frac{\partial V}{\partial c_t}$.

No expectations opertors appear in that definition, since the marginal utilities at t are known with certainty in period t. Nevertheless, formula (3.1), which applies under perfect certainty, cannot be correct under risk, since the interest rates in equation (3.1) are not known contemporaneously, so the right hand side of equation (3.1) is stochastic, while Definition 1 defines Π_{it} to be deterministic. In this section we derive the correct deterministic formula for the user cost defined by Definition 1.[10]

For notational convenience, we sometimes convert the nominal rates of return, r_{it} and R_t, to real total rates of return, $1 + r_{it}^*$ and $1 + R_t^*$, such that

$$1 + r_{it}^* = \frac{p_t^*(1 + r_{it})}{p_{t+1}^*} \text{ and } 1 + R_t^* = \frac{p_t^*(1 + R_t)}{p_{t+1}^*}, \tag{4.1}$$

where r_{it}^* and R_t^* defined in that manner are called the real rates of excess return. Under this change of variables and observing that current period marginal utilities are known with certainty, Euler equations (2.4a) and (2.4b) become:

$$\frac{\partial V}{\partial m_{it}} - \rho E_t\left[\left(R_t^* - r_{it}^*\right)\frac{\partial V}{\partial c_{t+1}}\right] = 0 \tag{4.2}$$

and

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[9] The nominal user cost is the real user cost multiplied by the price aggregate for consumer goods. Hence Definition 1 could be restated to be that the ratio of the nominal user cost of a monetary asset to the price aggregate for consumer goods equals the marginal rate of substitution between that monetary asset and consumer goods.

[10] Note that equation (4.1) remains correct for forward period s, so long as any expectations used in the definition of the user cost are defined relative to information available in that same period s, as is the case in the Euler equations (2.4a) and (2.4b). However, we shall produce our derivation for contemporaneous period t.

$$\frac{\partial V}{\partial c_t} - \rho E_t\left[\left(1+R_t^*\right)\frac{\partial V}{\partial c_{t+1}}\right] = 0 \tag{4.3}$$

We now can prove our user cost theorem under risk.

Theorem 1: The risk adjusted user cost of the services of monetary asset i under risk is $\Pi_{it} = \pi_{it} + \psi_{it}$, where

$$\pi_{it} = \frac{E_t R_t - E_t r_{it}}{1 + E_t R_t} \tag{4.4}$$

and

$$\psi_{it} = \rho\left(1-\pi_{it}\right)\frac{Cov(R_t^*, \frac{\partial V}{\partial c_{t+1}})}{\frac{\partial V}{\partial c_t}} - \rho\frac{Cov(r_{it}^*, \frac{\partial V}{\partial c_{t+1}})}{\frac{\partial V}{\partial c_t}}\,. \tag{4.5}$$

Proof: Equation (4.2) can be rewritten for current period t to be

$$\frac{\partial V}{\partial m_{it}} = \rho E_t\left[\left(R_t^* - r_{it}^*\right)\frac{\partial V}{\partial c_{t+1}}\right]. \tag{4.6}$$

If the marginal utility and the interest rates in the expectation on the right hand side of (4.6) were uncorrelated, we could write the expectation of the product as the product of the expectations. But under our assumption of weak separability in monetary assets, $\mathbf{m}_t$, the utility function V can be written in the form $V(\mathbf{m}_t, c_t) = F(M(\mathbf{m}_t), c_t)$, where the consumer is risk neutral if and only if F is linear in $M_t = M(\mathbf{m}_t)$ and in c_t. Hence under risk neutrality, V must be linear in c_t, so that the marginal utility of consumption must be a constant. But without risk neutrality and the resulting constancy of the marginal utility of consumption, we have no reason to expect the interest rates and marginal utility on the right hand side of (5.3) to be uncorrelated. The result is that (4.6) becomes

$$\frac{\partial V}{\partial m_{it}} = \rho E_t\left[\frac{\partial V}{\partial c_{t+1}}\right](E_t R_t^* - E_t r_{it}^*) + \rho Cov(R_t^*, \frac{\partial V}{\partial c_{t+1}}) - \rho Cov(r_{it}^*, \frac{\partial V}{\partial c_{t+1}}) \tag{4.7}$$

where the covariances would become zero, if we were to assume risk neutrality. Similarly, without risk neutrality, equation (4.3) becomes

$$\frac{\partial V}{\partial c_t} = \rho E_t\left[\frac{\partial V}{\partial c_{t+1}}\right] + \rho E_t\left[R_t^*\right]E_t\left[\frac{\partial V}{\partial c_{t+1}}\right] + \rho Cov(R_t^*, \frac{\partial V}{\partial c_{t+1}})\,. \tag{4.8}$$

