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# A REEXAMINATION OF THE RELATIONSHIP BETWEEN PREFERENCES AND MOMENT ORDERINGS BY RATIONAL RISK AVERSE INVESTORS

by

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## I. Introduction and Summary

Starting with the work of von Neumann and Morgenstern (1944), a typical approach to the quantification of the risk/reward relationship in investment decision making has been to postulate the existence of a utility function. In this formulation, the existence of a utility function U(X) is guaranteed by the von Neumann-Morgenstern characterization theorem, such that rational investors select from a group of random prospects in order to maximize their expected utility. The utility function, which is typically defined over the domain of possible values for end of period wealth, regards each investment alternative as a random˝variable yielding an *a priori* end of period final utility U(X). In this context, the mathematical properties of the utility function U(X) are important for developing stochastic dominance results and are subsequently important for the valuation of the distributions associated with alternative investment opportunities (see Bawa (1982)).

According to Pratt (1964) and Arrow (1971), a reasonable restriction set for the class of investors' utility functions includes $U'(x) > 0$ (non-satiation), $U''(x) < 0$ (risk aversion or a decreasing marginal utility for wealth) and decreasing absolute risk aversion. In a number of articles (e.g., Arditti (1967), Booth and Smith (1987), Kraus and Litzenberger (1976) and Tsiang (1972)), a negative second derivative is interpreted as implying an aversion to variance. A positive third derivative is also frequently interpreted as being indicative of skewness preference; that is, preferring a higher third moment for two random variables having equal means and variances. Two examples of such interpretations are:

> Risk averting investors prefer positive skew over no skew and over negative skew in the distribution of returns or wealth. (Scott and Horvath (1980,˝p. 917)), and
>
> Aversion to standard deviation and preference for (positive) skewness are general characteristics of all investors having utility functions displaying the desirable behavioristic attributes of decreasing marginal utility of wealth and non-increasing absolute risk aversion. (Kraus and Litzenberger (1976, p.1087))

Researchers have even developed asset pricing models similar to the CAPM that incorporate this assumed skewness preference idea (see Jean (1971) and Kraus˝and Litzenberger (1976)).

Since uniform consensus of preference among investors implies moment ordering properties of the decision variables, one might easily believe that, conversely, investors also uniformly prefer certain moment orderings. This view often leads to such˝statements as:

> ... a greater skewness would increase the expected utility.˝(Tsiang (1972, p. 359)), and
>
> On theoretical grounds, non-increasing absolute risk aversion dictates skewness preference. (Beedles (1979, p. 72))

In this paper, we shall show that the relationship between utility preferences and stochastic properties of the moment sequence of the variables involved is much more complex than these statements suggest. Indeed, expected utility preferences never universally translate into moment preferences -- even for utility functions that exhibit non-increasing absolute or constant proportional risk aversion.

When authors attempt to provide proof that justifies the connection between utility preference and moment preference, the following heuristic approach is one of the routes taken[1]: write a Taylor series expansion for U(X), truncate the series˝after a few (say three) terms and then take expected values. If the first two moments agree for two choice alternatives, and if U'"(x) > 0, then expected utility ordering is equivalent to skewness˝ordering. Although the logical holes in this approach are well known, the magnitude of this problem in the context of commonly used models has not been previously described. In Section 2 we show that, for distributions and utilities commonly postulated in finance and insurance research, the truncated part that is thrown away can be entirely unrelated to the true value when using this truncated Taylor series approximation (and hence, can even be larger in magnitude than the part kept). The heuristic

[1] The other two routes involve assuming multivariate normality for returns and arguing that a von Neumann-Morgenstern utility maximizing "rational" decision maker will then choose according to means and variances or appealing to empirical results that have found the desired relationship between mean and variance. We shall subsequently show these two approaches are also insufficient.

proof for this approach is not only incorrect, but the result˝itself is also false, as we demonstrate in Section 3. In fact, in Section 3, after careful analysis, we present the following result: for the class of utility functions U(X) possessing the properties $U'(x) > 0$, $U''(x) < 0$, $U'''(x) > 0$ and $U^{(4)}(x) < 0$, two random variables X and Y always exist with identical means and variances $\mu$ and $\sigma^2$ and $E(X-\mu)^3 > E(Y-\mu)^3$ whose preference relationship is exactly reversed; i.e., $EU(Y) > EU(X)$. This class of utility functions contains the utility functions commonly used in economics and finance (see Brockett and Golden (1987)).

