C Figure Captions

Fig.1 Time dependence of natural logarithm of Warsaw stock index WIG (bold line) in trading days n (from April 1991 till May 1995). Dashed and dotted curves are log-prices of two exemplary shares (Tonsil and Próchnik).

Fig.2 Cumulative distribution of relative price changes for 19 shares on Warsaw stock exchange 1991-1995: a) day to day changes $P(\delta_s)$ with mean $\langle\delta_s\rangle = 0.0098$ and rms $\sigma_1 = 0.069$, b) every third day changes $P(\delta_{3s})$ with $\langle\delta_{3s}\rangle = 0.0167$ and $\sigma_3 = 0.131$.

Fig.3 Residual risk R^* in the ternary model shown as function of the ratio between strike price and current price. Maturity time is $T = 20$ and the cut-off parameters are $q = 0.08$ (lower curve), 0.10 and 0.12 (upper curve).

Fig.4 Call option price in the ternary model with $T = 20$ and $q = 0.10$ as a function of the ratio k between strike price and current price. Black–Scholes formula is represented by the triangles, Bachelier's price $\mathcal{B}$ by solid line and the upper dashed line stands for the risk-corrected price given by eq. (73). Inset shows the ratio $\mathcal{B}/C_{BS}$ as a function of k.

Fig.5 Time dependence of Bachelier's price $\mathcal{B}$ (solid line) and Black-Scholes price C_{BS}, ($\triangle$) of the ternary model with $q = 0.10$ and $k = 1.115$. The dashed line represents the the risk-correction $\sqrt{R^*}$.

[9] Duffie D. , *Dynamical Asset Pricing Theory*, Princeton University Press (1992).

[10] Gnedenko B.V. & Kolmogorov A.N., *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley Press, 1954.

[11] Engle R., Econometrica **50** (1982), 987.

[12] Fama E., J. of Business **38** (1965), 34.

[13] Fama E., J. of Finance **25** (1970), 383.

[14] Gawor K. & Leśniak M., Arabski & Gawor - Biuro Maklerskie S.C., Ul. Lubicz 25, PL-31-503 Kraków, Poland, *(private communication)*.

[15] Lévy P., *Théorie de l'Addition des Variables Aléatoires*, Gauthier–Villars, Paris 1937:1954.

[16] Litner J., *J. of Finance* **20** (1965) 587; *Review of Economics and Statistics* **47** (1965) 13.

[17] Markowitz H., *Portfolio selection: Efficient Diversification of Investment*, John Wiley & Sons, New York 1959.

[18] Mandelbrot B., J. of Business, **36** (1963), 394.

[19] Mandelbrot B., J. of Business, **39** (1967), 242.

[20] Mandelbrot B. & Taylor H., Operations Research **15** (1967), 1057-62.

[21] Mantegna R.N. & Stanley H.E., *Nature* **376** (1995, July 6), 46.

[22] Mikheev L., *J. Phys. I (France)* **5** (1995) 217-218; Bouchaud J-P. & Sornette D., *J. Phys. I (France)* **5** (1995) 219-220.

[23] Müller U.A., Dacorogna M.M., Olsen R.B., Pictet O.V., Schwarz M. & Morgenegg C., J. Banking and Finance **14** (1990), 1189-1208. See also the later report from Olsen & Associates available from the WWW-page at http:www.olsen.ch .

[24] Press W., Flannery B., Teukolsky S. & Vetterling W., *Numerical Recipies*, Cambridge University Press (1988).

[25] See for instance the news article "Spełniony sen spekulanta" [The dream of the speculator come true] by Piotr Orlik and Pawel Wujec, Gazeta Wyborcza No 221, September 22 1995.

[26] Sharpe W., *J. of Finance* **19** (1964) 425; *J. of Finance* **20** (1965) 416.

[27] Sławomir Sadowski, Gdynia, *(private communication)*.

In fact, for the bank rate, volatility and rate of return to be finite as Δt tends to zero we must have to lowest order in Δt

$$\begin{aligned} r &= 1 + \Delta t \delta r \\ u &= 1 + q\sqrt{\Delta t}/p \\ d &= 1 - q\sqrt{\Delta t}/(1-p) \end{aligned}$$

Here q is some constant that does not depend on Δt and δr is the bank rate. It hence follows that actually the new 'probability' is very close to p, or more precisely that

$$\frac{r-d}{u-d} = p + \mathcal{O}(\sqrt{\Delta t}) \tag{112}$$

from which follows that

$$\tilde{\sigma}^2 = \sigma^2 + \mathcal{O}(\sqrt{\Delta t}). \tag{113}$$

Summing up, in the continuous limit the option price can be computed by integrating the pay-off function of the option against a transfer function as in (108). The transfer function is log-normal as in (107) and the volatility is the same as the price process, but the rate of return is adjusted to be the same as that of the bond. Substituting (107) into (108) and performing the integration we obtain the Black-Scholes price [3]

$$\begin{aligned} C_{BS} &= e^{-\delta r T}\int_{S_c}^{\infty}(S_T - S_c)P_{\mathrm{BS}}^{(\delta r,\sigma^2)}(S_0,0;S_T,T) \\ &= S_0\,N\Big(\frac{1}{\sqrt{\sigma^2 T}}[\log S_0/S_c + T(\delta r + \sigma^2/2)]\Big) \\ &\quad -e^{-\delta r T}S_c\,N\Big(\frac{1}{\sqrt{\sigma^2 T}}[\log S_0/S_c + T(\delta r - \sigma^2/2)]\Big) \end{aligned} \tag{114}$$

where the Gauss probability integral is $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}du$.

References

[1] Aurell E. & Simdyankin S., *in preparation.*

[2] Bachelier L., "Théorie de la Speculation", Annales Sci. de l'Ecole Normale Sup. **17** (1900), 21-88.

[3] Black F. & Scholes M., *J. Political Economy* **3** (1973) 637.

[4] Bollerslev T., Chou R. & Kroner K., J. of Econometrics **52** (1992), 5.

[5] Bouchaud J-P. & Sornette D., *J. Phys. I (France)* **4** (1994) 863-881.

[6] Bouchaud J.P., G. Iori & D. Sornette, "Real world options: smile and residual risk", Risk Magazine (in press).

[7] Bouchaud J-P. & Sornette D., Walter C. & Aguilar J.-P., "Taming large events: optimal portfolio theory for strongly fluctuating assets", preprint, 1995.

[8] Cox J. & Rubinstein M., *Options Markets*, Prentice-Hall (1985).

The relative error of this approximation is of order $1/N$ at the central peak in the distribution.

Let us include a factor Δt in the definition of the average and the variance of the price increments:

$$\begin{aligned} mS_i\Delta t &= < S_{i+1} - S_i >, & (102)\\ \sigma^2 S_i^2 \Delta t &= < (S_{i+1} - S_i)^2 > -(< S_{i+1} - S_i >)^2. & (103) \end{aligned}$$

The coefficient σ is now what in the finance literature is called volatility, while m is the mean rate of return of the stock per unit time. The price process can then be thought of as a stochastic discrete difference equation

$$\Delta S_i = mS_i\Delta t + \sigma S_i \Delta w_i, \tag{104}$$

where the Δw_i's are independent random variables of zero mean and variance Δt. For the binary price movement model we have

$$\begin{aligned} m\Delta t &= pu + (1-p)d - 1, & (105)\\ \sigma^2 \Delta t &= p(1-p)(u-d)^2. & (106) \end{aligned}$$

Taking the limits N tending to infinity and Δt to zero, such that T, m and σ^2 are constant, the price process is completely characterized by these two last numbers, and the transfer function tends to the log-normal distribution with parameters σ^2 and m:

$$\begin{aligned} P_{\mathrm{BS}}^{(m,\sigma^2)}(S_0, 0; S_T, T) &= \frac{1}{S_T}\frac{1}{\sqrt{2\pi\sigma^2 T}} \\ &\qquad \exp[-\frac{(\log S_T/S_0 - (m - \sigma^2/2)T)^2}{2\sigma^2 T}]. & (107) \end{aligned}$$

This is the kernel of the log-Brownian process, governed by the stochastic differential equation, $dS_t = mS_t dt + \sigma S_t dW_t$, where W_t is the Wiener process [3].

On the other hand, we can also rewrite the price at time zero of the European call option in the binary price movement model, (24), as

$$V(0, S_0) = r^{-N} \int_{S_c}^{\infty} (S_T - S_c)\tilde{P}_{\Delta t}(S_0 \to S_T) dS_T \tag{108}$$

using a transfer function

$$\tilde{P}_{\Delta t}(S_0 \to S_T) = \frac{1}{\Delta S_T}\binom{N}{l}(\frac{r-d}{u-d})^l(\frac{u-r}{u-d})^{T-l}. \tag{109}$$

Formally (109) is identical to (99) with a specific choice of the probability p. In the continuous limit we therefore expect (109) to be similar to (107) with only new values of the rate of return and the volatility. The analogous equations to (105) and (106) are

$$\begin{aligned} \tilde{m}\Delta t &= [\frac{r-d}{u-d}u + \frac{u-r}{u-d}d - 1] = (r-1) & (110)\\ \tilde{\sigma}^2 \Delta t &= \frac{r-d}{u-d}\frac{u-r}{u-d}(u-d)^2 = (r-d)(u-r). & (111) \end{aligned}$$

The probability distribution used to compute the price of the option therefore has the property that the mean rate of return of the stock is the same as that of the bond.

Using the short-hand notation we can write

$$P = \mu < \psi \cdot \mathbf{1} > \tag{92}$$
$$R = < \psi \cdot K\psi > \tag{93}$$

Performing the linear variation with a Lagrange parameter q we find an optimal strategy

$$G_i^*(S_i; q) = -\mu q (K^{-1}\mathbf{1})_i(S_i) \tag{94}$$

and inserting in the expression for the profit we find

$$q[P] = -P \frac{1}{\mu^2 < \mathbf{1} \cdot K^{-1}\mathbf{1} >}. \tag{95}$$

The optimal strategy as a function of P is thus

$$G_i^*(S_i; P) = P \frac{1}{\mu < \mathbf{1} \cdot K^{-1}\mathbf{1} >}(K^{-1}\mathbf{1})_i(S_i), \tag{96}$$

and the minimal risk is at given P is

$$R[M] = P^2 \frac{1}{\mu^2 < \mathbf{1} \cdot K^{-1}\mathbf{1} >} \tag{97}$$

which we can write $R[P] = \nu P^2$, with the same coefficient ν as in (68).