By eliminating $\rho E_t\left[\frac{\partial V}{\partial c_{t+1}}\right]$ between equations (4.7) and (4.8), we get

$$\frac{\partial V}{\partial m_{it}} = (\pi_{it} + \psi_{it})\frac{\partial V}{\partial c_t}, \tag{4.9}$$

where

$$\pi_{it} = \frac{E_t R_t^* - E_t r_{it}^*}{1 + E_t R_t^*} \tag{4.10}$$

and

$$\psi_{it} = \rho(1-\pi_{it})\frac{\mathrm{Cov}(R_t^*, \frac{\partial V}{\partial c_{t+1}})}{\frac{\partial V}{\partial c_t}} - \rho\frac{\mathrm{Cov}(r_{it}^*, \frac{\partial V}{\partial c_{t+1}})}{\frac{\partial V}{\partial c_t}}. \tag{4.11}$$

Using equation (4.1) to convert the real rates in equation (4.10) back to nominal rates, equation (4.10) becomes (4.4), while equation (4.11) is immediately identical to equation (4.5). Solving equation (4.9) for $\Pi_{it} = \pi_{it} + \psi_{it}$, Theorem 1 follows from Definition 1. **Q.E.D.**

Under risk neutrality, the covariances in (4.11) would all be zero, since the utility function would be linear in consumption. Hence the user cost would reduce to π_{it}, as defined in equation (4.4). The following corollary is immediate.

Corollary 1 to Theorem 1: Under risk neutrality, the user cost formula is the same as equation (3.1) in the perfect certainty case, but with all interest rates replaced by their expectations.

However, under risk aversion the utility function is strictly concave in consumption, so that marginal utility is inversely related to consumption. In principle, it is possible for the interest rate on a slightly risky investment to reduce the risk in the consumer's consumption stream if that interest rate and consumption are negatively correlated. Because of the inverse relationship between consumption and marginal utility, we conclude that risk is decreased by an investment if the rate of return is positively correlated with marginal utility. For monetary assets, with little or no principle risk and low volatility, the riskiness of the asset is likely to contribute relatively little to the riskiness of the household's consumption stream, and hence the sign of the covariance between the asset's rate of return and of the consumption stream is not easy to predict a priori. But with a very risky asset, such as common stock, it is far more likely that holding such a risky investment will increase risk, rather than decrease it. That occurs if the rate of return on the asset is positively correlated with consumption and thereby negatively correlated with marginal utility. This phenomenon is central to the consumption based capital asset pricing model (CCAPM).

Consider the interpretation of equation (4.5), which defines the adjustment for risk under risk aversion. Suppose we normalize relative to $\frac{\partial V}{\partial c_t}$, so that we need not consider the denominator of equation (4.5). Now consider first the second term on the right hand side of equation (4.5). Suppose that the own rate of return on monetary asset i is positively correlated with the marginal utility of consumption of goods, so that holding that monetary asset decreases risk. Since holding the asset decreases the consumer's consumption risk, we should expect that the risk-adjusted user cost price $\Pi_{it}=\pi_{it} + \psi_{it}$ that the consumer would have to "pay" to hold that asset would be decreased as that positive covariance increases, and that is precisely what the second term of equation (4.5) would do in that case. Conversely, if the covariance between the own rate and the marginal utility of consumption of goods is negative, so that holding the asset increases the risk of the consumer's consumption stream, the second term in equation (4.5) introduces a positive term into the risk-adjusted user cost $\Pi_{it}=\pi_{it} + \psi_{it}$ to reflect the increased cost of holding the asset as that covariance increases the consumer's risk. If the central bank were to introduce common stock or bond funds into monetary aggregates or other assets having substantial principal risk, we should expect to find the latter case would apply to those assets.

Now consider the first term on the right hand side of equation (4.5). The benchmark rate is the interest rate foregone by not holding the benchmark asset. If the benchmark rate decreases consumption risk through a positive covariance between the benchmark rate and the marginal utility of consumption of goods, then the opportunity cost of foregoing the benchmark asset by holding monetary asset i instead, is increased. Hence we should expect that such a positive covariance should increase the risk adjusted user cost Π_{it}, as indeed is the effect of the first term of equation (4.5). Conversely if that covariance is negative, so that holding the benchmark asset increases the consumer's risk, then foregoing the benchmark asset in favor of monetary asset i decreases risk and hence results in a subtraction from the risk adjusted user cost, Π_{it}, of holding asset i.[11]

4.2 The Generalized Divisia Index Under Risk Aversion

The ordinary Divisia index was derived by Francois Divisia from the first order conditons for rational consumer behavior under perfect certainty. In the case of risk aversion, the first order conditions are Euler equations, and we have found that those Euler equations for monetary assets demanded by consumers can be put into the form (4.2), which we now use to derive a generalized Divisia index, as follows.[12]