This heuristic approach to skewness preference is often justified by "*ceteris paribus*" conditions that imply equality among the higher order moments of X and Y and the first and second moments. In Section 4, we demonstrate that such conditions contradict the differing lower order moments, making the set of assumptions internally contradictory. We do this by showing that essentially <u>any</u> two random variables with equal higher order moments must, in fact, be <u>identical</u>. Section 5 then shows that the multivariate normality assumption for returns cannot justify the mean variance decision rule for any but the quadratic von Neumann-Morgenstern utility maximizer. In Section 6, we demonstrate that the "positivist" view, which asserts empirical verification of the use of mean-variance rules, is also logically flawed because such empirical examinations suffer from an econometric identification problem, thereby making it impossible to distinguish shifts along the risk/return curve from shifts in the risk/return relationship over time. Finally, Section 7 presents the conclusions of our analysis.

## II. Truncating the Taylor Series

When investigating the effect of variance or skewness preference on an investor's choice selection, one commonly used methodology is to write a Taylor series expansion for U(X), truncate the series after two or three terms and then proceed with the analysis as if the utility function were a polynomial utility (so that expected utility can be expressed in terms of moments alone). There are obvious drawbacks to using quadratic, cubic or more general polynomial utility functions. For example, the quadratic utility is not only limited in its range of applicability as a

utility, but also, within that range, the utility implies increasing absolute risk aversion.[2] Moreover, as shown by Ekern and Wilson (1974), all shareholders in this setting prefer a firm characterized by constant stochastic returns that operate so that their market value is zero. Additionally, Borch (1969) shows that, if preferences satisfy a monotonicity condition, then all indifference curves in the mean-variance plane reduce to a single point with the quadratic utility. For higher order polynomial utilities, Levy (1969) has shown that a cubic utility function does not exhibit decreasing marginal utility for all wealth levels. In fact, when the utility function has decreasing marginal utility for positive wealth levels near zero, it has increasing absolute risk aversion within that same range. According to Tsiang (1972), polynomial utilities cannot simultaneously satisfy the essential properties of $U' > 0$, $U'' < 0$ and non-increasing absolute and proportional risk aversion. At best, this truncation procedure must be considered to provide only an approximation.

Often a justification for eliminating the higher order moments from consideration is either not presented or is presented incorrectly. When a justification is presented, the logic behind why the remainder term can be thrown out as inconsequential in the expected utility analysis is sometimes logically flawed. For example, in Arditti (1967, p. 20), the remainder term is ignored "because the higher moments of r (the rate of return) add little, if any, information about the distribution's physical features." As noted by Levy (1969), this does not imply that the remainder term is small in magnitude and can be neglected. Similarly, when discussing the incorporation of higher moments, Kraus and Litzenberger (1976, p. 1087) claim˝that:

> ... there is no reason to do so from the perspective of a positive theory of valuation. Aversion to standard deviation and preference for (positive) skewness are general characteristics of all investors having utility functions displaying the desirable behavioristic attributes of decreasing marginal utility of wealth and non-increasing absolute risk aversion. Comparable *a priori* behavioristic arguments for general investor attitudes towards the fourth and higher moments have not been made."

[2] Increasing absolute risk aversion has been generally rejected as˝inappropriate by most economists.

The idea of ignoring higher terms because "comparable arguments have yet to be made" represents questionable logic that does not take into account the direction or importance of the unincorporated terms. Indeed, standard risk aversion and prudence concepts (see Kimball (1990, 1991)) provide such behavioral arguments. Conine and Tamarkin (1981, p. 1145) offer yet another example when they state "that the fourth and higher moments are irrelevant since their economic significance remains indeterminate."

Example 2.1 demonstrates that, even for a frequently used utility function and for a very popular distribution used in modeling, terms can be thrown away which are far larger in magnitude than those terms that have been kept. Moreover, the approximation obtained by using this truncation technique can be completely unrelated to the˝actual desired true utility value.

