Long-Lived Information and Intraday Patterns*

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Abstract

This paper studies the effect of clustering of liquidity trades on intraday patterns of volatility and market depth when private information is long-lived. The assumption of long-lived information allows us to distinguish between the patterns of information arrival and information use. Our results are: (i) volatility follows the same pattern as liquidity trading, (ii) there are no systematic patterns in the price impacts of orders, and (iii) the timing of information arrival is is unimportant. Result (i) is the same as that obtained by Admati and Pfleiderer (1988) in a model of short-lived private information, but (ii) and (iii) are different.

It is well documented that there are systematic intraday patterns in the volatility of security prices and the volume of trading. In a widely cited paper, Admati and Pfleiderer (1988) offer an explanation of these patterns: clustering in time of liquidity-motivated trades induces clustering of information acquisition and informed trading, which generates the same clustering of information-driven price changes. The clustering of information acquisition takes the form of more traders collecting information in some parts of the day than in others. When more traders collect (and use) information, the order process is more informative; thus, price volatility is greater.

It seems implausible to us that the number of agents collecting information varies systematically over the day. It seems more plausible that the intensity with which agents trade on their information may vary systematically. However, this distinction is impossible to make within the Admati-Pfleiderer model, because private information has a short lifespan in that model. Admati and Pfleiderer assume multiple public announcements occur each day, each of which reveals all the information collected privately earlier.

The purpose of the present paper is to examine the effect of clustering of liquidity trades in a model of long-lived private information, to see whether such clustering generates the same clustering of information use as opposed to information acquisition, and to determine the implications of such a model for volatility and market depth. In the model we investigate, which is a generalization of the continuous-time model studied in Kyle (1985) and Back (1992), there is only a single informed trader. The absence of competition gives this trader a great deal of flexibility regarding the timing of his trades. Thus, in a sense, our model is at the opposite extreme of the Admati-

Pfleiderer model, in which informed traders have no discretion regarding when they trade (because their information lasts only one period).

Our results can be summarized as follows: (i) the informativeness of orders and the volatility of prices follow the same pattern as liquidity trading, (ii) there are no systematic patterns in the price impacts of orders, and (iii) given the total amount of private information, the pattern of information flow to the informed trader is irrelevant. The first result is the same as Admati and Pfleiderer's, but the second and third are different. A particular consequence of (i) and (iii) is that clustering of liquidity trades induces clustering of information use (and hence volatility), without implying the same clustering of information acquisition.

We specify exogenously the information flow to the informed trader. One can think of the informed trader as committing to information collection at the beginning of the model. An implication of (ii) and (iii) is that the trader will be indifferent among information arrival processes that provide the same total amount of information. Market depth, volatility, and the costs of liquidity traders are also independent of the pattern of information arrival.

We also take the pattern of liquidity trading to be exogenously specified. However, result (ii) implies that the expected execution costs of liquidity traders do not depend on the timing of their trades. Hence, the liquidity traders can be interpreted as either discretionary or nondiscretionary.

Result (ii) is the implication of our model that differs most significantly from Admati and Pfleiderer's. In the Admati-Pfleiderer model, both liquidity traders and informed traders are motivated to concentrate their trades in certain periods because execution costs – the price impacts of trades – are low in those periods. Given the U-shaped patterns of volume and volatility found on the New York Stock Exchange,[1] this would imply an inverted U-shape for execution costs. However, bid-ask spreads have a U-shaped pattern (Wood and McInish, 1992). Accounting for the probability of transacting within the spread, Madhavan, Richardson and Roomans (1994) find that execution costs rise steadily over the day. Neither the Admati-Pfleiderer model nor our model can generate either a U-shaped pattern or an upward sloping pattern. Our result is consistent with Ferguson, Mann, and Schneck (1993), who find no relationship in foreign exchange futures markets between volume and volatility on the one hand and execution costs on the other. Our result also has some support from Foster and Viswanathan (1993), who find no association between volume and the adverse selection component of trading costs for two of the three sets of stocks they examine (the least actively traded and moderately actively traded deciles). However, Foster and Viswanathan find a positive association between volume and the adverse selection component for the most actively traded stocks, which is inconsistent with both our result and Admati and Pfleiderer's. Apparently, neither our model nor Admati and Pfleiderer's can explain the empirical patterns in execution costs.[2]

The model is described in Part I. Part II presents the results, and Part III concludes the paper.

I The Model

Trading occurs over an interval $[0, 1]$. This period can be interpreted as a single day, a part of a day, or several days. The risk-free rate is taken to be zero. During the period $[0, 1]$, orders for a risky asset are submitted by a single informed trader and uninformed "liquidity traders" to risk-neutral competitive market makers, who set prices and clear the market. An announcement at time 1 reveals the asset value, which is a finite-variance random variable $\tilde{v}$. Prior to that time, competition between the market makers forces the price to equal the conditional expectation of $\tilde{v}$, given the market makers' information. Their information consists of the history of combined informed and liquidity-motivated orders. We want to find the optimal trading strategy for the informed trader and the equilibrium price-adjustment rule for the market makers.