B Black-Scholes' model

It is well known that the celebrated continuous-time option pricing model of Black and Scholes [3] can be derived as the continuous limit of the discrete-time binary price movement model treated above in section 4. For completeness we include this derivation here.

We introduce a time step Δt such that T is equal to $N \cdot \Delta t$, and write the expectation value of the value upon maturing of the European call option in the binary price movement model as

$$< V(T, \cdot) > = \int_{S_c}^{\infty} (S_T - S_c) P_{\Delta t}(S_0 \to S_T) dS_T, \tag{98}$$

with the transfer function

$$P_{\Delta t}(S_0 \to S_T) = \frac{1}{\Delta S_T} \binom{N}{l} p^l (1-p)^{T-l}, \tag{99}$$

and l determined through

$$S_T = u^l d^{N-l} S_0. \tag{100}$$

The integral in (98) is at this point really a sum, and dS_T and ΔS_T are both equal to the smallest (finite) increment in S_T, that is $(u/d - 1)S_T$.

Approximating the factorials in the binomial coefficients by Stirling's formula we have

$$P_{\Delta t}(S_0 \to S_T) \approx \frac{1}{S_T} \frac{1}{\sqrt{2\pi N p(1-p)(\log(u/d))^2}} \exp[-\frac{(\log S_T/S_0 - N \log d - pN \log u/d)^2}{2Np(1-p)(\log u/d)^2}]. \tag{101}$$

$\{n_j, m_j, l_j\}$. Assume j to be greater than i. Then the conditional price increment is by convention zero if the price S_j cannot be reached from S_i, that is if $n_j - n_i$, $m_j - m_i$ and $l_j - l_i$ are not all non-negative and their sum equal to $j - i$. In the opposite case we obtain the average conditional price increment

$$\begin{aligned} < u_i - r >_{S_i S_j} &= \frac{1}{j-i}[(v_1 - r)(n_j - n_i) + (v_2 - r)(m_j - m_i) + \\ &\qquad (v_3 - r)(l_j - l_i)]. \end{aligned} \tag{83}$$

If j is less or equal to i the conditional price increment is equal to μ, again assuming that S_j can be reached from S_i.

The second moments of the conditional price increments are, if i is equal to j and S_i is equal to S_j,

$$< (u_i - r)^2 > \quad = \quad \sigma^2 + \mu^2. \tag{84}$$

If i is less than j then

$$< (u_i - r)(u_j - r) >_{S_i S_j} \quad = \quad \mu(< u_i - r >_{S_i S_j}) \tag{85}$$

and symmetrically if j is less than i. Let us note that according to above, the off-diagonal matrix elements of K satisfy

$$K_{ij}(n_i, m_i, l_i; n_j, m_j, l_j) = K_{i-j}(n_i - n_j, m_i - m_j, l_i - l_j), \tag{86}$$

and the diagonal elements are all equal to

$$K_{ii}(n_i, m_i, l_i; n_j, m_j, l_j) = \sigma^2 \delta_{n_i - n_j} \delta_{m_i - m_j} \delta_{l_i - l_j}. \tag{87}$$

K is therefore a Toeplitz matrix and the necessary contractions with its inverse, such as $K^{-1}\mathbf{1}$ or $K^{-1}\mathbf{F}$, can be computed by a fast recursive procedure[24].

The vector-valued function $\mathbf{F}$ is given by

$$\begin{aligned} F_i(n_i, m_i, l_i) &= < \{S_T - S_c\}^+ (u_i - r) >_{S_i = S_0 \cdot v_1^{n_i} v_2^{m_i} v_3^{l_i}} \\ &= S_0 \sum_{n_T + m_T + l_T = T} \left(\frac{(T-i)!}{(n_T - n_i)!(m_T - m_i)!(l_T - l_i)!}\right) \cdot \\ &\quad p_1^{n_T - n_i} p_2^{m_T - m_i} p_3^{l_T - l_i} \cdot \{v_1^{n_T} v_2^{m_T} v_3^{l_T} - v_1^{n_c} v_2^{m_c} v_3^{l_c}\}^+ \cdot \\ &\quad \frac{1}{T-i}[(v_1 - r)(n_T - n_i) + (v_2 - r)(m_T - m_i) + \\ &\qquad (v_3 - r)(l_T - l_i)]. \end{aligned} \tag{88}$$

In the special case where μ is zero K is diagonal. The risk can then be lowered by an amount

$$\Delta R[\mu = 0] \quad = \quad \frac{1}{\sigma^2} \sum_{i=0}^{T-1} \sum_{n_i + m_i + l_i = i} \left(\frac{i!}{n_i! m_i! l_i!}\right) \cdot p_1^{n_i} p_2^{m_i} p_3^{l_i} \cdot (F_i(n_i, m_i, l_i))^2. \tag{89}$$

A.3 Investing only in stock

A portfolio of stock is characterized by its profit and risk

$$P[\psi_i = G_i(S_i)] \quad = \quad \sum_{i=0}^{T-1} < \psi_i (u_i - r) >, \tag{90}$$

$$R[\psi_i = G_i(S_i)] \quad = \quad < (\sum_{i=0}^{T-1} \psi_i (u_i - r))^2 > - M^2. \tag{91}$$

To proceed further we want to expand K as

$$K = \sigma^2 \mathbf{1} + \mu Q \tag{77}$$

where Q is an operator of leading zeroth order in μ. Symbolically then

$$K^{-1} = \sigma^{-2}\mathbf{1} - \sigma^{-4}\mu Q + \sigma^{-6}\mu^2 Q^2 + \ldots \tag{78}$$

The optimal level of gain is given perturbatively as

$$\begin{aligned} M^* &= -\mathcal{B} + \frac{\mu}{\sigma^2} < \mathbf{F} \cdot \mathbf{1} > + \\ &\quad \mu^2 [\frac{T}{\sigma^2}(-\mathcal{B} + \lambda^{-1}\sqrt{R_c - \frac{1}{\sigma^2} < |\mathbf{F}^2| >}) \\ &\quad - \frac{1}{\sigma^4} < \mathbf{1} \cdot Q\mathbf{F} >] + \mathcal{O}(\mu^3) \end{aligned} \tag{79}$$

and the optimal strategy as

$$\begin{aligned} G^* &= \frac{1}{\sigma^2}\mathbf{F} + \mu[-\frac{1}{\sigma^4}(Q\mathbf{F}) + (-\mathcal{B} + \lambda^{-1}\sqrt{R_c - \frac{1}{\sigma^2} < |\mathbf{F}^2| >})\frac{1}{\sigma^2}\mathbf{1}] \\ &\quad + \mu^2[\frac{1}{\sigma^6}(Q^2\mathbf{F}) - (-\mathcal{B} + \lambda^{-1}\sqrt{R_c - \frac{1}{\sigma^2} < |\mathbf{F}^2| >})\frac{1}{\sigma^4}(Q\mathbf{1})] \\ &\quad - \frac{< (Q\mathbf{F})\cdot\mathbf{F} >}{2\lambda\sqrt{R_c - \frac{1}{\sigma^2} < |\mathbf{F}^2| >}}\frac{1}{\sigma^6}\mathbf{1}] + \mathcal{O}(\mu^3). \end{aligned} \tag{80}$$

A.2 Expressions for risk and price in the ternary model

The content of this subsection is explicit versions of various formulae for the ternary price movement model. Let us assume that the price of a share, initially equal to unity, takes the next trading day one of three values: v_1, v_2 or v_3.

A possible share price on day i is thus

$$S_i = S_0 \cdot v_1^{n_i} v_2^{m_i} v_3^{l_i}, \quad n_i + m_i + l_i = i$$

with just one choice of the three exponents n_i, m_i and l_i. Similarly the strike price will be written

$$S_c = S_0 \cdot v_1^{n_c} v_2^{m_c} v_3^{l_c}$$

Bachelier's option price is then

$$\begin{aligned} \mathcal{B} &= S_0 \sum_{n_T + m_T + l_T = T} (\frac{T!}{n_T! m_T! l_T!}) \cdot p_1^{n_T} p_2^{m_T} p_3^{l_T} \cdot \\ &\quad \{v_1^{n_T} v_2^{m_T} v_3^{l_T} - v_1^{n_c} v_2^{m_c} v_3^{l_c}\}^+ \end{aligned} \tag{81}$$

and the bare risk is

$$\begin{aligned} R_c &= S_0^2 [\sum_{n_T + m_T + l_T = T} (\frac{T!}{n_T! m_T! l_T!}) \cdot p_1^{n_T} p_2^{m_T} p_3^{l_T} \cdot \\ &\quad (\{v_1^{n_T} v_2^{m_T} v_3^{l_T} - v_1^{n_c} v_2^{m_c} v_3^{l_c}\}^+)^2] \quad - \quad \mathcal{B}^2 \end{aligned} \tag{82}$$

We wish to compute the first and second moments of the conditional price increments. Let S_i and S_j be given as above, with two sets of coefficients $\{n_i, m_i, l_i\}$, and with the following identification of the coefficients

$$\begin{aligned} \rho &= R_c - < \mathbf{F} \cdot K^{-1}\mathbf{F} > + 2\mathcal{B}\mu < \mathbf{1} \cdot K^{-1}\mathbf{F} > \\ &\qquad - \mathcal{B}^2\mu^2 < \mathbf{1} \cdot K^{-1}\mathbf{1} > \qquad (66) \\ \mathcal{A} &= \mathcal{B} - \mu < \mathbf{1} \cdot K^{-1}\mathbf{F} > + \mathcal{B}\mu^2 < \mathbf{1} \cdot K^{-1}\mathbf{1} > \qquad (67) \\ \nu &= \frac{1}{\mu^2 < \mathbf{1} \cdot K^{-1}\mathbf{1} >} \qquad (68) \end{aligned}$$

A.1 Perturbation when μ is small

First we will consider the case when the expected return of the stock precisely equals that of the bond, that is when μ vanishes. The gain of the option is then independent of the portfolio strategy, and we have

$$\begin{aligned} M &= -\mathcal{B} \qquad (\mu = 0) \qquad (69) \\ R[\psi_i = G_i(S_i); \mu = 0] &= R_c - 2 < \{S_T - S_c\}^+ \psi_i(u_i - r) > \\ &\quad + < (\sum_{i=0}^{T-1} \psi_i(u_i - r))^2 >, \qquad (70) \end{aligned}$$

where the bare risk is defined in (50). In this case the solution to the hedging problem is found directly by a minimization without constraints, and we have simply

$$\begin{aligned} G_i^*(S_i; \mu = 0) &= \frac{1}{\sigma^2} < \{S_T - S_c\}^+ (u_i - r) >_{S_i} \qquad (71) \\ R^*[M; \mu = 0] &= R_c - \frac{1}{\sigma^2} \sum_{i=0}^{T-1} < (< \{S_T - S_c\}^+ (u_i - r) >_{S_i})^2 > . \qquad (72) \end{aligned}$$