**Example 2.1**: Truncation of Taylor Series May Be Poor Approximation of˝Expected Utility

The logarithmic utility function $U(X) = \ln(X)$ is one of the most common utility functions whose use dates all the way back to the seminal work of Bernoulli in 1738. Similarly, the lognormal distribution is one of the most common stochastic models used in economic models. For logarithmic utility and the lognormal distribution, we show that the magnitude of the amount thrown away by truncating the Taylor series about the mean after three (or any number of) derivatives bears no relationship to the true expected utility;˝hence, the error of approximation can be arbitrarily large.[3]

Let X be a lognormally distributed random variable with logarithmic mean and variance $E[\ln(X)] = \eta$ and $E[(\ln(X) - \eta)^2] = \tau^2$, respectively.[4] Then the expected utility is $EU(X) = E[\ln(X)] = \eta$ and the mean ($\mu$), variance ($\sigma^2$) and skewness ($\rho$) measures for X are:

$$\mu = EX = \exp(\eta + \frac{\tau^2}{2}),$$

$$\sigma^2 = E(X-\mu)^2 = \exp(2\eta + \tau^2)[\exp(\tau^2) -1] \text{ and}$$

[3] This same pair of random choice distribution (lognormal) and decision maker preference function (logarithmic utility) was also discussed by Feldstein (1969) in a different context. He showed that for this choice pair, the mean-standard deviation indifference curves for a risk averter need not be convex downward; although upwardly sloping.

[4] The parameters $\eta$ and $\tau^2$ represent the location and scale parameters for the associated˝normal distribution for X.

$$\rho = E(X-\mu)^3 = \exp(3\eta + \frac{3\tau^2}{2})[\exp(3\tau^2)-3\exp(\tau^2)+2].$$

Consequently, the remainder term $\mathcal{R}$ after truncating the Taylor series expansion for U after three terms and taking expectations is:

$$\begin{aligned}\mathcal{R} &= EU(X) - U(\mu) - \frac{U''(\mu)}{2}\sigma^2 - \frac{U'''(\mu)}{6}\rho \\ &= \eta - \ln(\mu) + \frac{\sigma^2}{2\mu^2} - \frac{\rho}{3\mu^3} \\ &= -\frac{\tau^2}{2} + \frac{\exp(\tau^2) - 1}{2} - \frac{\exp(3\tau^2)-3\exp(\tau^2)+2}{3}.\end{aligned}$$

The preceding equation is solely a function of the scale˝parameter $\tau$.

Note that the true expected utility $EU(X) = \eta$ has no relationship to the remainder term $\mathcal{R}$, which is strictly a function of $\tau$. By appropriate choice of $\tau$, the absolute or relative errors of approximation (that is, $\mathcal{R}$ or $\mathcal{R}$/EU) can be made as large or as small as desired. Thus, the truncation of the Taylor series may, in fact, yield a very poor approximation of the true expected utility.[5,6] Indeed, by truncating and discarding the higher order terms as inconsequential, one could be inadvertently throwing out the baby with the˝bathwater!

## III. U'' < 0 Is Not Aversion to Variance, and U'''(x) > 0 Does Not Imply Skewness Preference[7]

A common approach to providing behavioral arguments concerning investors' attitudes towards moments is to view derivatives of U(X) as implying moment preference. That is to say, the assertion is often made that the risk averse investor will prefer the smaller variance when selecting between alternative investments having equal means. Similarly, it is widely believed that

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[5] This is actually true no matter how many terms are taken in the Taylor expansion. To see this, write $X = \exp(\tau Z + \eta)$ where Z is standard normal. The $n^{th}$ term in the Taylor series involves $E[\exp(\tau Z + \eta) - \exp(\tau + \eta^2/2)]^n/\mu^n = E[\exp(\tau Z) - \exp(\eta^2/2)]^n/\exp(n\eta^2/2)$, which is solely a function of $\tau$ and is unrelated to the true expected utility $\eta$.

[6] For further discussion of this point, see Loistl (1976). Loistl not only provides a number of examples of expected utility approximations obtained by truncating a Taylor expansion, but also shows that the approximation actually worsens as the number of terms in the Taylor series increases.