The model is more general than that studied by Kyle (1985) and Back (1992) in two respects: (i) the informed trader learns over time about $\tilde{v}$, and (ii) the volatility of liquidity trading is time-varying.

To model (i), assume there is a sufficient statistic $S(t)$ for the informed trader's information at each time t, in the sense that his conditional expectation of $\tilde{v}$ at time t equals $f(t, S(t))$ for some function f. Take f to be strictly monotone in S, so increases in S represent good news for the asset. Assume $S(0)$ is normally distributed with mean zero and that S follows a Gaussian process:

$$dS(t) = \sigma_s(t) dW_s(t), \tag{1}$$

where σ_s is a deterministic and continuous function of time, and W_s is a

Wiener process. A special case is $\sigma_s \equiv 0$, in which case our information structure is the same as Kyle's (1985).[3] The deterministic volatility $\sigma_s^2(t)$ is a feature of any Gaussian filtering model.

When $\sigma_s^2(t)$ is large, the agent is learning a lot. To understand this, note that the variance of $S(1)$ is $\operatorname{var} S(0) + \int_0^1 \sigma_s^2(t)\, dt$. This variance represents the uncertainty the market would have regarding the informed agent's signal at time 1, if the market learned nothing before time 1. When $\sigma_s^2(t)$ is large, the uncertainty is increasing at a high rate, reflecting the fact that the agent is learning at a high rate.

To model (ii), denote the cumulative orders of liquidity traders through time t by $Z(t)$, and assume the process Z is a Gaussian process:

$$Z(0) = 0 \quad \text{and} \quad dZ(t) = \sigma_z(t) dW_z(t), \tag{2}$$

where σ_z is a deterministic, strictly positive, continuous function of time, and W_z is a Wiener process independent of W_s and $\tilde{v}$.[4]

We now introduce an assumption regarding the parameters of the model. Assume that

$$\operatorname{var} S(0) > \sup_t \frac{\int_t^1 \sigma_s^2(u)\, du}{\int_t^1 \sigma_z^2(u)\, du} \cdot \int_0^1 \sigma_z^2(u)\, du - \int_0^1 \sigma_s^2(u)\, du. \tag{3}$$

Particular cases in which (3) is satisfied are when $\sigma_s^2 \equiv 0$ or when σ_s^2 and σ_z^2 are constants. In each of these cases, the right-hand side of (3) is zero. Each of these particular cases is a generalization of Kyle (1985). Assumption (3) does not seem too restrictive, but we will comment on its role in the concluding section of the paper.

The scale of the signal process S is arbitrary, so we will choose a scale that simplifies the notation. Given a signal process S as defined above, set

$\hat{S} = b_z S/b_s$, where $b_s^2 \equiv \operatorname{var} S(0) + \int_0^1 \sigma_s^2 dt$ and $b_z^2 \equiv \int_0^1 \sigma_z^2 dt$. Note that b_s^2 is the variance of $S(1)$ and hence a measure of the total amount of private information. Likewise, b_z^2 is the variance of $Z(1)$ and hence a measure of the total amount of liquidity trading.[5] For the rescaled signal process $\hat{S}$, we have $\operatorname{var} \hat{S}(1) \equiv \hat{b}_s^2 = b_z^2$. This rescaling also affects the function f. Specifically,

$$\hat{f}(x) \equiv E[\tilde{v}|\hat{S}(1) = x] = E[\tilde{v}|S(1) = b_s x/b_z] = f(b_s x/b_z).$$

Assumption (3) holds for $\hat{S}$ if and only if it holds for S. Henceforth, we will work with $\hat{S}$ and $\hat{f}$, but drop the "hats." Note that this convention means we have $b_s^2 = b_z^2$ (which allows us to avoid writing a factor b_s/b_z that would otherwise appear throughout the definition of the equilibrium).

The remainder of the model description essentially follows Back (1992). Let $X(t)$ denote the number of shares acquired by the informed trader during $[0, t]$, and set $Y(t) = X(t) + Z(t)$. Market makers observe Y, which is the combined informed and liquidity trades, so their information structure is the filtration $\mathbf{F}^Y = \{\mathcal{F}^Y(t)\}$, where $\mathcal{F}^Y(t) = \sigma\{Y(u)|0 \le u \le t\}$. Define a pricing rule to be a function $(t, y) \mapsto P(t, y) : [0, 1] \times \Re \to \Re$. Given a particular trading strategy for the informed trader, call a pricing rule P *competitive* if the condition

$$P(t, Y(t)) = E[\tilde{v}|\mathcal{F}^Y(t)] \tag{4}$$

is satisfied for all t. Note that we are taking the price at each time t to depend only on $Y(t)$ rather than on the entire history of Y through time t. We will prove that there is a rule of this form satisfying (4).

