Stochastic Dominance, Pareto Optimality, and Equilibrium Asset Pricing

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Abstract In this paper, we give a unified approach to equilibrium asset pricing theories. We define a factor subspace and develop a general equilibrium model with an infinite dimensional contingent claim space which will be applied to asset pricing models. We show that there exists a minimal factor subspace F in the sense that no proper subspace of F can serve a factor subspace. We discuss how the minimal F can be determined endogenously given a market structure. The analysis in this paper can be applied to: Economy without aggergate risk; CAPM with elliptical distributions; Equilibrium version of APT; Economy with call options.

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1. Introduction

The purpose of this paper is to study a general equilibrium model of asset markets with an infinite dimensional contingent claim space. We introduce a concept of a factor subspace. We give a proof of the existence of an equilibrium. Furthermore we characterize an equilibrium price system in terms of prices of contingent claims spanning the factor subspace. We then apply these results to some asset pricing models.

How prices of assets traded in markets are determined has been one of the most discussed subjects in financial economics. The study on this subject is known as the theory of asset pricing. The implication of the literature on asset pricing is that expected returns of assets traded in markets are linearly related to risks measured by the covariances of their returns with some fundamental sources of uncertainty. Two significant theories in this group are the Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT).

The CAPM says that expected returns of marketed assets are linearly related to a rate of riskfree return and the risk measured by the covariances of their returns with the market portfolio return under assumptions of either quadratic preferences of investors or normal distributions of asset payoffs. The APT gives us a similar pricing implication under the assumption that each asset payoff is affected by a finite number of fundamentals representing economy wide fluctuations, called factors, and idiosyncratic risks. It says that expected returns of assets are linearly related to a rate of riskfree return and factor loadings which are proportional to the covariances of their returns with factors. It is well known that both the CAPM and the APT can be studied either within a partial equilibrium approach or within a general equilibrium approach. (1)

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(1) For the CAPM with a general equilibrium approach, see Mossin (1966). For partial equilibrium approach APT, see Ross (1976). For equilibrium version of the APT, see Connor (1984).

One common feature shared by the CAPM and the APT is a linear pricing rule. However the number of fundamentals involved in princing assets is different. The APT recognizes the possibility that the market portfolio return alone might not explain all the variation in the return of an asset. There have been attempts to derive the pricing implication of the CAPM with the distributional assumption on asset returns via the APT approach using the market portfolio as the unique factor and thereby to give a unified approach to the CAPM and the APT. For example, Connor(1984) developed an equilibrium version of the APT and showed that if each asset payoff follows a joint normal distribution, then the pricing implication of the CAPM can be derived via the APT with one factor. However to treat the CAPM as a special case of the APT is not quite a satisfying attempt of unifying two theories. Factors in the APT are exogenously given. Thus how to find them is a question to be answered by empirical studies. On the other hand, the unique factor in the CAPM is identified by the market portfolio which is related to the exogenous market data, endowments of consumers.

The purpose of this paper is to develop a unified asset pricing theory within a general equilibrium framework. We derive a pricing implication such that the price of any marketed portfolio payoff can be expressed as a linear combination of prices of a certain collection of portfolios, the number of which is small relative to the number of assets traded in markets. The CAPM and the APT are examples of this pricing theory.

We consider markets in which a collection of financial assets is traded. The collection may be finite or infinite. Markets are frictionless, and short-sales of assets are permitted. Asset payoffs are random variables which are elements of a space of contingent claims. By forming porfolios of the marketed assets, payoffs in a certain linear subspace of the space of all contingent claims can be attained. We call this subspace a space of marketed claims. We say that a closed subspace of the space of marketed claims is a *factor subspace* if two conditions are satisfied; first, the market portfolio payoff (i.e, the payoff of an aggregate endowment of portfolios per capita) belongs to this subspace; second, every marketed contingent claim is dominated in the sense of second order stochastic dominance by some contingent claim in the factor subspace. We demonstrate that for each asset market economy, there is a well-defined minimal factor subspace in the sense that no proper subspace of it is a factor subspace.

Making use of the argument of second order stochastic dominance, we show that every constrained Pareto optimal portfolio allocation consists exclusively of portfolios which generate the factor subspace. The same holds for an equilibrium portfolio allocation. We show that there exists an equilibrium in which the price of a portfolio equals the price of the dominating portfolio with the same expected return in the factor subspace. Since the dominating portfolio can be attained by forming portfolios which generate the factor subspace, its price can be expressed in terms of prices of portfolios generating the factor subspace.

This paper gives a unified approach to such asset market models as an economy without aggregate risk, CAPM with elliptical distributions, an equilibrium version of the APT, and an economy with call options on the aggregate endowment per capita. These economies can be analyzed by the approach described above. They all have finite dimensional minimal factor subspaces. Therefore equilibrium pricing relations in these examples are the following; for all marketed claims x, $p(x) = \sum_{j \in A} c_j p(\alpha_j)$ for some $c_1, \ldots, c_A$, where $\{\alpha_i\}_{i \in A}$ is an orthonormal system in a factor subspace and the number of claims needed to span the factor subspace depends on the market structure.

The organization of the remainder of the paper is as follows. In section 3, we examine the relation between second order stochastic dominance and pareto optimality for an economy with a factor subspace F. We show that each consumer's consumption bundle in Pareto optimal allocations belongs to F (Theorem 1). Hence at equilibrium, each consumer consumes a linear combination of contingent claims which span F. If F is finite dimensional, then these contingent claims are called "mutual funds" (Corollaries 1-2). We show that there always exists a minimal factor subspace F* for any given market structure (Proposition 1).

In section 4, we show that, for an economy with a factor subspace, there exists an equilibrium such that the price of any contingent claim can be expressed in terms of the prices of mutual funds. We also give sufficient conditions for every equilibrium price system to have that property. We explain the relationship between the pricing relation in this paper and the standard finance model pricing formula in which prices are expressed in terms of the expected returns of the assets.

In section 5, we give some applications of the results in this paper. In each example, the closed linear subspace F of M which is a factor subspace is minimal. Examples include an economy without aggregate risk, the CAPM with elliptical distributions, and an equilibrium version of the APT. The most significant example is given in section 5.4. In this example which builds upon Breeden and Litzenberger (1978), we explain how to find a closed linear subspace F of M which can be regarded as a factor space given the market structure. We show that the space spanned by call options written on the market endowment works exactly the same way as the factor space does in the APT. Each marketed contingent claim can be expressed by a linear combination of call options and a contingent claim orthogonal to call options, which can be interpreted as an idiosyncratic risk in APT. At equilibrium the price of each contingent claim is explained by the prices of the call optons.

2. Model

We consider markets in which a collection of financial assets is traded. Markets are frictionless and short-sales of assets are permitted. Asset payoffs are random variables which are elements of a space of contingent claims. Formally, let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\mathcal{L}_2(\Omega, \mathcal{F}, P)$ be the space of square integrable random variables on $(\Omega, \mathcal{F}, P)$, i.e., $\mathcal{L}_2(\Omega, \mathcal{F}, P) = \{x : (\Omega, \mathcal{F}, P) \rightarrow \mathcal{R}; Ex^2 < \infty\}$. We shall identify random variables which are equal everywhere except on a set with probability zero. It is well known that $\mathcal{L}_2(\Omega, \mathcal{F}, P)$ is a Hilbert space under the inner product $:< x, y >= E(xy)$, with the associated norm : $||x|| = (Ex^2)^{\frac{1}{2}}$. We shall identify a contingent claim space with $X = \mathcal{L}_2$. Let $X_+ = \{x \in X : x \geq 0 \quad a.s\}$ and $X_{++} = \{x \in X : x > 0 \quad a.s\}$. By forming a portfolios of the marketed assets, payoffs in a certain linear space of X can be attained. We call this subspace a marketed contingent claim space denoted by M. We assume that there is not necessarily a sufficient number of marketed assets so as to make every contingent claim in X attainable. Markets are allowed to be incomplete in this sense.

There are $I < \infty$ consumers indexed by $i = 1, 2, \ldots, I$. Each consumer i is described by a preference relation $\succeq_i$ defined on $X \times X$ and an endowment $e_i \in M$. We refer to a tuple $E \equiv (M, \succeq, e)$, where $\succeq \equiv (\succeq_i)_{i=1,2,\ldots,I}, e \equiv (e_i)_{i=1,2,\ldots,I}$, as an exchange economy. Let X' be the space of continuous linear functionals on X induced by the norm topology. X' will constitute the space of prices with a generic element $p \in X'$. Since Hilbert space is self dual, $X' = \mathcal{L}_2$. An (feasible) *allocation* $(x_i)_{i=1,2,\ldots,I}$ is a collection of consumption bundles such that $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} e_i$.