In the notation of the previous section we have

$$\begin{aligned} G_i^*(S_i; M; \mu = 0) &= (K^{-1}\mathbf{F})_i(S_i) \\ R^*[M; \mu = 0] &= R_c - < \mathbf{F} \cdot K^{-1}\mathbf{F} > \end{aligned}$$

The ask price of the option is Bachelier's price plus a risk premium proportional to λ and the residual risk:

$$\tilde{C}^* = \mathcal{B} + \lambda\sqrt{R^*} \qquad (73)$$

Let us now assume that μ is small but not zero. The coefficients ρ, $\hat{M}$ and ν include various terms that all depend on μ, some explicitly and some implicitly through K. Let us first expand only in the explicit factors of μ, and treat K and its inverse as zeroth order. That gives

$$M^* = -\mathcal{B} + \mu < \mathbf{1} \cdot K^{-1}\mathbf{F} > + \mathcal{O}(\mu^2) \qquad (74)$$

where the dependence on λ only enters in quadratic and higher orders. Inserting (74) in (63) and (64) gives

$$\begin{aligned} G_i^*(S_i; M^*) &= (K^{-1}\mathbf{F})_i(S_i) + \mathcal{O}(\mu) \qquad (75) \\ R[M^*] &= R_c - < \mathbf{F} \cdot K^{-1}\mathbf{F} > + \mathcal{O}(\mu) \qquad (76) \end{aligned}$$

which shows that the optimal strategy and the residual risk both tend to their values when μ is zero.

The gain and the risk can then be written as

$$\begin{aligned} M &= -\mathcal{B} + \mu < \psi \cdot \mathbf{1} > && (54) \\ R &= R_c - 2 < \mathbf{F} \cdot \psi > + 2\mu\mathcal{B} < \psi \cdot \mathbf{1} > + < \psi \cdot K\psi >, && (55) \end{aligned}$$

where $\mathbf{1}$ is the unit vector and summation over repeated indices and integration over respective arguments is implied.

To perform the constrained minimization we introduce a Lagrange multiplier q and vary

$$Q[G_i; q] = R[\psi_i = G_i(S_i)] + 2q(M[\psi_i = G_i(S_i)] - M) \tag{56}$$

with respect to q and the variations of the G_i's. A particular convenient choice is to vary only one of the functions, say G_k, and only if the argument of that function lies in a small interval[5]. That means that we take

$$\begin{aligned} G_i(S) &= G_i^*(S) + \epsilon \qquad \text{if} \qquad i = k, S \in [S', S' + dS'] && (57) \\ G_i(S_i) &= G_i^*(S_i) \qquad \text{otherwise}, && (58) \end{aligned}$$

and do so in turn for all k and all possible values of the argument. Vanishing of the linear variations then leads to a system of equations for all values of i and all values of the argument S_i:

$$(K \cdot G^*)_i(S_i) = F_i(S_i) - \mu(\mathcal{B} + q)\mathbf{1}_i(S_i)\,. \tag{59}$$

Formally inverting (59) we have

$$G_i^*(S_i; q) = (K^{-1}\mathbf{F})_i(S_i) - \mu(\mathcal{B} + q)(K^{-1}\mathbf{1})_i(S_i). \tag{60}$$

Inserting this strategy in the expression for the gain we have

$$M = -\mathcal{B} + \mu < \mathbf{1} \cdot K^{-1}\mathbf{F} > -\mu^2(\mathcal{B} + q) < \mathbf{1} \cdot K^{-1}\mathbf{1} > \tag{61}$$

from which we can solve for the Lagrange multiplier:

$$q[M] = -\mathcal{B} - \frac{M + \mathcal{B} - \mu < \mathbf{1} \cdot K^{-1}\mathbf{F} >}{\mu^2 < \mathbf{1} \cdot K^{-1}\mathbf{1} >} \tag{62}$$

The optimal strategy as a function of M is thus

$$\begin{aligned} G_i^*(S_i; M) &= (K^{-1}\mathbf{F})_i(S_i) + \\ &\quad \frac{M + \mathcal{B} - \mu < \mathbf{1} \cdot K^{-1}\mathbf{F} >}{\mu < \mathbf{1} \cdot K^{-1}\mathbf{1} >}(K^{-1}\mathbf{1})_i(S_i), && (63) \end{aligned}$$

The minimal risk at given M is

$$\begin{aligned} R[M] &= R_c - < \mathbf{F} \cdot (K^{-1}\mathbf{F}) > + \frac{(< \mathbf{1} \cdot K^{-1}\mathbf{F} >)^2}{< \mathbf{1} \cdot K^{-1}\mathbf{1} >} \\ &\quad + 2(M + \mathcal{B})[\mathcal{B} - \frac{< \mathbf{1} \cdot K^{-1}\mathbf{F} >}{\mu < \mathbf{1} \cdot K^{-1}\mathbf{1} >}] \\ &\quad + (M + \mathcal{B})^2 \frac{1}{\mu^2 < \mathbf{1} \cdot K^{-1}\mathbf{1} >}. && (64) \end{aligned}$$

Equation (64) can finally be rewritten in the form

$$R[M] = \rho + \nu(M + \mathcal{A})^2 \tag{65}$$

A Hedging and option pricing: when return on the stock differs from that of the bond

We first consider the hedging problem: we are to choose portfolio strategy such as to minimize risk at constant gain, where the gain and the risk are defined by

$$
\begin{aligned}
M[\psi_i = G_i(S_i)] &= -\mathcal{B} + \sum_{i=0}^{T-1} < \psi_i(u_i - r) >, && (46)\\
R[\psi_i = G_i(S_i)] &= < (\{S_T - S_c\}^+)^2 > -2 \sum_{i=0}^{T-1} < \{S_T - S_c\}^+ \psi_i(u_i - r) > \\
&\quad + < (\sum_{i=0}^{T-1} \psi_i(u_i - r))^2 > \; - M^2. && (47)
\end{aligned}
$$

The notation in (46) and (47) is explained in section 3, in particular $\mathcal{B}$ denotes the expectation value $< \{S_T - S_c\}^+ >$.

The variables ψ_i in (46) and (47) are related to the number of stocks held by an option writer on day i: they are functions of S_i, the stock price on day i, but not on the previous history of stock prices. With $< \ldots >$ we mean average over all realizations of the price process. In the following we will use also conditional averages such as $< \ldots >_{S_i = S}$ which means averaging over all realizations of the price process with a given value of S_i.

Since price increments have been taken independent, one sees that the expression for the gain can be simplified to

$$
M[\psi_i = G_i(S_i)] = -\mathcal{B} + \mu \sum_{i=0}^{T-1} < \psi_i > \qquad \mu =< u_i - r > \qquad (48)
$$

It is also notationally convenient to separate the risk into one part that only depends on fallout of the option, the bare risk, and a remainder:

$$
\begin{aligned}
R[\psi_i = G_i(S_i)] &= R_c - \Delta R[\psi_i = G_i(S_i)] && (49)\\
R_c &= < (\{S_T - S_c\}^+)^2 > -(< \{S_T - S_c\}^+ >)^2 && (50)
\end{aligned}
$$

In the absence of trading the risk equals the bare risk.

The second term in (47) involves the share price on day T and the share price increment on day i. The average of this increment is conditioned by the share values on day i and day T, such that the term should be understood as three nested averages:

$$
< (< [< (u_i - r) >_{(S_i,i)->(S_T,T)}]\{S_T - S_c\}^+ >_{S_i})\psi_i > \qquad (51)
$$

and similarly the third term in (47). It is convenient to work with a short-hand notation and introduce a vector $\psi = \{\psi_i; i = 1, \ldots, T\}$, a vector-valued set of functions $\mathbf{F} = \{F_i; i = 1, \ldots, T\}$ with

$$
F_i(S_i) =< \{S_T - S_c\}^+(u_i - r) >_{S_i}, \qquad (52)
$$

and a covariance matrix

$$
K_{ij}(S_i, S_j) =< (u_i - r)(u_j - r) >_{S_i,S_j} \; - \mu^2. \qquad (53)
$$

traded over-the-counter at the Warsaw bank PBR. The WIG is an average, and the distribution of daily changes in the WIG are closer to a log-Gaussian than those of a single share. The difference between the normal price and the Black-Scholes' price are accordingly smaller for a WIG option than for a hypothetical option on Polish stock. On the other hand, we cannot compare with a market price, but only with the price offered on some particular day by the bank. Taking for instance the going price on September 21, we can interpret that as a Black-Scholes price with an unknown volatility. This procedure gives an implied volatility of the WIG option of $\sigma_{impl} \approx 3.1\%$, which is close to the historical value 3.01% computed from data from January to May 1995. In spite of this coincidence, the stock exchange in Warsaw can hardly be regarded as an established market. Further anecdotal support of this conclusion may be gleaned from the markedly speculative approach Polish investors seemed to take to options trading[25]. Indeed, at the request of the regulating authority of the Warsaw Stock Exchange, trading in WIG options has now ceased, at least temporarily.

The best prospects of comparing Black-Scholes' price and our proposed normal price would be in established markets, where the differences in price movements to the log-Brownian process are smaller, but small differences may be significant. We hope to return to this question.

A second possible objection to the approach taken in this paper, already investigated in [7], would be to postulate that price increments follow a Lévy process, or more generally a process with slowly decaying power-law tails in the distribution. The second moment would then be then formally infinite, and least-square minimization not possible. Again, this objection must be met on empirical grounds. On the Warsaw Stock Exchange trading rules disallow arbitrary large price increments, so in this market all moments are finite by regulation. Most recent studies that report power-law tails in the distribution of price increments in established markets also report that these distributions have cut-offs at sufficiently large increments. It is fairly natural that there would be such cut-offs. Even the largest financial markets in the world will have difficulties to carry on trade if price movements are large enough, as was demonstrated in the crash in 1987.

If least-square minimization is applicable is then a question of scale. If an operator is only prepared to loose an amount that is well within the limits where the distribution is a Lévy law, then for practical purposes the distribution is a Lévy law to her , and the approach of [5] and of this paper is not very useful. Operators with sufficiently large capitalization will however be able to reach the cut-off of the distribution, so to them least-square minimization is reasonable.