The informed trader observes the asset price and signal process. For technical reasons, it is convenient to suppose that he also observes the liquidity trades. We will justify this later by showing that the asset price can be inverted to compute the liquidity trades.[6] The information structure of the informed trader is therefore assumed to be the filtration $\mathbf{F} = \{\mathcal{F}(t)\}$, where $\mathcal{F}(t) = \sigma\{S(u), Z(u) | 0 \leq u \leq t\}$. In order to simplify the analysis we will restrict the informed trader to trading strategies that are absolutely continuous; i.e., trading in rates.[7] Consequently, we assume that $X(t) = \int_0^t \theta(u)\, du$, for some process θ adapted to $\mathbf{F}$. We need to impose some constraint on the informed agent's strategy in order to rule out the analogue of the "doubling strategies" that exist in competitive models (Harrison and Kreps, 1979). A constraint that suffices is:

$$\int_0^1 EY(t)^4\, dt < \infty \quad \text{and} \quad \int_0^1 Ef(X(t) + Z(1))^4\, dt < \infty. \tag{5}$$

Let P denote an arbitrary pricing rule. The end of period wealth accruing to the informed trader from an application of a trading strategy θ in the face of this pricing rule is

$$W(1) = \int_0^1 [\tilde{v} - P(u, Y(u))]\theta(u)\, du, \tag{6}$$

where, as explained before,

$$Y(u) = Z(u) + \int_0^u \theta(t)\, dt. \tag{7}$$

Given a price rule P, a trading strategy for the informed agent is *optimal* if it maximizes the expected value of $W(1)$ over all trading strategies satisfying (5). An *equilibrium* is a pair (P, θ) such that P is a competitive pricing rule given θ and such that θ is an optimal trading strategy given P.

II Results

The equilibrium is defined in the following theorem. It is unique within a certain class, which we will not describe explicitly here [see Back (1992)]. In this theorem, and henceforth, we write $f(s)$ for $f(1,s) \equiv E[\tilde{v}|S(1)=s]$.

Theorem. *Define*

$$\Sigma(t) = \int_t^1 [\sigma_z^2(u) - \sigma_s^2(u)]\, du, \tag{8}$$

$$\theta(t) = \sigma_z^2(t)\frac{S(t)-Y(t)}{\Sigma(t)}. \tag{9}$$

The trading strategy θ is well defined by (9) in the sense that there exists a unique solution (θ, Y) to the system consisting of (9) and (7). Given the trading strategy (9), the distribution of $S(t)$ conditional on $\mathcal{F}^Y(t)$ is normal with mean $Y(t)$ and variance $\Sigma(t)$. Let $\pi(t,y,\cdot)$ denote the density function for the normal distribution with mean y and variance $\int_t^1 \sigma_z^2(u)\, du$. Define

$$P(t,y) = \int_{-\infty}^{\infty} f(s)\pi(t,y,s)\, ds. \tag{10}$$

The pair (P,θ) is an equilibrium. In this equilibrium, the price evolves as

$$dP(t,Y(t)) = \lambda(t,Y(t))\, dY(t), \tag{11}$$

where

$$\lambda(t,y) \equiv \frac{\partial}{\partial y}P(t,y). \tag{12}$$

If f is continuously differentiable with $E[f'(S(1))] < \infty$, then the process $\lambda(t,Y(t))$ is a martingale relative to the market makers' information structure $\mathbf{F}^Y$.

Notice that $P(t,y)$ defined by (10) is strictly monotone in y, by virtue of the monotonicity of f. Therefore, the informed trader can invert $P(t,Y(t))$ at each time t to compute $Y(t)$. Since he knows his own orders $X(t)$, this reveals $Z(t)$, justifying the assumption made in Part I regarding his information.

The proof of the theorem is in the appendix, but we will explain the essence of the argument here. Market makers at each time t are trying to estimate $f(S(1))$. Since $S(1)$ equals $S(t)$ plus an independent increment, market makers need to estimate $S(t)$. It turns out that, when the informed agent trades according to (9), the distribution of $S(t)$ conditional on the market makers' information at time t is normal with mean $Y(t)$ and variance $\Sigma(t)$. This is proven via the Kalman filter. Hence, given the market makers' information, $S(1)$ is distributed normally with mean $Y(t)$ and variance $\int_t^1 \sigma_z^2(u)\,du$. Thus, the price defined by (10) is

$$P(t,Y(t)) \equiv \int_{-\infty}^{\infty} f(s)\pi(t,Y(t),s)\,ds = E[f(S(1))|\mathcal{F}^Y(t)],$$

which implies the pricing rule satisfies the equilibrium condition. As for the optimality of the trading strategy (9), the key fact is that $Y(t) - S(t) \to 0$ as $t \to 1$, when (9) is followed. This implies $P(t,Y(t)) \to f(S(1))$, so the price is "right" by time 1. We can show that this is the only requirement for optimality. This fact is implicit in Kyle's (1985) argument and is made explicit in Back (1992). Given the risk neutrality, it is not unexpected that many optima exist simultaneously.