[7] The results discussed in this section are drawn from the theoretical results presented in Brockett and Kahane (1991) and Brockett and Cox (1985). More detailed discussions and examples of these results may be found in the Brockett and Kahane article.

risk averse investors prefer greater positive skewness when choosing between alternatives having identical means and variances. Examples of these beliefs abound in the literature, as indicated by the following assertions:

> Thus, as a preference for positive skewness can be shown for those with power utility functions, .... Grauer (1985, p. 1392) and
>
> Aversion to standard deviation and preference for (positive) skewness are general characteristics of all investors having utility functions displaying the desirable behavioristic attributes of decreasing marginal utility of wealth and non-increasing absolute risk aversion. Kraus and Litzenberger (1976, p. 1087).

In the following paragraphs, we present simple examples that illustrate the incorrectness of such assertions.[8] Example 3.1 is presented to show that risk aversion does not imply aversion to variance.

**Example 3.1:** Risk Aversion Does Not Imply Aversion to Variance

We shall show that regardless of the utility function chosen, there are always situations in which the risk averse investor prefers higher variance when choosing between two assets having equal positive means and equal skewness measures. Illustrative of this fact are the random variables X and Y defined by their piecewise constant density˝functions:

$$f_X(x) = \begin{cases} 0 & x < 0 \\ .005924433 & 0 \le x \le 3.7304 \\ .07827427 & 3.7304 \le x \le 10 \\ .0690477 & 10 \le x \le 17.05483 \\ 0 & x \ge 17.05483 \end{cases}$$

and

[8] The results we provide can be extended to include any given number of moments. In particular, decreasing absolute prudence (cf. Kimball (1990, 1991)) is still not˝sufficient to justify these assertions.

$$f_Y(y)=\begin{cases} 0 & y < 2.9345 \\ .070341938 & 2.9345 \le y \le 10 \\ .07241876 & 10 \le y \le 16.86646 \\ .001842 & 16.86646 \le y \le 20 \\ 0 & y \ge 20 \end{cases}.$$

Both X and Y are continuous unimodal distributions supported over the interval [0, 20]. They were explicitly constructed using the theory of Tchebychev functions (see Brockett and Cox (1985) and Karlin and Studden (1966)) to have a common mean and mode of 10, a skewness of zero and to have equal fourth order central moments of 500.[9] Since the variance of X is 16 while the variance of Y is 16.5, the mean-variance (and also the mean-variance-skewness) decision rule would indicate that X would be considered less risky and˝therefore be strictly preferred to Y.

However, consider an expected utility ordering of these two alternatives. Using the common constant absolute risk aversion utility function $U(X) = 1000(.01 - e^{(-X)})$, we find that $EU(X) = 2.3407 < EU(Y) = 6.2608$, thereby indicating that the variable that is regarded as more risky and therefore less desired according to the mean-variance (as well as mean-variance-skewness) decision rule is actually preferred from an expected˝utility standpoint.[10]

The preceding result did not occur because of overpowering skewness or fourth order moments of Y over X. Rather, the example illustrates the inappropriateness of the common intuition that risk averse investors will seek to maximize the mean for fixed variance or minimize the variance for fixed mean. Even simpler discrete counter-examples can be given, as demonstrated next.[11]

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[9] The actual numbers selected for this illustrative example (mean of 10, skewness of zero and fourth central moment of 500) were arbitrarily selected subject to being consistent with a sequence of moments. Other choices of mean, skewness and kurtosis could also have been used and an˝example constructed using these.

[10] In terms of the certainty equivalents, X has a certainty equivalent of 4.872 while Y has a certainty equivalent of 5.589; thus an investor who works only with certainty˝equivalents and assumes that preferences can be summarized in terms of four moments can also make substantial errors, even for the constant absolute risk aversion utility function.

[11] These examples were also constructed using the general theory of Tchebychev systems of functions (see Karlen and Studden (1966)).