Theorem 2: In the share equations $s_{it} = \pi_{it}m_{it}/\Sigma_{j=1}^{k1}\pi_{jt}\, m_{jt}$, replace the unadjusted user costs ${}^{1}{}_{it}$, defined by (3.1), by the risk adjusted user costs Π_{it}, defined by Definition 1, to produce the adjusted shares $S_{it} = \Pi_{it}m_{it} \,/ {}_{j=1}^{k1}\Pi_{jt}\, m_{jt}$. Under our weak separability assumption, $V(\mathbf{m}_t,c_t) = F(M(\mathbf{m}_t),c_t)$, and our assumption that the monetary aggregator function M is linearly homogeneous, the following generalized Divisia index is true under risk:

$$d\log M_t = \Sigma\, S_{it} d\log m_{it} \tag{4.12}$$

Proof: Under our weak separability assumption, $V(\mathbf{m}_t,c_t) = F(M(\mathbf{m}_t),c_t)$, we have that

$$\frac{\partial V}{\partial m_{it}} = \frac{\partial F}{\partial M_t}\frac{\partial M_t}{\partial m_{it}}\ . \tag{4.13}$$

Substituting (4.10) into (4.13), we acquire

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[11] Note that the magnitude of the adjustment from the first term of (4.5) depends upon the size of the unadjusted user cost π_{it} of monetary asset i . The unadjusted user cost defined by (4.4) would equal 1.0 in the limit as the benchmark rate increases towards infinity, while the own rates on the component assets are held down at low levels by regulation. In that case the first term of equation (4.5) would produce no risk adjustment, although the second term still would. This special case is far from likely and should not be expected to be encountered with actual data. In another special case, the unadjusted user cost would equal zero, if the expected rate of return on monetary asset i equals that on the benchmark asset. In that case, the unadjusted user cost π_{it} would drop out of the adjustment term ψ_{it} defined by equation (4.5).

[12] Our generalized Divisia index should not be confused with Stahl's (1983) generalized Divisia index. Stahl's index under perfect certainty provides a parametric extension to the case of affine homothetic aggregator function, in contrast with the Caves, Christensen, and Diewert (1982a,b) nonparametric approximation to the Malmquist index aggregator, when the utility function is completely nonhomothetic. In our proofs under risk, we limit ourselves to the case of homothetic category utility function as aggregator, but the extension to nonhomotheticity could be produced by either Stahl's method or by Caves, Christensen, and Diewert's method.

$$\frac{\partial M_t}{\partial m_{it}} = (\pi_{it} + \psi_{it}) \frac{\frac{\partial V}{\partial c_t}}{\frac{\partial F}{\partial M_t}} . \tag{4.14}$$

Since the total differential of $M_t = M(\mathbf{m}_t)$ is,

$$dM_t = \sum_i \frac{\partial M}{\partial m_{it}} dm_{it} \tag{4.15}$$

we can substitute (4.14) into (4.15) to get

$$dM_t = \frac{\frac{\partial V}{\partial c_t}}{\frac{\partial F}{\partial M_t}} \sum (\pi_{it} + \psi_{it}) dm_{it} . \tag{4.16}$$

Using the linear homogeneity of M, we have from Euler's theorem for homogeneous functions that

$$M_t = \sum_i \frac{\partial M}{\partial m_{it}} m_{it} . \tag{4.17}$$

Substituting (4.14) into (4.17), we acquire

$$M_t = \frac{\frac{\partial V}{\partial c_t}}{\frac{\partial F}{\partial M_t}} \sum (\pi_{it} + \psi_{it}) m_{it} . \tag{4.18}$$

Dividing (4.16) by (4.18), we get equation (4.12). **Q.E.D.**

Hence we see that the exact tracking of the Divisia monetary index is not compromised by risk aversion, so long as the adjusted user costs $\pi_{it} + \psi_{it}$ are used in computing the index. As we have observed, the adjusted user costs reduce to the usual user costs in the case of perfect certainty and our generalized Divisia index (4.12) reduces to the usual Divsia index (3.2). Similarly the risk neutral case is acquired as the special case with $\psi_{it}=0$, so that equation (4.10) serves as the user cost . In short, our generalized Divisia index (4.12) is a true generalization in the sense that the risk neutral and perfect certainty cases are strictly nested special cases. Formally that conclusion is the following:

Corollary 1 to Theorem 2: Under risk neutrality, the generalized Divisia index (4.12) reduces to (3.2), where the user costs in the formula are defined by (4.10). Under perfect certainty, the user costs reduce to (3.1).