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In standard financial practice risk carries no price if it can be diversified away[26, 16], and in a remark in [3] Black and Scholes indeed state that risk in option trading is diversifiable. We are not sure that this is literally true, in particular in a market like the Warsaw Stock Exchange where equal-time stock-stock correlations are very significant. We do however agree on the consequence that in the typical market price there is no price for risk.

The approach of Bouchaud and Sornette to option theory amounts mathematically mainly to a least-square minimization of a linear functional of a random process. This is a standard procedure in control theory. Some problems that could be treated in a straight-forward way are then:

Many stocks, the price movements of which are correlated, and options on different stocks and combinations of these. If all price movements are log-Brownian, presumably arbitrage arguments could still be used. But least-square minimization is bound to be easier.

The bond price can also be random process. This reduces practically to the former case, since the bond is then effectively but one more share.

The characteristics of the random processes may change over time. This would give a rational procedure to price an option on a share the price of which follows an ARCH or a GARCH process[11, 4].

Effects of dividends, normal (deterministic) and extraordinary (stochastic) may be included.

Costs of trading may be included. This point has already been considered heuristically in [5].

Prices may be correlated over time. In this case the method will however give more involved results, with trading strategies that depend on the previous history of the share prices.

In real market it is evident that the conditions of the Black-Scholes theory do not hold exactly. Even if the underlying price movements really are log-Brownian, the hedging strategy of Black and Scholes demands portfolio adjustment infinitely often and leads to infinite transaction costs. Real-world applications of Black-Scholes' formula must always take a least time-interval between rehedging, and therefore do not implement the strategy to arbitrary accuracy. The Black-Scholes' price, computed from historical price volatility, is an estimate of the market price. It should become better and better the more nearly the conditions are fulfilled, but it is still only a sign-post, just as the normal price we have proposed.

One natural objection to the general usefulness of our interpretation of the Bouchaud-Sornette approach to option pricing would then be that Black-Scholes' price and our normal price are so close that their difference is smaller than the typical fluctuations on the market. This is an empirical statement that must be decided by looking at the data. In section 9 we investigated option pricing in a model of stock price movements on the Warsaw Stock Exchange. We found significant differences between the Balck-Scholes price and the normal price, even in the special case where the rate of return of the stock is the same as that of the bond. We believe that such price differences would be judged significant in established markets. Since until today options on common stock have not been traded in Poland, we can not judge if this difference would be significant in the market where they would actually be traded. Considering the large historical price volatility and quickly changing institutional conditions in Poland, a conservative estimate is that these differences between the normal price and the Black-Scholes' price are too small to be of practical value.

In September this year an option on the stock market index WIG was in fact

Figure 4 shows the dependence of the option price on the price ratio k in the symmetric ternary model with $q = 0.1$ and $T = 20$. Since daily volatility σ of the ternary price movement process is easily computable (it is equal to $q\sqrt{2/3}$) one may compare the present option pricing procedure with a direct application of the standard Black-Scholes formula (114). In figure 4 the price according to Black-Scholes is represented by triangles. The solid line denotes Bachelier's price ($\mathcal{B}$), while the dashed line gives the the risk-corrected price. The value of the Markowitz' parameter was taken to be 0.44. The difference between Black-Sholes and Bachelier prices is significant only for "far out of money" options. The inset of Fig.4 shows the ratio of both prices as a function of $k = S_c/S_0$.

The risk correction of the option price is important in the range $k > 1.5$, for which the correction $\sqrt{R_*}$ may be twice Bachelier's price. Figure 5 shows the time dependence of the risk correction $\sqrt{R_*}$ for $q = 0.1$ and $k = 1.15$ (lower, dashed line). For comparison we plotted Bachelier's price $\mathcal{B}$ (upper, solid line), which grows with time as $\sqrt{T}$. In the limit $T \to \infty$ the ternary process becomes Gaussian due to the central limit theorem and the relative weight of the correction $\sqrt{R_*}/\mathcal{B}$ tends to zero. For this value of k Black-Sholes price (triangles) coincides with Bachelier's price.

10 Summary, outlook and discussion

In this paper we have described share price movements on the Warsaw Stock Exchange. They differ from price movements in established markets. Relative fluctuations are much larger, although very large fluctuations are forbidden by trading rules. These rules are actually in effect most of the time. We introduced a simple model, which we call the ternary model, where share prices are allowed to go up, down, or stay the same, at each day of trading. In this simple model we showed analytically and numerically that risk cannot be entirely eliminated by trading.

We worked out in detail the recent approach of Bouchaud and Sornette [5] to (partial) hedging and option pricing when risk can not be eliminated. We find a qualitative new strategy which is adopted by sufficiently risk-willing operators. This strategy can be described as first investing in stock, and then writing options on top of that investment. To avoid paradoxical results, we are led to introduce a generalization of the Markowitz' portfolio criterion. The new criterion reduces to the Markowitz' criterion for low-risk operations, but depends for high-risk operations on the total amount of money an operator is prepared to loose in the market. We computed numerically the option price in the ternary model with parameters corresponding to data from the Warsaw Stock Exchange, and showed that the corrections to the ask price from off-setting risk are substantial.

When risk can not be eliminated an option does not have a definitive value. Different operators in the market, with different attitudes to risk and paying different fees for trading, will offer different prices to buy and sell. The market price of options is fixed by balancing supply and demand, and thus depends on the characteristics of the investor population active at any given time. We identified one special price, which is our candidate guess for the market price of options. For want of a better designation we call it here the normal price. When risk can be eliminated the normal price coincides with the price derived from arbitrage arguments. When risk cannot be eliminated we expect the instantaneous market price to fluctuate around the normal price. The normal price plus interest is equal to the gain of an operator using a special strategy, determined by the condition that risk is minimized over all possible strategies. There is no price for residual risk included in the normal price.

$$S_{i+1} = v_3 \cdot S_i \qquad \text{with probability } p_3$$

where v_1, v_2 and v_3 equal 0.9, 1.0 and 1.1, respectively. The three probabilities p_1, p_2 and p_3 are approximately equal, but not exactly since the average rate of return of shares on the WSE is positive.

We now wish to show that in the ternary price movement model the price of an option cannot be fixed by an arbitrage argument. To do so it is sufficient to just look at an option expiring tomorrow, that is, it expires on day T, and today is day $T-1$. Assume that we construct a portfolio containing χ shares and η bonds. If we want the portfolio to be worth the same as the option on day T, we must simultaneously satisfy three equations in two unknowns:

$$\begin{aligned} V(T, S_T = v_1 \cdot S_{T-1}) &= v_1 \chi S_{T-1} + r\eta b_{T-1} \\ V(T, S_T = v_2 \cdot S_{T-1}) &= v_2 \chi S_{T-1} + r\eta b_{T-1} \\ V(T, S_T = v_3 \cdot S_{T-1}) &= v_3 \chi S_{T-1} + r\eta b_{T-1} \end{aligned}$$

which is of course impossible. The arbitrage argument is therefore insufficient already over the time period of one day.

The general procedure of determining portfolio strategy and option price when residual risk cannot be made to vanish is in section 3 and appendix A. Detailed formulae for the ternary model are given in subsection A.2 of appendix A.

9 Numerical illustrations

In order to give a concrete example of a stock market for which a perfect hedging is not possible, we studied numerically a simple version of the symmetric ternary model $v_1 = 1-q, v_2 = 1, v_3 = 1+q$ with all three probabilities equal $p_1 = p_2 = p_3$. In the ternary price movement model it is straight-forward to write down explicit expressions for various quantities such as the residual risk and the option price. The technical details are carried out in subsection A.2 of appendix A. We will for simplicity only consider here the case when the mean rise of the stock is equal to the bond ($\mu = 0$). A numerical investigation of the more interesting case when μ is different from zero is in progress[1]. According to (69) the only possible value of the gain then is Bachelier's price (2), which for the ternary price movement model is given more explicitly in (81). The residual risk is $R^* = R_c - \Delta R$, where R_c and ΔR are given respectively in (82) and (89).

Figure 3 presents the residual risk as a function of the ratio $k = S_c/S_0$ between the strike price and the current price. We have taken $T = 20$ trading days, which corresponds to a month of real time. The cut-off parameter q, which limits the maximal daily relative price changes has been taken equal to 0.08 (lower curve), 0.10 and 0.12 (upper curve). Residual risk grows with the cut-off parameter q. Clearly visible oscillations of the risk are due to the discrete nature of the model. The period of the oscillations depends on q and k, however at $k = 1$ the period is approximately equal to q. Positiveness of the residual risk in the most interesting range of $k \sim 1$ makes the perfect hedging impossible. For very small or very large values of the price ratio R^* vanishes.

We checked that if the probability p_2 decreases while p_1 and p_3 are kept equal, the residual risk tends to zero. In this way we have smooth transition from the ternary model, where residual risk is non-zero, to the binary model where perfect hedging is possible.

In figure 2a we show the distribution of relative price changes (δ_S) of 19 shares with the longest trading record. The number of data points is on the order of 10^4. The solid line is a Gaussian with the same mean $\langle \delta_s \rangle = 0.98\%$ and the same standard deviation $\sigma = 6.91\%$. The distribution is obviously far from being a Gaussian. The effects of the market rules are clearly visible as two big peaks at up or down movements by 10 %, and another peak is situated right at the middle. Surprisingly enough, the most likely stock price behaviours on the WSE is for the market to go up or down as much as it is allowed to, or not move at all.

Another feature of the distribution in figure 2a, already mentioned above, that makes it different from a smooth distribution is the pronounced peak at $\delta_s = 0$. As far as we are aware this peak has not been noticed before. We conjecture that it is due to a psychological effect of the discrete nature of trading. Prices are settled once a day and strongly influence the orders investors prepare for the next trading session. Chances are greater that orders are executed when buying and selling at the current price, and therefore the number of events with a constant price rises far above the background distribution. On the other hand, the probability of very small fluctuations ($|\delta_s| < 0.3\%$) is small. This contrasts the case of continuous trading, common in established stock markets, where a temporary current price does not play such a role and the center of the distribution $P(\delta_s)$ is smooth.

The regulation of the market is less visible in the distribution of price changes during several consecutive trading days. Figure 2b presents a histogram made of the same data with a three days delay (the data points are not independent). The distribution is slightly closer to a normal one, but the peaks at $\pm 10\%, \pm 20\%$ and $+30\%$ persist, and we notice an evident asymmetry of the histogram. Note that the occurrence of the upper peak at 30% does not contradict observation that the auto–correlation functions vanish for the time delay equal to two trading days.