To interpret the equilibrium, we begin by observing that, as in Admati-Pfleiderer (1988), clustering of liquidity trades leads to clustering of informed trades, information flow to the market, and price volatility. That informed trades follow the same pattern as liquidity trades is evident from the factor $\sigma_z^2(t)$ in the trading strategy (9). The information flow to the market can be seen by looking at the change in the market's uncertainty about $S(t)$. As stated in the theorem, the conditional variance of $S(t)$ is $\Sigma(t)$, and from (8) we have

$$\Sigma'(t) = \sigma_s^2(t) - \sigma_z^2(t).$$

The two components of $\Sigma'(t)$ capture the two reasons that the market's uncertainty regarding $S(t)$ changes over time. If no information were communicated to the market via orders, the uncertainty would grow over time at rate $\sigma_s^2(t)$, because of the change in $S(t)$ itself. Since Σ actually changes at rate $\sigma_s^2(t) - \sigma_z^2(t)$, the term $\sigma_z^2(t)$ must represent the rate at which information is communicated via orders. Thus, it is the time pattern of liquidity trading that determines the rate at which information is communicated to the market. Furthermore, the rate of information flow to the market determines the volatility of prices. Indeed, the volatility of prices is given by

$$dP\,dP = \lambda^2\,dY\,dY = \lambda^2\,dZ\,dZ = \lambda^2\sigma_z^2\,dt.$$

Therefore, Admati and Pfleiderer's result that concentration of liquidity trades leads to the same concentration of volatility holds in our model also.

We now turn to the properties that distinguish our model from Admati and Pfleiderer's. The first difference is that the timing of information flow to the informed trader is irrelevant in our model. Notice that the pricing rule is completely determined by the joint distribution of $\tilde{v}$ and $S(1)$, acting through f, and by the function σ_z^2, acting through π. Hence, given the joint distribution of $\tilde{v}$ and $S(1)$, the pricing rule does not depend on any other characteristics of the information process S. The same is therefore true for $\lambda(t, Y(t))$, which is the reciprocal of what Kyle terms "market depth," and for the variance of price changes. Furthermore, the same must be true for the expected profits of the informed trader and the expected execution costs of liquidity traders. These quantities depend on the total amount of information but do not depend on the pattern of information arrival. The basis for this irrelevance result is that in equilibrium the informed trader "smooths" his use of information. The factor $(S(t) - Y(t))/\Sigma(t)$ in the trading strategy (9) represents the private information of the informed trader, normalized to have unit conditional variance. By virtue of this normalization, the trading of the informed agent does not depend on the amount of private information, as measured by the conditional variance. Thus, the pattern of information communication to the market does not depend on the pattern of information acquisition by the informed trader.

The second difference is the fact that in our model $\lambda(t, Y(t))$ is a martingale. In Admati and Pfleiderer, λ follows a pattern opposite to that of volume and volatility. The martingale property here follows from the fact that the pattern of informed trading precisely tracks that of liquidity trading, due to the factor $\sigma_z^2(t)$ in the trading strategy (9). Intuitively, the probability that an order is informed in our model does not change systematically over time, so the sensitivity of prices to orders does not change systematically.

A final difference is that, in the equilibrium of our model, liquidity traders are indifferent about when they trade. The aggregate execution cost of the liquidity traders is defined as

$$\int_0^1 dZ\,dP = \int_0^1 \lambda\,dZ\,dY = \int_0^1 \lambda\,dZ\,dZ = \int_0^1 \lambda\sigma_z^2\,dt.$$

The expected aggregate execution cost is

$$E\left[\int_0^1 \lambda(t,Y(t))\sigma_z^2(t)\,dt\right] = \bar{\lambda}\,\mathrm{var}\,Z(1),$$

where $\bar{\lambda}$ is the constant $E[\lambda(t,Y(t))]$. Therefore, the expected aggregate execution cost depends only on the total amount of liquidity trading, as measured by var $Z(1)$, and not on the pattern of liquidity trading.

If $\tilde{v}$ and $S(1)$ have a joint normal distribution, then market depth is constant, as in Kyle (1985). This is illustrated in the following.