The need for the generalization can be explained as follows. The consumer has a three dimensional decision, in terms of asset characteristics. The monetary assets having nonzero own rates of return produce an investment return, contribute to risk, and provide liquidity services. Our objective is to track the nested utility function, $M(\mathbf{m}_t)$, which measures solely liquidity and is the true economic monetary aggregate. To do so, we must remove the other two motives: investment yield and risk aversion. While those two motives are relevant to savings and intertemporal substitution, we seek to track the liquidity flow alone. The ordinary Divisia monetary aggregate removes the investment motive and would track the liquidity services, if there were no risk. The generalized Divisia index removes both the investment motive and and the aversion-to-risk motive to extract the liquidity service flow, when the data is produced by consumer's who in fact are making decisions that involve a three way tradeoff among mean investment return, risk aversion, and liquidity service consumption.[13]

5. The CCAPM Special Case

As a means of illustrating the nature of the risk adjustment ψ_{it}, we consider a special case, based upon the usual assumptions in CAPM theory of either quadratic utility or Gaussian stochastic processes. Direct empirical use of theorems 1 and 2, without any CAPM simplifications, would require availability of prior econometric estimates of the parameters of the utility function V and of the subjective rate of time discount. Under the usual CAPM assumptions, we show in this section that empirical use of theorems 1 and 2 would require prior estimation of only one property of the utility function: the degree of risk aversion, on which a large body of published information is available.

Consider first the following case of utility that is quadratic in consumption of goods, conditionally on the level of monetary asset service consumption:

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[13] The intent of the Divisia index is to track the category utility function, $M(\mathbf{m}_t)$, which reflects the liquidity services received from holding the components of the aggregate. It is not M, but rather the outer function, F, in the consumer's utility function that reflects the consumer's degree of risk aversion, and it is the integration in the resulting expected utility function that brings in the probability distributions defining the amount of risk in the economic environment of the consumer. But the market opportunity cost evaluated at the margin by the unadjusted user cost π_{it} of asset i reflects both the liquidity premium embedded in the expected rate of return and also the risk premium demanded by risk averse consumers, when an asset's own rate of return is correlated with the consumer's consumption stream. Hence the unadjusted user costs cannot be used to eliminate the marginal utilities in the total differential (4.15). The adjustment ψ_{it} to the opportunity cost in $\pi_{it} + \psi_{it}$ can be viewed as extracting the risk premium at the margin and leaving only the liquidity premium, as is needed to track the marginal utilities in the total differential (4.15) that produces the Divisia index's tracking capability.

Assumption 1: Let V have the form

$$V(\mathbf{m}_t, c_t) = F(M(\mathbf{m}_t), c_t) = A(M(\mathbf{m}_t))c_t - \frac{1}{2}B(M(\mathbf{m}_t))c_t^2, \qquad (5.1)$$

where A is a positive, increasing, concave function and B is a nonnegative, decreasing, convex function.[14]

The alternative assumption is Guassianity, as follows:

Assumption 2: Let (r_{it}^*, c_{t+1}) be a bivariate Gaussian process for each asset i = 1, . . . , k_1.

We also make the following conventional CAPM assumption:[15]

Assumption 3: The benchmark rate process is deterministic or already risk-adjusted, so that R_s^* is a risk free rate for all s³t.

Under this assumption, it follows that $\mathrm{Cov}(R_t^*, \frac{\partial V}{\partial c_{t+1}})$ equals zero.

We define $H_{t+1}=H(M_{t+1},c_{t+1})$ to be the well known Arrow-Pratt measure of absolute risk aversion,

$$H(M_{t+1},c_{t+1}) = \frac{-E_t[V'']}{E_t[V']}, \qquad (5.2)$$

where $V' = \partial V(\mathbf{m}_{t+1},c_{t+1}) / \partial c_{t+1}$ and $V'' = \partial^2 V(\mathbf{m}_{t+1}, c_{t+1}) / \partial c_{t+1}^2$. In this definition, risk aversion is measured relative to consumption risk, conditionally upon the level of monetary services produced by $M_{t+1}=M(\mathbf{m}_t)$. Under risk aversion, H_{t+1} is positive and increases as the degree of absolute risk aversion increases. The following lemma is central to our Theorem 3.

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[14] In CAPM applications, it also is necessary to assume that all observations are to the left of the quadratic maximum.

[15] It amounts to the assumption that the risk premium already has been extracted from the benchmark rate. In practice, this assumption is harmless, since the risk premia adjustments below are applied to all component assets before the benchmark rate is computed---usually as an upper envelope of the component rates. If other asset paths are also included among those used to produce the upper envelope, then this assumption requires that the same risk premia adjustments also be applied to those paths before the upper envelope is generated. Since the risk premia already have been extracted at the time that the envelope is produced, the benchmark rate automatically is risk adjusted.

Lemma 1: Under Assumption 3 and either Assumption 1 or Assumption 2, the user cost risk adjustment, ψ_{it}, defined by equation (4.5) reduces to

$$\psi_{it} = \frac{1}{1+R_t^*} H_{t+1} \mathrm{Cov}(r_{it}^*, c_{t+1}) \tag{5.3}$$

Proof: Assuming that R_s^* is a risk free rate for all s³t, equation (4.3) simplifies to

$$\frac{\partial V}{\partial c_t} = \rho(1+R_t^*) E_t\left[\frac{\partial V}{\partial c_{t+1}}\right], \tag{5.4}$$

and the risk adjustment term (4.5) simplifies to

$$\psi_{it} = -\ \rho \frac{\mathrm{Cov}(r_{it}^*, \frac{\partial V}{\partial c_{t+1}})}{\frac{\partial V}{\partial c_t}}. \tag{5.5}$$