Consumers in this economy trade contingent claims in a frictionless market M taking a price system as given. They maximize their preferences by choosing optimal contingent claims among consumption bundles whose values are less than or equal to the value of their endowments. We give a definition of an equilibrium for E as follows.

Definition 1 : A competitive equilibrium for an exchange economy $E \equiv (M, \succeq, e)$ consists of an allocation $(x_i)_{i=1,2,\ldots,I}$ and a non-zero price functional $p \in X'$ such that for all i, x_i is $\succeq_i$-optimal in $B_i(p)$, where $B_i(p) = \{x_i \in M : p(x_i) \leq p(e_i)\}$.

3. Stochastic Dominance and Pareto Optimality

Let F be a closed linear subspace of M. Let $F^{\perp}$ denote the orthogonal complement of F, i.e, $F^{\perp} = \{x \in X :< x, y >= 0 \text{ for all } y \in F\}$. By the Hilbert Space Projection Theorem, we can decompose any $x \in X$ in a unique way: $x = x^F + x^{F^{\perp}}$, where $x^F \in F$ and $x^{F^{\perp}} \in F^{\perp}$. Notice that F is a subspace of M, but $F^{\perp}$ is not. We have the following relation among F, $F^{\perp}$, and X. $F \subseteq M \subseteq F \oplus F^{\perp} = X$, where $\oplus$ denotes the direct sum.

Let $F^{\dagger}$ be the set of random variables such that

$$F^{\dagger} = \{x \in X : E(x|y) = 0 \ \forall y \in F\}.$$

It is clear that $F^{\dagger} \subseteq F^{\perp}$ since mean independence implies orthogonality. Recall that e_i is consumer i's endowment. Let μ be the aggregate endowment per capita, i.e, $\mu = \frac{1}{I} \sum_{i=1}^{I} e_i$, where I denote the number of consumers. We call μ the market endowment.

Definition 2 : Let F be a closed linear subspace of M. F is called a *factor subspace* if $F \subset M \subseteq F \oplus F^{\dagger}$ and $\mu \in F$.

It is worth noticing that $F^{\dagger} \cap M = F^{\perp} \cap M$ if F is a factor space. Since $F^{\dagger} \cap M \subset F^{\perp} \cap M$, therefore it suffices to show that $F^{\perp} \cap M \subset F^{\dagger} \cap M$. Suppose that there exists $x \in F^{\perp} \cap M$, but $x \notin F^{\dagger} \cap M$, which implies that $E(x|z) \neq 0$ for some $z \in F$. Since $x \in M$ and F is a factor subspace, $x = X^F + X^{F^{\dagger}}$, where $X^F \in F$ and $X^{F^{\dagger}} \in F^{\dagger}$. Furthermore $X^F \neq 0$, since $x \notin F^{\dagger}$. That $E(x^{F^{\dagger}}|y) = E(x - x^F|y) = 0$, for all $y \in F$ implies that $< x - x^F, y >=< x, y > - < x^F, y >= 0$ for all $y \in F$. Since $x \in F^{\perp}$, $< x, y >= 0$. Therefore $< x^F, y >= 0$ for all $y \in F$, which implies that either $x^F \in F^{\perp}$ or $x^F = 0$. This is a contradiction. Thus $F^{\dagger} \cap M = F^{\perp} \cap M$. Take an arbitrary marketed claim x. Then $x = x^F + x^{F^{\perp}}$. F is a factor subspace if and only if for all y in F, $E(x^{F^{\perp}}|y) = 0$ and $\mu \in F$.

Consumers in the economy E choose $x \in M$, which is a random variable. In other words, they have to choose a consumption bundle whose outcome is not certain. Therefore each consumer's choice depends on her attitude toward risks.

Since each contingent claim $x \in X$ induces a probability distribution, it will be very useful if we can describe consumer's risk attitudes with regard to relations among cumulative distributions of contingent claims.

Definition 3 : A contingent claim x is said to be *second order stochastically dominant*[(2)] over a contingent claim y, denoted by x SSD y, if y $\overset{d}{=}$ x + ϵ for some ϵ such that $E(\epsilon|x) \leq 0$ a.s, where A $\overset{d}{=}$ B means A is the same as B in distribution.

Definition 4 : A preference relation $\succeq$ on X is said to be risk averse if and only if

$$\text{for all } x, y \ \in X \ \big[x \text{ SSD } y \Rightarrow x \succeq y\big].$$

The preference relation $\succeq$ is strictly risk averse if for all $x, y \ \in X$ and $x \neq y$ $\big[x \text{ SSD } y \Rightarrow x \succ y\big]$.

If a contingent claim x second order stochastically dominates another contingent claim y, then x is chosen over y by all risk averse consumers provided that both are budget feasible. We make the following assumptions.

Assumption 1 : Each consumer i's preference relation $\succeq_i$ is strictly risk averse.

It is well known that if $\succeq_i$ is von Neumann-Morgenstern expected utility representable, then $\succeq_i$ is (strictly) risk averse if and only if a von Neumann-Morgenstern expected utility function is (strictly) concave. Furthermore by the property of the second order stochastic dominance it is monotone. In some cases, Assumption 1 shall be replaced by the following assumption 1$'$.

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(2) There are several statements which are equivalent to the concept of second order stochastic domenance. One of them is the following : A contingent claim x with a cumulative distribution function Fis said to be *second order stochastically dominant* over a contingent claim y with a cumulative distribution function G, denoted by x SSD y, if

$$\int_{-\infty}^{t} G(t)dt \geq \int_{-\infty}^{t} F(t)dt, \quad \forall t \in R.$$

For detailed discussions, see Hadar and Russell (1969).

Assumption 1′ Each consumer i 's preferences are representable by a strictly concave von Neumann-Morgenstern utility function.

Assumption 2: Each consumer i's preferences are continuous.

If the economy has a factor subspace F, then for any contingent claim $x \in M$, there exists a marketed contingent claim x^F which gives at least the same preference as x. As long as $x \notin F$, x^F will be strictly preferred to x by all strictly risk averse consumers. In the economy with a factor subspace, for any allocation (x_i) such that for some i, $x_i \notin F$, there exists an allocation (y_i), s.t $y_i \in F\ \forall i$ which Pareto dominates (x_i) relative to M. It should be noticed that a competitive equilibrium allocation is a Pareto optimal allocation relative to allocations which can be achieved by trading claims in a market. In this sense it should be distinguished from the usual concept of Pareto optimality. We say an allocation is *constrained* Pareto optimal if it is Pareto optimal relative to M. The meaning of an contrained Pareto optimal allocation coincides with that of pareto optimal portfolio allocation.

Theorem 1: Suppose an exchange economy has a factor subspace F and satisfies Assumptions 1. Let $(x_i)_{i=1,\ldots,I}$ be a constrained Pareto optimal allocation. Then $x_i \in F$ for all $i = 1, 2, \ldots, I$.

Proof : Suppose not. Then there exists some $j \in \{1, 2, \ldots, I\}$ such that $x_j \notin F$. Since $x_j \in M$, there exists a unique decomposition as $x_j = x_j^F + x_j^{F^\perp}$ such that $x_j^F \in F$, $x_j^{F^\perp} \in F^\perp \cap M$ and $E(x_j^{F^\perp}|x_j^F) = 0$. Then x_j^F SSD x_j. Since consumer j is strictly risk averse , $x_j^F \succ x_j$. Now for all $i \in \{1, 2, \ldots, I\}$, $x_i = x_i^F + x_i^{F^\perp}$ such that $x_i^F \in F$, $x_i^{F^\perp} \in F^\perp \cap M$ and $E(x_i^{F^\perp}|x_i^F) = 0$. Take $(y_j)_{i=1,2\ldots,I}$ such that $y_i = x_i^F\ \forall i = 1, 2 \ldots, I$.