**Example 3.2:** Variance is Not an Absolute Indicator of Preference

Consider the discrete variables $X_1$ and $X_2$ defined by:

$$X_1 = \begin{cases} 3.5858 & \text{with probability .5} \\ 6.4142 & \text{with probability .5} \end{cases} \qquad X_2 = \begin{cases} 0 & \text{with probability .02} \\ 5 & \text{with probability .96} \\ 10 & \text{with probability .02} \end{cases}$$

Both $X_1$ and $X_2$ are symmetrically distributed with mean 5 and all odd order central moments of order 3 or higher set equal to zero.[12] The variance of $X_1$ is 2 while the variance of $X_2$ is 1. The mean variance decision rule indicates that $X_2$ should be considered as only half as risky as $X_1$ and, because it has the same mean, should be preferred to $X_1$ using any common utility functions such as $U_1(X) = 1 - \exp\{-X\}$, $U_2(X) = \ln[1 + 2X]$ or $U_3(X) = X^{1/2}$. However, from an expected utility perspective, one can easily calculate that the reverse is true for each of these utility functions[13]:

| Variable | $EU_1(X)$ | $EU_2(X)$ | $EU_3(X)$ |
|---|---|---|---|
| $X_1$ | .98532 | 2.3637 | 1.20680 |
| $X_2$ | .97353 | 2.3629 | .98322 |

The preceding examples and others similar to them are relevant to our discussion since the intuition leading to the false conclusion that a lower variance is preferred by "rational risk averse investors" continues to be used to ostensibly justify "rational" investors' price reactions to changes in the variance (e.g., uncertain information and stock price “over-reaction” as explained by Brown, Harlow and Tiniç (1988)). Indeed, by using Tchebychev system theory[14], it is possible to construct examples involving multiple investors, all having decreasing absolute risk aversion, in which any finite number of "rational risk averse investors" will all simultaneously and uniformly prefer the larger variance alternative. In this very general model of a rational market, it is clear

[12] Note that this is similar to the properties enjoyed by the˝normal distribution.

[13] Since utility functions are only relevant up to transformations, the exact numerical differences between the utilities are irrelevant. These differences could be made arbitrarily large by taking a version $U^* = A + BU$ appropriate for the A and B. Moreover, as in Bernoulli, even the certainty equivalents can differ substantially in the opposite direction from that which a belief in moment˝preferences would imply.

[14] See Karlin and Studden (1966). Tchebychev system theory was˝used in the construction of examples 3.1 and 3.2 above.

that it is not a consequence of rational risk averse behavior for prices to necessarily fall with increases in uncertainty (variance), *ceteris paribus*. That is to say, if there are five billion risk averse investors in the world, all of whom will pay a higher price in order to purchase an asset that yields higher expected utility, then it is possible to construct two alternative asset distributions X and Y such that X has higher variance and equal mean compared to Y, and yet all investors derive greater utility from X. Accordingly, the market value of asset X will exceed that of Y, even though X has higher variance.

The next example (Example 3.3) demonstrates that even if an investor's utility function possesses all of the desirable properties commonly assumed to lead to parsimonious choice (e.g., risk aversion, decreasing absolute risk aversion, etc.), situations will still exist in which the investor will prefer a random prospect possessing simultaneously a lower mean, a higher variance and a lower skewness. The set of utility functions for which this result holds includes all the common utility functions such as the exponential, mixtures of exponential, logarithmic, power, etc. (see Brockett and Golden (1987)). Similar examples can be constructed for any of these common utility functions, showing that the counterexample is not due to any anomalous utility behavior, but is in fact, a property of all utility functions typically used in rational choice models (except of course, for the largely discredited quadratic˝function).

**Example 3.3**: Mean, Variance and Skewness Cannot Summarize Preference

Consider the continuous, unimodal positive random variables X and Y defined by their piecewise constant density functions, where X has the density function $f_X$ given in Example 3.1, and Y has the density function:

$$f_Y(y)=\begin{cases} 0 & y < 2.42 \\ .070 & 2.42 \le y \le 10 \\ .073 & 10 \le y \le 16.35 \\ .001 & 16.35 \le y \le 20 \\ 0 & y \ge 20 \end{cases}$$

These distributions were explicitly constructed using the theory of Tchebychev functions (see Brockett and Cox (1985) to possess the following moments:

| Variable | Mean | Mode | Variance | Skew | Fourth Central Moment |
|---|---|---|---|---|---|
| X | 10.00 | 10 | 16.00 | 0.00 | 500.00 |
| Y | 9.47 | 10 | 16.55 | -1.32 | 504.58 |