Substituting (5.4) into (5.5), we acquire

$$\psi_{it} = -\frac{1}{1+R_t^*} \frac{\mathrm{Cov}(r_{it}^*, \frac{\partial V}{\partial c_{t+1}})}{E_t\left[\frac{\partial V}{\partial c_{t+1}}\right]}. \tag{5.6}$$

Consider first the case in which we accept Assumption 1. Substituting the quadratic specification (5.1) into (5.6), we get

$$\psi_{it} = \frac{1}{1+R_t^*}\left[\frac{-EV''}{EV'}\right]\mathrm{Cov}(r_{it}^*, c_{t+1}), \tag{5.7}$$

which under our definition of H_{t+1} is identical to (5.3).

Now consider the alternative possibility of accepting Assumption 2 instead of Assumption 1. Applying Stein's lemma for bivariate normal distributions to equation (5.6), we again acquire (5.7) and thereby (5.3).[16]

Q.E.D.

Observe that equation (5.3) provides a CCAPM (consumption CAPM) type result, since the risk adjustment term ψ_{it} is very much like the risk premium on a risky asset in CCAPM. In CCAPM, as in our model,

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[16] For Stein's lemma, see Stein (1973). Alternatively see Ingersoll (1987, p. 13, eq. 62)) or Rubinstein (1976).

compensation for risk is proportional to the covariance of the asset's return with consumption through the factor $\mathrm{Cov}(r_{it}, c_{t+1})$ in (5.3) and also to the degree of risk aversion H_{t+1} in (5.3).[17]

In effect, what the adjustment does for very risky rates is to remove the the risk premium from $E_t r_{it}$ so that the adjusted user cost becomes positive. To see this more clearly, define $Z_t = H_{t+1} c_t$, where Z_t is a modified (time shifted) Arrow-Pratt relative risk aversion measure. Our theorem now follows immediately.

Theorem 3: Under the assumptions of Lemma 1, we have

$$\Pi_{it} = \frac{E_t R_t^* - \left(E_t r_{it}^* - \phi_{it}\right)}{1 + E_t R_t^*}. \tag{5.8}$$

where

$$\phi_{it} = Z_t \mathrm{Cov}\left(r_{it}^*, \frac{c_{t+1}}{c_t} \right) \tag{5.9}$$

Proof: Substitute (5.3) and (4.4) into $\Pi_{it} = \pi_{it} + \psi_{it}$ and substitute $Z_t = H_{t+1} c_t$. **Q.E.D.**

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[17] If own interest rates are positively correlated with consumption, (5.3) is positive, since H_{t+1} would be positive under risk aversion. Alternatively, if the asset's return is not sufficiently risky to dominate the direction of the net shocks to consumption from risk, the opposite could happen. The asset's rate of return could correlate negatively with the consumption stream in a manner tending to decrease the household's consumption risk, and hence (5.3) would be negative. In the CCAPM theory of finance, beta of a very risky asset is usually positive, where beta is defined to be $\beta_{ic} = \mathrm{Cov}(r_{it}^*, c_{t+1}) \big/ \mathrm{Var}(c_{t+1})$. The subscript c in β_{ic} designates "consumption based" beta, and the lack of a time subscript in the notation β_{ic} results from the assumption of stationarity of the interest rate and consumption bivariate process. Clearly the usual finance view of positive β_{ic} can hold if and only if $\mathrm{Cov}(r_{it}, c_{t+1})$ is positive.

This conclusion about the sign of the adjustment term, ψ_{it}, in the adjusted user cost $\pi_{it} + \psi_{it}$ of very risky assets is especially revealing, when the benchmark rate is defined to be riskless, as we have just done. Consider the definition of the unadjusted user cost in equation (4.4). Since we now are assuming that the benchmark rate is defined to be the maximum available rate of return on a risk free asset, we can conclude that the benchmark asset has no embedded liquidity premium and cannot be less than the own rate of return on any risk free monetary asset i. Hence (4.4) is nonnegative if monetary asset i is risk free. But suppose that consumers are risk averse and that monetary asset i is not risk free. Then $E_t r_{it}$ will contain a risk premium, despite the fact that R_t does not (or its risk premium has been removed). Hence the unadjusted user cost (4.4) could be negative. But as we have just observed, ψ_{it} in this case will be positive, and we would expect that in fact it will be sufficiently positive to offset the possible negativity of the unadjusted user cost to produce positive value of the adjusted user cost $\pi_{it} + \psi_{it}$.