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\begin{aligned}
\sum_{i=1}^{I} x_i = \sum_{i=1}^{I}(x_i^F + x_i^{F^\perp}) &= \sum_{i=1}^{I} x_i^F + \sum_{i=1}^{I} x_i^{F^\perp} = I\mu \\
&\Rightarrow \sum_{i=1}^{I} x_i^{F^\perp} = 0, \text{ since } < I\mu, \sum_{i=1}^{I} x_i^{F^\perp} >= 0 \\
&\Rightarrow \sum_{i=1}^{I} x_i^F = \sum_{i=1}^{I} y_i = I\mu
\end{aligned}
$$

Now for all $i = 1, 2\ldots, I \quad y_i \succeq x_i$ and $y_j \succ x_j$, which implies that $(y_j)_{i=1,2\ldots,I}$ Pareto dominates $(x_i)_{i=1,2\ldots,I}$. This is a contradiction. ∎

Theorem 1 directly implies Corollary 1, since competitive equilibrium is constrained pareto optimal.

Corollary 1: Suppose an exchange economy has a factor subspace F and satisfies Assumptions 1. Then for every competitive equilibrium allocation $(x_i)_{i=1,2\ldots,I}$, $x_i \in F \quad \forall i = 1, 2\ldots, I$.

Since F is a closed linear subspace of a Hilbert space, there exists a complete orthonormal system in F. Let $\{\alpha_i\}_{i \in A \subset \mathcal{N}}$ be such a system, where $\mathcal{N}$ is the set of natural numbers. The cardinality of A equals the dimension of F. Then Corollary 1 implies that in equilibrium each consumer holds a consumption bundle which is a linear combination of contingent claims which span F. If F is K-dimensional subspace of M, then Corollary 1 implies that each consumer's equilibrium consumption bundle consists of a linear combination of K contingent claims. We call those K contingent claims "mutual funds".

Corollary 2 : Suppose dim(F) = K $< \infty$. Suppose F is a factor subspace of an economy and Assumptions 1-2 are satisfied. Then each consumer's equilibrium consumption bundle is a linear combination of K mutual funds.

Proof : Let $(x_i)_{i=1,2\ldots,I}$ be an equilibrium allocation. By Corollary 1, for all

$i = 1, 2 \ldots, I$ $x_i \in F$. Let $\{\alpha_i\}_{i=1}^{K}$ be a complete orthonormal system in F. Then

$$x_i = \sum_{i=1}^{K} \beta_i \alpha_i, \quad \beta_i \in R \text{ where } \beta_i =< x_i, \alpha_i >,$$

which implies that each consumer holds a linear combinaton of α_i's, i.e, "mutual funds". ∎

By virtue of Corollary 2, it is easy to analyze equilibrium allocations for an economy with a factor subspace satisfying Assumptions 1 and 2. Suppose we are given a market structure, by which we mean available marketed contingent claims. Since the marketed contingent claim space M itself can serve a factor subspce, there always exists one. However this choice does not give us any useful information about equilibrium allocations other than the market structure. Whether we can find a proper factor subspace F of M depends on the market structure. In section 5.1, we will discusse an example of an economy with no aggregate risk and show that such an economy has one-dimensional factor subspace. In general, a space which gives an exchange economy a factor subspace has a dimension more than one. If there are multiple choices of factor subspaces, then we choose F* such that F* $\subseteq$ G, where G is an arbitrary factor subspace.

Definition 5 : F* is said to be a *minimal* factor subspace if F* is a factor subspace and F* $\subseteq$ G, where G is an arbitrary factor subspace.

The following theorem says that there exists a minimal F* for any given market structure. Recall that $F^{\dagger} \cap M = F^{\perp} \cap M$ if F is a factor subspace of M.

Propositon 1: There exists a minimal F* for any given market structure M.

Proof : Let $\{F_i : i \in \mathcal{D}\}$ be a collection of closed linear subspaces of M such that each F_i is a factor subspace, where $\mathcal{D}$ is an index set. It suffices to show that $\cap_{i \in \mathcal{D}} F_i$ belongs to $\{F_i : i \in \mathcal{D}\}$. Clearly $\cap_{i \in \mathcal{D}} F_i$ is a closed linear subspace of M. First, we show that $(\cap_{i \in \mathcal{D}} F_i)^{\perp} \subseteq cl(\sum_{i \in \mathcal{D}} F_i^{\perp})$. Take an arbitrary $x \in (cl(\sum_{i \in \mathcal{D}} F_i^{\perp}))^{\perp} = (\sum_{i \in \mathcal{D}} F_i^{\perp})^{\perp}$. Then for all $i \in \mathcal{D}$, $\Big[y \in$

$F_i^{\perp} \Rightarrow < x, y >= 0$ because $F_i^{\perp} \subset \sum_{i \in \mathcal{D}} F_i^{\perp}$ and $x \in (\sum_{i \in \mathcal{D}} F_i^{\perp})^{\perp}\Big]$. This implies that for all $i \in \mathcal{D}$, $x \in (F_i{}^{\perp})^{\perp} = F_i$ and thus $x \in \cap_{i \in \mathcal{D}} F_i$. Therefore $(cl(\sum_{i \in \mathcal{D}} F_i^{\perp}))^{\perp} = ((\sum_{i \in \mathcal{D}} F_i^{\perp}))^{\perp} \subseteq \cap_{i \in \mathcal{D}} F_i$, which implies $(\cap_{i \in \mathcal{D}} F_i)^{\perp} \subseteq cl(\sum_{i \in \mathcal{D}} F_i^{\perp})$. Let $F^* \equiv \cap_{i \in \mathcal{D}} F_i$. For all $x \in M$, $x = x^{F^*} + x^{F^{*\perp}}$ and $x^{F^{*\perp}} \in M$. If $x^{F^{*\perp}} \in \sum_{i \in \mathcal{D}} F_i^{\perp} \cap M$, then $x^{F^{*\perp}} = \sum_{i \in \mathcal{D}} \tilde{x}_i$ for some $\tilde{x}_i \in F_i^{\perp} \cap M = F_i^{\dagger} \cap M$ and

$$E(x^{F^{*\perp}}|y) = E(\sum_{i \in \mathcal{D}} \tilde{x}_i^{F_i^{\perp}}|y) = \sum_{i \in \mathcal{D}} E(\tilde{x}_i^{F_i^{\perp}}|y) = 0, \text{ for all } y \in F^*$$

since each F_i is a factor subspace. If $x^{F^{*\perp}} \in \Big[cl(\sum_{i \in \mathcal{D}} F_i^{\perp}) \backslash \sum_{i \in \mathcal{D}} F_i^{\perp}\Big] \cap M$, then we can take a sequence$\{x_j\}_{j \in \mathcal{A}}$ in $\sum_{i \in \mathcal{D}} F_i^{\perp}$ converging to $x^{F^{*\perp}}$, where $\mathcal{A}$ is a directed set. Since each $j \in \mathcal{A}$, $x_j = \sum_{i \in \mathcal{D}} \hat{x}_i$, for some $\hat{x}_i \in F_i^{\perp} \cap M$, we can use a similar argument to show that $E(x^{F^{*\perp}}|y) = 0$, for all $y \in F^*$. This implies that $F^* \subseteq M \subseteq F^* \oplus (F^*)^{\dagger}$. Thus $F^* \in \{F_i : i \in \mathcal{D}\}$. $\blacksquare$

If there exists a finite dimensional factor subspace F of M, then the minimal F* is a subspace of F and itself is finite dimensional. In section 5, we give examples of asset market economies with finite dimensional minimal factor subspaces.

Remark 1: Notice that if we take the consumption set as $M \cap X_+$ instead of taking M and define $\succeq_i$ on $X_+ \times X_+$, then most arguments in this section fail unless the closed linear subspace F of M satisfies the following; $\big[x \in M \cap X_+\big] \Rightarrow \big[x^F \in F \cap X_+\big]$. In order to apply second order stochastic dominance arguments, x^F should be a well defined consumption bundle which is not true in general even if $F \cap X_+ \neq \emptyset$. Whether $\big[x \in M \cap X_+\big] \Rightarrow \big[x^F \in F \cap X_+\big]$ depends on the subspace F, which itself depends on the market structure. If we can find such an F and take $M \cap X_+$ as a consumption set, then the results in this section and afterwards are valid without changes. In section 5.4, we discuss how to find an F which endows the economy a factor space given the market structure and which has the property described above. For more discussion, see section 5.4 and Appendix I.

4. Existence of Exact Pricing Equilibrium

In this section, we prove the existence of a competitive equilibrium for an economy defined in section 2. Moreover we characterize equilibrium price systems.