Example. Assume $\tilde{v} = \bar{v} + \tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is distributed normally with mean zero and variance σ_ε^2. Assume the signal process reveals $\tilde{\varepsilon}$ at time 1. Then, as defined in Part I, the normalized signal at time 1 is $S(1) = b_z\tilde{\varepsilon}/\sigma_\varepsilon$, and $f(s) = \bar{v} + \sigma_\varepsilon s/b_z$. Set $\lambda = \sigma_\varepsilon/b_z$. Then the equilibrium pricing rule (10) is

$$P(t,y) = \bar{v} + \lambda y.$$

Since λ is constant, actual as well as expected execution costs are independent of the pattern of liquidity trading. Furthermore, volatility is a constant multiple of σ_z^2. Let $\Omega(t)$ denote the conditional variance of the random variable $E[\tilde{v}|\mathcal{F}(t)]$ given $\mathcal{F}^Y(t)$. In this model, we have $E[\tilde{v}|\mathcal{F}(t)] = \bar{v} + \lambda S(t)$, so $\Omega(t) = \lambda^2\Sigma(t)$. Therefore,

$$\frac{1}{\lambda} = \frac{\sqrt{\Sigma(t)}}{\sqrt{\Omega(t)}}.$$

As in Kyle (1985, p. 1317), market depth (the reciprocal of λ) is "inversely proportional to the amount of private information (in the sense of an error variance) which has not yet been incorporated into prices." In Kyle, "market depth is proportional to the amount of noise trading," but here it is proportional to

$$\sqrt{\Sigma(t)} = \sqrt{\int_t^1 [\sigma_z^2(u) - \sigma_s^2(u)]\, du}.$$

III Conclusion

We have shown that in a dynamic Kyle model with a single risk-neutral informed trader, the volatility of prices follows the same pattern as liquidity trading and is unaffected by the pattern of information arrival. The reciprocal of market depth (Kyle's λ parameter) is a martingale. The model delivers the observed coincidence of high volatility with high volume without requiring the number of traders collecting information to vary systematically over the day.

We assumed there is nontrivial private information at the beginning of the trading period [see (3)]. The reason for this is as follows. The informed trader will want to make unbounded trades unless the λ parameter is a martingale, so in equilibrium there cannot be systematic changes in λ over time. In order for this to be consistent with equilibrium, the information content of orders must be constant over time. This is apparently impossible if there is no private information, or too little private information, at the beginning of the trading period.

The willingness to use information at a constant rate is one manifestation of the patience of the informed trader in this model. If the trader faced competition from other informed traders, or if he were risk averse, it might be consistent with equilibrium for the λ parameter to vary in systematic ways. This is an interesting issue for future research.

Appendix

The purpose of this appendix is to prove the theorem. Throughout the proof, we will use the notation $\xi(t) = S(t) - Y(t)$.

Lemma 1. *For each t, $\Sigma(t) > 0$. Furthermore, $\sigma_z^2(1) > \sigma_s^2(1)$.*

Proof. Invoking the normalization $b_z^2 = b_s^2$, assumption (3) can be restated as

$$1 > \sup_t \frac{\int_t^1 \sigma_s^2(u)\, du}{\int_t^1 \sigma_z^2(u)\, du}. \tag{A.1}$$

This implies $(\forall t)\ \Sigma(t) > 0$. Furthermore, by the mean-value theorem, the ratio of integrals in (A.1) equals

$$\frac{\sigma_s^2(a(t))}{\sigma_z^2(b(t))}$$

for some $t < a(t), b(t) < 1$. Taking the limit as $t \to 1$ (A.1) implies

$$1 > \frac{\sigma_s^2(1)}{\sigma_z^2(1)}.$$

□

Consider the claim that θ is well defined by (7) and (9). Obviously, (9) defines θ uniquely if Y is well defined, so the issue is the existence of a solution Y to the stochastic differential equation defined by (7) and (9). This is a linear stochastic differential equation with coefficients that are locally bounded on $[0, 1)$ and adapted to $\mathbf{F}$. Hence, it has a unique solution on $[0, 1)$ (Karatzas and Shreve, 1988, §5.6.C). We can define Y by continuity at $t = 1$ if $\lim_{t\to 1} Y(t)$ exists a.s. This is established in the following lemma, which will also be used subsequently.

Lemma 2. *Given the trading strategy (9),* $\lim_{t\to 1} Y(t) = S(1)$ *a.s.*

Proof. The claim is that $\lim_{t\to 1}\xi(t) = 0$ a.s. The process ξ satisfies the stochastic differential equation

$$d\xi(t) = \frac{-\sigma_z^2(t)}{\Sigma(t)}\xi(t)\,dt + \sigma_z(t)\,dW_z(t) - \sigma_s(t)\,dW_s(t).$$

This may be rewritten as

$$d\xi(t) = \frac{-\sigma_z^2(t)}{\Sigma(t)}\xi(t)\,dt + \sigma(t)\,dW(t), \tag{A.2}$$

where $\sigma^2(t) \equiv \sigma_z^2(t) + \sigma_s^2(t)$, and W is a standard Wiener process.