As is evident from this theorem, the risk premium adjustment is ϕ_{it}, where c_{t+1}/c_t is a measure of the consumption growth rate. Hence the risk adjustment depends upon relative risk aversion and the covariance between the consumption growth path c_{t+1}/c_t and the real rates of excess return r^*_{it}. Hence we see that the adjusted user cost $\Pi_{it} = \pi_{it} + \psi_{it}$ can be written in the same form as the unadjusted user cost (4.4), if the benchmark rate is defined to be risk free and if the risk premium adjustment ϕ_{it} is subtracted out of the expected value of the real rates of excess return r^*_{it}. As we have observed, that adjustment should be expected to decrease the expected own rate of return, if the asset is very risky and thereby contributes positively to consumption risk.

6. Conclusion

We have, for the first time, extended index number theory to the case of risk. The CAPM theory deals with the tradeoff between risk and return in two dimensional space. Barnett's Divisia monetary aggregates under perfect certainty deal with the tradeoff between liquidity and return in two dimensional space. Each of those two theories can be viewed as modeling a two dimensional section through a three dimensional space, containing risk, return, and liquidity as its three dimensions. In this paper, we produce the three dimensional extension containing the other two theories as special cases.

This paper creates a new field: index number theory under risk. As such, our results connect with finance at its most fundamental level: elementary consumption CAPM theory. This connection opens the way to introduction of the many variations of CAPM theory appearing at the present time in the field of finance, and we strongly believe that those variations should indeed be incorporated into this new field, since the literature on the equity premium puzzle makes clear that the covariances between the consumption and return processes are not large enough to explain some risk premia. In particular, a large and important literature now exists on habit persistent intertemporal nonseparability and on aggregation over heterogeneous economic agents.[18] By using those growing advances in finance, this new literature on index number theory under risk should increase in empirical explanatory power, as the finance literature grows in explanatory power.

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[18] See, e.g., Constantinides (199)) and Daniel and Marshall (1995).

APPENDIX: Aggregation Theorem

It is clear that equation (2.3) does define the exact monetary aggregate in the welfare sense, since M_s measures the consumer's subjective evaluation of the services that he receives from holding $\mathbf{m}_s$. However it also can be shown that equation 2.6 defines the exact monetary aggregate in the aggregation theoretic sense. In particular, the stochastic process M_s , s^3t, contains all of the information about $\mathbf{m}_s$ that is needed by the consumer to solve the rest of his decision problem. This conclusion is based upon the following theorem, which we call the consumer's aggregation theorem.

$$\text{Let } D_s = I_s + \sum_{i=1}^{k_1} [(1+r_{i,s-1}) \overset{*}{p}_{s-1} m_{i,s-1} - \overset{*}{p}_s m_{is}] ,$$

and let

$$D(s) = \{ (\mathbf{h}_s, \mathbf{x}_{s,}, A_s) \in Y: \sum_{i=1}^{n} p_{is} x_{is} =$$

$$+ \sum_{i=1}^{k_2} [(1+\mathbf{i}_{i,s-1}) \overset{*}{p}_{s-1} h_{i,s-1} - \overset{*}{p}_s h_{is}] + (1+R_{s-1})\overset{*}{p}_{s-1} A_{s-1} - \overset{*}{p}_s A_s + D_s\}. \qquad \text{(A.1)}$$

Let $(\overset{*}{\mathbf{a}}_s ,\overset{*}{\mathbf{x}}_s ,\overset{*}{A}_s)$, $s\ ^3\ t$, solve Problem 1 , and assume that the utility function $u(\mathbf{m}_s,\mathbf{h}_s, \mathbf{x}_s)$ is weakly separable in $\mathbf{m}_s$, so that there exists aggregator function M and utility function U such that $U(M(\mathbf{m}_s),\mathbf{h}_s,\mathbf{x}_s)=u(\mathbf{m}_s,\mathbf{h}_s,\mathbf{x}_s)$. Consider the following decision problem, which is conditional upon prior knowledge of the aggregate process $\overset{*}{M}_s$ $=M(\overset{*}{\mathbf{m}}_s)$, although not upon the component processes $\overset{*}{\mathbf{m}}_s$.

<u>**Problem 2**</u>: Choose the deterministic point ($\mathbf{h}_t$, $\mathbf{x}_t$, A_t) and the stochastic process $(\mathbf{h}_s,\mathbf{x}_s,A_s)$, $s=t+1, ..., \infty$, to maximize

$$U(\overset{*}{M}_t , \mathbf{h}_t, \mathbf{x}_t) + E_t [\sum_{s=t+1}^{\infty} (\frac{1}{1+\xi})^{s-t} U(\overset{*}{M}_s , \mathbf{h}_s, \mathbf{x}_s)] \qquad \text{(A.2)}$$

subject to $(\mathbf{h}_s, \mathbf{x}_s, A_s) \in D(s)$ for $s\ ^3\ t$, and also subject to

$$\lim_{s\to\infty} E_t (\frac{1}{1+\xi})^{s-t} A_s = 0,$$

with the process $\overset{*}{M}_s$ given for $s\ ^3\ t$.