Let E be an exchange economy and let $((\tilde{x}_i), p)$ be an equilibrium for E. In section 3, we showed that equilibrium consumption plan $(\tilde{x}_i)$ of consumer i belongs to F, if E has a factor subspace F and satisfies Assumptions 1 and 2. Since F is a closed linear subspace of M, there exists a complete orthonormal system $\{\alpha_i\}_{i \in A}$ in F. This implies that $p(\tilde{x}_i) = p(\sum_{j \in A} < \tilde{x}_i, \alpha_j > \alpha_j) = \sum_{j \in A} < \tilde{x}_i, \alpha_j > p(\alpha_j)$ by the linearity and continuity of p. In equilibrium, the same holds for any contingent claim in F. For a contingent claim $x \notin F$, we have $p(x) = p(x^F + x^{F^\perp}) = \sum_{j \in A} < x_i, \alpha_j > p(\alpha_j) + p(x^{F^\perp})$. Thus it is not enough to know prices of α_i's to value x which is not in F unless $p(x^{F^\perp}) = 0$. If an equilibrium $((\tilde{x}_i), p)$ is such that $p(x) = p(x^F)\ \forall x \in M$, then the price of any marketed contingent claim can be expressed in terms of prices of α_i's. If F is finite dimensional, then prices of marketed contingent claims can be expressed in terms of prices of mutual funds. So the question which natually arises is the following : "Is there an equilibrium $((\tilde{x}_i), p)$ for E such that $p(x) = p(x^F)\ \forall x \in M$?" We show that there exists such an equilibrium.

Let F be a factor subspace. Let $a_i = e_i^F \quad \forall i = 1, 2, \ldots, I$, where $e_i^F = e_i - e_i^{F^\perp}$. Then $a \equiv (a_i)_{i=1,2\ldots,I}$ is a feasible allocation. Consider an economy in which consumers only trade contingent claims in F taking a_i's as their initial endowments. We call such an economy a *reduced* economy denoted by $E' = (F, \succeq, a)$ for a given factor subspace F. An equilibrium for E' consists of an allocation $(x_i)_{i=1,2\ldots,I}$ and a non zero price functional $p \in X'$ such that for all i, x_i is $\succeq_i$ optimal in $B_i(p) = \{x \in F : p(x_i) \leq p(a_i)\}$. Consider a price system $p \in X'$ such that $p(x) = p(x^F)\ \forall x \in M$. Then consumer i who is strictly risk averse will not choose a contingent claim which is not in F, since for all $x \in M$, $x^F \succ_i x$ if $x \notin F$ and $p(x) = p(x^F)$. Therefore under such a price system, x^F is chosen over x by every consumer. Let $((\tilde{x}_i), p)$ be an equilibrium for E'. Consider a price system q given by $\forall x \in M \quad q(x) = q(x^{F^\perp}) = p(x^{F^\perp})$ for $x \in M$. Consumers facing the price system q will not change their consumption behavior from $(\tilde{x}_i)$ even though they are allowed to trade any marketed claims.

Theorem 2 : Suppose an exchange economy E has a factor subspace F and satisfies Assumptions 1-2. Let $((\tilde{x}_i), p)$ be an equilibrium for the reduced economy E$'$. Then $((\tilde{x}_i), q)$ is an equilibrium for E, where q is given by

$$q(x) = q(x^F + x^{F^\perp}) = q(x^F) = p(x^F) \text{ for all } x \in M.$$

Proof :

1) Budget feasibility : $q(\tilde{x}_i) = p(\tilde{x}_i) = p(a_i) = p(e_i^F) = q(e_i)$, since $\tilde{x}_i \in F \ \forall i$ by Corollary 1.
2) Optimality : Suppose $((\tilde{x}_i), q)$ is not an equilibrium for E. Then there exists some $j \in \{1, 2, \ldots, I\}$ and $y_j \in M$ such that $y_j \succ_j (\tilde{x}_j)$ and $q(y_j) \leq q(e_i)$. Suppose $y_j \in F$. Then $p(y_j) = q(y_j) \leq q(e_i) = p(a_j)$, which contradicts that $((\tilde{x}_i), p)$ is an equilbrium for E'. Suppose $y_j \notin F$. Then $y_j = y_j^F + y_j^{F^\perp}$ such that $E(y_j^{F^\perp} | y_j^F) = 0$. Thus $y_j^F \succ_j y_j$ and $p(y_j^F) = q(y_j^F) = q(y_j) \leq q(e_j) = p(a_j)$. This leads to another contradiction. Therefore $((\tilde{x}_i), q)$ is an equilibrium for E. This completes the proof. ∎

By Theorem 2, it suffices to show the existence of an equilibrium for E$'$ to prove the existence of a competitive equilibrium for E. Suppose an economy has a finite dimensional factor subspace. Consider a reduced economy E$'$ with an F as a consumption set. If the consumption set F of the reduced economy is bounded below, then an equilibrium exists. However F is not bounded below and it is well known that there may be no equilibrium if the consumption set is not bounded below. Much work has been done on the existence of equilibrium with an unbounded consumption set, and sufficient conditions for existence are well known. Assumption 3 is sufficient for this purpose with Assumptions 1$'$ and

2. Assumption 3 is known as the "positive semi-independent of the directions of improvement" condition.(3)

Let B be a nonempty, closed, and convex set in F. The recession cone of B, denoted by $\mathcal{A}B = \{\bar{x} \in F : x + \lambda\bar{x} \in B \ \forall\lambda \geq 0\}$ for some $x \in B$. $\bar{x}$ is called a direction of recession on B. d_i is said to be a direction of improvement for a consumver i if and only if d_i is a direction of recession of $\{y \in F : U_i(y) \geq U_i(a_i)\}$.

Assumption 3 : If for each i, d_i is a direction of imprevement for consumer i, and if $\sum_{i=1}^{I} d = 0$, then $d_i = 0$ for all i.

Assumption 4 : $F \cap X_{++} \neq \emptyset$.

It is known that satiation can occur in the expected utility model if each asset has negative returns with positive probability. To avoid this, we assume that $F \cap X_{++} \neq \emptyset$. If a riskless asset is traded, then it can be shown that Assumption 4 is satisfied.(4)

Theorem 3 : If an exchange economy $E \equiv (M, \succeq, e)$ with a finite dimensional factor subspace F, satisfies Assumption $1'$, 2, 3 and 4, then there exists a competitive equilibrium $((\tilde{x}_i), q)$ such that $q(x) = q(x^F) \ \forall x \in M$.

Proof : By Nielsen(1989) there exists an equilibrium $((\tilde{x}_i), p)$ for E'. Then by Theorem 2, $((\tilde{x}_i), q)$ is an equilbrium for E, where q is given by $q(x) = q(x^F + x^{F^\perp} = q(x^F) = p(x^F)$ for all $x \in M$. This completes the proof

The next question we will examine is what conditions gurantee that every equilibrium $((\tilde{x}_i), q)$ for E is such that $q(x) = q(x^F) \ \forall x \in M$.

It is well known that by Assumption 2, each consumer's preference relation $\succeq_i$ is representible by a real valued function $U_i : X \to R$. Let T be a function defined on an open domain D in X and having range in R. If, for fixed $x \in D$ and

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(3) See Nielsen (1989).

(4) In other words, a riskless asset always belongs to F.

for each $h \in X$, there exists $\delta T(x)$ which is linear and continuous with respect to h such that

$$lim_{||h|| \to 0} \frac{||T(x+h) - T(x) - \delta T(x)h||}{||h||} = 0$$

then T is said to be Fréchet differentiable at x and $\delta T(x)$ is said to the Fréchet differential of T at x. In our case where $T = U_i$ is a functional, we have $\delta U_i(x) \in$ the dual of X. Let $T' : x \longrightarrow \delta T(x)$. Then T' defines a transformation from D into the dual of X. T' is called the Fréchet derivative T' of T. Thus by definition $\delta T(x)h = T'(x)h$. If T' is continuous on some open sphere S, we say that T is continuously Fréchet differentiable on S. A point at which $\delta T(x)h = 0 \; \forall h \in X$ is called a *stationary* point. It is easy to see that if x is an extremum of a functional which is Fréchet differentiable, then x is a stationary point. The analogous arguments hold true for constrained optimization problems. x_0 is said to be a *regular* point of the inequality $g(x) \leq 0$ if $g(x_0) \leq 0$ and there is $h \in X$ such that $g(x_0) + \delta g(x_0) \cdot h < 0$. If x_0 is an extremum of the functional f subject to the constraint $g(x) \leq 0$ and x_0 is a regular point of the inequality $g(x) \leq 0$, then there exists a scalar $\lambda \geq 0$ that renders the functional $f(x) + \lambda g(x)$ stationary at x_0 and $\lambda \cdot g(x_0) = 0$.(5)

The following Theorem tells us that if there exists some consumer i with a utility function which is continuously Fréchet differentiable on the open sphere of $\hat{x}_i$ such that $\hat{x}_i$ is the optimal choice given the equilibrium price system p. Then $p(x) = 0 \; \forall x \in F^{\perp}$.