Coming back to more traditional measures of stock price behaviour we observe that both the mean rate of return and the volatility have historically been much larger on the WSE than commonly observed in established markets. The large rate of return is essentially a reflection of the bull market in 1993-94, and in the last year, from mid-1994 to mid-1994 prices have generally increased more slowly. The average daily increase of the WIG index is equal to 0.37% for the entire history of WSE, while the average taken over last 100 data of 1995 equals 0.09%. The fluctuations of the share prices and the index WIG are described by a huge volatility, much larger than these characteristic to established market. The daily volatility of WIG computed out of all data equals 4.06%. During the last year (mid-1994 to mid-1995) the fluctuations of WIG became milder and its volatility obtained from last 100 data is equal to 3.01%. It has also been suggested that the WSE is characterized by a relatively large number of investors willing to take large risks and of the presence of a few powerful investors, that can exert a non-negligible influence on the market [14, 27]. We believe that these observations, typical of WSE, can be also characteristic to other emerging markets.

It follows from the previous discussion that a good model of stock price movements on the WSE is that stock prices change in just three ways on each day of trading. We will call this the *ternary* model, as it is a simple generalization of the classic binary model investigated above in section 4. Prices then move as

$$\begin{aligned} S_{i+1} &= v_1 \cdot S_i \qquad \text{with probability } p_1 \\ S_{i+1} &= v_2 \cdot S_i \qquad \text{with probability } p_2 \end{aligned}$$

8 Price movements on the Warsaw Stock Exchange

The Warsaw Stock Exchange (WSE) started to operate in March 1991. During the first year only 11 different shares where exchanged and trading occurred once a week. Since then the frequency of trading has risen to 5 sessions per week. In June 1995 there were 42 different shares exchanged on the so-called principal market, the average number of orders for each trading session exceeded 30 000 and the average daily turnover was of order of 30 M USD. Macroeconomical and general socio-economical conditions have changed quickly in Poland during the 90'ies, and, accordingly, some experts distinguish three main periods in the history of the WSE [14, 27]. The first one, which lasted until mid-1993 or about 150 trading events, was characterized by a small number of investors and a small turnover. The next phase was a bull market lasting until mid-1994, or up to about the 350'th trading event, which attracted a large number of fresh investors. The last, current, period is is characterized by a slower overall growth of the share prices, smaller trading volume and a smaller number of active investors.

The main trading at the WSE is organized daily on the clearinghouse auction principle: offers to buy and sell a certain share (bids and asks) are collected, and then prices are set such that turnover is maximized. Later in the day there is a second trading event, where additional orders can be placed at the now fixed price of the day. In order to stabilize the market the following regulation has been introduced: relative price changes are not allowed to be more than 10% per day (up to round-off errors). In the history of the WSE this rule has only been lifted by the board a few times, and can for all practical purposes be taken to be strict.

The general state of the WSE is measured by a capital index WIG (*Warszawski Indeks Giełdowy*), which is approximately equal to the average of stock prices weighted according to the market value of the outstanding shares in each stock. In figure 1 we show the time evolution of the WIG in logarithmic scale, together with the prices of two shares with the longest history (563 data points). All three curves are normalized with their initial value in 1991. The WIG is an average, and therefore displays smaller fluctuations than the two shares. Nevertheless it is visually quite clear that both prices are strongly correlated with the index and with each other. These results are in no way exceptional: the normalized equal-time cross-correlation coefficients between different shares computed on data for the last two years is reported to vary from 0.3 to 0.9 [27].

Let $\delta_l = (S_l - S_{l-1})/S_{l-1}$ denotes the relative change of the price of a share and let $\sigma^2 = \langle\delta^2\rangle - \langle\delta\rangle^2$. The daily mean increment $\langle\delta\rangle$ for a majority of stocks is of order of 0.5%, what reflects an overall growth of the prices, while the daily volatility σ varies from (5% to 7%) for different stocks.

The auto-correlation function of the price changes of the same share at different times defined by

$$H(n) = \sum_{l=1}^{N} (\delta_l \delta_{n+l} - \langle\delta\rangle^2)/N\sigma^2 \tag{45}$$

generally decays fast with the time delay. Auto-correlation $H(n)$ is practically zero for time delays of two days or more. Only one-day correlations are of some importance: $H(1)$ for different branches of industry vary from 0.04 to 0.26 and the average over all stocks is 0.15. The positiveness reflects an inertia of the stock market: if the price of a share is rising (or falling), this process is likely to persist for another day. On time scales more than one day the WSE is hence weakly efficient in the sense of Fama [13], but on a day-to-day basis some deviation from efficiency is observed.

A comparison reveals that maximum price of any buyer is less or equal to the minimum price of any seller. The only price on which they can theoretically agree is $\mathcal{A}$, and this price in turn is only acceptable to sufficiently risk-willing operators who nevertheless only write out a few options.

7 Why is there option trading at all?

The results of the preceding section indicate that the volume of trade in options should be small, as all asks are higher or at least equal to all bids. This contrafactual conclusion needs an explanation. There are several possible explanations, most rather self-evident, but for completeness listed here.

Inhomogeneous expectations: We have assumed that all investors agree on the characteristics of the price process. In reality this is not the case, and one can be fairly sure that the buyer of an option tends to take a more optimistic view of future price movements. This effect is sufficient that some bids will be higher than some asks.

Trading costs: In reality an operator pays a fee on all transactions. This effect may easily be included in the wealth equation (7). The result of including such market friction is that hedging operations become more expensive, which in turn raises both bid and ask prices. Typically very large investors pay proportionally smaller fees. Their asks can then be lower than bids of smaller investors.

Market liquidity: It may be difficult to realise a portfolio containing a negative number of shares, as is typically required in the hedging strategy of the buyer of an option. This tends to raise the bids.

Other utility functions: Plausibly investors may have use very different utility functions that those of the mean-variance type. An example would be if the investors are very sensitive to losses. If so, writing an option or buying stock would be discouraged, and the bid would rise.

Further liabilities: Commonly options are used as insurance for another investment. Then the wealth process does not include only the options and buying and selling stock, but also the primary investment, whatever that is. Different operators with different a priori liabilities will then offer different bids and asks.

Investor bias: The popularity of the Black-Scholes option price prescription probably influences at least some investors to offer bids or asks at that price, using estimates of volatility and expected return on the stock from price history, even if this is above or below their rational choice of highest or lowest price.

The conclusion from all these points is that option price is set by balancing supply and demand. Most of the effects tend to raise the bids and asks, such that we expect that the market price should perhaps be somewhat above $\mathcal{A}$.

Let us end by saying that if the market price of an option is given, the logic of the optimization problems studied above in sections 3, 5 and 6 is turned around. If an operator would be prepared to sell an option for a lower price than the going market price she could in principle increase her utility further by opting for a more risky strategy than the one with which she can offer the lowest price. As long as we stay within mean-variance utility arguments the results do however not change. If utility as a function of gain is given by $\tilde{C} + M - f(R[M])$, where f is some function of risk and the price $\tilde{C}$ is fixed, utility is maximized by maximizing $M - f(R[M])$. This is equivalent to minimizing $-M + f(R[M])$, which was our previous procedure.

Summing up, we have previously shown that if the dimensionless combination $\lambda^2\nu$ tends to one from above, the ask price tends to a minimum value $\mathcal{A}$. This price is simply described in the following way: profit of an imaginary operator adopting this price, and a special trading strategy that minimizes risk inconditionally, is zero. Let us remark that the special strategy will only be adopted by an operator who is very sensitive to risk. Her ask price will contain a large insurance against risk, and so be much larger than $\mathcal{A}$. The special strategy and the minimum price are therefore in fact not chosen together by one operator at the same time.

In this section we have shown that if $\lambda^2\nu$ is less than one and the operator invests in stock but chooses to write few options, she will again settle for the same price. In addition we have shown that if ρ is zero, that is if risk can be made to vanish at a certain level of gain, every operator will settle for $\mathcal{A}$ irrespective of λ.

6 The buyers' side of the market

In this section we repeat the analyses of sections 3 and 5 above, but taking instead the perspective of the buyer of the option. In analogy with (7) her wealth at time T will have increased by an amount

$$\Delta W_{\text{buyer}} = -C \cdot r^T + \{S_T - S_c\}^+ + \sum_{i=0}^{T-1} [\phi_i S_i u_i - \phi_i S_i r] r^{T-i-1}. \tag{38}$$

If we introduce the new variables

$$\psi_i = -\phi_i S_i r^{T-i-1} \tag{39}$$

we can simply write $\Delta W_{\text{buyer}} = -\Delta W_{\text{writer}}$. An optimal trading strategy of the buyer at given level of gain is then simply related to the one previously determined of the writer:

$$G^*_{\text{buyer}}(S_i; M) = G^*(S_i; -M) \tag{40}$$

Let us remark that due to the minus sign in (39), an optimal portfolio for a risk-sensitive buyer ($\psi_i = G^*_{\text{buyer}}(S_i, M)$) typically includes a negative amount of shares, i.e. going short.

All the previous analysis now goes through. The risk is

$$R_{\text{buyer}}[M] = \rho + \nu(M - \mathcal{A})^2, \tag{41}$$

with the same meaning of the coefficients as before.