<u>**Theorem A1**</u> **(Consumer's Aggregation Theorem):** Let the deterministic point $(\mathbf{m}_t,\mathbf{h}_t,\mathbf{x}_t,A_t)$ and the stochastic process $(\mathbf{m}_s, \mathbf{h}_s, \mathbf{x}_s , A_s)$, s=t+1, ..., t+T solve Problem 1. Then the deterministic point $(\mathbf{h}_t,\mathbf{x}_t,A_t)$ and the stochastic process $(\mathbf{h}_s,\mathbf{x}_s,A_s)$, s = t+1, ..., t+T, will solve Problem 2 conditionally upon $M_s^* = M(\mathbf{m}_s)$ for s = t, ... , t+T.

<u>**Proof**</u>**:** Let $(\mathbf{m}_s,\mathbf{h}_s,\mathbf{x}_s, A_s)$, s^3t solve problem 1, but let $(\mathbf{h}_s,\mathbf{x}_s,A_s)$, $s^3 t$, not solve problem 2 conditionally upon the process $M_s^* = M(\mathbf{m}_s)$ given for s = t, ... , t+T. Then there exist $(\tilde{\mathbf{h}}_s, \tilde{\mathbf{x}}_s, \tilde{A}_s) \in D(s)$, s^3t, satisfying the transversality condition, such that (A.2) evaluated at $(\tilde{\mathbf{h}}_s, \tilde{\mathbf{x}}_s, \tilde{A}_s)$, s^3t, is strictly greater than (A.2) evaluated at $(\mathbf{h}_s,\mathbf{x}_s,A_s)$, $s^3 t$, conditionally upon $M_s^* = M(\mathbf{m}_s)$.

Hence (2.1) evaluated at $(\mathbf{m}_s, \tilde{\mathbf{h}}_s, \tilde{\mathbf{x}}_s, \tilde{\mathbf{A}}_s)$, s^3t. is strictly greater than (2.1) evaluated at $(\mathbf{m}_s,\mathbf{h}_s, \mathbf{x}_s,A_s)$, s^3t. But since $(\tilde{\mathbf{h}}_s, \tilde{\mathbf{x}}_s, \tilde{A}_s)$, s^3t, is feasible for problem 2 conditionally upon $M_s=M(\mathbf{m}_s)$, it follows that $(\mathbf{m}_s, \tilde{\mathbf{h}}_s, \tilde{\mathbf{x}}_s, \tilde{\mathbf{A}}_s)$, s^3t, is feasible for problem 1. Our assumption that $(\mathbf{m}_s,\mathbf{h}_s,\mathbf{x}_s, A_s)$, $s^3 t$, solves problem 1 is contradicted.

Q.E.D.

Clearly this proof by contradiction applies not only when M_s is produced by voluntary behavior, but also when the M_s process is exogenously imposed upon the consumer, as through a perfectly inelastic supply function for M_s imposed by central bank policy. In that case, Problem 2 describes optimal behavior by the consumer in the remaining variables. Clearly the information about M_s is needed in the solution of problem 2 for the processes $(\mathbf{h}_s,\mathbf{x}_s,A_s)$. Alternatively information about the usual simple sum monetary aggregate over the components of $\mathbf{m}_s$ is of no use in solving either problem 1 or 2. unless the monetary aggregator function M happens to be a simple sum. In other words, the simple sum aggregate contains useful information about behavior only if the components of $\mathbf{m}_s$ are perfect substitutes in identical ratios (linear aggregation with <u>equal</u> coefficients).

REFERENCES

Arrow, K. J. and F. Hahn (1971), *General Competitive Analysis* . San Francisco, Holden-Day.

Barnett, William A. (1978), " The User Cost of Money," *Economic Letters*, 1, 145-149.

Barnett. William A. (1980), " Economic Monetary Aggregates: An Application of Index Number and Aggregation Theory," *Journal of Econometrics*, 14, 11-48.

Barnett, William A. (1987), " The Microeconomic Theory of Monetary Aggregation." In Barnett, William A. and K. Singleton (Eds.), *New Approaches to Monetary Economics*, 115-168, Cambridge University Press.

Barnett, William A. (1995). "Exact Aggregation under Risk." In Barnett, William A., Moulin, Herve, Salles, Maurice, and Norman Schofield (Eds.), *Social Choice, Welfare, and Ethics*, Cambridge University Press, pp. 353-374.

Barnett, William A., D. Fisher and S. Serletis, (1992), " Consumer Theory and the Demand for Money," *Journal of Economic Literature*, 92, 2086-119.

Barnett, William A., Hinich, Melvin, and Piyu Yue (1991), "Monitoring Monetary Aggregates under Risk Aversion," in Michael T. Belongia (ed.), *Monetary Policy on the 75th Anniverary of the Federal Reserve System,* Proceedings of the Fourteenth Annual Economic Policy Conference of the Federal Reserve Bank of St. Louis, Kluwer, pp. 189-222.

Barnett, William A. and Piyu Yue (1991), " Exact Monetary Aggregation under Risk," Working Paper #163, Washington University.