Theorem 4 : Consider an economy $E = (M, \succeq, e)$ with a factor subspace F satisfying Assumption 1,2. Let $((\hat{x}_i), p)$ be an equilibrium for E. Suppose there exists a consumer i with a utility function which is continuously Fréchet differentiable on the open sphere of $\hat{x}_i$. Then p is a price system such that $p(x) = 0 \; \forall x \in F^{\perp}$.

Proof : By Corollary 1, for each $i \in \{1, 2, \ldots, I\}$, $\hat{x}_i \in F$. Consider the following

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(5) See Luenberger for further discussion.

optimization problem for consumer i: $\max_{x\in X} U(x)$ such that $p(x) \leq p(e)$. Then $\hat{x}_i$ is regular and $u(x)+\lambda p(x)$ is stationary at $\hat{x}_i$ for some $\lambda > 0$, since $\hat{x}_i$ solves the optimization problem and the constraint holds with equality. This implies that

$$\begin{aligned}\delta(U(\hat{x}) + \lambda p(\hat{x})) &= \delta U(\hat{x}) + \lambda \delta p(\hat{x}) \\ &= \delta U(\hat{x}) + \lambda p, \text{ since } p : X \to R \text{ linear} \\ &= 0 \\ &\Rightarrow p = -\frac{1}{\lambda}\delta U(\hat{x}).\end{aligned}$$

Let $\{\alpha_i\}_{i\in\mathcal{A}}$ be a complete orthonormal system in F. Since U is continuously Fréchet differentiable and $\hat{x}_i \in F$, $p(h) =< \pi, h >= E(\delta U(\hat{x}) \cdot h) = E(E(\delta U(\hat{x}) \cdot h)|\alpha_i,\ i \in \mathcal{A})) = E(\delta U(\hat{x}) E(h|\alpha_i,\ i \in \mathcal{A})) = 0$. This completes the proof. ∎

In the rest of this section, we will examine the relationship between the pricing relation in this paper and that in typical finance model. In finance models, the pricing relation is usually given in terms of the expected return of an asset.(6) The pricing relation in this paper also can be expressed in terms of expected returns. Let p be an equilibrim price system for an economy with a factor subspace F such that $p(x) = 0\ \forall x \in F^{\perp}$. Let π be the asset which is uniquely associated with such a price system. Then $\pi \in F$. Let x be an asset which has a non-zero market value. Then $< \pi, x > \neq 0$. Since the expectation operator E is a continuous linear functional, there exists a unique asset $m \in M$ such that $\forall x \in M,\ E(x) =< m, x >$. Take an arbitrary $y \in F^{\perp}$. Then $E(y|x) = 0$ for all $x \in F$. This implies that $E(y) = 0$. Let $\Gamma = \{M : E(y) = 0\}$. Then it is easy to see that $m \in \Gamma^{\perp} \subseteq F$. Let $R_x \equiv \frac{x}{<\pi, x>}$. Then R_x is an asset with unit cost. Let $\{\alpha_i\}_{i\in\mathcal{A}}$ be an orthonormal system in F, where the cardinality of A is the dimension of F. For any traded asset with non-zero market value, $E(R_x) = \varphi + < m - \varphi\pi, R_x >$ for any $\varphi \in R$. Since m and π are in F, $m - \rho\pi \in F$. This implies that $m - \rho\pi = \sum_{i\in\mathcal{A}} \tau_i \alpha_i$. Therefore

$$\forall x \in M \text{ with nonzero market value } E(R_x) = \varphi + \sum_{i\in\mathcal{A}} \tau_i < \alpha_i, R_x >$$

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(6) A contingent claim x can be thought of an asset with a payoff $x(\omega)$ in state $\omega \in \Omega$

where $\tau_i =< m - \varphi\pi, \alpha_i >$.

If a riskless asset denoted by x_0 is traded, then $m = \frac{x_0}{\rho}$, where $\rho = E(x_0)$. Take $\alpha_1 = m = \frac{x_0}{\rho}$ and $\varphi = \rho$. Without loss of generality, we normalize π such that $< \pi, x_0 >= 1$. Then $m - \rho\pi = \sum_{i \in \mathcal{A} \setminus \{1\}} \tau_i \alpha_i$, since $E(x_0) = \rho + < m - \rho\pi, x_0 >= \rho + \tau_1 < m, x_0 >= \rho$ implies that $\tau_1 = 0$. From this relation we can express the expected return of an asset with an unit cost in terms of the gross return of a riskless asset and covariances with assets which span the space F.

$$\forall x \in M \text{ with nonzero market value } E(R_x) = \rho + \sum_{i \in \mathcal{A} \setminus \{1\}} \tau_i < \alpha_i, R_x >$$

where $\tau_i =< m - \rho\pi, \alpha_i >$. This proves the following Corollary.

Corollary 3 : Suppose an economy E has a factor subspace F and satisfies Assumptions 1,2 and 5. Let $((\hat{x}_i), p)$ be an equilibrium for E. Suppose there exists consumer i with a continuously Fréchet differentiable u_i on an open sphere of $\hat{x}_i$. Then $\forall x \in M$ s.t $p(x) \neq 0$, $E(R_x) = \varphi + \sum_{i \in \mathcal{A}} \tau_i < \alpha_i, R_x >$ for any $\varphi \in R$, where $\tau_i =< m - \varphi\pi, \alpha_i >$.
If a riskless x_0 is traded with $< \pi, x_0 >= 1$ and $E(x_0) = \rho$, then for all x in M with non-zero market value $E(R_x) = \rho + \sum_{i \in \mathcal{A} \setminus \{1\}} \tau_i < \alpha_i, R_x >$,
where $\tau_i =< m - \rho\pi, \alpha_i >$ and π is a unique element associated with p.

5. Examples

In this section, we give examples to which the analysis made in previous sections can be applied. For the given market structure, we find a closed linear subspace F of M which is a factor subspace. Furthermore in each case, F is minimal.

5.1. Economy without Aggregate Risks

Suppose we are given an exchange economy $E = (M, \succeq, e)$. We maintain the assumption that each consumer i's endowment is traded in the market, denoted by M. Suppose that there is no aggregate risk. In other words, $\frac{1}{I}\sum_{i=1}^{I} e_i \equiv \mu =$ constant a.s. Consumers are assumed to have preferences which satisfy assumptions 1, 2, and 3. Let F be the space spanned by the market endowment which is deterministic. Then $F \cap X_{++} \neq \emptyset$ and $F \subset M$. $F^{\perp}$ is the collection of contingent claims with mean zero. Furthermore for all $x \in F^{\perp}$ $E(x|\mu) = 0$, which implies that F is a factor subspace. Suppose a price system is given such that for all $x \in M$ $p(x) = E(x)$. Then each consumer will consume a riskless claim equal to the expected value of his/her initial endowment. This allocation is feasible. It is an equilibrium in which all consumers hold riskless claims and the equilibrium price system values only the riskless part of a contingent claim.

5.2. Equilibrium Version of Arbitrage Pricing Theory

In this section we apply our results to the equilibrium version of the Arbitrage Pricing Theory (hereafter, APT). The APT relates the price of an asset to its factor exposures. We borrow most of notations and assumptions from Connor(1984).

Suppose there are countably many assets traded in a market. In APT, it is assumed that each asset's payoff shows the following structure known as a "factor structure". Let y_i denote asset i's payoff which is a random variable in $\mathcal{L}_2$.

$$y_i = c_i + \sum_{j=1}^{K} \beta_{ij} f_j + \epsilon_i, \text{ where}$$

i) c_i is a constant.

ii) $\{f_1, f_2, \ldots, f_K\}$ is a set of mutually orthonormal random variables such that $E(f_j) = 0$ for all $j = 1, 2, \ldots, K$.

iii) $\{\epsilon_i\}_{i=1}^{\infty}$ is a set of mutually orthonormal random variables such that $E(\epsilon_i) = 0$ and $< \epsilon_i, f_j >= 0$ $\forall j = 1, 2, \ldots, K$.