The maximum price the option buyer is prepared to pay using the Markowitz' portfolio criterion and a trading strategy of gain M is given by

$$-\tilde{C}[M] + M = \lambda\sqrt{\rho + \nu(M - \mathcal{A})^2}. \tag{42}$$

Maximizing $\tilde{C}$ yields the optimal trading strategy and the highest price the buyer is prepared to pay:

$$M^* = \mathcal{A} + \sqrt{\frac{\rho}{\nu(\lambda^2\nu - 1)}} \tag{43}$$

$$\tilde{C}[M^*] = \mathcal{A} - \sqrt{\frac{\rho(\lambda^2\nu - 1)}{\nu}} \tag{44}$$

avoid this turn of events. We can model such an effect by assuming that the operator maximizes a utility function of the form

$$U(P,R) \quad = \quad P - \lambda\sqrt{R} - \frac{1}{2}\frac{1}{W_0}R \tag{31}$$

where W_0 sets a scale on the amount of money this operator can loose with impunity. Further, any smooth utility function can be expanded in powers of the risk. Equation (31) then gives the next approximation beyond the linear. Inserting (29) in (31) in the interesting parameter range where $\lambda^2\nu$ is less than one, we find that profit is maximized at

$$P^* \quad = \quad W_0\frac{1-\lambda\sqrt{\nu}}{\nu} \tag{32}$$

and the maximal utility of investing only in stock is

$$U_0 \quad = \quad \frac{1}{2}W_0(1-\lambda\sqrt{\nu})^2 \tag{33}$$

We now switch to an option writer. It is convenient to consider the case of a writer writing x number of options. If M is the gain from trading and possible fall-out of x options, and the coefficients ρ, ν and $\mathcal{A}$ are as in section 3, the minimal risk at given level of gain is

$$R[x,M] \quad = \quad x^2\rho + \nu(M + x\mathcal{A})^2 \tag{34}$$

A post-Markowitz utility analogous to (31) then reads

$$\begin{aligned} U(\tilde{C},M) \quad &= \quad \tilde{C} + M - \lambda\sqrt{x^2\rho + \nu(M+x\mathcal{A})^2} \\ &\quad -\frac{1}{2}\frac{1}{W_0}(x^2\rho + \nu(M+x\mathcal{A})^2) \end{aligned} \tag{35}$$

The minimal price of x options at a given level of gain is fixed by the condition that the utility to the option writer can not be less than the utility of investing only in stock. That leads to

$$\begin{aligned} \tilde{C}[x,M] \quad &= \quad -M + \lambda\sqrt{x^2\rho + \nu(M+x\mathcal{A})^2} \\ &\quad +\frac{1}{2}\frac{1}{W_0}(x^2\rho + \nu(M+x\mathcal{A})^2) + \frac{1}{2}W_0(1-\lambda\sqrt{\nu})^2 \end{aligned} \tag{36}$$

It is not difficult to now find the optimum gain by minimizing the left hand side in (36), and the value of that minimum will be the minimal price offered for writing x options by this writer.

Let us for simplicity consider the situation where the operator is investing in stock and then writes a few options on the margin. That corresponds to the limit when $x\mathcal{A}$ is much smaller than W_0. In that limit we may expand the square root and find

$$\tilde{C}[x,M] \quad \sim \quad x\mathcal{A} + \mathcal{O}(x\mathcal{A}/W_0). \tag{37}$$

The price per option in the small x limit is $\mathcal{A}$.

We can also consider the situation when ρ vanishes, as in the binary price movement model. Then (36) simplifies considerably. The minimum is attained at M equal to $\frac{W_0(1-\lambda\sqrt{\nu})}{x\nu}$, and the price of x options is always fixed by $\tilde{C}[x] - \mathcal{A}x = 0$.

It is not hard to show that ΔW_{binary} in (25) always vanishes. Therefore the risk is zero. Returning to the discussion in the beginning of this section we have hence shown that a sufficiently risk-aversive option writer, choosing her trading strategy with the method of Bouchaud-Sornette, will opt for the perfect hedge. In the following section we will show that the same thing holds also for very risk-willing option writers.

5 Risk-willing operators, and a digression on investing in stock

In the two preceding sections we run into the problem that sufficiently risk-willing operators prefer to invest an unlimited amount in stock. This is an unreasonable result, and calls for a generalization of the Markowitz' portfolio criterion.

To arrive at a solution of the option pricing problem in this case we first show that something very similar occurs to a presumptive investor who is only interested in buying stock on the margin. The utility in investing writing options should then be compared with the utility of investing in stock, and this will fix for us the option price and the hedging strategy.

In analogy with (7) the wealth of such an investor has at time T increased by the following amount:

$$\Delta W_{\text{stock}} = \sum_{i=0}^{T-1} [\phi_i S_i u_i - \phi_i S_i r] r^{T-i-1} \tag{26}$$

where ϕ_i is the amount of shares in the portfolio on day i, and S_i is the price of one share on that day. The profit of this operation is the expectation value of the wealth process and the risk is the variance:

$$\begin{aligned} P_{\text{stock}} &= < \Delta W_{\text{stock}} > && (27) \\ R_{\text{stock}} &= < \Delta W^2_{\text{stock}} > -(< \Delta W_{\text{stock}} >)^2 && (28) \end{aligned}$$

At equal profit the investor prefers less risk. At given level of profit we can therefore identify an optimal strategy and a minimal level of risk:

$$R_{\text{stock}}[P] = \nu P^2 \tag{29}$$

The coefficient ν is the same as that found in section 3. The actual portfolio will be directly proportional to P, and similar to buy-and-hold if T is sufficiently large. Formula (29) and the portfolio are derived below in subsection A.3 of appendix A.

Assume now that the investor tries to maximize a linear utility function, that is

$$U(P, R) = P - \lambda\sqrt{\nu P^2} \tag{30}$$

If $\lambda^2\nu$ is greater than one utility is maximized at P equal to zero, i.e. the investor does not buy stock. On the other hand, if $\lambda^2\nu$ is less than one utility grows ever larger as the investor buys more and more stock. We therefore have the same transitions between investing small amounts and investing a formally infinite amount as in the case of options.

The problem is obviously with the linear utility function. Every real operator is finitely wealthy, and if she looses all her wealth she goes bankrupt and is no longer allowed to continue investing in the market. Any operator is supposed to always

Take the point of view of a buyer of the option. Instead of holding the option, she could have invested in a portfolio containing χ shares and η bonds. We allow η to be negative, meaning that the buyer borrowed η bonds. At time $T-1$ such a portfolio would be worth

$$P(T-1, S_{T-1}) = \chi S_{T-1} + \eta B_{T-1}.$$

Now make the ansatz that at time T the portfolio is worth the same as the option:

$$\begin{aligned} V(T, S_T = u \cdot S_{T-1}) &= \chi u S_{T-1} + \eta r B_{T-1} \\ V(T, S_T = d \cdot S_{T-1}) &= \chi d S_{T-1} + \eta r B_{T-1}, \end{aligned}$$

which yields the solutions:

$$\begin{aligned} \chi(T-1, S_{T-1}) &= \frac{1}{S_{T-1}(u-d)}[V(T, S_T = u \cdot S_{T-1}) \\ &\qquad -V(T, S_T = d \cdot S_{T-1})] \\ \eta(T-1, S_{T-1}) &= \frac{1}{rB_{T-1}(u-d)}[uV(T, S_T = d \cdot S_{T-1}) - dV(T, S_T = u \cdot S_{T-1})] \\ P(T-1, S_{T-1}) &= r^{-1}[V(T, S_T = u \cdot S_{T-1})\frac{r-d}{u-d} \\ &\qquad +V(T, S_T = d \cdot S_{T-1})\frac{u-r}{u-d}] \\ V(T-1, S_{T-1}) &= P(T-1, S_{T-1}). \end{aligned}$$

The last equality follows because the option and the portfolio are by construction always worth precisely the same to the presumptive option buyer at time T.

Since the value of the option at time $T-1$ is now determined for whatever stock price, we can work backwards in time to arrive at the value of the option and the portfolio on any day down to day zero.

$$\chi(0, S_0) = \frac{1}{S_0(u-d)}[V(1, S_1 = u \cdot S_0) - V(1, S_1 = d \cdot S_0)] \tag{23}$$

$$V(0, S_0) = r^{-T} \sum_{l=0}^{T} \binom{T}{l} (\frac{r-d}{u-d})^l (\frac{u-r}{u-d})^{T-l} V(T, S_T = u^l d^{T-l} \cdot S_0), \tag{24}$$

We note that the probability p dropped out of the expressions for the portfolio and the option price. This is so because prices change in just two ways, and we have two securities (the stock and the bond) in which we can invest. By choosing the portfolio properly we can always offset the effects of stock price movements, and the probabilities with which these happen play no role.

Let us now switch to the perspective of the option writer. Suppose she sells the option to the buyer on day zero for a fee $V(0, S_0)$, and chooses the trading strategy $\chi(i, S_i)$ on consecutive trading days, i ranging from zero to $T-1$. The total increase in her wealth, as in (7), is then

$$\begin{aligned} \Delta W_{\text{binary}} &= V(0, S_0) \cdot r^T - V(T, S_T) + \sum_{i=0}^{T-1} [\frac{1}{u-d}(V(i+1, S_{i+1} = u \cdot S_i) \\ &\qquad -(V(i, S_{i+1} = d \cdot S_i))(u_i - r)] \cdot r^{T-i-1} \end{aligned} \tag{25}$$

where each of the u_i's is either u or d in a particular realization of the price process.

Different operators at different λ opt for different values of optimal gain, and consequently for different trading strategies. When λ decreases operators tend to use strategies that give them more and more gain, and eventually they will be prepared to settle for a price that is lower than Bachelier's price (2). The lowest price we find is in the limit when $\lambda^2\nu$ tends to one:

$$\tilde{C}[\infty] \quad = \quad \mathcal{A}. \tag{20}$$

This price $\mathcal{A}$ is our candidate for the market price. It is such that the profit of an operator using a strategy which minimizes risk over all possible strategies is zero. The residual risk that still remains is not priced.

A somewhat special case is when the mean rise of the stock is the same as that of the bond. In this limit it is easy to see that the gain can only take one value, and all operators therefore opt for that, and use the same strategy. The price offered by any of them will be Bachelier's price, plus a risk premium proportional to λ. In particular, nobody will offer a price lower than (2). For details of this case see subsection A.1 of appendix A.

4 The binary walk price process revisited

We will first investigate what happens to the Bouchaud-Sornette option pricing procedure if perfect hedging is possible, and then verify that such is the case in the discrete binary stock price movement model [8]. The continuous-time model of Black and Scholes is treated below in appendix B.

Risk is by construction a non-negative quantity. If, at a given level of gain, the risk can be made to vanish using a certain strategy, that strategy is also optimal according to Bouchaud and Sornette. Furthermore, risk using an optimal strategy is a quadratic polynomial in the gain. Among the possible optimal trading strategies at different levels of gain there is therefore at most one which is a perfect hedge.

Let therefore risk as function of gain be

$$R[M] = \nu(M + \mathcal{A})^2 \tag{21}$$

with some positive coefficient ν. At the value M equal to $-\mathcal{A}$ the risk vanishes, that is, at this value of trading gain the optimal strategy is a perfect hedge. The price according to the Markowitz' portfolio criterion is

$$\tilde{C}[M] \quad = \quad -M + \sqrt{\lambda^2\nu}|M + \mathcal{A}|. \tag{22}$$

If $\lambda^2\nu$ is less than one, the minimum is formally obtained at arbitrary large gains. In this case we will have to go beyond the Markowitz' criterion, just as in the general case. If on the other hand $\lambda^2\nu$ is greater than one, the minimum is always obtained at $-\mathcal{A}$, and we have determined that the perfect hedge is the optimal trading strategy for all these values of λ.