The solution of the stochastic differential equation (A.2) with initial condition $\xi(0) = S(0)$ is (Karatzas and Shreve, 1988, §5.6C)

$$\xi(t) = \phi^{-1}(t)\left[S(0) + \int_0^t \sigma(u)\phi(u)\,dW(u)\right],$$

where

$$\phi(t) \equiv \exp\left(\int_0^t \sigma_z^2(u)\Sigma^{-1}(u)\,du\right).$$

Here, and in the following argument, the inverse notation $^{-1}$ denotes the reciprocal. The desired result $\xi(t) \to 0$ a.s. as $t \to 1$ will follow from showing

$$\phi^{-1}(t) \to 0 \tag{A.3}$$

and

$$\phi^{-1}(t)\int_0^t \sigma(u)\phi(u)\,dW(u) \to 0 \text{a.s.} \tag{A.4}$$

as $t \to 1$. In the proofs of these results, we will use without further comment the facts that $\sigma_z^2(\cdot)$ and $\sigma^2(\cdot)$ are continuous, strictly positive functions and hence are bounded above and bounded away from zero. Likewise, $\sigma_z^2(\cdot) -$

$\sigma_s^2(\cdot)$ is bounded above, and it is bounded away from zero on a neighborhood of $t = 1$ by virtue of continuity and Lemma 1.

To prove (A.3), it suffices to show that $\int_0^t \Sigma^{-1}(u)\, du \to \infty$. An application of the mean-value theorem gives, for each u and some $u^* \in [u, 1]$,

$$\Sigma(u) = -\Sigma'(u^*)(1-u) = [\sigma_z^2(u^*) - \sigma_s^2(u^*)](1-u) \leq a(1-u)$$

for some constant a. Therefore

$$\int_0^t \Sigma^{-1}(u)\, du \geq a^{-1} \int_0^t (1-u)^{-1}\, du \to \infty.$$

It remains to establish (A.4). It suffices to show that

$$\phi^{-1}(t) \int_0^t \phi(u)\, dW(u) \to 0 \text{a.s.}$$

Applying the law of the iterated logarithm to the continuous local martingale $\int_0^t \phi(u)\, dW(u)$ (Durrett, 1984, p. 77) gives

$$\limsup_{t \to 1} \frac{\int_0^t \phi(u) dW(u)}{\sqrt{2 \int_0^t \phi^2(u)\, du \cdot \log\log \int_0^t \phi^2(u)\, du}} = 1.$$

Hence, it suffices to show that

$$\phi^{-1}(t) \sqrt{2 \int_0^t \phi^2(u)\, du \cdot \log\log \int_0^t \phi^2(u)\, du} \to 0,$$

equivalently,

$$\frac{2 \int_0^t \phi^2(u)\, du \cdot \log\log \int_0^t \phi^2(u)\, du}{\phi^2(t)} \to 0. \tag{A.5}$$

This follows from (A.3) unless

$$\int_0^t \phi^2(u)\, du \to \infty, \tag{A.6}$$

so suppose (A.6) holds.

The proof of (A.5) now follows from repeated application of l'Hôpital's rule. A first application, combined with dropping bounded factors and terms known to converge to zero, shows that it suffices to establish

$$\frac{\log \log \int_0^t \phi^2(u)\, du}{1/\Sigma(t)} \to 0.$$

A second application, again dropping bounded factors, shows that it suffices to establish

$$\frac{\phi^2(t)\Sigma^2(t)}{\int_0^t \phi^2(u)\, du \cdot \log \int_0^t \phi^2(u)\, du} \to 0.$$

A third application shows that it suffices to establish

$$\frac{\sigma_s^2(t)\Sigma(t)}{1 + \log \int_0^t \phi^2(u)\, du} \to 0,$$

and this is clearly true.

□

We now establish the claim regarding the $\mathcal{F}^Y(t)$-conditional distribution of $S(t)$.

Lemma 3. *For each t, the $\mathcal{F}^Y(t)$-conditional distribution of $S(t)$ is normal with mean $Y(t)$ and variance $\Sigma(t)$.*

Proof. We have

$$\begin{aligned} d\xi(t) &= -\alpha(t)\xi(t)\, dt - \sigma_z(t)\, dW_z(t) + \sigma_s(t)\, dW_s(t), \\ dY(t) &= \alpha(t)\xi(t)\, dt + \sigma_z(t)\, dW_z(t), \end{aligned}$$

where $\alpha(t) = \sigma_z^2(t)/\Sigma(t)$. These are the state (or signal) and observation equations, in the terminology of filtering. Conditional on $\mathcal{F}^Y(t)$, $\xi(t)$ is normally distributed. We will denote its conditional mean by $\hat{\xi}(t)$ and its conditional variance by $\Sigma^*(t)$. Therefore, $S(t)$ is conditionally normal with mean $\hat{\xi}(t) + Y(t)$ and variance $\Sigma^*(t)$.