This characterization says that the assets' payoffs are all closely related through the constant term and the f_j's. We can think of these terms as economy wide fluctuations. Each has its own idiosyncratic noise term, and ϵ_i's represent these asset specific fluctuations. In APT, the constant term and the f_j's are called factors, and the ϵ_i's are called idiosyncratic risks. Thus c_i and β_{ij} denote y_i's factor exposures. Connor (1984) called the economy with traded assets satisfying the above characterization "a factor economy". Let M be the closure of span of $\{y_i\}_{i=1,2,\ldots,\infty}$, and let $F =$ span$\{x_0, f_1, f_2, \ldots, f_K\}$, where x_0 denote a riskless asset.

Assumption 5.2.1 : $F \subset M$.

Connor (1984) defines an economy satisfying Assumption 5.2.1 as "an insurable factor economy." By the definition of the ϵ_i's, it is clear that the closure of the span of $\{\epsilon_i\}_{i=1,2,\ldots,\infty} \subseteq F^{\perp}$. The following assumption is conventional in the equilibirum version of APT.

Assumption 5.2.2 : For all i, $E(\epsilon_i | x_0, f_1, f_2, \ldots, f_K) = 0$.

By Assumption 5.2.1 and 5.2.2

$$F \subset M \subseteq F \oplus F^{\dagger} \subseteq F \oplus F^{\perp} = X.$$

The equilibrium version of APT also assumes that the market endowment is spanned by factors. Thus the factor economy with Assumption 5.2.1 and 5.2.2 possesses a factor subspace F. Since F is the set of orthonormal vectors which are not necessarily mutually mean independent, F is minimal. Thus the equilibirum version of APT satisfies Assumption 1-4 in our model. Therefore the results in section 3 and 4 with a finite dimensional factor subspace are applicable. By Corollary 2, each consumer holds a consumption bundle which is a linear combination of (K+1) mutual funds. This result is consistent with Connor's Corollary 2.1. Theorem 3 tells that there is an equilibrium at which the price of each asset can be expressed "exactly" in terms of the prices of (K+1) mutual funds. If preferences are representible by differentiable von Neumann-Morgenstern utilities, then Theorem 3 and Corollary 4 implies that every equilibrium price is linear with regard to the prices of the factors.

5.3. CAPM with Elliptical Distributions

In this example, we study the CAPM from the point of view of general equilibirum theory. The CAPM is a model of asset pricing which implies that the expected return on any portfolio payoff (here, a contingent claim) in excess of the riskless payoff is the *beta* of the portfolio multiplied by the excess return of the maket portfolio (here, the market endowment). This pricing relation can be derived after placing restrictions either on consumer preferences or on the probability distributions of the asset payoffs. In this example, we show that the CAPM pricing formula can be derived from the general equilibrium framework developed in sections 3 and 4. Most equilibrium versions of the CAPM establish the existence of the CAPM equilibrium by imposing restrictions on preferences. We, instead, restrict the probability distribution of contingent claims. This enables us to study both the existence question and the pricing at the same time. Let $X = \mathcal{L}_2(\Omega, \mathcal{F}, P)$ be a space of contingent claims. There are consumers with preferences which satisfy assumptions 1,2, and 3. Let M be the marketed claim space and assume that any contingent claim in M has an elliptical distribution.(7) The class of elliptical distributions contains the mutivariate normal distribution as well as other distributions. This class of distribution has many interesting properties. Most importantly, 1) any linear combination of elliptical distributions has an elliptical distribution, 2) if two random variables are uncorrelated and have elliptical distributions, then these random variables are mean independent of each other.(8) Each

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(7) Let Δ be a fixed p-component vector and Ω a $(p \times p)$ positive definite symmetric matrix. Then a p-component random vector $X = (X_1, X_2 \ldots, X_p)$ is said to be distributed elliptically if and only if for all nonzero p-component scalar vector α, all the univariate random variables $\alpha \cdot X$ such that VAR$(\alpha \cdot X)$ is constant, follow the same distribution. There are several equivalent difinitions of elliptical distributions of random variables. For further discussions, see Owen and Rabinobitch(1983)

(8) For detailed discussions, see Owen and Rabinobitch (1983).

consumer i has an endowment e_i in M, which implies that $\mu \equiv \frac{1}{I}\sum_{i=1}^{I} e_i \in M$. If a riskless claim x_0 is in M, take $F =$ span$\{x_0, \mu\}$. If there is no riskless asset, then take $F =$ span$\{m, \mu\}$, where m is an asset which is uniquely associated with an expectation operator, i.e, $\forall x \in M, \quad E(x) =< m, x >$. Now we have an economy $E = (M, \succeq, e)$ which satisfies Assumption 1,2, and 3. Let $F^{\perp}$ be the orthogonal complement of F. It is clear that $F^{\perp} \subset \{x \in X : E(x) = 0\}$. Take an arbitary $x \in F^{\perp}$, then $< x, y >= 0 = E(x) \; \forall y \in F$. Since x, y have elliptical distributions, this implies that $E(x|y) = 0 \; \forall y \in F$. Therefore $E = (M, \succeq, e)$ has a factor subspace F. If there exists a riskless asset, then theorem 3 is applicable since $F \cap X_{++} \neq \emptyset$. However if a riskless asset is not traded, then satiation can occur on the marketed claim space with elliptical distributions. This was pointed out by Nielsen (1989). Thus Theorem 3 is not readily applicable without making an additional assumption,e.g (condition e) used in Nielsen (1989). For a more detailed discussion about the existence of the CAPM equilibrium without a riskless asset, we refer the reader to Nielsen (1989).

Now Corollary 3 is applicable to the CAPM economy. Thus there exists a price system identified with a pricing asset π such that $\pi \in F$. Corollary 3 implies the *beta* model of rates of return in CAPM.

5.4. Equilibrium with Call Options

In this section, we apply the results of sections 3 and 4 to an economy with call options on the aggregate endowment per capita. In section 5.1 and 5.3, we saw that for a given market structure, we can find a relevant subspace of M which is a factor subspace. In this section we suggest a way to find such an F for the general case. And the F can be regarded as an endogenously determined factor space in the APT.

Suppose there are finitely many states. It is well known that any efficient allocation can be achieved through trading a full set of Arrow securities. We, however, do not observe these types of securities in the real world. Breeden and Litzenberger (1978) suggest that an efficient allocation can also be achieved if call options on the market endowment are traded. In this section, we extend these arguments to the case in which the marketed claim space is infinite dimensional and potentially incomplete. In fact, we already examined one example without explicitly mentioning this line of arguments in section 5.1. We saw that if there is no aggregate risk, then only one contingent claim is needed to span the set of efficient allocations. This contingent claim can be interpreted as a call option on the market endowment with zero striking price. In this section we show that the space spanned by call options on the market endowment is a factor subspace of an economy if the market endowment takes countably many distinct values.

Assume that μ takes finitely many distinct values.(9) Let $\{\mu_1, \mu_2, \ldots, \mu_N\}$ be the set of distinct values of μ. Without loss of generality, we assume that $\mu_i \geq \mu_j$ if $i > j$. Define

$$\delta_i = \{\omega : \mu_{i+1} > \mu(\omega) \geq \mu_i\} \ \forall i = 1, 2, \ldots, N-1 \text{ and } \delta_N = \{\omega : \mu(\omega) \geq \mu_N\}.$$

Then it is easy to see that $\{\delta_i\}_{i=1,2,\ldots,N}$ forms a partition of Ω. Suppose that *call options*(10) on μ are traded. A call option gives its holder the right to purchase the underlying contingent claim at a specified price, called the exercise price. Let $c(\mu_{i-1})$ be a payoff of a call option on the aggregate endowment with an exercise price of μ_{i-1} $\forall i = 1, 2, \ldots, N$, where $\mu_0 = 0$. Then the payoff distribution of $c(\mu_{i-1}),\ 1 \leq i \leq N$ is the following;

$$c(\mu_{i-1})(\omega) = \begin{cases} 0, & \text{if } \omega \in \delta_j, j \leq i-1; \\ \mu_j - \mu_i, & \text{if } \omega \in \delta_j, j > i-1. \end{cases}$$

We shall assume that each consumer's endowment is traded. Let M denote the

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(9) The analysis hereafter is also valid if μ takes countably many distinct values. This corresponds the case where dim F $= \infty$. If μ takes countably many values , then the partition generated by the market endowment consists of countably many sets

(10) In this model, the European call option is equivalent to the American one. For this reason, we use "call option."

marketed claim space, which includes e_i's, call options with payoffs $c(\mu_{i-1}),\ 1 \geq i \leq N$. Since $c(0) = \mu$, the market endowment is in M.