Let us now turn to the binary price movement model. It is assumed that stock prices can change at discrete time intervals in just two ways:

$$\begin{aligned} S_{i+1} &= u \cdot S_i \qquad \text{with probability } p \\ S_{i+1} &= d \cdot S_i \qquad \text{with probability } 1 - p. \end{aligned}$$

Consider time $T - 1$ and a given share price at that time, S_{T-1}. At striking time the option will have the values:

$$\begin{aligned} &V(T, S_T = u \cdot S_{T-1}) \qquad \text{with probability } p, \\ &V(T, S_T = d \cdot S_{T-1}) \qquad \text{with probability } 1 - p. \end{aligned}$$

Within the Markowitz' portfolio criterion the minimum profit an option writer is prepared to settle for, if she has decided to use a trading strategy with gain M, is

$$\tilde{C} + M = \lambda\sqrt{R[M]} \tag{13}$$

Generally speaking, M could well be negative, but the price must in the end be high enough that the profit is not negative. Putting M on the right hand side we determine the minimum acceptable price to this operator as a function of gain:

$$\tilde{C}[M] = -M + \lambda\sqrt{R[M]} \tag{14}$$

The option writer competes on the market with other operators that have similar predilections to take risks. All of these can choose different strategies of the same utility by opting for different gain. By choosing M such that $\tilde{C}$ determined by (14) is as small as possible, the writer has an advantage on the market, and this fixes M^*, the optimal level of gain for this writer. We have thus determined the optimal portfolio strategy and the minimum price asked by a writer with a given value of λ.

The general structure of the solution of the constrained minimization problem is that the optimal portfolio will be a linear function of gain (63), and the risk a quadratic function (64). Let us therefore see how the option price and optimal level of trading gain is determined if the risk is written as

$$R[M] = \rho + \nu(M + \mathcal{A})^2, \tag{15}$$

where the coefficients in the polynomial are computed in appendix A and given in (66), (67) and (68) respectively. We assume that ρ is positive. The case when ρ is zero, and thus risk entirely eliminated at a level of gain equal to $-\mathcal{A}$, is treated below in section 4.

The price according to the Markowitz' portfolio criterion is then

$$\tilde{C}[M] \quad = \quad -M + \lambda\sqrt{\rho + \nu(M + \mathcal{A})^2}. \tag{16}$$

At infinity the two functions M and $|M + \mathcal{A}|$ grow at the same rate. Depending on the values of λ and ν the minimum of $\tilde{C}$ is therefore either obtained at a value of M close to $-\mathcal{A}$, or at arbitrarily large values of M.

The second case, which occurs when $\lambda^2\nu < 1$, logically leads to the writer being willing to charge zero or negative price for the option. We consider that this paradoxical result calls for a generalization of the Markowitz' criterion, and we discuss this, and what it means for hedging and option pricing, in section 5 below.

In the first case however, and using formula (16) for the price, minimizing with respect to M leads to:

$$M^* \quad = \quad -\mathcal{A} + \sqrt{\frac{\rho}{\nu(\lambda^2\nu - 1)}} \tag{17}$$

which, inserted in (63), gives the portfolio strategy. The resultant risk is

$$R[M^*] \quad = \quad \frac{\rho\lambda^2\nu}{\lambda^2\nu - 1} \tag{18}$$

and the ask price of the option is

$$\tilde{C}[M^*] \quad = \quad \mathcal{A} + \sqrt{\frac{\rho(\lambda^2\nu - 1)}{\nu}}. \tag{19}$$

profit of $\phi_{T-1}S_{T-1}(u_{T-1}-r)$, and then faces the prospect of selling a share to the buyer for the striking price S_c. If the market price S_T is lower than S_c the buyer will not want to exercise the option, and the writer looses nothing. If however it is higher, the writer has to buy at S_T, sell at S_c, and looses $(S_T - S_c)$. In (7) we could as well have included transaction costs [5, 6].

To simplify the notation we introduce new variables $\psi_i = \phi_i S_i r^{T-i-1}$ and $\tilde{C} = C r^T$, and assume that the writer chooses her portfolio in a deterministic way[3]

$$\psi_i = G_i(S_i) \tag{8}$$

The set of functions $\{G_i\}$ will be called a portfolio strategy or a trading strategy. It may depend on the characteristics of the price process relative to the bond, partly given by r, m and σ, on the parameters of the particular option (S_c and T) and on S_i, the actual value of the share on day i in this particular realization of the price process. Since we assume that the price increments are independent, no knowledge is gained by taking into account the previous price history before day i.

In the time period between day zero and day T the sum gained by the writer when she sells the option has been invested in bonds, and differs in no way from other capital she possesses. The portfolio strategy should therefore not be influenced by the price the option was sold for, but of the price process actually in course. Our presentation here differs slightly from [5]. As important characteristics of the portfolio strategy we therefore take the average M and the variance R of the wealth process due only to trading and possible fallout of the option:

$$\begin{aligned} M[\psi] &= < \Delta W_{\text{trading}} > \\ &= - < \{S_T - S_c\}^+ > + \sum_{i=0}^{T-1} < \psi_i(u_i - r) > \qquad (9) \\ R[\psi] &= < (\Delta W_{\text{trading}})^2 > - (< \Delta W_{\text{trading}} >)^2 \\ &= < (\{S_T - S_c\}^+)^2 > -2 \sum_{i=0}^{T-1} < \{S_T - S_c\}^+ \psi_i(u_i - r) > \\ &\quad + < \left(\sum_{i=0}^{T-1} \psi_i(u_i - r) \right)^2 > - \quad M^2 \qquad (10) \end{aligned}$$

We will refer to M and R as the gain and the risk. The profit equals the sum of the gain and the price of the option plus interest:

$$P = \tilde{C} + M \tag{11}$$

For portfolio strategies with equal gain, the best one for the writer is one for which the risk is minimal, that is

$$R[M] = \text{Min}_{G_i} R[\psi_i = G_i(S_i)] \tag{12}$$

where the minimization is taken over all sets of functions $\{G_i\}$ such that M in (9) is fixed. The minimization is conveniently carried out with the method of Lagrange multipliers. See appendix A for details.

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[3] Conceivably the writer could also choose her portfolio randomly, or a moment of randomness in the portfolio choice could model the writer lacking knowledge of the state of the market.

unit of time will be called 'days'.

The main idea of the method is to solve the hedging and option pricing problems by balancing risk against profit. This calls for a definition of risk. We follow [5] in measuring subjective risk taken by a writer by the variance of the wealth process. The profit of the writer is the expectation value of the wealth process.

Let us therefore assume that a presumptive option writer tries to maximize a utility function that depends only on risk and profit. In the simplest case we assume a linear relationship between expectation value and standard deviation:

$$U(P,R) = P - \lambda\sqrt{R} \tag{3}$$

where P is the profit, R is the risk and λ is a dimension-less constant which characterizes the risk-aversiveness of this particular operator. The maximization is performed over all possible actions by the operator that include writing one option and then buying and selling shares, and also not writing an option at all. Zero, the utility of doing nothing, is therefore always the minimum value of maximized utility to be found from (3). This means that for any positive action the minimum acceptable profit at given level of risk is given by

$$P = \lambda\sqrt{R}, \tag{4}$$

which is known as Markowitz' portfolio criterion[17]. Later we will consider more general utility functions and portfolio criteria, see section 5.

Let us now turn to a more detailed description of the wealth process. In practice options are generally written for large numbers of shares, and we will therefore assume from now on that the number of shares held by the writer against the option can take any real value. In the time interval $[0,T]$ the writer can buy and sell shares once a day.

We take the price process as:

$$S_{i+1} = u_i S_i \tag{5}$$

where S_i is the price at day i and u_i is the relative price change. The variables u_i are independent, identically distributed random variables with mean m and variance σ^2. In addition we assume there exists a risk-less security called a bond or a bank account, which always prices up by a constant factor r per day:

$$B_{i+1} = rB_i \tag{6}$$

The number of shares held by the writer between trading on days i and $i+1$ is written ϕ_i, and the price charged for the option by the writer is C. The wealth of the writer has then at time T increased by the following amount [5, 6]:

$$\Delta W_{\text{writer}} = C \cdot r^T - \{S_T - S_c\}^+ + \sum_{i=0}^{T-1} [\phi_i S_i u_i - \phi_i S_i r] r^{T-i-1}. \tag{7}$$

The logic of (7) is that on day zero the writer sells the option for C and invests that sum in bonds. Then she takes out an amount $\phi_0 S_0$ of bonds and invests that in shares. On day one these shares are worth $\phi_0 S_0 u_0$, compared to $\phi_0 S_0 r$, which would have been the value of the bonds. The writer realizes this positive or negative profit, invests that sum in bonds, and again takes out a sum $\phi_1 S_1$ of bonds and invests in shares. This process is continued until day T, when the writer first realizes the price is called the *option pricing problem.*

It is instructive to recall here the approach to option pricing of Bachelier [2]. He considered European call options[1], which in the language of his day were called "operations à prime" [2]. Bachelier did not consider hedging. Presumably such operations were not done at the Paris stock exchange at that time, or were too cumbersome. Neither did he consider that the sum for which the option was sold could be put in the bank by the writer to yield interest up to time T.

The wealth gained by the option writer at time T in one particular option transaction was then

$$\Delta W = C - \{S_T - S_c\}^+ \qquad \text{(Bachelier's option)} \tag{1}$$

where C was the price of the option, S_T the share price at time T and S_c the strike price. The short-hand $\{S_T - S_c\}^+$ denotes the positive part, that is $(S_T - S_c)$ if S_T is greater than S_c, and otherwise zero. This notation expresses in a compact way the pay-off function of a European call option: if S_T is larger than the strike price, the writer has to buy at S_T, sell at S_c, and so looses $(S_T - S_c)$. However, in the opposite case, the buyer does not exercise her right to the share, and the writer just pockets the selling price C.

Bachelier postulated that the average profit of an operator is zero:

$$C = \mathcal{B} \qquad \mathcal{B} = < \{S_T - S_c\}^+ > \qquad \text{(Bachelier's option price)} \tag{2}$$

where the expectation value is taken over all realizations of the price process. We introduce a special letter $\mathcal{B}$ since this expectation value will reappear later many times. If hedging operations are possible, (2) is not an appropriate procedure to fix the price. To take the extreme case, if perfect hedging is possible, which one can easily see that it is in the Gaussian price movement model explicitly investigated by Bachelier, then writing an option is a risk-free operation, similar to putting money in the bank. The return on an investment in options must then be the same as the bank rate, since otherwise there would be two different bank rates available, and an operator could make a risk-less profit on the difference. This argument for fixing the option price is called the principle that there should be no *arbitrage* in the market[3].