Barnett, William A. and Ge Zhou (1994), "Commentary," Federal Reserve Bank of St. Louis *Review*, November/December, vol. 76, no. 6.

Belongia, Michael T. (1993), "Measurement Matters: Recent Results from Monetary Economics Re-examined," *Journal of Political Economy*, forthcoming.

Blackorby, Charles, Daniel Primont, and Robert R. Russell (1978), *Duality, Separability, and Functional Structure: Theory and Economic Applications*, New York, North-Holland.

Blackorby, Charles and R. Robert Russell (1994), "The Conjunction of Direct and Indirect Separabilty," *Journal of Economic Theory*, vol. 62, April, pp. 480-98.

Blackorby, Charles, Russell Davidson, and William Schworm (1991), "Implicit Separabilty: Characterization and Implications for Consumer Demands," *Journal of Economic Theory*, vol. 55, December, pp. 364-99.

Blackorby, Charles and William Schworm (1984), "The Structure of Economies Aggregate Measures of Capital: A Complete Characterizatrion," *Review of Economic Studies*, vol. 51, October, pp. 633-50.

Caves, Douglas, Laurits R. Christensen, and Erwin W. Diewert (1982a), "Multilateral Comparisons of Output, Input, and Productivity Using Superlative Index Numbers," *Economic Journal* 92, pp. 73-86.

Caves, Douglas, Laurits R. Christensen, and Erwin W. Diewert (1982b), "The Economic Theory of Index Numbers and the Measurement of Input, Output, and Productivity," *Econometrica* 50, pp. 1393-1414,

Chrystal, K. Alec and Ronald MacDonald (1994), "Empirical Evidence on the Recent Behavior and Usefulness of Simple-Sum and Weighted Measures of the Money Stock," Federal Reserve Bank of St. Louis *Review*, March/April, pp. 73-109.

Constantinides, George M. (1990), "Habit Formation: A Resolution of the Equity Premium Puzzle," *Journal of Political Economy*, vol. 98, pp. 519-543.

Daniel, Kent and David Marshall (1995), "Consumption-Based Modeling of Long-Horizon Returns," In Barnett, William A., Lars Hansen, George Tauchen, and Andrew Lo (eds.), *Computation and Estimation in Finance and Economics*, Cambridge University Press, forthcoming.

Divisia, Francois (1925), "L'indice Monétaire et la Théorie de la Monnaie," *Revue d'Economie Politique*, vol 39, pp. 980-1008.

Feenstra, Robert C. (1986), "Functional Equivalence Between Liquidity Costs and the Utility of Money," *Journal of Monetary Economics,* March, 271-291.

Feldstein, Martin and James H. Stock (1994), "Measuring Money Growth when Financial Markets are Changing," NBER Working Paper No. 4888, National Bureau of Economic Research, 1050 Massachusetts Avenue, Cambridge, MA 02138.

Fischer, Stanley (1974), "Money and the Production Function," *Economic Inquiry*, vol 12, pp. 517-33.

Fisher, Franklin M. and Karl Shell (1972), *The Ecooomic Theory of Price Indices*, Academic Press, New York.

Hulten, C. R. (1973), "Divisia Index Numbers," *Econometrica* 63, pp. 1017-26.

Ingersoll, Jonathan E. (1987), *Theory of Financial Decision Making*, Rowman & Littlefield.

Phlips, Louis and Frans Spinnewyn (1982), "Rationality versus Myopia in Dynamic Demand Systems," in R. L. Basmann and G. F. Rhodes (eds.), *Advances in Econometrics* , JAI Press, pp. 3-33.

Poterba, James M. and Julio J. Rotemberg (1987), "Money in the Utility Function: An Empirical Implementation," in William A. Barnett and Kenneth J. Singleton (eds.), *New Approaches to Monetary Economics*, Cambridge, Cambridge University Press, 1987, pp. 219-240.

Quirk, James and Rubin Saposnik (1968), *Introduction to General Equilibrium Theory and Welfare Economics*, McGraw-Hill, New York.

Rotemberg, Julio J., Driscoll, John C., and James M. Poterba (1995), " Money, Output, and Prices: Evidence from a New Monetary Aggregate," *Journal of Business and Economics Statistics*, January, vol. 13, pp. 67-84.

Rubinstein, Mark (1976), "The Valuation of Uncertain Income Streams and the Pricing of Options," *Bell Journal of Economics*, 7, pp. 407-425.

Samuelson, Paul (1948), *Foundations of Economic Analysis*, Harvard University Press, Cambridge, Mass.

Stahl, Dale O. (1983), "Quasi-homothetic Preferences, the Generalized Divisia Quantity Index, and Aggregation," *Economica*, vol. 50, February, pp. 87-93.

Stein, C. (1973), "Estimation of the Mean of a Multivariate Normal Distributin,"*Proceedings of the Prague Symposium on Asymptotic Statistics*, September.