The Kalman-Bucy filtering equation for $\hat{\xi}(t)$ is[8]

$$d\hat{\xi}(t) = -\alpha(t)\hat{\xi}(t)dt + \frac{1}{\sigma_z(t)}[\alpha(t)\Sigma^*(t) - \sigma_z^2(t)]\, d\nu(t), \qquad \text{(A.7)}$$

with initial condition $\hat{\xi}(0) = E[S(0)] = 0$. In (A.7), ν denotes a Wiener process (the innovations process). The variance process Σ^* satisfies the differential equation

$$\frac{d}{dt}\Sigma^*(t) = -2\alpha(t)\Sigma^*(t) + \sigma_z^2(t) + \sigma_s^2(t) - \frac{1}{\sigma_z^2(t)}[\alpha(t)\Sigma^*(t) - \sigma_z^2(t)]^2$$

and initial condition

$$\Sigma^*(0) = \operatorname{var}\xi(0) = \sigma_0^2.$$

This Riccati equation is well known to have a unique solution. It is easy to check that it is solved by $\Sigma^* = \Sigma$, which shows that Σ is the conditional variance of $S(t)$.

Substituting $\Sigma^* = \Sigma$ into (A.7), it becomes

$$d\hat{\xi}(t) = -\alpha(t)\hat{\xi}(t)\, dt.$$

The unique solution of this with initial condition $\hat{\xi}(0) = 0$ is $\hat{\xi} \equiv 0$. Hence, the conditional mean of $S(t)$ is $Y(t)$.

□

The next two lemmas establish that (P, θ) is an equilibrium.

Lemma 4. *Given the trading strategy (9), the pricing rule (10) is competitive.*

Proof. Using successively the facts that $Y(1) = S(1)$ a.s., the fact that S has independent zero-mean increments, and Lemma 3, we obtain

$$\begin{aligned} E[Y(1)|\mathcal{F}^Y(t)] &= E[S(1)|\mathcal{F}^Y(t)] \\ &= E[S(t)|\mathcal{F}^Y(t)] \\ &= Y(t). \end{aligned}$$

Hence, Y is a martingale relative to the filtration $\mathbf{F}^Y$. Lévy's theorem (Karatzas and Shreve, 1988, p. 157) therefore implies that the processes $(Y, \mathbf{F}^Y)$ and $(Z, \mathbf{F}^Z)$ are equivalent in law. Successively applying the law of iterated expectations, the fact that $Y(1) = S(1)$ a.s., and this equivalence in law gives

$$\begin{aligned} E[\tilde{v}|\mathcal{F}^Y(t)] &= E[f(S(1))|\mathcal{F}^Y(t)] \\ &= E[f(Y(1))|\mathcal{F}^Y(t)] \\ &= \int_{-\infty}^{\infty} f(y)\pi(t, Y(t), y)\, dy, \end{aligned}$$

which completes the proof.

□

Lemma 5. *Given the pricing rule (10), the trading strategy (9) is optimal.*

Proof. Let θ denote a generic trading strategy for the informed trader. The proof depends on establishing a sharp upper bound on the expected wealth of the informed trader. Define

$$j(s, y) = \int_y^s (f(s) - f(x))\, dx.$$

Let $\phi(t, s, \cdot)$ denote the density function for the normal distribution with mean s and variance $\int_t^1 \sigma_s^2(u)\, du$. For $t < 1$, define

$$J(t, s, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j(s', y') \phi(t, s, s') \pi(t, y, y')\, ds'\, dy'. \tag{A.8}$$

Set $J(1, s, y) = j(s, y)$. The function J is continuous at $t = 1$ (Karatzas and Shreve, 1988, p. 255).

In the following, we omit writing the arguments of the various functions and use subscripts on J to denote partial derivatives. Obviously,

$$(\forall\, y) \;\; J(1, s, y) > J(1, s, s) = 0, \tag{A.9}$$

with equality holding if and only if $y = s$. Also, one can differentiate (A.8) to obtain[9]

$$J_t + \frac{1}{2}\sigma_s^2 J_{ss} + \frac{1}{2}\sigma_z^2 J_{yy} = 0. \tag{A.10}$$

Applying Itô's Lemma and making use of equation (A.10) we find that

$$\begin{aligned} &J(1, S(1), Y(1)) - J(0, S(0), 0) = \\ &\qquad \int_0^1 J_s \sigma_s\, dW_s + \int_0^1 J_y \theta\, dt + \int_0^1 J_y \sigma_z\, dW_z. \end{aligned}$$

Upon applying the expectation operator, the two stochastic integrals against the Wiener processes vanish,[10] and this equation becomes

$$E\left[-\int_0^1 J_y \theta\, dt | \mathcal{F}(0)\right] = J(0, S(0), 0) - E\left[J(1, S(1), Y(1)) | \mathcal{F}(0)\right].$$