Let $F = \text{span}\{c(\mu_0), c(\mu_1), \ldots, c(\mu_{N-1})\}$. F is the linear subspace of X spanned by the set of call options. Since there are only finite μ_i's, F is closed and finite dimensional. Furthermore $c(0, \omega) = \mu$ implies that $\mu \in F$. We assume that there is a finite number of consumers with preferences satisfying assumption 1, 2, and 3.(11)

Suppose $\mu_1 = 1, \mu_2 = 2, \ldots, \mu_N = N$. Then it is easy to see that $c(0) - c(1) = 1$ and $c(1) - c(2)$ gives 0 if δ_1 happens and 1 otherwise. Hence $[c(0) - c(1)] - [c(1) - c(2)]$ takes 1 on δ_1 and 0 otherwise. By repeating this procedure, for any one of the partitions generated by μ we can generate a contingent claim which gives one unit of a good on a given partition and 0 otherwise. Notice that the collection of these contingent claims forms a complete orthogonal system in the space spanned by call options. In general, we have Lemma 5.4.1.

Lemma 5.4.1 : $F = \text{span}\{1_{\delta_1}, 1_{\delta_2}, \ldots, 1_{\delta_N}\}$, where 1_{δ_i} is a contingent claim which pays

$$1_{\delta_i}(\omega) = \begin{cases} 1, & \text{if } \omega \in \delta_i; \\ 0, & \text{if otherwise.} \end{cases}$$

Proof : It suffices to show that for all $i = 1, 2, \ldots, N$, 1_{δ_i} can be generated by call options. Let $\Delta_i = \mu_i - \mu_{i-1}$. Then

$$1_{\delta_i} = \begin{cases} \frac{1}{\Delta_i}[c(\mu_{i-1}) - c(\mu_i)] - \frac{1}{\Delta_{i+1}}[c(\mu_i, \omega) - c(\mu_{i+1})], & \text{if } i \leq N-1; \\ \frac{1}{\Delta_i} c(\mu_{i-1}), & \text{if } i = N. \end{cases}$$

And it is clear that $1_{\delta_i}\ \forall i = 1, 2, \ldots, N$ is an orthogonal system in F. This completes the proof. ∎

Let $F^{\perp}$ be the orthogonal complement of F. The following Lemma shows that $F^{\perp}$ is the set of contingent claims with zero means on the partition $\{\delta_i\}_{i \in N}$.

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(11) We can dispense with Assumption 3 in this example. Assumption 3 was sufficient to show the existence of equilibrium. But we can prove the existence in this economy without it by replacing the consumption set M with $M \cap X_+$. See Appendix I for the detailed discussion.

Lemma 5.4.2: $F^{\perp} = \{x \in \mathcal{L}_2 : E(x \cdot 1_{\delta_i}) = 0 \ \forall i \in N\}$.

Proof : straightforward. ∎

Consider a contingent claim z such that

$$z(\omega) = \begin{cases} 0, & \text{if } \omega \notin \delta_1; \\ 1, & \text{if } \omega \in s_1; \\ -1, & \text{if } \omega \in s_2. \end{cases}$$

where $s_1 \cup s_2 = \delta_1$, $s_1 \cap s_2 = \emptyset$, and $P(s_1) = P(s_2)$. By construction $z \in F^{\perp}$. The expected value of z is zero. Suppose we know which one of the δ_i's will be realized. What is the expected value given this information? The conditional expectation is still zero even though more information is available. The reason is the following; Suppose we know that $\delta_{j,j\neq 1}$ is the event which will be realized. Then we know that z will give nothing. Suppose we know that δ_1 will happen. Even though we know that in this event z will yield an outcome which is nonzero, we can do no better than saying that z will give nothing on average since δ_1 is coarser than the s_is. In other words, z is mean independent of the σ-field generated by the δ_is. It is easy to see that this σ-field is the one generated by 1_{δ_i}s. Therefore z is mean independent of any contingent claim in F. In the following lemma, we show that this is true for all contingent claims in $F^{\perp}$.

Lemma 5.4.3: For all $x \in F^{\perp}$, $E(x | 1_{\delta_i}, \ \forall i = 1, 2, \ldots, N) = 0$

Proof : Let $\sigma(1_{\delta_i}, \ \forall i = 1, 2, \ldots, N)$ denote the σ-field generated by $1_{\delta_i}, \ \forall i = 1, 2, \ldots, N$. Since the σ-field generated by each 1_{δ_i} forms a partition $\{\delta_i, \cup_{j\neq i}\delta_j\}$, it is easy to see that $\sigma(1_{\delta_i}, \ \forall i = 1, 2, \ldots, N) = \sigma(\delta_i, \ \forall i = 1, 2, \ldots, N)$, where $\sigma(\delta_i, \ \forall i = 1, 2, \ldots, N)$ is the σ-field generated by $\delta_i, \ \forall i = 1, 2, \ldots, N$. We shall denote $E(x | 1_{\delta_i}, \ \forall i = 1, 2, \ldots, N)$ by $E(x|\sigma)$.

Claim 1 : $E(x \cdot 1_{\delta_i} | \sigma)$ is a step function on $\{\delta_i\}_{i=1,2,\ldots,N}$. In other words,for some $y_j \in R$ $E(x \cdot 1_{\delta_i}|\sigma)(\omega) = y_j$ a.s. if $\omega \in \delta_j$

Proof : Suppose not. Then there exists ω_1, $\omega_1 \in \delta_j$ for some j such that $E(x \cdot 1_{\delta_i}|\sigma)(\omega_1) \neq E(x \cdot 1_{\delta_i}|\sigma)(\omega_2)$. Without loss of generality, we assume that $E(x \cdot$

$1_{\delta_i}|\sigma)(\omega_1) > E(x \cdot 1_{\delta_i}|\sigma)(\omega_2)$. Take $z \in R$ such that $E(x \cdot 1_{\delta_i}|\sigma)(\omega_1) \geq z > E(x \cdot 1_{\delta_i}|\sigma)(\omega_2)$. Since $E(x \cdot 1_{\delta_i}|\sigma)$ is measurable with regard to σ, $A \equiv \{\omega : E(x \cdot 1_{\delta_i}|\sigma)(\omega_1) \geq z\} \in \sigma$. By construction $B \equiv A \cap \delta_j \neq \emptyset$ and $B \subset \delta_j$ and $B \neq \delta_j$, since $\omega_2 \notin B$. But σ is a σ-field generated by mutually disjoint sets. Therefore $B \notin \sigma$, which contradicts the assumption that $E(x \cdot 1_{\delta_i}|\sigma)$ is measurable with respect to σ. This completes the proof of claim 1. ∎

Claim 2: $E[E(x \cdot 1_{\delta_i}|\sigma)] = y_i p(\delta_i)$

Proof : By Claim 1,

$$E(x \cdot 1_{\delta_i}|\sigma) = \begin{cases} 0, & \text{if } \omega \notin \delta_i; \\ j_i, & \text{if } \omega \in \delta_i; \end{cases}$$

Therefore $E[E(x \cdot 1_{\delta_i}|\sigma)] = y_i p(\delta_i)$ This completes the proof of claim 2. ∎

Claim 3: $E(x \cdot 1_{\delta_i}|\sigma) = 0$ a.s

Proof :

$$\begin{aligned} E[E(x \cdot 1_{\delta_i}|\sigma)] &= E(E(x \cdot 1_{\delta_i})) \\ &= 0 \text{ ,since } x \in F^{\perp} \\ &= y_i p(\delta_i) \text{ by Claim 2} \\ &\Rightarrow y_i = 0 \text{ if } p(\delta_i) \neq 0 \\ &\Rightarrow E(x \cdot 1_{\delta_i}|\sigma) = 0 \text{ a.s.} \end{aligned}$$

This completes the proof of claim 3. ∎

By Claim 1 - 3, $E(x|\sigma) = \sum_{i=1,2,\ldots,N} E(x \cdot 1_{\delta_i}|\sigma) = 0$ a.s. This complets the proof of Lemma 5.4.3. ∎

Now the exchange economy $E = (M, \succeq, e)$ with call options satisfies assumptions 1 - 4 and the next proposition shows that $E = (M, \succeq, e)$ has a factor subspace.