Using the version of the common CARA utility function U(X) = 1000(.1 - exp{-X}) from Example 3.1 for illustrative purposes, we find EU(X) = 2.3407 < 3.515 = EU(Y). Thus, despite the fact that this utility function displays decreasing marginal utility of wealth and non-increasing absolute risk aversion, and despite the fact that X has higher expected value, lower variance, higher positive skew, and lower fourth central moment than Y,˝the investor prefers Y over X.[15]

## IV. The Impossibility of "*Ceteris paribus*" Moment Preference Analysis

In an attempt to isolate the effect of variance (or skewness) on investors' preferences, authors often assume not only that the first moments of the variables X and Y are as proscribed, but in addition, by appending the words "*ceteris paribus*," that the fourth and higher order moments agree as well (e.g., they might state "$\sigma^2(X) < \sigma^2(Y)$" or "$\rho(X) > \rho(Y)$," *ceteris paribus*). For example, in explaining the implications of their results, Scott and Horvath (1980, p. 917) claim that

> " ... the investor would be willing to accept a lower˝expected value from his investment of **w** in portfolio A, than in portfolio B if both portfolios have the˝same variance, $\mu_2^A = \mu_2^B$, portfolio A has greater positive skewness $\mu_3^A > \mu_3^B$ and all higher moments are the same."

Similarly, Tsiang (1972, p. 359) states

> "... a positive skewness of the distribution is a desirable feature, and all other things being equal, a greater skewness would increase the expected utility."

[15] Although not presented in this paper, extensions to more moments are possible using the theory of Tchebyshev systems of functions as outlined in Brockett and Cox (1985).

In this section, we demonstrate that constraints on the moment relations creates an internal contradiction (i.e., the set of variables to which this analysis applies is the empty set). This is accomplished by showing that if the means and certain higher order moments are equal, then the two random variables are <u>identical</u> in distribution (which implies that they could not possibly satisfy the condition $\mu_3 > \mu_3$ or have unequal expected utilities to begin with).

We state our results for bounded random variables for simplicity and also because, in fact, the distribution of investment returns do have <u>some</u> upper bound (it is not necessary or important that the exact bounds be known). The important fact of this presentation is that the distributions are uniquely determined by their moments (as is always the case with bounded random variables) rather than by the actual bounds on the variables themselves.

The key to the proof of our theorem is the following Lemma that can be found in Feller (1968, Lemma 4).

<u>Lemma</u> If $\nu$ is a measure on $[0, \infty)$, then the Laplace transform of $\nu$:

$$f(t) = \int_0^\infty e^{-tx} \nu dx ,$$

is uniquely determined by its values $\{f(t\,j), j = 1, 2, ...\}$, provided that only $\sum t_j^{-1} = \infty$.

Using this lemma we may prove Theorem 1.

**<u>Theorem 1</u>** Suppose that X and Y are two bounded variables with equal means and with central moments that are equal above some arbitrarily prespecified integer $k_0 \geq 2$. Then X and Y have identical distributions.

**<u>Proof</u>** Suppose for notational purposes that $a \leq X \leq b$ and $a \leq Y \leq b$, and that all moments of X and Y of order higher than some $k_0 \geq 2$ are equal. Let N denote the first odd integer past $k_0$, and define the random variables:

$$Z = \frac{(X - \mu)^N - (a - \mu)^N}{(b - \mu)^N - (a - \mu)^N}, \text{ and}$$

$$W = \frac{(Y - \mu)^N - (a - \mu)^N}{(b - \mu)^N - (a - \mu)^N}.$$

Since Z and W are bounded between 0 and 1, we can write $Z = \exp\{-V_1\}$ and $W = \exp\{-V_2\}$ for random variables $V_1$ and $V_2$. Accordingly, the $t^{th}$ moment of Z and W correspond to the Laplace transform of $V_1$ and $V_2$ , respectively, evaluated at the point t. Thus, since $EZ^j = EW^j$, we have $f_{V_1}(j) = f_{V_2}(j)$, j = 1, 2 ..., where f denotes the Laplace transform. Because $\sum t_j^{-1} = \infty$, the entire Laplace transforms of $V_1$ and $V_2$ must be equal according to the Lemma. However, since the Laplace transformation exists and uniquely determines the distribution of a bounded random variable, the distributions of $V_1$ and $V_2$ must also be equal. Accordingly, the same must also be true for $(X - \mu)^N$ and $(Y - \mu)^N$. When we take the $N^{th}$ roots (which are unique since N was chosen to be an odd integer), we see that X and Y must also have identical distributions. This completes the proof.