Alternatively, there will then be a gain from fall-out of the option and hedging, which is certain, not random. The price of the option is adjusted such that the wealth gained by a writer using the perfect hedge is zero. In this sense the arbitrage argument fits well with the fair-game argument of Bachelier.

In general we would like to call any option pricing procedure to be of the Bachelier type if a special hedging strategy is first found by some argument that does not involve the price, and then afterwards the price is determined such that the expected profit of an operator using the special hedge is zero. One conclusion of the present work is that a typical market price of options should be determined this way. This is a natural generalization of the arbitrage arguments[3] and a validation of a brief statement by Bouchaud and Sornette in [22].

3 Option pricing according to Bouchaud-Sornette

The content of this section is a repetition of the integrated-time approach to option price setting presented in [5], done here for the case of discrete time processes. The

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[1] On 'coupons', or French treasury bonds, but this difference is not essential.

[2] In contrast to "operations à terme fermes", which corresponded to what today is called futures contracts.

can not then be predicted only by their previous history.

There does not seem to be similar wide agreement on the actual distribution of the individual increments in stock prices. The Gaussian and the Lévy distributions differ mainly in the distribution of large and rare events, and it is therefore not evident to differentiate between them using limited data sets. In addition a number of empirical modifications of the log-Brownian price process has been proposed, typically by letting the parameters in the Gaussian be time-dependent (c.f [11, 4]), and it is clear that sufficiently elaborate models of this kind can be fitted to a very wide class of data. The recent studies referred to above favor that price increments in several different markets follow a Lévy law, except far in tails where the distributions fall off more quickly. Since sums of random variables with cut-offs in the tails eventually tend to the normal Gaussian distribution if sufficiently many terms are included, the fit to a Lévy law should be best for short time intervals, and then progressively grow less evident for longer times [23, 21]. Similar results where also obtained recently by Bouchaud *et al* using a rank ordering technique on a more limited data set [7].

The data on the Warsaw Stock Exchange that we describe more fully below in section 8 gives a different picture from either a Gaussian or a Lévy distribution, with or without cut-offs. In this market a trading rule disallows relative price changes of more than 10 % per day, which in itself is a large change. Nevertheless, price movements of up or down by 10 % are among the most frequently occurring, and perhaps more surprising, there is in addition a preference for the price not moving at all. A simple qualitative model that we study below, is therefore to assume that prices may either move up, down, or stay the same on every day of trading, which may also be thought as one of the simplest generalizations of the binary price movement model [8] (see section 4 below). A similar model was considered in another context in [19].

Given that stock prices are hard to predict, it is not surprising that different ways have been invented to insure a potential investor against adverse price movements: such contracts are called *options* [8, 9]. The theoretically simplest example is a *European call option*, which is a contract sold by one operator, the 'writer', to another operator, the 'buyer', by which the writer undertakes to sell to the buyer one share of a certain stock at an agreed 'strike price', S_c, on an 'expiration date', T, in the future. The buyer is not obliged to buy the share if the market price at T happens to be lower than S_c and for this insurance she pays a fee to the writer.

During the time interval from now up to T the writer can buy and sell shares on the market. If, for instance, the share price quickly rises up to or above the striking price, it seems intuitively obvious that the writer does better in buying a share at an early date, instead of waiting until expiration time and then probably buying the share at a higher price. This buying and selling shares by the writer is called *hedging*, and the problem of finding the best hedging strategy is called the *hedging problem.* If a strategy can be found by which the writer entirely offsets the risk to which she exposes herself from adverse price movements, one says that *perfect hedging* is possible. Two classic results are that in the binary price movement model, and in the continuous-time log-Brownian model, perfect hedging is possible [8, 3] (*see* section 4 and appendix B below).

The writer is prepared to sell the option to the buyer at a certain price, which depends on what the writer knows or guesses about future price movements, the parameters of the option being sold (the present share price, the strike price and the expiration date), and on the hedging strategy used by the writer. Determining this a descriptive analysis of the Warsaw Stock Exchange, which has been in operation since 1991. The data suggest a simple model of price movements that is nevertheless sufficiently rich that risk can indeed not be eliminated. We work out the details of the hedging and option pricing in this model. We also review the latest available high-frequency and high-quality data on price movements in developed markets, and point out that in cases documented in the literature price movements are such that option risk can not be eliminated here either.

The paper is organized as follows: in section 2 we briefly review different statistical models of the stock price movements, define what is an option on common stock, and formulate the twin problems of hedging and option pricing. In section 3 we repeat the integrated-time approach to option pricing from [5] with due attention to that the expected rate of return of the stock may be different from that of the bond. In section 4 we repeat the derivation of option pricing within the binary price movement model, and show that for sufficiently risk-aversive operators the general approach of section 3 leads to the classic result. In section 5 we investigate the qualitatively new strategy available to sufficient risk-willing investors, and in section 6 we look at the buyer's side of the market. In section 7 we list reasons for the bids and asks to actually vary away from the Bouchaud-Sornette prescription, such that there is room for trading. In section 8 we describe stock price movements on the Warsaw Stock Exchange and introduce a simple model in which risk can not be eliminated. Section 9 contains numerical illustrations. In section 10 we sum up and discuss our results. Appendix A contains details of the derivations, and in appendix B we repeat for completeness the option pricing prescription of Black and Scholes by deriving it from the discrete-time binary price movement model.

2 Stock prices, options, hedging and option pricing

The first *descriptive* theory of the markedly erratic behaviour of speculative prices was proposed by Bachelier in 1900. He assumed that prices perform what is today called a Brownian walk [2]. His model is clearly not fully realistic, since prices may become negative, and in the subsequent economic literature the standard model has instead been that the logarithm of the price performs a random walk. One attractive feature of this the *log-Brownian model* follows from the central limit theorem: if relative price increments in logarithmic scale over short time-intervals are identically distributed independent random variables of finite variance, then their sum, that is the relative price increments in logarithmic scale over larger time-intervals, is a random variable the distribution of which tends to the Gaussian distribution.

Sums of random variables having distributions with power-law tails, such that the variance is not finite, do however instead tend to the Lévy laws [15, 10]. These are stable laws like the Gaussian, but with the same power-law tails as the the original random variable being summed. Mandelbrot in 1963 proposed that prices in commodity markets, in particular prices of cotton in the United States at different times from 1880 to 1958, follow one of the laws of Lévy [18, 20]. The hypothesis that stock price increments follow a Lévy law was subsequently taken up by Fama [12], and recently by the group of Olsen[23] and by Mantegna and Stanley[21].

Most published studies of stock markets indicate that price increments at different times are independent, or become practically independent after short time delays. Loosely speaking one may refer to this feature as the *random walk* hypothesis. In the terminology of Fama [13], a stock market where stock prices perform random walks is said to be (at least) *weakly efficient*, since stock prices in the future

1 Introduction

In Scholastic thought the *natural price* of a commodity is assumed to be its *value*. If a commodity is sold at a lesser or higher price this is deemed incorrect or dishonest behaviour on the part of the buyer or the seller.

On the other hand, in economic thought, price is set by market forces that balance supply and demand. This definition contrasts with the first in that prices are determined by what the majority of economic agents operating today are able to understand. If we consider goods or commodities that are to be consumed or delivered in the future, some agent might make a better judgment of future value, and eventually make a profit out of her less fortunately endowed colleagues.

The two definitions can be combined if it is assumed that the word *value*, or *definite value* is reserved for commodities which an agent can predict with certainty. In the real world most often there is an unpredictable element, and *price* contains as well an insurance against unpleasant surprises. Different economic agents have different attitudes to risk, different access to information and different ability to evaluate that information and predict the future. They will therefore offer different prices to buy or to sell a commodity, and the market mechanism decides which of these buy and sell orders get executed.

The present paper contains an illustration to the above distinctions. We study the classic problem of pricing of an option on common stock. Within two simple models of stochastic stock price movements, the discrete-time binary price movement model [8] and the continuous-time log-Brownian model of Black and Scholes [3], by buying and selling stock the operator selling the option can completely eliminate her risk. The option in this case therefore has a value. For more general stock price movements risk can not be eliminated entirely and the option does not have a definitive value. The bid and ask prices of the option can nevertheless be fixed in a rational way by balancing risk with an increased expected profit.

The inspiration for our work is a recent paper by Bouchaud and Sornette [5]. Using very intuitive arguments they showed how to rationally fix the ask price of an option when risk can not be eliminated. For simplicity Bouchaud and Sornette concentrated on how to find an optimal strategy for buying and selling stocks against the option, and postulated that the price was fixed beforehand by a fair game argument. This does not give the correct price in the continuous-time log-Brownian model, but for the case when the expected rate of return of the stock is the same as that of the bond. It is however not at all necessary to first fix the price, and the logic of the approach of Bouchaud and Sornette stands out more clearly if first the hedging strategy is determined by minimizing risk, and then the price by a fair-game argument. In this way one finds the correct price in the log-Brownian model in all cases, as was pointed out by Bouchaud and Sornette in their short reply to the critique of Mikheev[22]. We here work out the details when risk can not be entirely eliminated and the expected rate of return of the stock may differ from that of the bond. In the process we encounter two types of strategies: one that is qualitatively similar to the Black-Scholes strategy, but with an insurance against residual risk, and one that is quite different. The latter is available to operators that are sufficiently willing to take on risk and invest in stock directly.

We identify a natural candidate for the market price of options that differs from the option pricing prescription of Black and Scholes.

We use the idea of a "Polish Option" as a simile for any option such that risk can not be eliminated by trading in the underlying security. A second line in our work is

OPTION PRICING & PARTIAL HEDGING: THEORY OF POLISH OPTIONS

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Abstract

The twin problems of hedging and pricing of options in discrete-time markets are analyzed. We consider trading strategies consisting of one stock and one bond. The bond price rises deterministically over time while the stock price can change in several (more than two) ways at each instant of trading. Given such stock price movements, perfect hedging is not possible, and arbitrage arguments alone are not sufficient. We determine hedging and bid and ask prices by balancing expected gain against risk. Using a recent approach of Bouchaud and Sornette, we work out in detail the case where the mean rate of return of the stock differs from that of the bond. We identify a new kind of strategy open to operators that are sufficiently insensitive to risk. We find a candidate for market price of risky options, which reduces to the Black-Scholes prescription when risk can be eliminated. We report on data on stock price movements on the Warsaw Stock Exchange, and show that they are well described by a simple model where prices on each day can either increase, decrease or stay the same. We work out the details of the option pricing and hedging problems in this case.

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