Proposition 2: Let F, M, and $F^{\perp}$ be defined as above. Then F is a factor subspace of $E = (M, \succeq, e)$. Furthermore F is minimal.

Proof : By construction $F \subset M$, and $M \subset F \oplus F^{\perp} = X$. Furthermore $x_0 \in F$, $\mu \in F$. By above Lemmata, all claims in $F^{\perp}$ are mean independent of call options on the aggregate endowment. By Lemma 5.4.1, F is spanned by orthogonal vectors which are measurable with regard to each other. Thus F is minimal. This completes the proof. ∎

Since the economy with call options has a factor subspace F which is spanned by call options, we can apply Theorem 1-3 and corollaries to the economy with call options.

Theorem 5: Let $E = (M, \succeq, e)$ be an exchange economy satisfying Assumption $1'$, 2, 3, and 4. Suppose call options on the market endowment are traded. If the market endowment takes finitely many distinct values, then there exists an equilibrium $((\tilde{x}_i), p)$ for E. Furthermore each consumer i's consumption bundle in equilibrium is a linear combination of call options. and the price of any marketed claim can be expressed in terms of the prices of the call options.(12)

In this example call options on the market endowment play the same role as factors do in APT. In other words, any Pareto optimal consumption bundle can be generated by the linear combination of call options on the aggregate endowment. And all marketed contingent claims can be valued in terms of the prices of call options on the aggregate endowment in an equilibrium. This example, however, is very different from conventional APT theory. The most significant difference is that we do not assume that assets show a factor structure but instead derive a factor subspace from the given market structure. In APT, each asset's payoff is assumed to show a factor structure. And the space spanned by factors plays a crucial role in pricing assets. But APT theory does not tell how the selection of factors should be made. If we have an economy like the one described in this example, then this model explains how the factor subspace is endogenously determined. The explicit pricing formula is given in the following.

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(12) For the statement of Theorem 5 without Assumption 4, see Appendix II.

Let $x \in M$. Then $x = x^F + x^{F^\perp}$, where $x^F \in F$ and $x^{F^\perp} \in F^\perp$. At equilibria, $p(x) = p(x^F)$ by Theorem 3. Now by Lemma 5.4.1, $x^F = \sum_{i=1}^N \alpha_i 1_{\delta_i}$, where $\alpha_i = \frac{<x, 1_{\delta_i}>}{||1_{\delta_i}||}$. Furthermore by Lemma 5.4.1, 1_{δ_i} can be generated by call options,i.e,

$$1_{\delta_i} = \begin{cases} \frac{1}{\Delta_i}[c(\mu_{i-1}) - c(\mu_i)] - \frac{1}{\Delta_{i+1}}[c(\mu_i, \omega) - c(\mu_{i+1})], & \text{if } i \leq N-1; \\ \frac{1}{\Delta_i} c(\mu_{i-1}), & \text{if } i = N. \end{cases}$$

Let p_{i-1} be the price of call option with striking price μ_{i-1}. Then the price of 1_{δ_i} in terms of the prices of the call options, is

$$p(1_{\delta_i}) = \begin{cases} \frac{1}{\Delta_i}(p_{i-1} - p_i) - \frac{1}{\Delta_{i+1}}(p_i - p_{i+1}), & \text{if } i \leq N-1; \\ \frac{1}{\Delta_N} p_{N-1}, & \text{if } i = N. \end{cases}$$

Thus the price of a contingent claim x in terms of prices of call options is the following ;

$$p(x) = p(x^F) = \sum_{i=1}^{N-1} \alpha_i [\frac{1}{\Delta_i}(p_{i-1} - p_i) - \frac{1}{\Delta_{i+1}}(p_i - p_{i+1})] + \frac{\alpha_N}{\Delta_N} p_{N-1}.$$

This example also can be viewed as an infinite dimensional generalization of Breeden and Litzenberger (1978) with abstract preferences. The pricing relation of this paper is valid not only for the contingent claims in F but also for those not in F.

6. Concluding Remarks

In this paper we studied a general equilibrium approach to the asset pricing model. Using the concept of a factor subspace, we explained how second order stochastic dominance theory and the concept of pareto optimality, and competitive equilibrium are interrelated to each other. The pricing implication of this model is potentially useful in finance theory since the price of any asset can be expressed in terms of prices of finitely many assets spanning a factor subspace. We show that there exists a minimal factor subspace in the sense that no proper subspace of that space is a factor subspace. Connor (1984) obtains the similar pricing relation as we have. But our approach in this paper differs from Connor's in many respects. The most significant difference is that we do not take a set of contingent claims regarded as a set of factors in the APT as exogenously given . Instead we analyze how a (minimal) factor subspace is endogenously determined given the market structure. In fact, we show that if call options on the market endowment are trade and the space spanned by those call options is taken as F, then the economy has a factor subspace provided that the market endowment takes finitely many distict values. This implies that call options can be regarded as endogenously determined factors. In this sense, this paper attempts to build an asset pricing theory which narrows the gap between the CAPM and the APT. Future research will be needed to extend these results to mutiperiod models.

7. Appendix I

We show that there exists an equilibrium with call options as described in section 5.4 without Assumption 3 and 4 if we define $M \cap X_+$ as a consumption set and define $\succeq_i$ on $X_+ \times X_+$. The proofs of most theorems in this paper are based on the general equilibrium application of stochastic dominance thoery. In other words, $x^F \succeq x$ if $x = x^F + x^{F^\perp}$ such that $E(x^{F^\perp}|x^F) = 0$. To apply this argument, it must be the case that both x^F and x are well defined consumption bundles. Assumption 4 says that $F \cap X_+ \neq \emptyset$. If we take $x \in M \cap X_+$, then in general $x^F \notin F \cap X_+$, which implies x^F is not a consumption bundle if consumption set is defined by $M \cap X_+$. This is the reason why we can not take the positive orthant of the marketed claim space as a consumption set. But in the economy with call options, this problem does not appear.

Lemma : Let X, M, and F be as described in section 5.4. Then for all $x \in M$ $[x \in M \cap X_+] \Rightarrow [x^F \in F \cap X_+]$.

Proof : Take an arbitrary $x \in M \cap X_+$. Then $x = x^F + x^{F^\perp}$ such that $E(x^{F^\perp}|x^F) = 0$, and $x^F = \sum_{i=1}^N \alpha_i 1_{\delta_i}$, Thus it suffices to show that $\alpha_i \geq 0 \ \forall i$. But $\alpha_i = \frac{<x, 1_{\delta_i}>}{||1_{\delta_i}||} = \frac{E(x \cdot 1_{\delta_i})}{||1_{\delta_i}||} \geq 0$, since $x \geq 0$. Hence $x^F \in F \cap X_+$. This completes the proof. ∎

Due to lemma, we can take $M \cap X_+$ as a consumption set and the main results in this paper except Theorem 3 are valid under Assumption 1-3. Since we take the positive orthant as a consumption set, it is reasonable to assume that each consumer has a consumption bundle as an initial endowment.

Assumption 3′ : For all i, $e_i > 0$.

The reduced ecocnomy correspoding to $E = (M \cap X_+, \succeq, e)$ is $E' = (F \cap X_+, \succeq, a)$. Clearly E' has a finite dimensional bounded below consumption set. And $a_j = e_j^F = \sum_{i=1}^N \alpha_i i_{\delta_i} > 0$, since $e_j > 0$ implies $\alpha_i = \frac{<e_j, 1_{\delta_i}>}{||1_{\delta_i}||} = \frac{E(e_j \cdot 1_{\delta_i})}{||1_{\delta_i}||} \geq 0 \ \forall i$ with at least one strict inequality. It is well known that there exists an equilibrium for this reduced economy. By the same method as that used in the proof of Theorem 3, the equilibrium for the reduced economy can be extended to the original economy $E = (M \cap X_+, \succeq, e)$. We have the following theorem for the economy with call options.

Theorem 6: Let $E = (M \cap X_+, \succeq, e)$ be an exchange economy satisfying Assumptions 1′, 2 and 3′. Suppose call options on the market endowment are traded. If the market endowment takes finitely many distinct values, then there exists an equilibrium $((\tilde{x}_i), p)$ for E. Furthermore at the equilibrium each consumer i's consumption bundle is a linear combination of call options and the price of any marketed claim can be expressed in terms of the prices of call options on the market endowment.

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