A direct calculation shows that

$$-J_y = E[\tilde{v} | \mathcal{F}(t)] - P.$$

By iterated expectations, we can write the informed trader's optimization problem as

$$\max_{\theta} \; E\left[\int_0^1 \{E[\tilde{v}|\mathcal{F}(u)] - P(u, Y(u))\}\theta(u)\, du\right]. \tag{A.11}$$

Therefore, the objective function takes the value

$$J(0, S(0), 0) - E[J(1, S(1), Y(1))|\mathcal{F}(0)] \;\le\; J(0, S(0), 0),$$

with, from (A.9), equality obtaining if and only if $S(1) = Y(1)$ a.s. Since, by Lemma 2, this equality holds for the strategy (9), that strategy is optimal.

□

Equation (11) follows from Itô's Lemma, noting that the dt terms cancel (the explanation is the same as in footnote 9). We can write $P(t, y)$ as

$$\int_{-\infty}^{\infty} f(y + u)\, \pi(t, 0, u)\, du.$$

The assumptions regarding the derivative of f guarantee that we can differentiate this with respect to y under the integral operator. This gives

$$\lambda(t, y) = \int_{-\infty}^{\infty} f'(y + u)\pi(t, 0, u)\, du.$$

It follows that $\lambda(t, Z(t)) = E[f'(Z(1))|\mathcal{F}^Z(t)]$. Given the equality of the laws of $(Z, \mathbf{F}^Z)$ and $(Y, \mathbf{F}^Y)$ (see the proof of Lemma 4), this implies $\lambda(t, Y(t))$ is the martingale $E[f'(Y(1))|\mathcal{F}^Y(t)]$.

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Notes

[1]See Wood, McInish, and Ord (1985) for the volatility pattern and Jain and Joh (1988) for volume.

[2]For an approach based on imperfect competition in market making, see Brock and Kleidon (1992), who, however, do not attempt to explain volatility patterns.

[3]Actually, we should also assume $S(0) = \tilde{v}$ to be fully consistent with Kyle's presentation, but whether the informed trader actually observes $\tilde{v}$ or only has a signal about $\tilde{v}$ is a trivial distinction, because the solution in any case only depends on his expectation of $\tilde{v}$. It is harmless to interpret this expectation as the true value, since all agents in the model are risk neutral.

[4] A trivial generalization of the model would be to allow S and Z to have deterministic drifts. A deterministic drift would add nothing to the informativeness of S; hence, one might as well work with the "detrended" process, as we are doing here. As for a drift in Z, the market here is so "deep" that a predictable component of liquidity trades has no effect on prices: market makers are willing to take the opposite side of any trade that is known to have no information content, without requiring a price concession. Again, one can work with the detrended liquidity-trade process, as we are implicitly doing here. Of course, if market makers were risk-averse, or imperfectly competitive, or had position limits (such as margin requirements), then even predictable liquidity trades could affect prices.

[5]More precisely, b_z^2 represents the amount of liquidity trading only after netting out all offsetting trades occurring during $[0,1]$. The gross amount of liquidity trading is infinite: if one adds up the absolute changes in Z over increasingly finer partitions of $[0,1]$, the sum is unbounded (and even this unbounded sum is net of offsetting instantaneous trades, because $dZ(t)$ represents only the net liquidity trades at time t).

[6]Our approach is the same as looking at the artificial symmetrically-informed economy to prove the existence of fully revealing rational expectations equilibria; see Grossman (1981).

[7] As Back (1992) has demonstrated, there is no gain in generality from allowing the informed trader general semimartingale trading strategies.

[8] See Kallianpur (1980, §10.2). The state and observation equations are written in Kallianpur's notation as $A_0(t) = 0$, $A_1(t) = -\alpha(t)$, $A_2(t) = 0$, $C_0(t) = 0$, $C_1(t) = \alpha(t)$, $C_2(t) = 0$, $B(t) = [-\sigma_z(t), \sigma_s(t)]$, and $D(t) = [\sigma_z(t), 0]$.

[9] See Karatzas and Shreve (1988, p. 254). Note that their regularity condition (4.3.3), which justifies differentiating (A.8) through the integral operator, follows from $Ef(Z(1))^2 = Ef(S(1))^2 = E\tilde{v}^2 < \infty$ and the bound $j(s, y) \leq (s - y)(f(s) - f(y))$.

[10] As explained in footnote 9, we can calculate J_y and J_s by differentiating through the integral operator in (A.8). The conditions $E\int_0^1 J_y^2\,dt < \infty$ and $E\int_0^1 J_s^2\,dt < \infty$ follow from the bound on j in footnote 9 and the constraint (5), and these conditions imply that the stochastic integrals are martingales.