Actually, although bounded random variables were assumed in˝Theorem 1, the results of our analysis generally apply to random variables for which the moment-generating function exists. A proof similar to that previously given can be adapted to this situation as well.[16] Thus, one cannot isolate variance or skewness preference for decision makers by˝the ordering of moments.

## V. Normality of Return Distributions Does Not Imply˝Mean-Variance Preference Ordering Except For the Case of Quadratic Utility

In discussions concerning preference orderings, it is commonly asserted that a rational von Neumann-Morgenstern utility maximizer with utility function U will rank prospects in accordance with a mean-variance ordering, provided that asset returns are multivariate normally distributed. However, as noted by Baron (1977), this assertion is incorrect except in the case of the quadratic utility function. The reason for this fallacy is due to the von Neumann-Morgenstern axiomatic

[16] Of course, if the random variables are not uniquely determined by their moments, (as happens with the lognormal distribution), then no results concerning moment˝preference and expected utility can be expected to hold since we can construct two variables with all identical moments but with different expected utilities.

description of rationality, which states that the investor must˝be able to rank not only all individual prospects and portfolios, but also lotteries between prospects. While the individual and portfolio distributions are normal (and hence, may be summarized in terms of means and variances), lotteries of normal variates have a mixture of normal and non-normal distributions.[17] Thus, these lotteries can have decidedly non-normal distributions. To illustrate this fact, Figure 1 exhibits the plot of the density function of a simple lottery of two univariate normal distributions. Baron correctly points out that these lotteries of normal variates constitute a mixed normal distribution that cannot generally be described in terms of the means and the variances of the constituent normal variates. Further, he proves that all prospects and lotteries of prospects are ranked consistently with the mean-variance rule only for the quadratic˝utility function.

[Figure 1 about here]

In summary, one cannot simply supplement the von Neumann-Morgenstern axioms regarding rational decision-making with normality assumptions in˝order to assert that risk aversion implies aversion of variance, or that $U'''(x) > 0$ implies skewness preference. If rationality is desired (in the sense of von Neumann and Morgenstern, implying consistent preference among lotteries of variables), then the mean variance rule must either be discarded, or the very inappropriate quadratic utility function must be used. This analysis also follows from the results of Thorp and Whitley (1972), who prove that utility functions are uniquely determined by their optimal strategies. If the strategy for a rational investor is consistent with mean variance ordering (as results from the quadratic utility), then the utility must, in fact, be quadratic. Otherwise, choice situations will exist in which the optimal strategy is other than a mean variance rule. Thus, even normality assumptions (which are at odds with empirical data) do not justify the application of mean variance analysis for rational decision making.

[17] Obviously, a sum of normal variates is itself normally distributed. However, the rationality axioms of von Neumann and Morgenstern require a consistency of preferences among lotteries that involves mixtures of normal variates and says nothing regarding sums of normal variates.

## VI. A Reexamination Of Empirical Results Concerning Risk and˝Return Relations

As a final argument circumscribing the development of relationships between risk and return (and mean variance analysis in particular), researchers often˝refer to empirical studies that claim to have found a positive (or in some cases a negative) relationship between risk and return. Attempts have been made to empirically estimate the relative importance of the various higher order moments by using a regression of the mean rate of return on investments with the sample estimates of moments of higher order.

Empirical studies by Arditti (1967), Levy and Sarnat (1972), Kraus and Litzenberger (1976) and others found the coefficient for the second moment to be positive and statistically significant. This finding was interpreted as indicating that a higher return˝tends to go together with a prospect that has a higher variance; i.e., a higher variance seems to be˝an undesirable property that must be compensated for by having a higher mean. In these studies, the regression coefficient for the third moment (skewness) was negative and analysts interpreted this coefficient as evidence for investor preference for a positive asymmetry. The coefficient for the fourth moment (kurtosis) was significant only in a few cases, and the coefficients for higher moments were always insignificant. Considering the results presented in Sections 2 and 4, it is unclear how one should interpret these empirical results since the logic of regressing the sample higher order moments on the sample mean is fraught with theoretical difficulties and internal contradictions. If the return distributions were normally distributed, then independence of the sample mean and variance is a consequence, and, in fact, is a characterizing property of the normal distribution (see Mathai and Pederzoli (1977)) so positive coefficients can only arise as a sampling artifact in this situation. On the other hand, if the distribution is not normally distributed, then the sample mean and variance are necessarily dependent; thus, correlation may be not so much a statement indicative of a relationship between risk and return as much as a statement concerning the lack of normality of the distributions. Moreover, from Section 3, it is clear that the totality of all moments is globally dependent (since one moment cannot be adjusted without simultaneously changing the other

moments in one direction or another), while truncating the Taylor series for use in a regression equation can lead to the difficulties pointed out in Section 2.

Ruefli (1990) has shown that the direct empirical use of the mean and variance of the same variable as data in the fitting of a model leads to a version of the econometric identification problem in which movements along a fixed risk-return curve cannot be distinguished from shifts in the underlying curve modeling the relationship. It is, in fact, possible that the risk-return relationship could be negative in each subinterval but positive over the larger subinterval considered. For example, suppose that during period i with observed data $\mathbf{x}_i$ the relationship between risk $R_i(\mathbf{x}_i)$ and return $Re_i(\mathbf{x}_i)$ is modeled as a function $f_i$ (which may even depend on certain (possibly random) parameters and variables $\theta_i$); i.e., $R_i(\mathbf{x}_i) = f_i[Re_i(\mathbf{x}_i), \theta_i]$. (It matters not how risk and return are quantified here). Ruefli's work highlights the fact that it is impossible to tell whether the relationship $f_i$ is constant over i and an observed positive (or negative) relationship between risk $R_i(\mathbf{x}_i)$ and return $Re_i(\mathbf{x}_i)$ is indicative of a true positive (or negative) time invariant relationship f. Instead, it may be the case that each individual relationship $f_i$ is exactly the opposite to that which was postulated to be true, namely negative (respectively positive), but the relationship $f_i$ itself changes with i in such a manner that the global empirical impression is the opposite of the actual relationship for each i. We might have a truly negative relationship such as $Re_i(\mathbf{x}_i) = \alpha_i - \gamma R_i(\mathbf{x}_i) + \varepsilon$ with $\gamma > 0$ over every subinterval, however, if the intercept term $\alpha_i$ increases with i sufficiently rapidly, then a more simple regression such as $Re = \alpha + \gamma R + \varepsilon$ of Re on R would lead to a positive estimate for $\gamma$, indicating exactly the opposite of the true direction for the relationship.

An analogous situation in statistics that uses probabilities instead of rates or expectations is known as the Simpson's paradox. This paradox exhibits that it is possible for all the conditional probabilities to satisfy $P[A|B] > P[C|B]$ for all i, and yet the global (unconditional) relationship may be exactly the reverse: $P[A] < P[C]$ (see Freedman, Pisani˝and Purves (1978)). This situation is not merely a theoretical possibility since the strategic management literature contains many studies in which the risk return relationship has been shown to be negative (the so-called

Bowman Paradox - see Bowman (1980) and Ruefli (1990) for references, discussion and a multitude of examples). The point we wish to make here is that often an empirically derived relationship can shed no light on the actual theoretical risk return relationship. Although the relationship may be positive or negative, the empirical analysis cannot logically identify one versus the other as representing truth.

## VII. Conclusions

This paper has shown that a flaw exists in the logic behind the previously stated theoretical connections between utility theory and moment preferences. In fact, no such relationship exists. There is also a flaw in the logic that postulates that approximate normality can justify moment (e.g., mean variance) methods for investors' preferences (it cannot), or even that exact normality can justify the use of mean variance analysis consistent with the von Neumann-Morgenstern rationality axioms except for the quadratic utility. A flaw also exists in attempts to analyze the effect of moment changes "*ceteris paribus*" since "all else the same" is "all the same." Finally, arguments that claim that empirical evidence supports the use of moment approximations to investors' preferences are logically questionable. In our discussion, we have questioned the logical underpinning of the empirics and provided references to˝numerous empirical illustrations in which the exact opposite relationship has been found.

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